5.01 Show that the taxicab metric on $\mathbb{R}^2$ satisfies the properties of a metric.
(i) Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Notice, $|x_1 - x_2| + |y_1 - y_2| \geq 0$ as absolute values are always non-negative and addition of non-negatives is always non-negative. Thus, condition (i) holds.
(ii) Let $x, y \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x,y) &= |x_1 - y_1| + |x_2 - y_2| \\ &= |y_1 - x_1| + |y_2 - x_2| \\ &= d(y,x) \end{aligned}$$

Thus, (ii) holds.
(iii) Let $x, y, z \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x,z) &= |x_1 - z_1| + |x_2 - z_2| \\ &= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2| \\ &\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| \\ &= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &= d(x,y) + d(y,z) \end{aligned}$$

Thus, (iii) holds.

Thus, all three conditions of a metric are met.
Therefore, the taxicab metric is a metric.

5.02 (a) Show that the max metric on $\mathbb{R}^2$ satisfies the properties of a metric.
(i) Notice, we are taking the max value of an absolute value which are non-negative. Hence, for $x, y \in \mathbb{R}^2, d(x,y) \geq 0$.
Thus, (i) holds.
(ii) Let $x, y \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x,y) &= max\{|x_1 - y_1|, |x_2 - y_2|\} \\ &= max\{|y_1 - x_1|, |y_2 - x_2|\} \\ &= d(y,x) \end{aligned}$$

Thus, (ii) holds.
(iii) Let $x, y, z \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x,z) &= max\{|x_1 - z_1|, |x_2 - z_2|\} \\ &= max\{|x_1 - y_1 + y_1 - z_1|, |x_2 - y_2 + y_2 - z_2|\} \\ &\leq \max\{|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|\} \\ &= |x_i - y_i| + |y_i - z_i| \\ &\text{where } i \text{ with value 1 or 2 holds the maximum value} \end{aligned}$$

$$\begin{aligned} |x_i - y_i| &\leq max\{|x_1 - y_1|, |x_2 - y_2|\} \\ |y_i - z_i| &\leq max\{|y_1 - z_1|, |y_2 - z_2|\} \end{aligned}$$

So,

$$d(x,z) \leq max\{|x_1 - y_1|, |x_2 - y_2|\} + max\{|y_1 - z_1|, |y_2 - z_2|\} = d(x,y) + d(y,z)$$

Thus, (iii) holds.
Thus, all three conditions of a metric are met.
Therefore, the max metric is a metric.

(b) Explain why $d(p,q) = \min\{|p_1 - q_1|, |p_2 - q_2|\}$ does not define a metric on $\mathbb{R}^2$.

The Triangle inequality does not hold.
(1,0)(2,0) Let $p, q, r \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(p,r) &= min\{|p_1 - r_1|, |p_2 - r_2|\} \\ &\geq min\{|p_1 - q_1| + |q_1 - r_1|, |p_2 - q_2| + |q_2 - r_2|\} \\ &= |p_i - q_i| + |q_i - r_i| \\ &\text{where } i \text{ with value 1 or 2 holds the minimum value} \end{aligned}$$

$$|p_i - q_i| \leq min\{|p_1 - q_1|, |p_2 - q_2|\}$$
$$|q_i - r_i| \leq min\{|q_1 - r_1|, |q_2 - r_2|\}$$

So,

$$d(p,r) \geq min\{|p_1 - q_1|, |p_2 - q_2|\} + min\{|q_1 - r_1|, |q_2 - r_2|\} \geq d(p,q) + d(q,r)$$

Thus, the triangle inequality does not hold in general.

5.05 Let $X$ be a nonempty set. Define $d$ on $X \times X$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Show that $d$ is a metric, and determine the topology on $X$ induced by $d$

(i) *WTS:* $\forall x, y \in X, d(x,y) \geq 0$
Let $x, y \in X$. Consider the following two cases:

Case 1: $x = y \implies d(x,y) = 0$
Case 2: $x \neq y \implies d(x,y) = 1$

Thus, in both cases $d(x,y) \geq 0$

(ii) *WTS:* $\forall x, y \in X, d(x,y) = d(y,x)$
Let $x, y \in X$. Consider the following two cases:

