2.23 Let $\mathcal{T}$ be the collection of subsets of $\mathbb{R}$ consisting of the empty set and every set whose complement is countable.

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$. (It is called the countable complement topology.)

(i) *WTS: $\varnothing \in \mathcal{T}, R \in \mathcal{T}$*
By definition, $\varnothing \in \mathcal{T}$ and $R^{\complement} = \mathbb{R} - \mathbb{R} = \varnothing$.Hence, $\mathbb{R} \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$

(ii)*WTS: The finite intersection of open sets is an open set*
Let $\{U_n\}_{n \in \mathbb{Z}^+}$ be an finite collection of non-empty open sets in $\mathcal{T}$.
Notice, $\mathbb{R} - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n \mathbb{R} - U_i$. Since each $\mathbb{R} - U_i$ is countable, $\bigcup_{i=1}^n \mathbb{R} - U_i$ must be countable.
Thus, $\mathbb{R} - \bigcap_{i=1}^n U_i$ is countable.
Therefore, The finite intersection of open sets is an open set.

(iii)*WTS: The union of arbitrary open sets is an open set*
Let $\{U_i\}_{i \in I}$ be an arbitrary collection of non-empty open sets in $\mathcal{T}$.
Notice, $\mathbb{R} - \bigcup_{i \in I} U_i = \bigcap_{i \in I} \mathbb{R} - U_i$. Since the intersection of countable sets is countable, we have that $\bigcap_{i \in I} \mathbb{R} - U_i$ is countable.
Thus, $\mathbb{R} - \bigcup_{i \in I} U_i$ is countable.
Therefore, the union of arbitrary open sets is an open set.

Therefore, $\mathcal{T}$ is a topology on $\mathbb{R}$.
(b) Show that the point 0 is a limit point of the set $A = \mathbb{R} - \{0\}$ in the countable complement topology.
WTS: if $\forall U \in \mathcal{T}$ such that $0 \in U$, $U \cap (A - 0) \neq \varnothing$, then 0 is a limit point of A.
Let $U \in \mathcal{T}$ such that $0 \in U$ and define $A := \mathbb{R} - \{0\}$. Note, A is countable and so $A \in \mathcal{T}$. Recall, the set of irrational numbers. In $\mathbb{R}$, the set of irrational numbers is uncountable and cannot be a complement of A or U. So, A and U contain uncountably many irrational numbers. Hence, $\exists i \in \mathbb{I}$ such that $i \in U \cap A$.
Thus, 0 is a limit point of A.
Therefore, the point 0 is a limit point of the set $A = \mathbb{R} - \{0\}$ in the countable complement topology.

(c) Show that in $A = \mathbb{R} - \{0\}$ there is no sequence converging to 0 in the countable complement topology.
Let (x_n) be a sequence in $\mathbb{R} - \{0\}$ and define $U := \mathbb{R} - \{x_n | n \in \mathbb{Z}^+\}$. Note, (x_n) is countable and so $U \in \mathcal{T}$. We also know that $0 \in U$ and U does not have any elements from (x_n) from our definition of the sequence $\mathbb{R} - \{0\}$.
Thus, (x_n) does not converge to 0.
Therefore, in $A = \mathbb{R} - \{0\}$ there is no sequence converging to 0 in the countable complement topology.

2.26 Determine the boundary of each of the following subsets of $\mathbb{R}^2$ in the standard topology:

(a) $A = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$

$$\begin{aligned} \partial A &= Cl(A) - Int(A) \\ X\text{-Axis} &= X\text{-Axis} - \varnothing \end{aligned}$$

(b) $B = \{(x, y) \in \mathbb{R}^2 | x > 0, y \neq 0\}$

$$\begin{aligned} \partial B &= Cl(B) - Int(B) \\ \{(x, y)|x = 0\} \cup \{(x, y)|x > 0, y = 0\} &= \{(x, y)|x \geq 0\} - B \end{aligned}$$

(c) $C = \left\{\left(\frac{1}{n}, 0\right) \in \mathbb{R}^2 | n \in \mathbb{Z}_+\right\}$

$$\begin{aligned} \partial C &= Cl(C) - Int(C) \\ \{(x, 0)|0 < x \leq 1\} &= \{(x, 0)|0 < x \leq 1\} - \varnothing \end{aligned}$$

(d) $D = \{(x, y) \in \mathbb{R}^2 | 0 \leq x^2 - y^2 < 1\}$

$$\begin{aligned} \partial D &= Cl(D) - Int(D) \\ D &= D - \varnothing \end{aligned}$$

2.28 Prove Theorem 2.15 : Let A be a subset of a topological space X.

(a) ∂A is closed.
Observe,

$$\begin{aligned} \partial A &= Cl(A) - Int(A) \\ \partial A &= Cl(A) \cap (X - Int(A)) \end{aligned}$$

