1 Basic Concepts in Vibrations

Vibrations, within the broader field of classical mechanics, is the investigation of oscillations that occur about an equilibrium point. Vibrations, both desired and undesired, are present in all mechanical systems and can be helpful (e.g. a soil sieve, rotary sander) or destructive (e.g. an aircraft frame in resonance). The oscillations that form a vibrating system may be periodic (e.g., pendulum) or random (e.g. turbulence in an airplane), or a combination of the two.

Vibrations impact our daily lives in a variety of ways, from the sound made by banjo strings that vibrate between 140 and 400 Hz to the vibrations felt by a passenger in a car seat that are typically under 6 Hz.

The consideration of vibrations, and their associated mathematical modeling, are important factors in the design of mechanical systems. In this text, the fundamental theories of vibration are presented and modeled using basic physical principles such as Newton's three laws of motion. These models and analyzed using the mathematical tools of calculus and differential equations.

Vibration Case Study 1.1 TSR-2 and the Resonance of the Human Eye

Why study vibrations? One day, it could save your life! The British Aircraft Corporation (BAC) TSR-2 was a strike and reconnaissance aircraft developed during the Cold War by BAC, for the Royal Air Force (RAF).

During the second flight test of air-frame XR219, vibration from one of the plane's fuel pumps caused vibration at the resonant frequency of the human eyeball. As you may expect, a human eye experiencing high levels of vibrations will distort, causing blindness. Test pilot Roland Beamont was blinded by the vibrations that originated in the fuel pump and transmitted to his head. Roland just happened to be an expert in vibrations and had the knowledge to throttle back one engine. This led to a reduction in the vibrations and a restoration of his full vision.

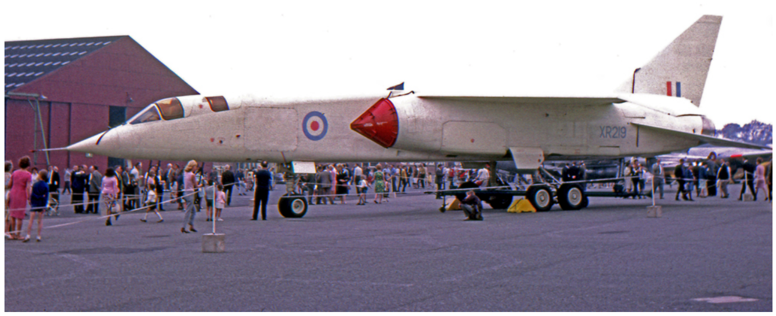


Figure 1.1: The only BAC TSR-2 prototype to fly, picture taken in 1966 at what is now BAE Warton Lancashire.[a]

Roland gained his expertise in vibrations during WW II. During this time he led the vibration program of the Hawker Typhoon. He fit vibrographs to airplanes to determine the effectiveness of propeller balancing. He also led the testing of seats with vibration isolators to limit vibrations transmitted from the airframe into the pilot.

Review 1.1 Newton's Laws of Motion
Newton's three laws of motion:

1. In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity unless acted upon by a force.
2. In an inertial reference frame, the vector sum of the forces F on an object is equal to the mass m of that object multiplied by the acceleration of the object: $F = ma$. (It is assumed here that the mass m is constant)
3. When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

1.1 Single Degree-of-Freedom Systems

In its simplest form, the phenomenon of vibration is the exchange of energy between potential and kinetic energy. Therefore, a vibrating system must have a component that stores potential energy. This component must also be capable of releasing the energy as kinetic energy. This kinetic energy is stored in the movement of a mass where the measure of this movement is the velocity of the system. The continuous interchange between potential and kinetic energy is the vibration of the system. The simplest vibrating systems can be modeled as a single-degree-of-freedom (1-DOF) system. In a 1-DOF system, one variable can describe the motion of a system. Potential examples of 1-DOF systems include:

- yo-yo
- pogo stick
- door swinging on axis
- throttle (gas pedal)

Variables often used for describing 1-DOF systems are $x(t)$, $y(t)$, $z(t)$, and $\theta(t)$. Examples of 1-DOF systems are presented in figure 1.2 where the assumption of small displacements is made.

NOTE
We will often drop the "(t)" for simplicity in this text, such that x, y, z, and θ become the notation for the variables of interests