Case 1: $d(x,y) = 1 = d(y,x)$

Case 2: $d(x, y) = 0 = d(y, x)$

Thus, in both cases $d(x, y) = d(y, x)$

(iii) *WTS:* $\forall x, y, z \in X, d(x, y) + d(y, z) \geq d(x, z)$
Let $x, y, z \in X$. Consider the following:

Case 1: $x \neq y \neq z \implies d(x, y) + d(y, z) = 2 \geq d(x, z) = 1 \implies 2 \geq 1$
Case 2: $y = x$ or $y = z \implies d(x, y) + d(y, z) = 1 \geq d(x, z) = 1 \implies 1 \geq 1$
Case 3: $x = z \implies d(x, y) + d(y, z) = 2$ or $1 \geq d(x, z) \implies 2$ or $1 \geq 0$
Case 4: $x = y = z \implies d(x, y) + d(y, z) = 0 \geq d(x, z) = 0 \implies 0 \geq 0$

Thus, all cases hold.

Therefore, the three conditions of a metric are satisfied and $d$ is a metric.

This is the discrete metric.

5.09 Prove Theorem 5.6: Let $(X, d)$ be a metric space. A set $U \subset X$ is open in the topology induced by $d$ if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$
Let $(X, d)$ be a metric space.
*WTS: $U \subset X$ is open in $(X, d)$ then $\forall y \in U \exists \delta > 0$ such that $B_d(y, \delta) \subset U$*
Let $U \subset X$ be open in the topology induced by $d$ and $y \in U$. By theorem 1.9 we can find $B_d(y, \delta) \subset U$ such that $y \in B_d$.

*WTS: $y \in U$ with $\delta > 0$ then $U$ is open in the topology induced by $d$.*
Let $U \subset X$ and $y \in U$ with $\delta > 0$ such that $B_d(y, \delta) \subset U$. Notice by theorem 1.9, we can find bases element at each point in $U$ which follows that $U$ is open.

5.10 (a) Let $(X, d)$ be a metric on a space. For $x, y \in X$, define

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Show that $D$ is also a metric on $X$

Let $x, y, z \in X$.
(i) Since $d(x, y) \geq 0$, $D(x, y) \geq 0$ as addition on non-negatives will be non-negative. Thus, (i) holds.

(ii) Observe.

$$\begin{aligned} D(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\ &= \frac{d(y,x)}{1+d(y,x)} \\ &= D(y,x) \end{aligned}$$

Thus, (ii) holds.
(iii) Observe.

$$\begin{aligned} D(x,y)+D(y,z) &= \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \\ &= \frac{d(x,y)(1+d(y,z))}{(1+d(x,y))(1+d(y,z))} + \frac{d(y,z)(1+d(x,y))}{(1+d(x,y))(1+d(y,z))} \\ &\geq \frac{d(x,y)+d(y,z)}{(1+d(x,y))(1+d(y,z))} \\ &\geq \frac{d(x,z)}{(1+d(x,y))(1+d(y,z))} \\ &\geq \frac{d(x,z)}{1+d(x,z)} \end{aligned}$$

As $d(x,z) = d(x,z)$ and $(1+d(x,y))(1+d(y,z)) \geq 1+d(x,z)$

Thus, (iii) holds.

Thus, the three conditions of a metric hold.
Therefore, $D$ is a metric on $X$

(b) Explain why no two points in $X$ are distance one or more apart in the metric $D$.
The numerator is always smaller than the denominator, so the distance will always be less than 1 apart.

5.14 Let $(X,d)$ be a metric space.
(a) Show that the closed balls in the metric $d$ are closed sets in the topology on $X$ induced by $d$
Let $B_d(x,\varepsilon)$ be a closed ball for some $x \in X$ and $\varepsilon > 0$. Suppose $y \in X - B_d(x,\varepsilon)$. Then, $d(y,x) > \varepsilon$ and so $d(y,x) - \varepsilon > 0$. Define $\delta := d(y,x) - \varepsilon$ and let $z \in B_d(y,\delta)$. Notice, $d(x,y) \leq d(x,z) + d(z,y) \implies d(z,x) \geq d(x,y) - d(z,y) > d(x,y) - \delta = \epsilon$. Hence, $B_d(y,\delta) \subset X - B_d(x,\varepsilon)$ and so must be open.
Therefore, closed balls in the metric $d$ are closed sets in the topology on $X$ induced by $d$
(b) Provide an example demonstrating that in general the closed ball $\bar{B}_d(x,\varepsilon)$ is not the closure of the open ball $B_d(x,\varepsilon)$