Notice, the complement of $Int(A)$ is closed and $Cl(A)$ is closed by definition. Thus, the intersection of closed sets is a closed set.
Therefore, ∂A is closed.
(b) $\partial A = Cl(A) \cap Cl(X - A)$
Observe,

$$\begin{aligned} \partial A &= Cl(A) - Int(A) \\ &= Cl(A) \cap (X - Int(A)) \\ &= Cl(A) \cap Cl(X - A) \text{ by theorem 2.6(ii)} \end{aligned}$$

Therefore, $\partial A = Cl(A) \cap Cl(X - A)$

(c) $\partial A \cap \mathrm{Int}(A) = \varnothing$
Observe,

$$\begin{aligned} \partial A &= Cl(A) - Int(A) \\ \partial A \cap Int(A) &= (Cl(A) - Int(A)) \cap Int(A) \\ \partial A \cap Int(A) &= \varnothing \end{aligned}$$

Therefore, $\partial A \cap \mathrm{Int}(A) = \varnothing$

(d) $\partial A \cup \mathrm{Int}(A) = Cl(A)$
Observe,

$$\begin{aligned} \partial A &= Cl(A) - Int(A) \\ \partial A \cup Int(A) &= (Cl(A) - Int(A)) \cup Int(A) \\ \partial A \cup Int(A) &= Cl(A) \end{aligned}$$

Therefore, $\partial A \cup \mathrm{Int}(A) = Cl(A)$

(e) $\partial A \subset A$ if and only if A is closed.
Observe,

$$\partial A \subset A \Leftrightarrow Cl(A) - Int(A) \subset A \Leftrightarrow Cl(A) = A \Leftrightarrow A \text{ is closed}$$

Therefore, $\partial A \subset A$ if and only if A is closed.

(f) $\partial A \cap A = \varnothing$ if and only if A is open.
Observe,

$$\partial A \cap A = \varnothing = \partial A \cap Int(A) \Leftrightarrow Int(A) = A \Leftrightarrow A \text{ is open}$$

Therefore, $\partial A \cap A = \varnothing$ if and only if A is open.

(g) $\partial A = \varnothing$ if and only if A is both open and closed.
Observe,

$$\partial A = \varnothing \Leftrightarrow Cl(A) - Int(A) = \varnothing \Leftrightarrow Cl(A) = Int(A) \Leftrightarrow A \text{ is open and closed}$$

Therefore, $\partial A = \varnothing$ if and only if A is both open and closed.

3.01 Let $X = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$, the x -axis in the plane. Describe the topology that X inherits as a subspace of $\mathbb{R}^2$ with the standard topology.
The topology is comprised of open intervals on the x-axis. The subspace is $\mathbb{R}$ with the standard topology.

3.02 Let $Y = [-1, 1]$ have the standard topology. Which of the following sets are open in Y and which are open in $\mathbb{R}$?

$$A = (-1, -1/2) \cup (1/2, 1) :: \text{open in } \mathbb{R}; \text{ open in } Y$$
$$B = (-1, -1/2] \cup [1/2, 1) :: \text{not open in } \mathbb{R}; \text{ not open in } Y$$
$$C = [-1, -1/2) \cup (1/2, 1] :: \text{not open in } \mathbb{R}; \text{ open in } Y$$
$$D = [-1, -1/2] \cup [1/2, 1] :: \text{not open in } \mathbb{R}; \text{ not open in } Y$$
$$E = \bigcup_{n=1}^{\infty} \left(\frac{1}{1+n}, \frac{1}{n} \right) :: \text{open in } \mathbb{R}; \text{ open in } Y$$

3.03 Prove Theorem 3.4 : Let X be a topological space, and let $Y \subset X$ have the subspace topology. Then $C \subset Y$ is closed in Y if and only if $C = D \cap Y$ for some closed set D in X.

Proof. Let X be topological space, $Y \subset X$ have the subspace topology, and $C \subset Y$
($\Rightarrow$) *WTS: if C is closed in Y then $C = D \cap Y$*
Assume C is closed in Y. Then $Y - C$ is open in Y. Since $Y - C$ is open, there exists an open set U in X, such that $Y - C = Y \cap U$. As U is open, it follows that $X - U$ is closed. Define $D = X - U$ and hence D is closed in X. Notice,

$$\begin{aligned} Y - C &= Y \cap U \\ C &= Y - (Y \cap U) \\ C &= Y - U \\ C &= Y \cap (X - U) \\ C &= Y \cap D \end{aligned}$$

Hence, $C = D \cap Y$
Thus, if C is closed in Y, then $C = D \cap Y$

($\Leftarrow$) *WTS: if D is a closed set in X and $C = D \cap Y$, then C is closed in Y*
Assume D is a closed set in X and $C = D \cap Y$. Since D is closed in X, $X - D$ is open in X. Which means, by definition of a subspace topology, $Y \cap (X - D)$ is open in Y. Then, $Y - (Y \cap (X - D))$ is closed in Y. Observe,

$$Y - (Y \cap (X - D)) = (Y - Y) \cup (Y - (X - D)) = Y - (X - D) = Y \cap D = C$$

It follows that C is closed in Y.