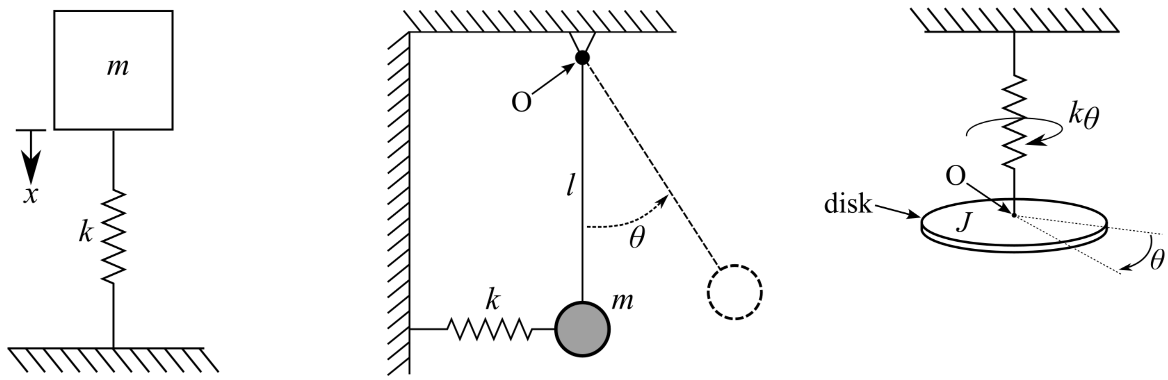


Figure 1.2: Examples of single degree of freedom (DOF) systems showing: (a) a vertical spring-mass system; (b) a simple pendulum; and (c) a rotational spring-mass system.

Review 1.2 Assumption of Small Displacements
The assumption of small displacements states that any displacement in a system is considerably smaller the the initial geometry of the system. This means that any 2$^{\text{nd}}$-order effects caused by displacements within the system are ignored. These 2$^{\text{nd}}$-order effects could be loads or angles at the point of linkage/spring connections.

1.1.1 Spring-Mass Model

Newtonian physics describes the motion of particles in terms of displacement x, velocity $\dot{x}$, and acceleration $\ddot{x}$ vectors. Moreover, Newton's second law of motion says that the change in the velocity of mass in motion is a product of the force acting on the mass. A simple way to express this phenomenon is through a spring-mass model as presented in figure 1.3. These spring-mass models neglect the mass of the spring and concentrate all the mass of the system into a single point. Note that in this case the force vector and mass-acceleration vectors lie on the same axis and as such are collinear. Therefore, these vectors can be easily treated as scalers simplifying the math used in the modeling of the system.

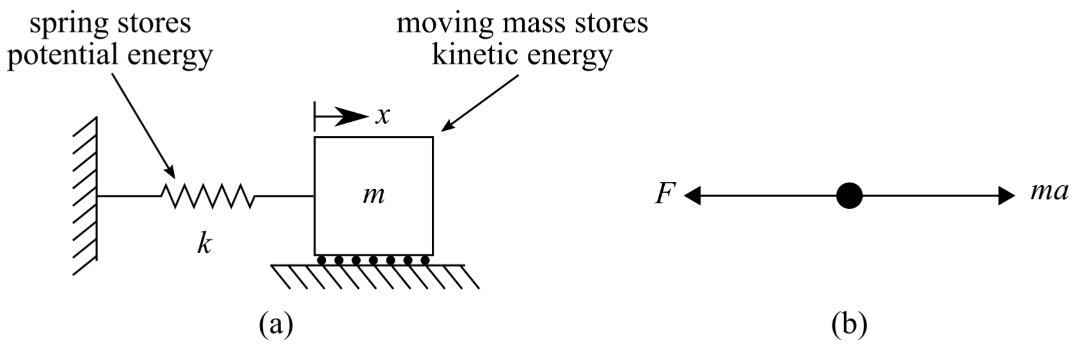


Figure 1.3: A single-degree-of-freedom (1-DOF) spring-mass model showing: (a) annotated schematic of a mass-spring system; and (b) the equivalent free-body diagram represented as a point-mass system.

1.1.2 Linear Springs

Springs are mechanical devices that store energy, moreover, an ideal spring is a theoretical representation of this mechanical device that is massless and responds with a linear increase in force for a unit increase in displacement (i.e. $F = kx$). For simplicity, the springs in the spring-mass models considered in this text are always assumed to be ideal linear springs. A graphical representation of the idealized linear spring is presented in figure 1.4 where a unit force F applied to the free end of the spring results in a unit displacement x of the spring. The resulting mathematical relationship, $F = kx$, is known as Hooke's Law. Nonlinear springs add considerable complexity to the modeling of spring-mass systems, therefore, these are not considered in this introductory work.

