8 Structural Dynamics

The dynamic response of civil infrastructures, including buildings, bridges, and towers, can be studied by applying fundamental vibration concepts studied in the previous chapters.

8.1 Single-story frame

Let's start by considering the single-story frame shown in figure 8.1 (a) in free vibration (no external load is applied to the structure). The frame has height H and bay width L. As shown in figure 8.1, the frame consists of two columns with a modulus of elasticity E and moment of inertia (second moment of the cross-sectional area) I. The columns are fixed at the base. The frame in figure 8.1 (a) can be modeled as a single-degree-of-freedom (SDOF) system under the following assumptions:

- Shear building: flexible columns ($EI \neq 0$), beam infinitely rigid ($EI_b = \infty$), axial deformations of beams and columns negligible ($EA = 0$);
- Lumped mass system: floor-mass concentrated at the floor level.

Figure 8.1 (b) illustrates an SDOF with mass m and stiffness k that can be used to model the dynamic behavior of the single-story frame, considering no damping ($\zeta = 0$).

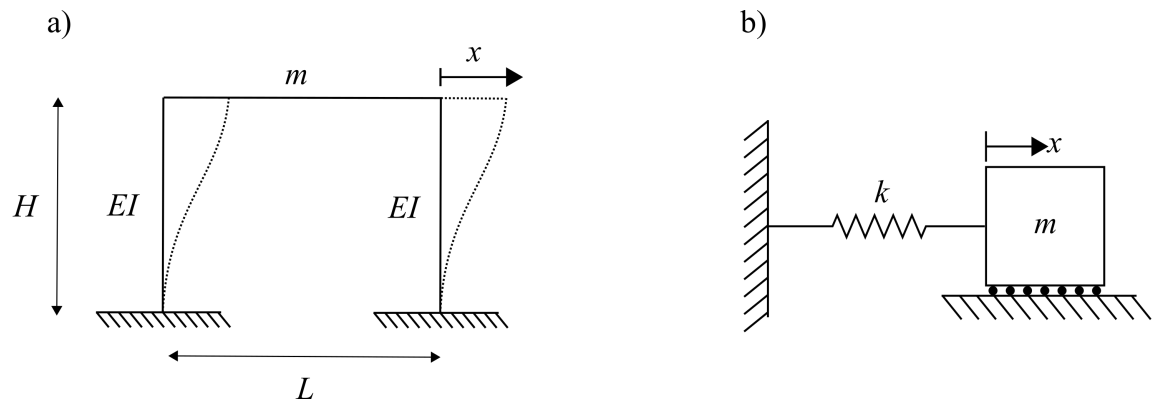

Figure 8.1: (a) Single story frame; (b) undamped single degree of freedom system.

The response of an SDOF system can be written in general notation as:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{8.1}$$

where ω_n is the natural frequency of the frame, x_0 and v_0 are the initial conditions. In order to find ω_n, we need to calculate the stiffness of the system. The mass is usually given.

The stiffness of the system can be found by applying Hooke's law: $F = kx$. To find k, let's imagine applying an arbitrary lateral force F to the frame and analyzing a single column. At the top, the column will be subjected to a force F and to a moment M_0, as schematically shown in figure 8.2 (a). Applying the equilibrium equations to the column, it can be found that $M_0 = \frac{FH}{2}$.

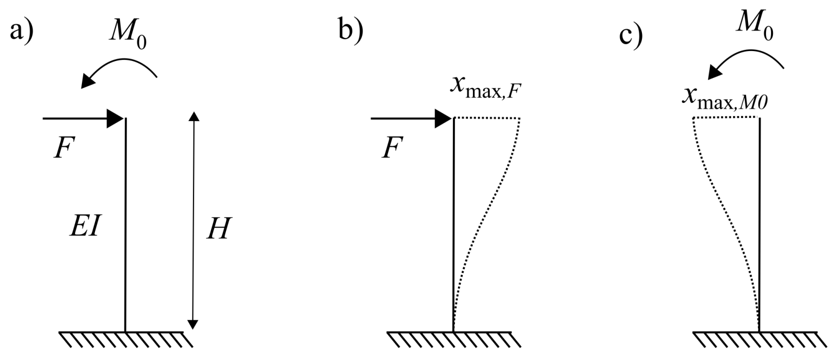

Figure 8.2: Single column subjected to: (a) force and moment; (b) force only; (c) moment only.