The metric on $\mathbb{R}$ given by $d(x, y) = |x - y|$ would be an example that shows that in general the closed ball $\bar{B}_d(x, \varepsilon)$ is not the closure of the open ball $B_d(x, \varepsilon)$

5.15 Let $(X, d)$ be a metric space and assume that $A \subset X$. Prove that $x \in \mathrm{Cl}(A)$ if and only if there exists a sequence in $A$ converging to $x$.
$(\implies)$ *WTS: $x \in Cl(A) \implies \exists$ a sequence in $A$ converging to $x$*
Let $x \in Cl(A)$. Notice, all metric spaces are Hausdorff. By theorem 2.12, every convergent sequence in $A$ converges to a unique point in $A$.
Thus, there exists a sequence in $A$ converging to $x$.
$(\impliedby)$ *WTS: There is a sequence in $A$ converging to $x \implies x \in Cl(A)$*
Since the sequence in $A$ converging to $x$ is a limit point, we have that $x \in A'$. Notice, $Cl(A) = A \cup A'$.
Thus, $x \in Cl(A)$.

Therefore, $x \in \mathrm{Cl}(A)$ if and only if there exists a sequence in $A$ converging to $x$

5.23 Let $(X, d)$ be a metric space. Let $A$ and $B$ be disjoint subsets of $X$ that are closed in the topology induced by $d$. Prove that there exist disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$
*WTS: $\forall$ closed $A \subset X, B \subset X$ such that $A \cap B = \varnothing$, $\exists U \subset A, V \subset B$ such that $U \cap V = \varnothing$*
Let $A \subset X, B \subset X$ be closed sets such that $A \cap B = \varnothing$.
Define $U := (X - A) \cap (X - B)$ which is open and $V := (X - B) \cap (X - A)$ which is also open.
Therefore, there exist disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$

5.24 Prove Theorem 5.13: Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is continuous in the open set definition if and only if for each $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x' \in X$ and $dx\,(x, x') < \delta$ then $d_Y\,(f(x), f\,(x')) < \varepsilon$. (Hint: Consider Exercise 4.3 and the proof of Theorem 4.6.)

$(\implies)$ Suppose $f : X \to Y$ is continuous in the open set definition and let $x \in X$, $\varepsilon > 0$, and $f(x) \in Y$ with $B(f(x), \varepsilon)$ be open in $Y$. Notice, by Theorem 4.6 there exists a $B(x, \delta)$ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. Let $x' \in X$ such that $d_x(x, x') < \delta$. It follows that $x' \in B(x, \delta)$. Then, $f(x') \in f(B(x, \delta))$ and so $d_y(f(x), f(x')) < \delta$. Since $f(B(x, \delta)) \subset B(f(x), \varepsilon)$, we must have $d_y(f(x), f(x')) < \varepsilon$.

$(\impliedby)$ Suppose $x \in X$,$\varepsilon > 0$, and there exists a $\delta > 0$ such that $x' \in X$ and $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \varepsilon$. Let $U \subset Y$ be open and $x \in f^{-1}(U)$. We define $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset U$. Then, $x' \in B(x, \delta)$ by our supposition. Notice $f(x') \in B(f(x), \varepsilon) \subset U$. Since $x' \in B(x, \delta)$, then $x' \in f^{-1}(U)$.
Thus, $B(x, \delta) \subset f^{-1}(U)$.
Hence, by Theorem 1.4 $f^{-1}(U)$ is open in $X$.
Thus, $f$ is continuous

Therefore, a function $f : X \to Y$ is continuous in the open set definition if and only if for each $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x' \in X$ and $dx\left(x, x'\right) < \delta$ then $d_Y\left(f(x), f\left(x'\right)\right) < \varepsilon$.

5.28 Let $(X, d)$ be a metric space. The function

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a bounded metric on $X$ . (See Exercise 5.10.) Show that the topologies induced by $D$ and $d$ are the same.