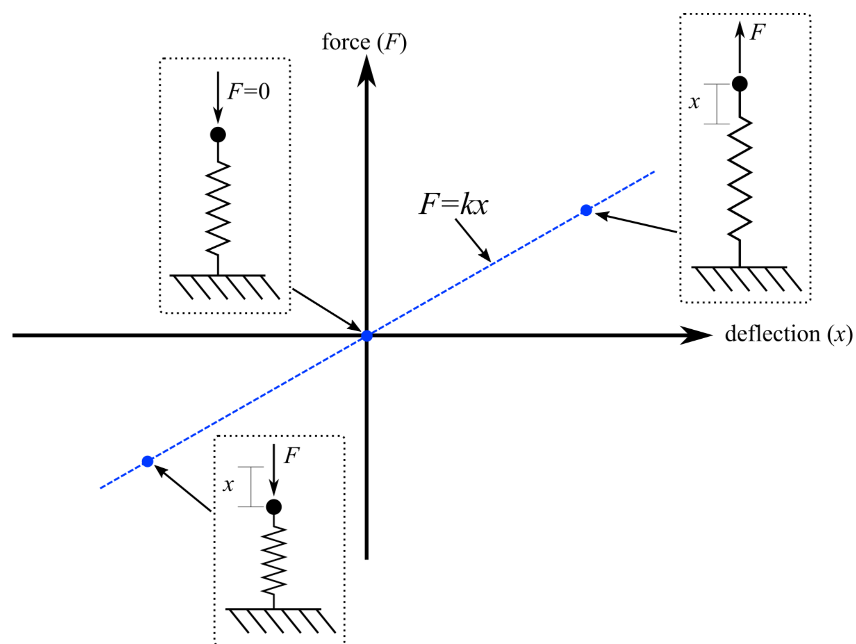


Figure 1.4: Force-displacement plot for a linear spring.

1.1.3 Linear Point-mass Models

Combining linear springs and point masses we get *linear point-mass models*; to which we will add dampers in Chapter 2. An important thing to consider is that the linear point-mass models used throughout this text are only a representation of real-world systems. Moreover, this representation removes any concept of non-linearity that is always present in physical systems. While these models are a gross under-representation of how a system would oscillate in the real world; they can capture enough of the system's dynamics to be incredibly useful in engineering and design. leading to the famous quote:

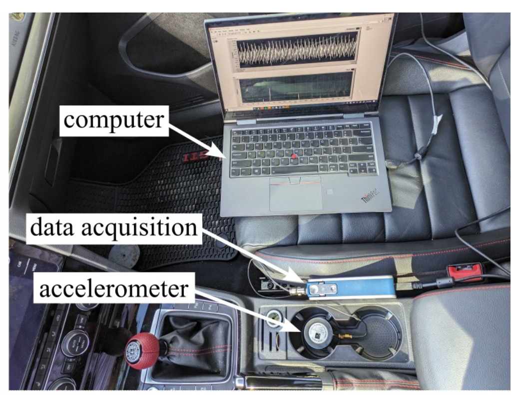
> "All models are wrong, but some are useful"
>
> George E.P. Box (1919 - 2013)

Vibration Case Study 1.2 Adjustable Vehicle Suspension

Why study vibrations? Because vibrations form an integral part of how we interact with our world and as such, are an important consideration in product. For example, vibrations in the automotive industry fall within a field of expertise termed Noise Vibration and Harshness (NVH). NVH is important because, within a single company, different levels of NVH will be desired for different market segments and products.

With a proper understanding of NVH, engineers can design cars that can adapt to their environment or desired use case. Consider the 2019 VW Golf GTI shown in figure 1.5(a) equipped with a dynamic suspension system where the driver can select between 'comfort' 'normal' and 'sport' suspension options. To investigate the effect of these suspension settings, an engineer can install an accelerometer (a sensor used for measuring acceleration) as shown in figure 1.5(b). An important consideration in measuring acceleration is where and how to mount the accelerometer. Here, the accelerometer is mounted in the cup holder to measure the vertical acceleration in the center of the car.

(a)



(b)

Figure 1.5: VW Golf GTI with three suspension modes, showing: (a) the car, and; (b) the accelerometer and data acquisition system used for measuring vibrations.

Figure 1.6 shows the measured acceleration in both the time and frequency domains for the three suspension modes during 5 minutes of interstate driving. Note that in the time domain, the responses of the three suspension modes are indistinguishable. However, in the frequency domain, the sport mode is shown to have greater vibrational energy. Later in this text we will delve into the technical aspects of power spectral density, for now, consider the area under the curve to be representative of the measured energy for each suspension setting.