Since the system is linear, we can calculate the effects of F and M_0 separately and then sum them together (superposition principle). The maximum deflection due to F occurs at the top of the column, as shown in figure 8.2 (b), and it is equal to:

$$x_{\max,F} = \frac{FH^3}{3EI} \tag{8.2}$$

while the maximum deflection caused by M_0 (figure 8.2 (c)) is:

$$x_{\max,M0} = \frac{M_0 H^2}{2EI} \tag{8.3}$$

The displacements in equation 8.2 and 8.3 were found using engineering tables. The total displacement x at the top of the column is obtained from the sum of the two displacements:

$$x = \frac{FH^3}{3EI} - \frac{M_0 H^2}{2EI} \tag{8.4}$$

where the $x_{\max,M0}$ is negative in sign because the displacement caused by M_0 goes in opposite direction to $x_{\max,F}$. Replacing $M_0 = \frac{FH}{2}$ in equation 8.4:

$$x = \frac{FH^3}{3EI} - \frac{FH^3}{4EI} = \frac{FH^3}{12EI} \tag{8.5}$$

Applying Hooke's law:

$$F = k_c x = k_c \frac{FH^3}{12EI} \tag{8.6}$$

where k_c is the stiffness of the column. Therefore:

$$k_c = \frac{12EI}{H^3} \tag{8.7}$$

Since the frame has two columns, the total stiffness of the SDOF system will be:

$$k = \sum_{columns} k_c = \sum_{2} \frac{12EI}{H^3} \tag{8.8}$$

where k is also called *lateral stiffness*. Note that the lateral stiffness of the frame is independent of the length of the bay L, and it depends only on the properties of the columns (E, I, and H). It is possible at this point to calculate the natural frequency of the frame:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{\sum_2 \frac{12EI}{H^3}}{m}} \tag{8.9}$$

If the columns have the same properties, equation 8.9 becomes:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{24EI}{H^3 m}} \tag{8.10}$$

Finally, the response of the system to initial conditions x_0 and v_0 can be obtained:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{8.11}$$

Example 8.1 Single-story Frame
Let's consider the single-story frame shown in figure 8.1 with mass m = 0.15 kip s^2/ft, L = 12 ft, EI= 1800 kip ft^2. a) Determine the EOM and the natural period of the frame; b) assume that the moment of inertia of the right column is $2I$. Will the EOM change?
Solution a) :
The frame can be modeled as a single degree of freedom in free vibration. Therefore, the EOM is:

$$m\ddot{x} + kx = 0 \tag{8.12}$$

The lateral stiffness of the system is:

$$k = \sum_{columns} k_c = \sum_2 \frac{12EI}{H^3} = \frac{24EI}{H^3} \tag{8.13}$$

Thus, the natural frequency and period are:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{24EI}{mH^3}} = \sqrt{\frac{1800}{0.15 \cdot 12^3}} = 12.91 \frac{rad}{s} \tag{8.14}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.48s \tag{8.15}$$

Solution b) :
The EOM won't change, but the lateral stiffness of the system will be:

$$k = \sum_{columns} k_c = \frac{12EI}{H^3} + \frac{24EI}{H^3} = \frac{36EI}{H^3} \tag{8.16}$$

The same principle can be applied to a single-story frame with a damping ratio $\zeta \neq 0$. In this case, the displacement of the frame will be given by:

$$x(t) = e^{(-\zeta\omega_n t)}\left(\frac{(v_0 + x_0)\omega_n}{\omega_d}\cos(\omega_d t) + x_0\sin(\omega_d t)\right) \tag{8.17}$$

where ω_d is the damped natural frequency of the system:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{8.18}$$

8.2 Duhamel's Integral

In Chapter 4, the frequency response method was used to solve the EOM of an SDOF system subjected to an arbitrary force. Here, an alternative method widely employed in structural dynamics to find the solution of the EOM is presented. This method exploits a specific integral, named Duhamel's integral.

