1.25 Prove that, in a topological space X, if U is open and C is closed, then $U - C$ Is open and $C - U$ is closed.

Proof. Let X be a topological space with U being open and C being closed.
Notice, $U - C$ can be rewritten to $U \cap C^{\complement}$. By definition of closed set we have that the complement is open. Thus, as U is open and finite intersections of open sets are open we must have that $U - C$ is open.
Notice, $C - U$ can be rewritten to $C \cap U^{\complement}$. By definition of open set we have that the complement is open. Thus, as C is closed and arbitrary unions of closed sets are closed we must have that $C - U$ is closed. □

1.26 Prove that closed balls are closed sets in the standard topology on $\mathbb{R}^2$.

Proof. If the closed ball $\mathcal{B}(x, \epsilon)$ is equal to X, then $\mathcal{B}$ must be closed as the complement is the empty set which by definition must be open. Thus, $\mathcal{B}$ must be closed.
Suppose that the closed ball $\mathcal{B}$ is not equal to X and hence not the empty set. Then there must exist an element $y \in \mathcal{B}^{\complement}$. Let the $d(x, y) = h > \epsilon$.
WTS: $\mathcal{B}'(y, h - \epsilon) \subset \mathcal{B}^{\complement}$
Suppose by way of contradiction, the open ball $\mathcal{B}'(y, h - \epsilon) \not\subset \mathcal{B}^{\complement}$. Then, there exists a $z \in \mathcal{B}'$ such that $z \in \mathcal{B}$.
Notice, $d(x, y) \leq d(x, z) + d(z, y)$. Which gives us $d(x, z) \leq \epsilon$, $d(z, y) < h - \epsilon \Rightarrow r + d(z, y) < h - \epsilon + \epsilon < h$, and $d(x, z) + d(z, y) < h$.
Then by transitivity, we have that $d(x, y) < h$. But this is a contradiction as $d(x, y) = h$. Thus, we have that $z \notin \mathcal{B}'$ which then gives us $z \in \mathcal{B}^{\complement}$. Hence, $\mathcal{B}' \subset \mathcal{B}^{\complement}$. And so, z is an interior point of $\mathcal{B}^{\complement}$ and implies that $\mathcal{B}^{\complement}$ must be open.
Therefore, by definition of closed we must have that $\mathcal{B}$ is closed. □

1.27 The infinite comb C is the subset of the plane illustrated in Figure 1.17 and defined by

$$C = \{(x, 0) | 0 \leq x \leq 1\} \cup \left\{ \left(\frac{1}{2^2}, y \right) | n = 0, 1, 2, \dots \text{ and } 0 \leq y \leq 1 \right\}$$

(a) Prove that C is not closed in the standard topology on $\mathbb{R}^2$. *WTS: $\mathbb{R}^2 - C$ is not open*
Consider the point $p = (0, 1/2) \in \mathbb{R}^2 - C$ and ball centered at point p with radius $\epsilon > 0$. We can find some $\frac{1}{2^n} < \epsilon$. Hence, the point $(\frac{1}{2^n}, \frac{1}{2})$ has distance to p less than ϵ. Thus, we have 0 as a limit point of C, but $(0, \frac{1}{2}) \notin C$. Hence, C is not closed in the standard topology on $\mathbb{R}^2$. (b) Prove that C is closed in the vertical interval topology on $\mathbb{R}^2$.

RECALL: Vertical Interval Topology is generated by $\{\{a\} \times (b, c) \subset \mathbb{R}^2 | a, b, c \in \mathbb{R}\}\}$

WTS: $\mathbb{R}^2 - C$ is open

1.33 Prove theorem 1.17: Let X be a topological space.

Proof. (a) Prove that $\varnothing$ and X are closed sets.
Notice, $\varnothing, X \subseteq X$ and $X - \varnothing = X$. Thus, as $\varnothing$ is open,X is closed. Similarly, $\varnothing - X = \varnothing$. Thus, $\varnothing$ and X are closed sets.

(b) Prove that the intersection of any collection of closed sets in X is a closed set
Let $\cap_{i \in I} U_i$ be the intersection of a indexed collection of closed sets of X. Taking the complement, we have $\cap_{i \in I} X - U_i$. Since, each U_i is closed for each $i \in I$, we have $X - U_i$ is open for each $i \in I$. Thus, we have a intersection of arbitrary open sets which is open. Therefore, $\cap_{i \in I} U_i$ is closed. The intersection of any collection of closed such that in X is a closed set.