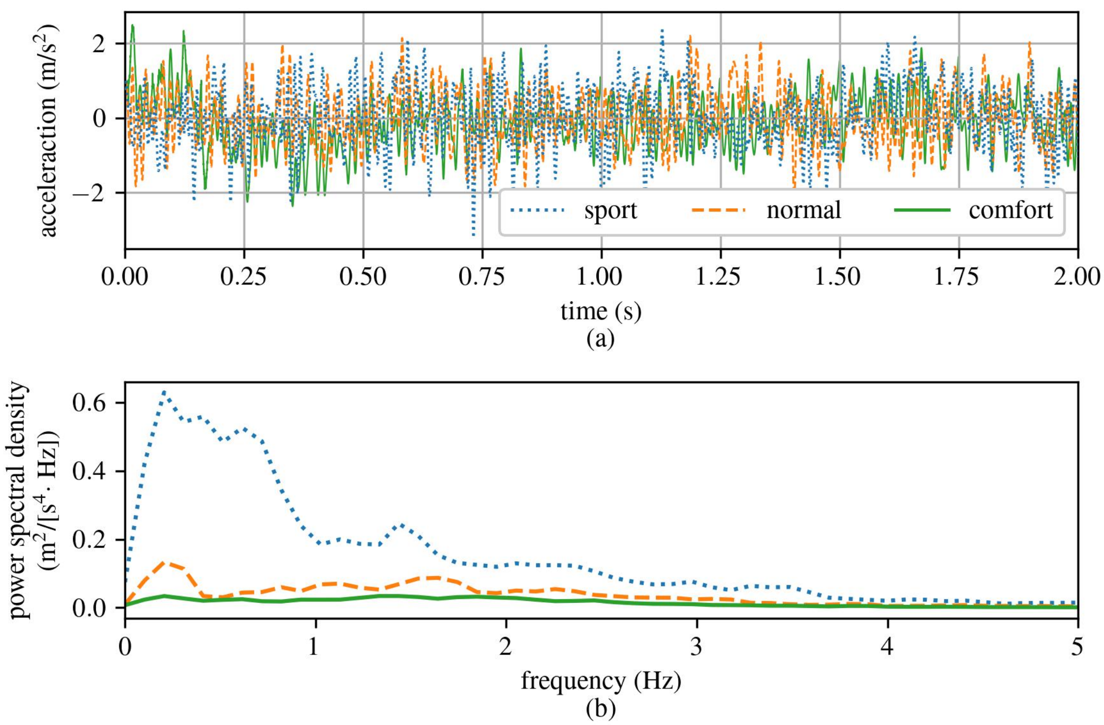


Figure 1.6: Response measured using the experimental setup shown in figure 1.5(b), showing (a) time-series data, and b) spectrum in the frequency domain.

The sport mode is by far the suspension mode with the firmest ride and the highest amount of measured vibration energy. While a stiff ride is beneficial during spirited driving on a track, the associated NVH level is tiring during prolonged driving. However, the comfort mode adds a considerable amount of damping to the suspension, resulting in a ride quality that is much more amenable to everyday driving. An engineer, using their knowledge of vibrations, could develop systems that enable a single product (such as an automobile or an airplane) to function well in multiple use cases; thereby increasing its usefulness and marketability.

1.2 Equivalent Stiffness

The generalized concept of stiffness can be directly related to mechanical systems and structural components through Hooke's law.

Review 1.3 Hooke's Law
Hooke's Law states that the force (F) needed to extend or compress a spring by some distance x scales linearly with respect to that distance. This law can be extended to the tensional stress of a uniform and elastic bar where the length, area, and Young's modulus of the bar are represented by l, A, and E, respectively. Knowing the tensile stress in the bar:

$$\sigma = \frac{F}{A} \tag{1.1}$$

and the definition of strain:

$$\varepsilon = \frac{\Delta l}{l} \tag{1.2}$$

Hooke's law can be expanded to represent a uniform and elastic bar:

$$\sigma = E\varepsilon \tag{1.3}$$

It follows that the change in length Δl can be expressed as:

$$\Delta l = \varepsilon l = \frac{Fl}{AE} \tag{1.4}$$

Hooke's law is often expressed using the convention that F is the restoring force exerted by the spring on the applied force at the free end. Defining the stiffness and displacement as $k = \frac{AE}{l}$ and $\Delta l = x$, respectively. The equation for Hooke's Law becomes:

$$F = -kx \tag{1.5}$$

since the direction of the restoring force is opposite the spring displacement.

1.2.1 Equivalent Stiffness of Structural Systems

For a rod with a uniform cross-section, a direct representation of the system can be developed as expressed in figure 1.7 where the vibration along the axis of the rod is to be considered. The stiffness of the rod, k, is a measure of the resistance offered by an elastic body to deformation.

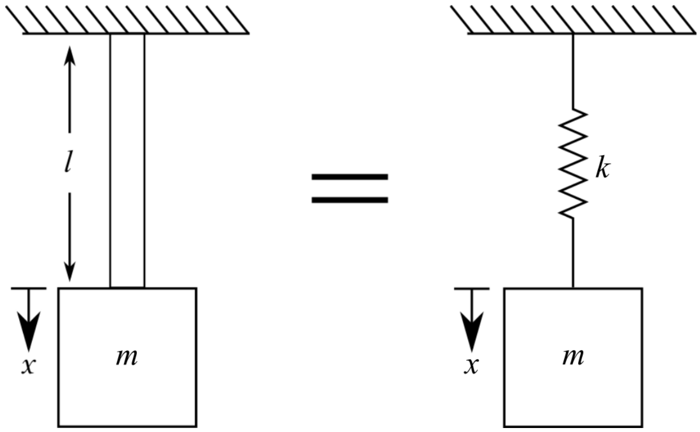


Figure 1.7: Equivalency between a vertical bar with a mass attached to the bottom and a spring-mass model of the system.