Let's consider an underdamped SDOF system subjected to an arbitrary force $F(t)$. The EOM is:

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{8.19}$$

Let's assume that the system is at rest: $x(0) = 0$ and $\dot{x} = 0$. The assumption underlying Duhamel's integral method is that a generic force $F(t)$ can be expressed as a sequence of impulses of very small duration, and the response of the system as the sum of the response to individual unit impulses.

An impulsive force can be defined as a very large force applied in a very short time interval. figure 8.3 (a) shows an impulsive force $F(t) = \frac{1}{\varepsilon}$ applied at time $t = \tau$. Assuming to apply an impulsive force to a generic mass m and applying Newton's second law:

$$m\ddot{x} = F(t) \tag{8.20}$$

and integrating both sides between two generic time instants t_1 and t_2 yields:

$$\int_{t_1}^{t_2} F(t)dt = m(\dot{x}_1 - \dot{x}_2) \tag{8.21}$$

where the left-hand side of the equation represents the magnitude of the force and the right-hand side the change in momentum.

In the limit case in which ε tends to 0, $F(t)$ tends to 1, and the impulsive force is called *unit impulse*. In the case of a unit impulse, $\int_{t_1}^{t_2} F(t)dt = 1$ and t_1 tends to t_2. Therefore, the velocity of the mass can be found as:

$$\dot{x}(\tau) = \frac{1}{m} \tag{8.22}$$

A similar concept applies to an SDOF system. Since the impulse is applied in a very short time interval, the spring and the damper do not have the time to react. When we apply a unit impulse to an underdamped SDOF, the system will start vibrating with velocity $\dot{x}(\tau)$ given by equation 8.22 and displacement $x(\tau) = 0$. The response of the system is given by the following equation:

$$x(t) = h(t-\tau) = \frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\tau)}sin(\omega_d(t-\tau)) \tag{8.23}$$

where τ is the time instant at which the impulse is applied.

NOTE

The Dirac delta function $\delta(t-\tau)$ mathematically defines a unit impulse centered at $t=\tau$.

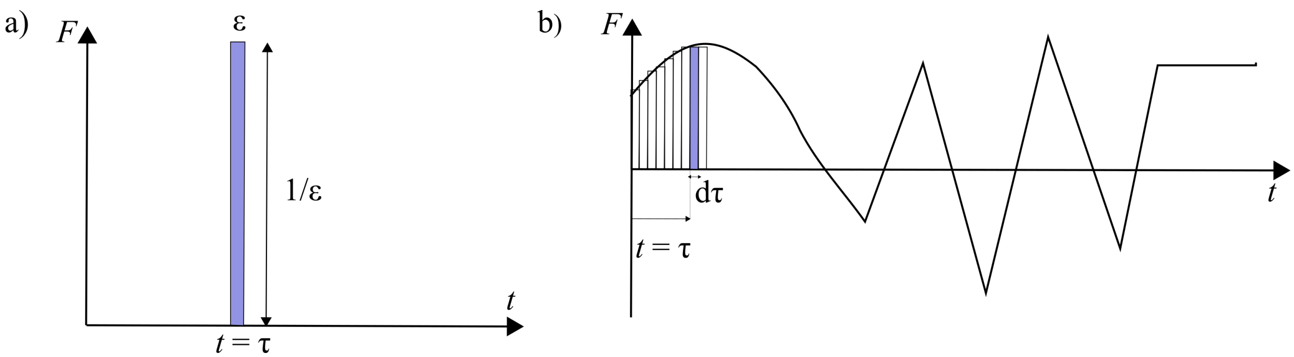

Figure 8.3: (a) Impulsive force; (b) arbitrary force decomposed in a series of impulses.

