Reading:

- Section on *Illustrations* in the preface.
- **Chapter 0** in full.
- **Sections 1.1 and 1.2** in **Chapter 1**.

As mentioned in class you must actively read (do not skip anything in the text). Reread:
– How to Read Mathematics by Shai Simonson and Fernando Gouvea
– Tom Forde's Tips for Reading Your Mathematics Textbook
Even read *all* the HW exercises! Try your best to attempt all the problems, though you don't have to submit them. Also, look over the study the problems with an eye to understanding why/how the problems were created by the authors. This will help you start creating your own questions and producing your own conjectures regarding the material.

The following problems are due by 11:30pm Tuesday 2/18. Submit both LaTeX and pdf files to the appropriate Canvas Dropbox.

Please name the files using the following format:

$$\boxed{\text{LastName\_FirstName\_MTH415\_Spring2020\_HW\_01}}$$

You may discuss the problems with your classmates, but your write-up must be your own. Problems with an asterisk (*) are problems you can not discuss with anyone except for me.

Please include the statements of the problems in your HW submissions. For the Extra problems you can copy the statements from the LaTeX file that generated this pdf. However, you will have to transcribe the remaining problems from our textbook.

HW #1 Problems:

1. Define the binary relation $\sim$ on $\mathbb{R}$ in the following way: for $a, b \in \mathbb{R}$, we say that $a \sim b$ if[1] $a - b \in \mathbb{Z}$. Prove that $\sim$ is an equivalence relation.

   Proof. *WTS:* $\sim$ *is reflexive, symmetric, and transitive*

Reflexivity *WTS:* $\forall a \in \mathbb{R}, a - a \in \mathbb{Z}$
Let $a \in \mathbb{R}$. Notice, $a - a = 0 \in \mathbb{Z}$. Thus, $\sim$ is reflexive.

Symmetry *WTS:* $\forall a, b \in \mathbb{R}, a - b = b - a$
Let $a, b \in \mathbb{R}$. Notice, $a - b = c$ for some $c \in \mathbb{Z}$. We can rewrite as $c = (-1)(b - a) \in \mathbb{Z}$. So, $b - a \in \mathbb{Z}$. Thus, $\sim$ is symmetric.

[1] In a definition, an "if" is always an "if and only if".

Transitivity *WTS: $\forall a, b, c \in \mathbb{R}$, if $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$, then $a - b - (b - c) \in \mathbb{Z}$*
Let $a, b, c \in \mathbb{R}$ such that $a - b = i \in \mathbb{Z}$ and $b - c = j \in \mathbb{Z}$. Notice, $i - j = k \in \mathbb{Z}$. Thus, $\sim$ is Transitive.

So, $\sim$ is reflexive, symmetric, and transitive. Hence, $\sim$ is a binary relation. □

2. Let $\mathcal{M}_3(\mathbb{R})$ be the set of all 3×3 matrices with real entries. Define the function $\phi : \mathcal{M}_3(\mathbb{R}) \to \mathbb{R}$ by the rule $\phi(A) = \sqrt{2}\det(A)$. Prove ϕ is surjective, but not bijective.

Proof. *WTS: ϕ is surjective, and not bijective (i.e not injective)*
WTS: $\forall y \in \mathbb{R}, \exists M \in \mathcal{M}_3(\mathbb{R})$ such that $y = \phi(M)$ and $\neg(\forall M, N \in \mathcal{M}_3(\mathbb{R}), \phi(M) = \phi(N) \Rightarrow M = N)$
WTS: $\forall y \in \mathbb{R}, \exists M \in \mathcal{M}_3(\mathbb{R})$ such that $y = \phi(M)$ and $\exists M, N \in \mathcal{M}(\mathbb{R}), \phi(M) = \phi(N)$ and $M \neq N$

Let $y \in \mathbb{R}$ and $M \in \mathcal{M}_3(\mathbb{R}) := \begin{bmatrix} y & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Observe,

$$\phi(M) = \sqrt{2} \cdot det(M) = \sqrt{2} \cdot \frac{y}{\sqrt{2}} = y$$

Thus, ϕ is surjective.