For this 1-DOF system, the equation of a spring can be rearranged such that the stiffness can be defined as:

$$k = \frac{F}{x} \tag{1.6}$$

The stiffness of the spring can be more closely related to material properties of the bar A, E, and l considering that Hooke's Law for the uniform tension on a bar can be expressed as:

$$\sigma = E\varepsilon \tag{1.7}$$

This expression can be expanded into the form:

$$\frac{F}{A} = E\left(\frac{x}{l}\right) \tag{1.8}$$

rearranging the terms and recalling the expression $k = \frac{F}{x}$ leads to:

$$k = \frac{EA}{l} \tag{1.9}$$

In a similar fashion, we can also solve the equivalent system for a mass at the end of a cantilever beam.

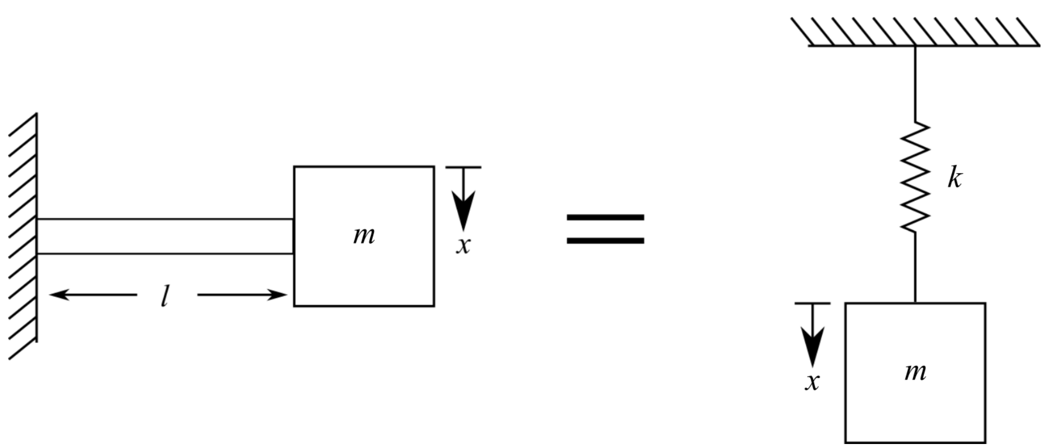


Figure 1.8: Equivalency between a cantilever beam and a spring mass system.

From engineering mechanics, we can compute the deflection at the point of a beam δ with a point load P. This expression is typically expressed as:

$$\delta = \frac{Pl^3}{3EI} \tag{1.10}$$

If we transform this equation into our variable system by exchanging P for F and δ for x. Thereafter, the point load is replaced with the equivalent force F generated by the mass and the pull of gravity(mg). As before, knowing that the stiffness of the system can be expressed as $k = F/x$ we can show that:

$$k = \frac{3EI}{l^3} \tag{1.11}$$

Example 1.1 Axial Rod Vibrations
Considering the rod diagrammed below; calculate an equivalent spring constant for the rod using the length of the rod l, its area A, and Young's modulus E for a compressive force F that compresses the rod a distance x. Additionally, is a linear spring a useful model for a rod under compression? What if the rod is under tension?

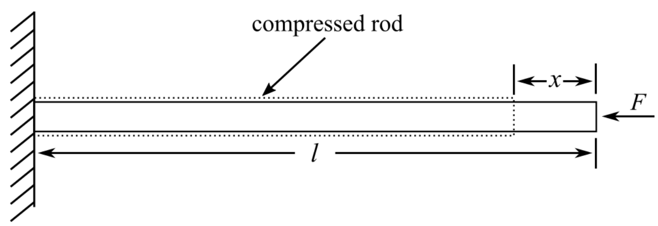


Figure 1.9: Compressed cantilever rod.

Solution:

The rod shortens by a distance x under the axial force F, this can be related to the equation of a linear spring $F = kx$ by recalling from solid mechanics that the elongation (or shortening) of a rod is expressed as

$$x = \frac{x}{l}l = \varepsilon l = \frac{\sigma}{E}l = \frac{Fl}{AE} \tag{1.12}$$

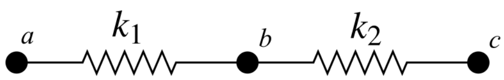
where $\varepsilon = \frac{x}{l}$ is the strain value and $\sigma = F/A$ is the stress induced in the rod. Combining this expression with the equation of a linear spring yields:

$$k = \frac{F}{x} = \frac{AE}{l} \tag{1.13}$$

As per the usefulness of the linear spring to represent an axial rod under compression or tension, this would be application-specific but could generally be considered an excellent first-order approximation.

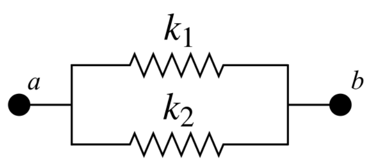
1.2.2 Springs in Series and Parallel

In many cases, it becomes necessary to model a mechanical system as a set of springs (e.g., a composite material, a table with multiple legs). For these systems, or for systems with more than one spring acting on a body, equivalent stiffness can be calculated as:



$$k_{ac} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

(a)



$$k_{ab} = k_1 + k_2$$

(b)

Figure 1.10: Equations for calculating the equivalent stiffness of two springs (k_1 and k_2); (a) in series; and (b) in parallel.