Let's now consider a force $F(t)$ varying arbitrarily with time. As shown in figure 8.3 (b), $F(t)$ can be represented as a sequence of infinitesimally short impulses. The response of a linear system to $F(t)$ can therefore be expressed as the response to a series of impulses, following:

$$x(t) = \int_0^t p(\tau)h(t-\tau)d\tau \tag{8.24}$$

where $h(t-\tau)$ is the response to a unit impulse and $p(\tau)$ is the magnitude of the actual impulse. For the case of an underdamped SDOF system, equation 8.24 can be rewritten as:

$$x(t) = \frac{1}{m\omega_d}\int_0^t p(\tau)e^{-\zeta\omega_n(t-\tau)}sin(\omega_d(t-\tau))d\tau \tag{8.25}$$

equation 8.25 represents the *Duhamel's integral*.

Similarly, the response of an undamped SDOF system to an arbitrary force can be expressed through Duhamel's integral as:

$$x(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)sin(\omega_n(t-\tau))d\tau \tag{8.26}$$

If $F(t)$ is characterized by a simple function, Duhamel's integral can be evaluated in closed form. If the equation of $F(t)$ is complicated, Duhamel's integral can be solved with numerical methods. Equation 8.25 and 8.26 apply when the initial conditions are zero (the system is at rest). If the initial conditions are different than zero, we need to add the free vibration response of the system to equation 8.25 and (8.26), respectively.

Example 8.2 Solving for Response due to Step Function Loading

Let's consider an undamped SDOF system subjected to a step function force with constant amplitude F_0, as schematically represented in figure 8.4. Assume that the system is at rest (initial conditions: $x(0) = \dot{x}(0) = 0$) and compute the system response $x(t)$.

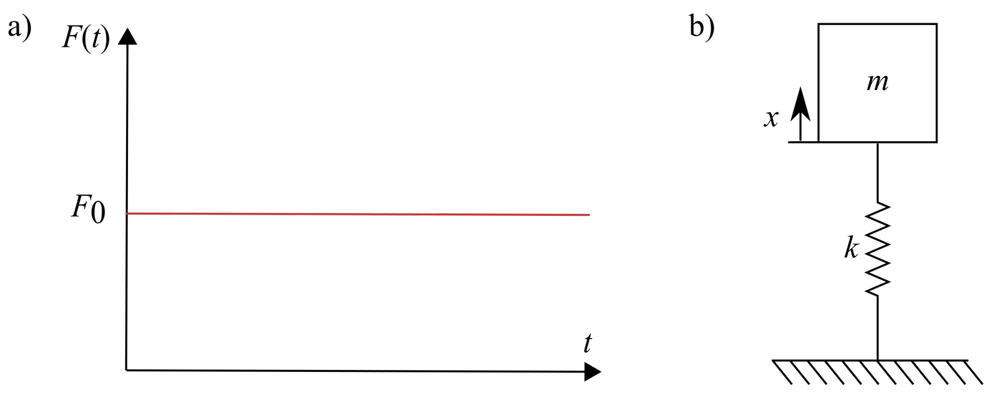

Figure 8.4: (a) Step function force; (b) undamped SDOF system.

Solution:

The system is undamped, therefore we can use Duhamel's integral in equation 8.26 to find $x(t)$:

$$x(t) = \frac{1}{m\omega_n} \int_0^t F_0 sin(\omega_n(t-\tau))d\tau \tag{8.27}$$

Considering that F_0 is constant:

$$x(t) = \frac{F_0}{m\omega_n} \left[\frac{cos(\omega_n(t-\tau))}{\omega_n} \right]_0^t = \frac{F_0}{m\omega_n^2}[1 - cos(\omega_n t)] \tag{8.28}$$

Reminding that $\omega_n^2 = k/m$, $x(t)$ becomes:

$$x(t) = \frac{F_0}{k}[1 - cos(\omega_n t)] \tag{8.29}$$

where $\frac{F_0}{k}$ is the displacement that the system would undergo if the force F_0 were applied statically. In the case of an underdamped SDOF system, the response becomes:

$$x(t) = \frac{F_0}{k}\left[1 - e^{-\zeta\omega_n t}\left(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t)\right)\right] \tag{8.30}$$