Let $y \in \mathbb{R}$, $M := \begin{bmatrix} y & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $N := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & y \end{bmatrix}$. Notice, $\phi(M) = y$ and $\phi(N) = y$. But $M \neq N$. Hence, ϕ is not injective. Thus, ϕ is not bijective
Therefore, ϕ is surjective, but not bijective. □

3. For each $n \in \mathbb{N}$ let $A_n := \{(n+1)k \mid k \in \mathbb{N}\}$. Assuming $0 \notin \mathbb{N}$.

   (a) What is $A_1 \cap A_2$?
   $A_1 \cap A_2$ is $\{6k | k \in \mathbb{N}\}$

   (b) Determine the sets $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$.
   $\bigcup_{n \in \mathbb{N}} A_n$ is $\mathbb{N}$ as every possible number will be generated their union will be the entire set $\mathbb{N}$.
   $\bigcap_{n \in \mathbb{N}} A_n$ is $\varnothing$ as there will always exist some value that doesn't exist in the other sets.

4. Show that if $f : A \to B$ and E, F are subsets of A, then

 (a) $f(E \cup F) = f(E) \cup f(F)$

 Proof. *WTS: $f(E \cup F) \subseteq f(E) \cup f(F)$ and $f(E \cup F) \supseteq f(E) \cup f(F)$*
 Let $y \in f(E \cup F)$. Hence, $\exists x \in E \cup F$ such that $f(x) = y$. Then, $x \in E$ or $x \in F$ and so $y \in f(E)$ or $y \in f(F)$. Thus, $y \in f(E) \cup f(F)$. Therefore, $f(E \cup F) \subseteq f(E) \cup f(F)$

 Let $y \in f(E) \cup f(F)$. So, $y \in f(E)$ or $y \in f(F)$. Then, $\exists x \in E$ or $x \in F$ such that $f(x) = y$. Hence, $x \in E \cup F$. Thus $y \in f(E \cup F)$. Therefore, $f(E \cup F) \supseteq f(E) \cup f(F)$

 As the two are subsets of each other they must be equal. Therefore, $f(E \cup F) = f(E) \cup f(F)$ $\square$

 (b) $f(E \cap F) \subseteq f(E) \cap f(F)$ Let $y \in f(E \cap F)$. So, $\exists x \in A$ such that $x \in E$, $x \in F$, and $f(x) = y$. Since $x \in E$ and $x \in F$, we can say $y \in f(E)$ and $y \in f(F)$. Thus, $y \in f(E) \cap f(F)$.
 Therefore, $f(E \cap F) \subseteq f(E) \cap f(F)$

Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ and $E, F \subseteq \mathbb{R}$ such that the $\subseteq$ in (b) is in fact a $\subsetneq$. Next, can you think of a property of functions so that if f possessed that property then it would follow that $f(E \cap F) = f(E) \cap f(F)$.

Consider the function $f = x^2$ with subsets $E := [-1, 0]$ and $F := [0, 1]$. Notice, $f(E \cap F) = \{0\}$ where as $f(E) \cap f(F) = (0, 1)$. If our function was injective then it would follow that $f(E \cap F) = f(E) \cap f(F)$.