These are derived considering the displacement δ of the systems. For two springs in series:

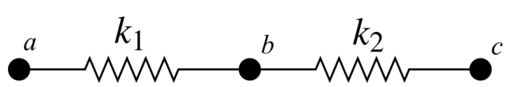


Figure 1.11: Two springs k_1 and k_2 combined in series.

where the total displacement is

$$\delta_{ac} = \delta_{ab} + \delta_{bc} \tag{1.14}$$

Using the equation for stiffness $k = F/\delta$, this converts to:

$$\frac{F}{k_{ac}} = \frac{F}{k_1} + \frac{F}{k_2} \tag{1.15}$$

As F is the same throughout the system, we can cancel out F. Solving for the equivalent stiffness yields:

$$k_{ac} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} \tag{1.16}$$

Similarly for a system of springs in parallel:

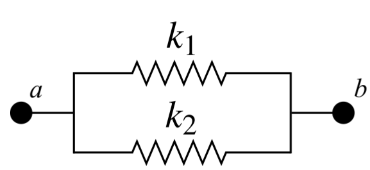


Figure 1.12: Two springs k_1 and k_2 combined in parallel.

The displacement in both springs is the same, so the total displacement is

$$\delta_{ab} = \delta_1 = \delta_2 = \delta \tag{1.17}$$

The forces in the direction of spring elongation sum to zero, therefore:

$$F_{ab} = F_1 + F_2 \tag{1.18}$$

Substituting the displacement and stiffness into the force equation yields:

$$\delta k_{ab} = \delta k_1 + \delta k_2 \tag{1.19}$$

this simplifies to:

$$k_{ab} = k_1 + k_2 \tag{1.20}$$

Example 1.2 Springs in Parallel and Series Configurations
Calculate the equivalent stiffness of the following system:

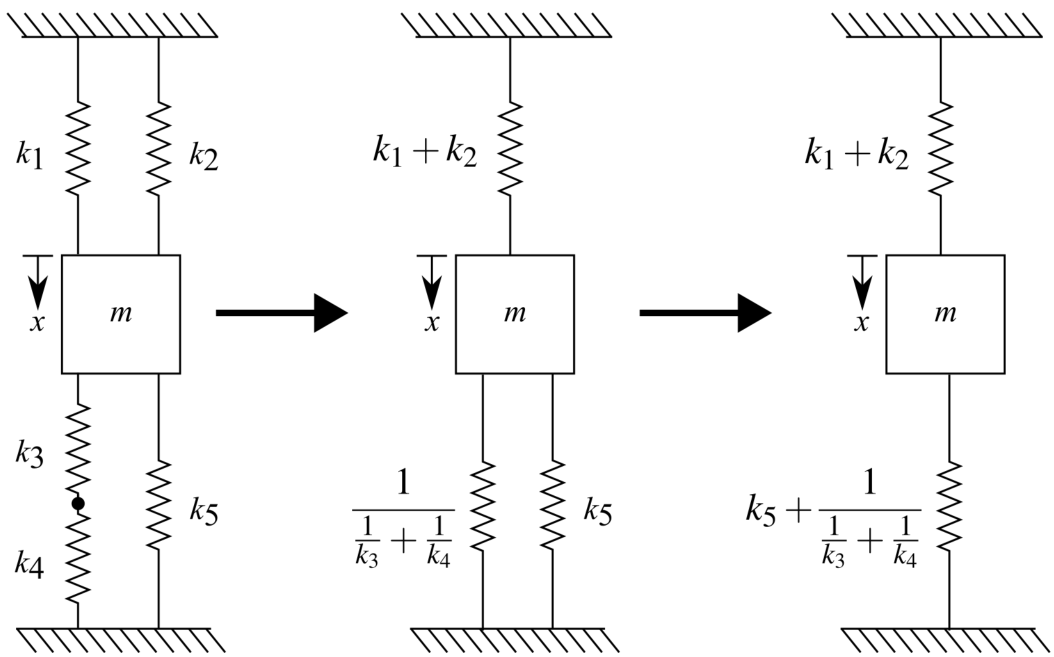


Figure 1.13: Equivalent stiffness for springs in series and parallel.

The springs are combined as shown, using the equations defined before. Now, considering that the displacement (δ) of the top spring, and the bottom spring are the same we can state the total stiffness k, which is the summation of the two. Therefore,

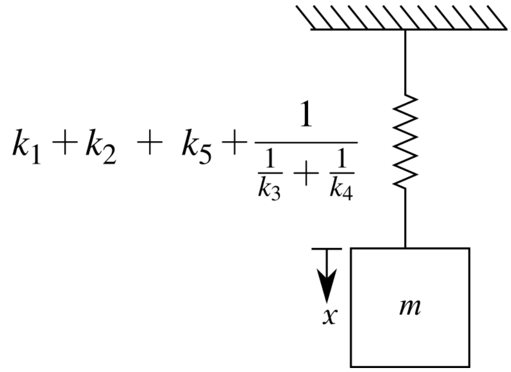


Figure 1.14: A spring-mass system simplified down form springs in series and parallel..