8.3 Two-story frame

The concepts discussed in Section 8.1 can be extended to the 2-story frame represented in Figure 8.5. In fact, a 2-story frame can be modeled as a 2-DOF system under the following assumptions:

- shear building: flexible columns ($EI \neq 0$), beam infinitely rigid ($EI_b = \infty$), axial deformations of beams and columns negligible ($EA = 0$);
- lumped mass system: floor-mass concentrated at the floor level.

Under such assumptions and free vibrations, we expect that the building moves following the deformed shape reported in figure 8.5 (dotted line). Let's call the degrees of freedom of the frame $x_1(t)$ and $x_2(t)$.

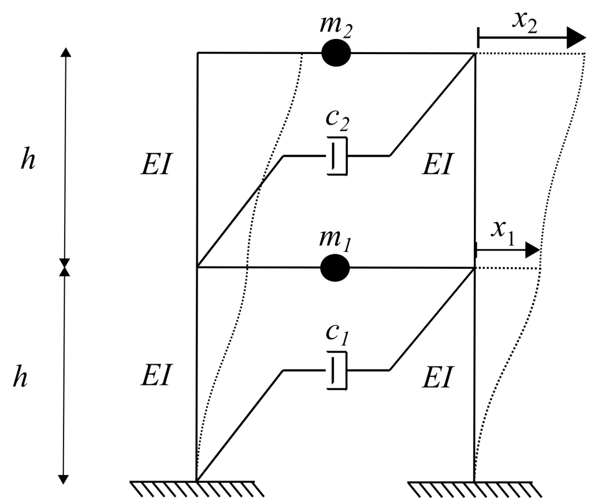

Figure 8.5: 2-story frame with lumped masses.

The forces acting on the 2-DOF system are reported in figure 8.6. It follows that the equations of motion of the two masses are:

$$\begin{aligned} m_1\ddot{x}_1 + k_1 x_1 + k_2(x_2 - x_1) + c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) = 0 \\ m_2\ddot{x}_2 - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) = 0 \end{aligned} \tag{8.31}$$

In matrix notation, these two equations become:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.32}$$

where we can define the mass matrix M as:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{8.33}$$

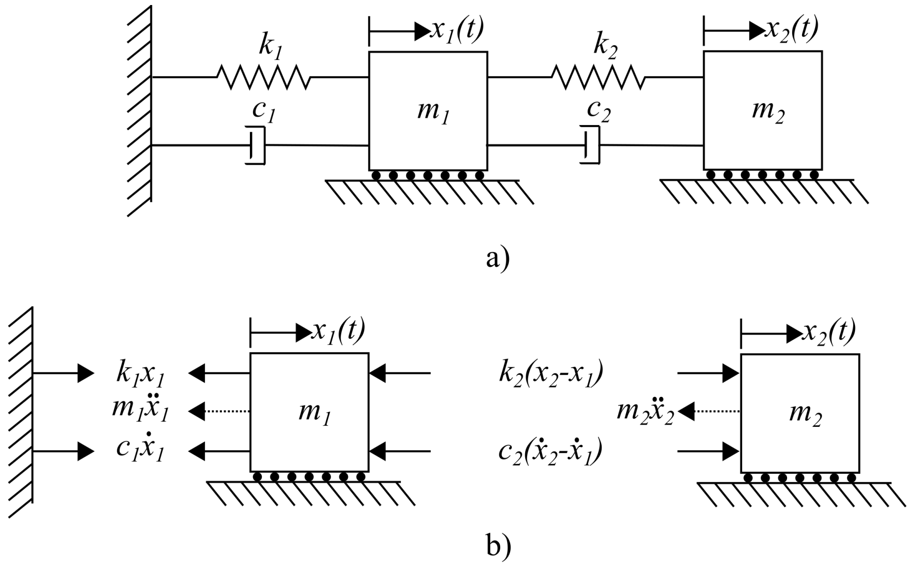

Figure 8.6: (a) 2-DOF system used to model the 2-story frame; (b) free body diagram of the two masses.