(c) Prove that the union of finitely many closed sets in X is a closed set.
Let $\cup_{i=1}^{n} U_i$ be a union of a finite number of closed sets in X. Taking the complement, we have $\cap_{i=1}^{n} X - U_i$. Since, each U_i is closed, we must have $X - \cup_i$ is open. Therefore, the complement is a finite union of open sets, which is open. Thus, $\cup_{i=1}^{n} U_i$ is closed. Thus, the union of finitely many closed sets in X is a closed set.

□

1.35 Show that $\mathbb{R}$ in the lower limit topology is Hausdorff.

Proof. Suppose a, b are distinct point in the lower limit topology. Assume without loss of generality, $a < b$. Notice, $[a, b)$ and $[b, b + 1)$ are disjoint open neighborhoods of a and b. Therefore, $\mathbb{R}$ is Hausdorff in the lower limit topology. □

1.36 Show that $\mathbb{R}$ in the finite complement topology is not Hausdorff.

Proof. By way of contradiction, suppose U, V are disjoint open sets. Then, $V \subset (\mathbb{R} - U)$. Notice, $\mathbb{R} - U$ is a finite set and so V is finite. But $\mathbb{R} - V$ is infinite, which is a contradiction to V is open.
Therefore, $\mathbb{R}$ in the finite complement topology is not Hausdorff. □

2.02 Prove theorem 2.2: Let X be a topological space and A and B be subsets of X.

(a) If C is a closed set in X and $A \subset C$, then $\mathrm{Cl}(A) \subset C$
Let C be a closed set in X and $A \subset C$. Notice the $Cl(A)$ is the finite intersection of closed sets. Thus, $Cl(A) \subset C$ as C is either the smallest closed set in the intersection or larger than the $Cl(A)$.

(b) If $A \subset B$ then $\mathrm{Cl}(A) \subset \mathrm{Cl}(B)$
Let $A \subset B$. Notice, that the $Cl(A)$ and $Cl(B)$ are the smallest closed sets containing A and B respectively. Since, A is contained in $Cl(B)$ we must have that $Cl(A) \subset Cl(B)$

(c) A is closed if and only if $A = \mathrm{Cl}(A)$
Assume A is closed. Then, A is in the intersection of all closed sets and $Cl(A)$ is the smallest closed set containing A. Thus, the intersection will be equal to A. Therefore $A = Cl(A)$

Suppose $A = Cl(A)$. Notice, $Cl(A)$ is closed as the finite intersection of closed sets is closed.
Therefore, A is closed.

2.07– Let $B = \left\{ \frac{a}{2^n} \in \mathbb{R} | a \in \mathbb{Z}, n \in \mathbb{Z}_+ \right\}$. Show that B is dense in $\mathbb{R}$

Proof. Let $\epsilon > 0, x \in \mathbb{R}$, and $a_1, a_2 \in B$ such that $a_1 < x < a_2$. Define $a_1 := \frac{m-1}{2^n}$ and $a_2 := \frac{m+1}{2^n}$. Notice, $\frac{m-1}{2^n} < x < \frac{m+1}{2^n}$ implies $\frac{-1}{2^n} < x - \frac{m}{2^n} < \frac{1}{2^n}$. We can then manipulate this further for $|x - \frac{m}{2^n}| < \frac{1}{2^n} < \epsilon$. Thus, we have $Cl(B) = \mathbb{R}$
Therefore, B is dense in $\mathbb{R}$. □

2.10 Prove Theorem 2.5: Let X be a topological space, A be a subset of X, and y be an element of X. Then $y \in \mathrm{Cl}(A)$ if and only if every open set containing y intersects A.

Proof. Let $(X, \mathcal{T})$ be a topological space, $A \subset X$ and $y \in X$.
($\Rightarrow$) *WTS: If $y \in Cl(A)$ then $\forall U \in \mathcal{T}$ such that $y \in U$ we will have the following $U \cap A \neq \varnothing$*
By way of contradiction, suppose $y \notin Cl(A)$. Then there exists a closed set C such that $y \notin C$. Thus, $X - C$ is open and $y \in X - C \subset X - A$. Notice, $(X - C) \cap A = \varnothing$. This is a contradiction as $X - C$ is an open set containing y, yet $(X - C) \cap = \varnothing$. Therefore, if $y \in Cl(A)$ then every open set containing y intersects A.