where the final addition, $(k_1 + k_2) + (k_5 + \frac{1}{\frac{1}{k_3} + \frac{1}{k_4}})$ is applied at two springs in parallel as each spring is connected between the mass and the fixity. Rearranging this new expression to get a common denominator:

$$k = \frac{(k_1 + k_2 + k_5)(k_3 + k_4) + k_3 k_4}{k_3 + k_4} \tag{1.21}$$

1.3 Equation of Motion for an Oscillating System

An Equation of Motion (EOM) is an equation that provides a basis for modeling a vibrating system about its equilibrium point and relates the transfer of the potential energy from the spring to the kinetic energy mass. In developing the EOM we assume that any surfaces are frictionless and as such, no energy is extracted from the vibrating system. Referencing the 1-DOF system in figure 1.15(a), and assuming the mass only moves in the x direction, the only force acting on the mass in the x direction is the force that results from the elongation of the spring as annotated in figure 1.15(b). Therefore, the sum of forces along the x axis must equal the mass (m) times the acceleration of the mass ($a\dot{x}$).

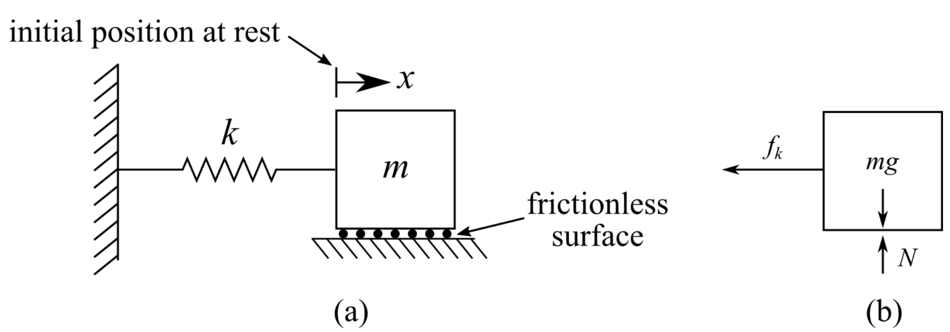


Figure 1.15: A spring-mass model of a 1-DOF system showing: (a) a schematic of the system; (b) a free-body diagram of the system at its initial position.

Considering that positive displacements are to the right, the standard form of the equation of motion for an undamped system without any excitation is expressed as:

$$s_1\ddot{x} + s_2 x = 0 \tag{1.22}$$

where s_1 and s_2 are constants to be determined for the specific system. A systematic approach to obtaining the free-body diagram (FBD) of a system under vibration can be expressed in three steps:

1. Draw a free-body diagram (FBD) at the system's equilibrium and displaced position (without a displacing force).
2. Apply Newton's second law to both FBDs (equilibrium and displaced).
3. Combine the equations to write the EOM in standard form with the forcing component on the right-hand side. For free vibration, the forcing component is 0.

Solving these three steps for 1-DOF system presented in figure 1.15 results in the EOM:

$$m\ddot{x} + kx = 0 \tag{1.23}$$

Review 1.4 Differential Equation
A second-order linear homogeneous differential equation has the form:

$$a\ddot{x} + b\dot{x} + cx = 0 \tag{1.24}$$

The EOM for a 1-DOF system under a free vibration is a second-order differential equation due to acceleration ($\ddot{x}$) being the second derivative of displacement (x) and homogeneous as the forcing function (right-hand side of the equations) is zero. In EOM's current form, $a = m$, $b = 0$, and $c = k$. In future work, b will account for damping in the vibrating system.

Example 1.3 Deriving Equation of Motion
Considering the system:

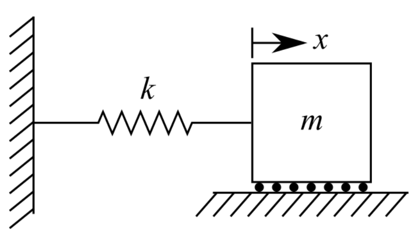


Figure 1.16: A 1 DOF spring-mass system with movement in the horizontal direction

Step-1 Define the direction of displacement, and draw the FBD for the equilibrium and displaced state.