the stiffness matrix K as:

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{8.34}$$

and the damping matrix C as:

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \tag{8.35}$$

While mass and damping of a frame are usually given, the stiffness values k_1 and k_2 need to be calculated as a function of the columns' properties (EI) and geometry (h). As demonstrated in Sec. 1, the stiffness of a column with clamped ends can be determined as:

$$k_c = \frac{12EI}{h^3} \tag{8.36}$$

The lateral stiffness of each floor can be computed as the sum of the stiffness of the columns at that floor:

$$k = \sum_{columns} k_c = \sum_{2} \frac{12EI}{h^3} \tag{8.37}$$

Therefore, for the frame in figure 8.5, the stiffness values are:

$$k_1 = k_2 = \frac{24EI}{h^3} \tag{8.38}$$

The solution of the EOM in Eq.(8.32) was derived in Chapter 5 and can be summarized as:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}, \qquad \omega_1 \text{ or } \omega_2 \neq 0 \tag{8.39}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors (or mode shapes), ω_1 and ω_2 are the natural frequency of vibration, ϕ_1, ϕ_2, A_1, and A_2 are constants that can be found based on the initial conditions (see Chapter 5 for more details).

Example 8.3 Finding the Structural Response of a Two-Story Building
Consider the frame in figure 8.7. Determine the natural frequency of vibration and mode shapes of the system.

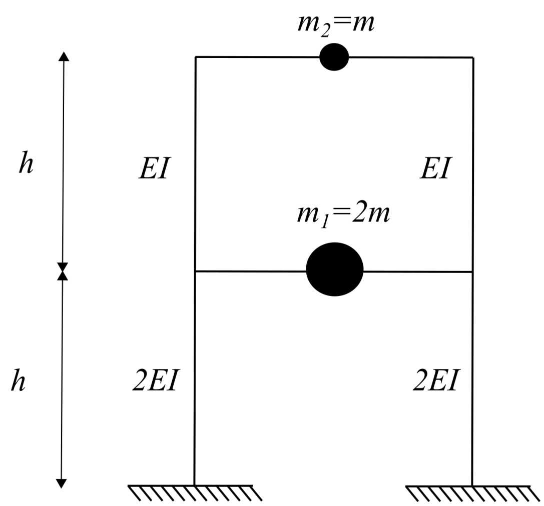

Figure 8.7: Example of a 2-story frame with floors with different dynamic properties.

Solution:

Assumption: the frame can be modeled as a shear building with mass lumped at the floor levels. The lateral stiffness at the first floor is:

$$k_1 = 2\frac{12(2EI)}{h^3} = \frac{48EI}{h^3} \tag{8.40}$$

The lateral stiffness at the second floor is:

$$k_2 = 2\frac{12(EI)}{h^3} = \frac{24EI}{h^3} \tag{8.41}$$

Therefore, the stiffness matrix can be written as:

$$K = \begin{bmatrix} \frac{48EI}{h^3} + \frac{24EI}{h^3} & -\frac{24EI}{h^3} \\ -\frac{24EI}{h^3} & \frac{24EI}{h^3} \end{bmatrix} = \frac{24EI}{h^3}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \tag{8.42}$$

The EOM of the system is:

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \frac{24EI}{h^3}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{8.43}$$