($\Leftarrow$) *WTS: If $\forall U \in \mathcal{T}$ such that $y \in U$ and $U \cap A \neq \varnothing$ then $y \in Cl(A)$*
Let U be open and $U \cap A \neq \varnothing$. By theorem 2.4, We have that $U \in Int(A)$. Recall, by definition $Int(A) \subset A \subset Cl(A)$. Since, $y \in U \in Int(A)$ it follows that $y \in Cl(A)$
Therefore $\forall U \in \mathcal{T}$ such that $y \in U$ and $U \cap A \neq \varnothing$ then $y \in Cl(A)$ □

2.11 Prove Theorem 2.6: For sets A and B in a topological space X, the following hold:

(a) $\mathrm{Cl}(X - A) = X - \mathrm{Int}(A)$
If we take the complement of both sides we have the following:

$$X - Cl(X - A) = Int(A)$$

Notice, a point $x \notin Int(A) \Leftrightarrow \forall U \in \mathcal{T}$ such that $x \in U, U \not\subset A$ and $U \cap (X - A) \neq \varnothing \Leftrightarrow x \in Cl(X - A)$.
Thus, $x \in Int(A) \Leftrightarrow x \in X - Cl(X - A)$.
Therefore, $Cl(X - A) = X - Int(A)$

(b) $\mathrm{Int}(A) \cap \mathrm{Int}(B) = \mathrm{Int}(A \cap B)$
Let $n \in Int(A) \cap Int(B)$. Then $n \in Int(A)$ and $n \in Int(B)$. Which gives us that A is a neighborhood of n and B is a neighborhood of n. Thus, $A \cap B$ is a neighborhood of n. Hence, $n \in Int(A \cap B)$.
Therefore, $Int(A) \cap Int(B) \subset Int(A \cap B)$

Let $n \in Int(A \cap B)$. Then, $A \cap B$ is a neighborhood of n. It follows that $n \in Int(A)$ and $n \in Int(B)$. Thus, $n \in Int(A) \cap Int(B)$.
Therefore, $Int(A \cap B) \subset Int(A) \cap Int(B)$,

Therefore, $Int(A) \cap Int(B) = Int(A \cap B)$

2.13 Determine the set of limit points of A in each case.

(a) $A = (0, 1]$ in the lower limit topology on $\mathbb{R}$.
$A' = [0, 1)$

(b) $A = \{a\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$
$A' = \{b, c\}$

(c) $A = \{a, c\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$
$A' = \{b, c\}$

(d) $A = \{b\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$
$A' = \{a, c\}$

(e) $A = (-1, 1) \cup \{2\}$ in the standard topology on $\mathbb{R}$
$A' = [-1, 2]$

(f) $A = (-1, 1) \cup \{2\}$ in the lower limit topology on $\mathbb{R}$ $A' = [-1, 2)$

(g) $A = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the standard topology.
$A' = \mathbb{R}$

(h) $A = \{(0, x) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the topology generated by the basis in Exercise 1.19

(i) $A = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the topology generated by the basis Exercise $1, 19$
$A' = \mathbb{R}$

2.20 Prove Theorem 2.11 : Let A be a subset of R^n in the standard topology. If x is a limit point of A, then there is a sequence of points in A that converges to x.

Proof. *WTS:* $x \in A' \Rightarrow \exists(x_n)$
Let $x \in A$ be a limit point and let $n \in \mathbb{Z}_+$. Then, for each $x_n \in B(x, \frac{1}{n}) \cap A - x$. Notice, $x_n \neq x$ and $x_n \in A$. Since, $d(x_n, x) \leq \frac{1}{n}$, we have x_n converges to x.
Therefore, If x is a limit point of A, then there is a sequence of points in A that converges to x. □

2.21 Determine the set of limit points of the set

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \in \mathbb{R}^2 | 0 < x \leq 1 \right\}$$

as a subset of $\mathbb{R}^2$ in the standard topology. (The closure of S in the plane is known as the topologist's sine curve.)

Proof. Let $y \in [-1, 1]$ and $p = (0, y)$. Notice, for every neighborhood of radius r, $B(p, r) - \{p\}$ contain points in S. Let $n \in \mathbb{R}$ such that $\frac{1}{2\pi n} < r$. Then, $sin(\frac{1}{x})$ maps to all values of $[-1, 1]$, for some $x \in (\frac{1}{2\pi(n+1)}, \frac{1}{2\pi n})$.
Thus, every point in S is a limit point $\square$