Shortest Path with Variable Edge Failure Financial Optimization

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1 Problem Setup

Given a directed graph $G = (V, E)$. Let s and t both exist on V. In this case, s is 0, and t is the final node value. The nominal travel time on a given edge (ij) is c_{ij}. If the edge fails (gets congested), then the travel time becomes $c_{ij} + d_{ij}$. There are at most L allowable simultaneous edge failures.

The goal is to minimize the worst-case travel cost from s to t. Vary L to observe its affect on the model.

2 Derivation

The initial setup of the shortest path involves creating a simple minimization mixed integer linear program, using the binary variable x.

$$\min c^T x$$
$$\text{s.t.} \quad x \in \{0, 1\}$$
$$\text{for} \quad c_j \in [\overline{c_{ij}}, \overline{c_{ij}} + d_{ij}]$$

Since we're minimizing the worst case scenario, we have to also consider the summation of edge failures.

$$\sum_{j=1}^{n} d_j x_j$$

This worst case scenario is determined by the maximum allowable edge failures. Therefore, this summation can be turned into a maximization problem, with decision variables S and L.

$$\max_{S \in 1 \ldots n, |S| = L} \sum_{j \in S} d_j x_j$$

From here, this maximization of the worst case can now be added to our simple linear program defined at the beginning, allowing us to solve for the optimal worst-case travel scenario.

$$\min_{x\in\{0,1\}}[c^T x + \max_{S\in 1\ldots n, |S|=L} \sum_{j\in S} d_j x_j]$$

In its current state as a minimzation of a maximization, the program is unsolvable. To deal with this, we first need to solve the inner problem. The initial setup for the inner problem solution is shown below.

$$\max \sum_{j=1}^{n} x_j d_j s_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} s_j = L$$

$$s_j \in \{0,1\} \quad \forall j$$

Now that the inner problem has been laid out, we need to solve for the dual. This will set the maximization problem into a minimization, which can then be reinserted into the initial minimization program in a solvable form. In order to solve for the dual, there has to be a relaxation of the binary variable s.

$$\max \sum_{j=1}^{n} x_j d_j s_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} s_j = L$$

$$0 \leq s_j \leq 1 \quad \forall j$$

Now we can formulate the dual of the relaxation.

3 Final Formulation

Now that the derivation is complete, we can state the final equation for the shortest path model.

$$\min \bar{c}^T x + L\lambda_0 + \sum_{j=1}^{n} z_j x_j$$

$$\text{s.t.} \quad x \in \{0,1\}$$

$$\lambda_0 \quad \text{u.r.}$$

$$z_j \geq 0 \quad \forall j$$

$$z_j \geq d_j - \lambda_0 \quad \forall j$$

$$\sum_j x_{0j} = 1$$

$$\sum_i x_{i0} = 0$$

$$\sum_i x_{it} = 1 \quad \text{for} \quad t = \text{end node}$$

$$\sum_j x_{tj} = 0 \quad \text{for} \quad t = \text{end node}$$

$$\sum_{i \neq 0, j \neq t} [x_{ij} - x_{ji}] = 0$$

In this final formulation, x is a binary decision variable, representing whether a path edge will be taken. λ_0 is a single continuous, unrestricted decision variable that amplifies the magnitude of the failure value L. The final decision variable, z, is used to determine which edge will have the congestion, so as to minimize the worst-case scenario. The summations of x values at the end are used to determine the start and end location of the path, and ensure the MILP doesn't end on a node somewhere in the middle.