In order to determine the natural frequency of vibration and the mode shapes of the system, we need to solve the characteristic equation:

$$\det(-\omega^2 M + K) = 0 \tag{8.44}$$

leading to:

$$2m^2\omega^4 - 5km\omega^2 + 2k^2 = 0 \tag{8.45}$$

This equation has two solutions:

$$\omega_1^2 = \frac{k}{2m} \tag{8.46}$$

$$\omega_2^2 = \frac{2k}{m} \tag{8.47}$$

Therefore, the two natural frequencies of vibration of the system are:

$$\omega_1 = \sqrt{\frac{k}{2m}} \tag{8.48}$$

$$\omega_2 = \sqrt{\frac{2k}{m}} \tag{8.49}$$

where $k = \frac{24EI}{h^3}$. The mode shapes of the frame can be found by solving the following equation:

$$(-\omega_1^2 M + K)\mathbf{u}_1 = 0 \tag{8.50}$$

Replacing the mass and stiffness matrix, the equation becomes:

$$\left(-\frac{k}{2m}\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} + k\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.51}$$

simplified to

$$\begin{bmatrix} 2k & -k \\ -k & \frac{k}{2} \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.52}$$

leading to two equations:

$$2ku_{11} - ku_{21} = 0, \text{ and } -ku_{11} + \frac{k}{2}u_{21} = 0 \tag{8.53}$$

It follows that:

$$2u_{11} = u_{21}, \text{ and } u_{11} = \frac{1}{2}u_{21} \tag{8.54}$$

To obtain a numerical value, we arbitrarily assign a value to one of the elements. Here, let $u_{21} = 1$ so let $u_{11} = 1/2$. Therefore,

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \tag{8.55}$$

Similarly, $\mathbf{u}_2$ can be obtained by solving the following equation:

$$(-\omega_2^2 M + K)\mathbf{u}_2 = 0 \tag{8.56}$$

leading to:

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{8.57}$$

Figure 8.8 represents the two-mode shapes of the building.

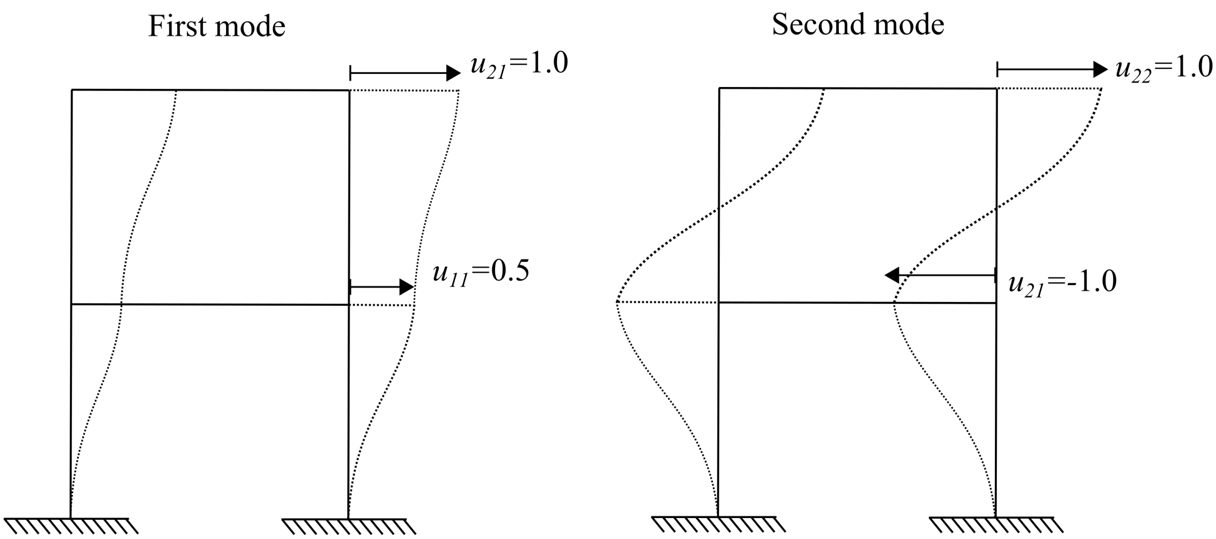

Figure 8.8: Mode shapes of the 2-story frame.

The temporal response of the system is given by:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}, \qquad \omega_1 \text{ or } \omega_2 \neq 0 \tag{8.58}$$

where $\mathbf{u} = [\mathbf{u}_1, \mathbf{u}_2]$ is the time invariant part of the equation.