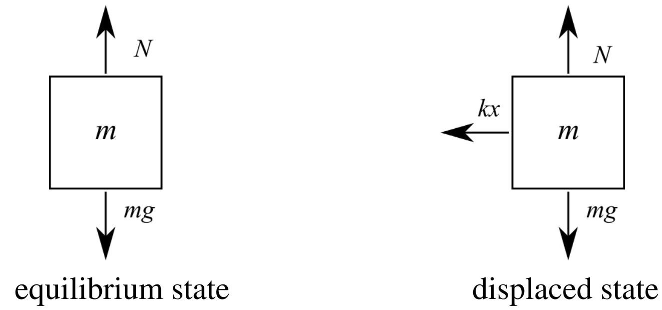


Figure 1.17: Equivalent forces for a 1 DOF spring-mass system with movement in the horizontal direction

The equation for the equilibrium state is:

$$\overset{+}{\rightarrow} \sum F_x = 0 \tag{1.25}$$

and in the displaced state:

$$\overset{+}{\rightarrow} \sum F_x = -kx \tag{1.26}$$

This equation does not equal zero as the FBD does not account for the restoring force.
Step-2 Apply Newton's second law (we want to store energy in the kinetic state) of motion to the sum of forces for the displaced position we get:

$$ma = m\ddot{x} = \overset{+}{\rightarrow} \sum F_x = -kx \tag{1.27}$$

$$m\ddot{x} = -kx \tag{1.28}$$

Step-3 Rearrange in the Equation to construct an EOM:

$$m\ddot{x} + kx = 0 \tag{1.29}$$

Example 1.4 Deriving Equation of Motion Considering Initial Displacement
Some systems will have an initial displacement, as the system will oscillate around this position we need to define the EOM about this position. Considering the system:

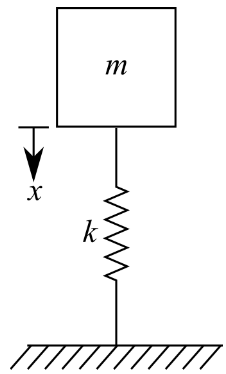


Figure 1.18: A 1 DOF spring-mass system with movement in the vertical direction.

Step-1 Define the direction of displacement (if needed, it is given in this problem) and draw the FBD for the equilibrium and displaced state.

Figure 1.19: Equivalent forces for a 1 DOF spring-mass system with movement in the vertical direction

The equation for the equilibrium state is:

$$+\downarrow\sum F_x = mg - k\delta = 0 \tag{1.30}$$

and in the displaced state:

$$+\downarrow\sum F_x = mg - k(\delta + x) \tag{1.31}$$

This equation does not equal zero as the FBD does not account for the restoring force.
Step-2 Apply Newton's second law (we want to store energy in the kinetic state) of motion to the sum of forces for the displaced position we get:

$$m\ddot{x} = +\downarrow\sum F_x = mg - k\delta - kx \tag{1.32}$$

We can then use the information from the equilibrium state to cancel out some terms, this becomes:

$$m\ddot{x} = -kx \tag{1.33}$$

Step-3 Rearrange in the Equation to construct an EOM:

$$m\ddot{x} + kx = 0 \tag{1.34}$$

Example 1.5 Deriving Equation of Motion Considering Torsional Stiffness
Equations of motion can also be developed for systems with torsional stiffness. Considering the system in figure 1.20 where k is the stiffness in the rotational direction and the shaft is perfectly rigid in the vertical direction. Moreover, consider the polar moment of inertia of the disk (J) that spins about the origin defined as point O.

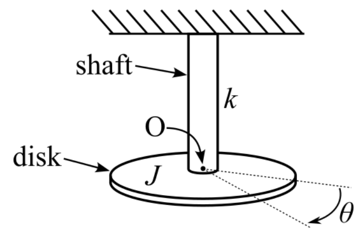


Figure 1.20: A 1 DOF system with a mass-less shaft and a disk where the direction of movement results in a torsional loading of the shaft.

Step-1 Draw the FBD for the equilibrium and displaced state.

Figure 1.21: Equivalent moments for a 1 DOF torsional system.

Considering that $\circlearrowright +$, the equation for the equilibrium state is:

$$\circlearrowright + \sum M_{\mathrm{O}} = 0 \tag{1.35}$$

as there is no initial displacement due to gravity this expression gives no useful information in this case and is ignored in this example. Next, the displaced state is:

$$\circlearrowright + \sum M_{\mathrm{O}} = -k\theta \tag{1.36}$$

This equation does not equal zero as the FBD does not account for the restoring force that is present where the shaft connects with the fixity.

Step-2 Apply Newton's 2nd law given the fact that we were given the moment of inertia of a disk as J,

$$\circlearrowleft + \sum M_O = J\ddot{\theta} = -k\theta \tag{1.37}$$

Step-3 Derive EOM:

$$J\ddot{\theta} + k\theta = 0 \tag{1.38}$$

Vibration Case Study 1.3 Design Considerations in Vibrations

Why study vibrations? One day, it could save your job! For a project to be successful it needs to be completed on time and within budget.

Consider the Ling-Temco-Vought (LTV) XC-142 which was a tilt-wing experimental aircraft developed in the 1960s for the US military and later turned over to NASA. During testing, the cross-linked driveshaft produced excessive vibration and noise which resulted in a high pilot workload. In general, the aircraft's cross-linked driveshaft was the main technical issue that caused the military to lose interest in the project.

Figure 1.22: A Ling-Temco-Vought XC-142A tested at the NASA Langley Research Center in 1969. [a]

[a]NASA, Photograph published in Winds of Change, 75th Anniversary NASA publication, by James Schultz, Public domain, via Wikimedia Commons