Therefore, $C \subset Y$ is closed in Y if and only if $C = D \cap Y$ for some closed set D in X. □

3.15 Prove Theorem 3.9: Let X and Y be topological spaces, and assume that $A \subset X$ and $B \subset Y$. Then the topology on $A \times B$ as a subspace of the product $X \times Y$ is the same as the product topology on $A \times B$, where A has the subspace topology inherited from X, and B has the subspace topology inherited from Y.

Proof. Let X and Y be topological spaces, and assume that $A \subset X$ and $B \subset Y$. Using definition 3.6 and theorem 3.7, we have that $\mathcal{B}_{X\times Y} = \{U \times V \mid U \in X, V \in Y\}$. Also, by definition of a subspace we have $\mathcal{B}_{A\times B} = \{(U \times V) \cap (A \times B)\}$. Observe,

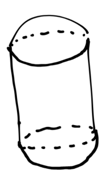
$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

Note, $U \cap A$ and $V \cap B$ are bases elements for the subspace on A and B respectively. Then, $\mathcal{B} = \{(U \cap A) \times (V \cap B)\}$ is our basis for the product topology on $A \times B$. Hence, by rewriting we see they are equivalent.
Therefore, the topology on $A \times B$ as a subspace of the product $X \times Y$ is the same as the product topology on $A \times B$, where A has the subspace topology inherited from X, and B has the subspace topology inherited from Y.

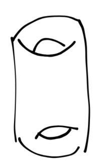
$\square$

3.16 Let S^2 be the sphere, D be the disk, T be the torus, S^1 be the circle, and $I = [0, 1]$ with the standard topology. Draw pictures of the product spaces $S^2 \times I, T \times I, S^1 \times I \times I$, and $S^1 \times D$

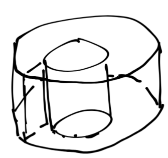
$S^2 \times I$



$T \times I$



$S^1 \times I \times I$



$S^1 \times D$

3.18 Show that if X and Y are Hausdorff spaces, then so is the product space $X \times Y$.

Proof. Let X and Y be Hausdorff spaces and $X \times Y$ be their product space.
WTS: $X \times Y$ is Hausdorff
WTS: $\forall (x, y), (x', y') \in X \times Y$ such that $x \neq x'$ or $y \neq y'$, $\exists U, V \in \mathcal{T}_{X\times Y}$ such that $(x, y) \in U$, $(x', y') \in V$ and $U \cap V = \varnothing$
Let $(x, y), (x', y') \in X \times Y$ with $x \neq x'$ or $y \neq y'$. Without loss of generality, assume $x \neq x'$. Since X is Hausdorff, $\exists U, U' \in \mathcal{T}_X$ such that $x \in U$, $x' \in U'$ and $U \cap U' = \varnothing$. Notice, $U \times Y, U' \times Y \in \mathcal{T}_{X\times Y}$ and $(x, y) \in U \times Y$, $(x', y') \in U'$. Observe,

$$(U \times Y) \cap (U' \times Y) = (U \cap U') \times Y = \varnothing \times Y = \varnothing$$

Thus, we have found two disjoint neighborhoods of two points in $X \times Y$.
Hence, by definition of Hausdorff, $X \times Y$ is also Hausdorff.
Therefore, if X and Y are Hausdorff spaces, then so is the product space $X \times Y$. $\square$

3.19 Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Proof. Let A be a closed subset in X and B be a closed subset in Y.
WTS: $A \times B$ is closed in $X \times Y$
WTS: $X \times Y - A \times B$ is open
WTS: $((X - A) \times Y) \cup (X \times (Y - B))$ is open
Then, $X - A$ is open in X and $Y - B$ is open in Y.
Let $(x, y) \in X \times Y - A \times B$. Then $(x, y) \in X \times Y$ and $(x, y) \notin A \times B$. So, $x \notin A$ or $y \notin B$. Thus, $x \in X - A$ or $y \in Y - B$.Since $X - A$ is open in X, $(X - A) \times Y$ is open in $X \times Y$. Similarly, $Y - B$ is open in Y, $X \times (Y - B)$ must be open in $X \times Y$.
So, $(x, y) \in ((X - A) \times Y) \cup (X \times (Y - B))$ Hence, $X \times Y - A \times B$ is open
Thus, $A \times B$ is closed in $X \times Y$
Therefore, if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

$\square$