($\subset$) Let $x \in X$, $\varepsilon > 0$, and $y \in B_d(x, \delta)$, where $\delta = \frac{\varepsilon}{1-\varepsilon}$.
*WTS:* $y \in B_D(x, \varepsilon) \implies D(x, y) < \varepsilon$
Note $d(x, y) < \delta$. Observe.

$$\begin{aligned} D(x, y) &= \frac{d(x, y)}{1 + d(x, y)} \\ &< \frac{\left(\frac{\varepsilon}{1-\varepsilon}\right)}{1 + \left(\frac{\varepsilon}{1-\varepsilon}\right)} \\ &< \varepsilon \end{aligned}$$

Thus, $y \in B_D(x, \varepsilon)$ and $B_d(x, \delta) \subset B_D(x, \varepsilon)$.
Hence, by theorem 5.15 the topology induced by $d$ is finer than the topology induced by $D$.

($\supset$) Let $x \in X$, $\varepsilon > 0$, and $y \in B_D(x, \delta)$, where $\delta = \frac{\varepsilon}{1+\varepsilon}$.
*WTS:* $y \in B_d(x, \varepsilon) \implies d(x, y) < \varepsilon$
Note $D(x, y) < \delta$ and so $\frac{d(x,y)}{1+d(x,y)} < \delta \implies d(x, y) < \frac{\delta}{1-\delta}$. Observe.

$$\begin{aligned} d(x, y) &< \frac{\delta}{1 - \delta} \\ &= \frac{\left(\frac{\varepsilon}{1+\varepsilon}\right)}{1 - \left(\frac{\varepsilon}{1+\varepsilon}\right)} \\ &< \varepsilon \end{aligned}$$

Hence, $y \in B_d(x, \varepsilon)$ and $B_D(x, \delta) \subset B_d(x, \varepsilon)$
Thus, by theorem 5.15 the topology induced bv $D$ is finer than the topology induced by $d$.

Thus, the two induced topologies are finer than each other.
Therefore, the topologies induced by $D$ and $d$ are the same

5.30 Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Show that if $f : X \to Y$ is such that $d_X\left(x, x'\right) = d_Y\left(f(x), f\left(x'\right)\right)$ for all $x, x' \in X$, then $f$ is injective.

5.31 Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \to Y$ be an isometry between them. Show that $f$ is a homeomorphism between the corresponding metric spaces.

Let $(X, d_X)$ and $(y, d_Y)$ be metric spaces and $f : X \to Y$ be an isometry between them. $f$ is continuous by definition of isometry.
*WTS: $f, f^{-1}$ are continuous*
*WTS: $f$ is continuous*
Let $x \in X$ and $V \subset X$ such that $f(x) \in V$. Let $z \in Y$ and $\varepsilon' > 0$ with $B_d(z, \varepsilon')$. Using theorem 1.9, we define $B_z := B_d(z, \varepsilon')$ such that $f(x) \in B_z$ and $B_z \subset V$. Then by lemma 5.4, $\exists \varepsilon > 0$ such that $B_d(f(x), \varepsilon) \subset B_d(z, \varepsilon')$. It follow that there exists a $\delta > 0$ such that $f(B_d(x, \delta)) \subset B_d(f(x), \varepsilon)$ by theorem 5.13.
Thus, $f(B_d(x, \delta)) \subset B_d(f(x), \varepsilon) \subset B_z \subset V$
Therefore, $f$ is continuous.

*WTS: $f^{-1}$ is continuous.*
Let $y \in Y$ and $U \subset Y$ such that $f^{-1}(y) \in X$. Let $z \in X$ and $\varepsilon' > 0$ with $B_d(z, \varepsilon')$. Using theorem 1.9, we define $B_z := B_d(z, \varepsilon')$ such that $f^{-1}(y) \in B_z$ and $B_z \subset U$. Then by lemma 5.4, $\exists \varepsilon > 0$ such that $B_d(f^{-1}(y), \varepsilon) \subset B_d(z, \varepsilon')$. It follow that there exists a $\delta > 0$ such that $f^{-1}(B_d(y, \delta)) \subset B_d(f^{-1}(y), \varepsilon)$ by theorem 5.13.
Thus, $f^{-1}(B_d(y, \delta)) \subset B_d(f^{-1}(y), \varepsilon) \subset B_z \subset U$

Therefore, $f$ is a homeomorphism between the corresponding metric spaces