5. Show that if $f : A \to B$ and G, H are subsets of B, then

 (a) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$
 WTS: $f^{-1}(G \cup H) \subset f^{-1}(G) \cup f^{-1}(H)$ and $f^{-1}(G \cup H) \supset f^{-1}(G) \cup f^{-1}(H)$
 Let $x \in A$ such that $f(x) \in G \cup H$. Then, $x \in f^{-1}(G \cup H)$. Notice, $f(x) \in G$ or $f(x) \in H$. Taking the preimage of each set we have that, $x \in f^{-1}(G)$ or $x \in f^{-1}(H)$. Hence, $x \in f^{-1}(G) \cup f^{-1}(H)$. Thus, $f^{-1}(G \cup H) \subset f^{-1}(G) \cup f^{-1}(H)$

 Let $x \in A$ such that $x \in f^{-1}(G) \cup f^{-1}(H)$. Then, $f(x) \in G$ or $f(x) \in H$. Hence, $f(x) \in G \cup H$. Taking the preimage of this, we have that $x \in f^{-1}(G \cup H)$. Thus, $f^{-1}(G \cup H) = f^{-1}(G) \supset f^{-1}(H)$

 Thus, $f^{-1}(G \cup H) \subset f^{-1}(G) \cup f^{-1}(H)$ and $f^{-1}(G \cup H) \supset f^{-1}(G) \cup f^{-1}(H)$
 Therefore, $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

 (b) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$
 WTS: $f^{-1}(G \cap H) \subset f^{-1}(G) \cap f^{-1}(H)$ and $f^{-1}(G \cap H) \supset f^{-1}(G) \cap f^{-1}(H)$
 Let $x \in A$ such that $f(x) \in G \cap H$. Then, $x \in f^{-1}(G \cap H)$. Notice, $f(x) \in G$

and $f(x) \in H$. Taking the preimage of each set we have that, $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$. Hence, $x \in f^{-1}(G) \cap f^{-1}(H)$. Thus, $f^{-1}(G \cap H) \subset f^{-1}(G) \cap f^{-1}(H)$

Let $x \in A$ such that $x \in f^{-1}(G) \cap f^{-1}(H)$. Then, $f(x) \in G$ and $f(x) \in H$. Hence, $f(x) \in G \cap H$. Taking the preimage of this, we have that $x \in f^{-1}(G \cap H)$. Thus, $f^{-1}(G \cap H) = f^{-1}(G) \supset f^{-1}(H)$

Thus, $f^{-1}(G \cap H) \subset f^{-1}(G) \cap f^{-1}(H)$ and $f^{-1}(G \cap H) \supset f^{-1}(G) \cap f^{-1}(H)$
Therefore, $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

6. (a) Show that if $f : A \to B$ is injective and $K \subseteq A$, then $f^{-1}(f(K)) = K$. Give an example to show that equality need not hold if f is not injective.

Proof. Let $f : A \to B$ be an injective function and $K \subseteq A$. Let $k \in K$. Then, $f(k) \in f(K)$ and so $k \in f^{-1}(f(K))$. Thus, $K \subseteq f^{-1}(f(K))$. We define, $b := f(k) \in f(K)$. Since $b \in f(K), \exists k' \in K$ such that $f(k') = b$. Since f is injective and $f(k) = b = f(k')$, we have that $k = k'$. Thus, $f^{-1}(f(K)) \subseteq K$. Therefore, if f is injective, then $f^{-1}(f(K)) = K$ □

Consider the function $f := x^2$. Let $K := \{-1, 0, 1\}$. Then $f(K) = \{0, 1\}$ and so $f^{-1}(f(K)) = \{0, 1\}$. Thus, $K \neq f^{-1}(f(K))$ when f is not injective.

(b) Show that if $f : A \to B$ is onto and $L \subseteq B$, then $f(f^{-1}(L)) = L$. Give an example to show that equality need not hold if f is not onto.

Proof. Let $f : A \to B$ be an onto function and $L \subseteq B$. Let $l \in f(f^{-1}(L))$. Then, $\exists x \in f^{-1}(L)$ such that $f(x) = l \in L$. Thus, $f(f^{-1}(L)) \subseteq L$. As f is onto, $l = f(x) \in f(f^{-1}(L))$. Thus, $L \subseteq f(f^{-1}(L))$
Therefore, if f is onto, then $f(f^{-1}(L)) = L$ □

Consider $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 2x$. Notice, f is not onto as all odd integers cannot be mapped from $\mathbb{N} \to \mathbb{N}$. Let $L = \{1, 2\}$. Thus, $f^{-1}(L)$ cannot be represented. That is $f(f^{-1}(L)) \subsetneq L$ as L contains 1 while $f(f^{-1}(L))$ does not.

7. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Show that if $g \circ f$ is one-to-one, then f is one-to-one.
WTS: f is one-to-one
WTS: If $f(a) = f(a')$ then $a = a'$ for some $a, a' \in A$
Let $g \circ f : A \to C$ be a one-to-one function and $a, a' \in A$ such that $g(f(a)) = g(f(a'))$. Notice,

$$g(f(a)) = g(b) = c$$
$$g(f(a')) = g(b') = c'$$

Then, $g^{-1}(c) = g^{-1}(c')$ That is $b = b'$. So, $f^{-1}(b) = f^{-1}(b')$. Hence, $a = a'$. Thus, $f(a) = f(a') \Rightarrow a = a'$
Therefore, f is one-to-one

(b) Show that if $g \circ f$ is surjective, then g is surjective. *WTS: g is surjective*
WTS: $\forall c \in C, \exists b \in B, g(b) = c$
Let $g \circ f : A \to C$ be a surjective function and $c \in C$. Since, $g \circ f$ is surjective, $\exists a \in A$ such that $g(f(a)) = c$. Notice, $f(a) = b$ for some $b \in B$. So, $g(b) = c$.
Therefore, g is surjective.

8. #1.7 in Section 1.1
Let X be a set and assume $p \in X$. Show that the collection T, consisting of $\varnothing, X$, and all subsets of X containing p, is a topology on X. This topology is called the particular point topology on X, and we denote it by PPX_P.

Proof. *WTS: $\mathcal{T}$ is a topology*
WTS: i.) $X, \varnothing \in \mathcal{T}$
ii.) If $p \in X$ and $S_1, S_2, \ldots, S_n \in \mathcal{T}, \bigcap_{i=1}^n p \in S_i \in \mathcal{T}$
iii.) If $p \in X$ and $\{U_\alpha\}$ be a collection of open sets on $X, \bigcup p \in U_\alpha \in \mathcal{T}$
Let $p \in X$ and $\mathcal{T}$ be a collection consisting of $\varnothing, X$ and the open sets $S_1, S_2, \ldots, S_n$ in $\mathcal{T}$ such that $p \in S_1, p \in S_2, \ldots, p \in S_n$
Then, by definition $X, \varnothing \in \mathcal{T}$. Since every open set in $\mathcal{T}$ has the common element p, we must have the $p \in \bigcap_{i=1}^n S_i \in \mathcal{T}$. Similarly, let I be an indexing set. Then, $p \in \bigcup_{\alpha \in I} S_\alpha \in \mathcal{T}$, since each S_α has the point p, we have that any arbitrary union will have p as well.
Therefore, $\mathcal{T}$ is a topology on X □

9. #1.8 in Section 1.1
Let X be a set and assume $p \in X$. Show that the collection T, consisting of $\varnothing, X$, and all subsets of X that exclude p, is a topology on X. This topology is called the excluded point topology on X, and we denote it by EPX_p.

Proof. Let $p \in X$ and $\mathcal{T}$ be a collection consisting of $\varnothing, X$ and the open sets $S_1, S_2, \ldots, S_n$ in $\mathcal{T}$ such that $p \notin S_1, p \notin S_2, \ldots, p \notin S_n$
Then, by definition $X, \varnothing \in \mathcal{T}$. Since every open set in $\mathcal{T}$ does not have the common element p, we must have the $p \notin \bigcap_{i=1}^n S_i \in \mathcal{T}$. Similarly, let I be an indexing set. Then, $p \notin \bigcup_{\alpha \in I} S_\alpha \in \mathcal{T}$, since each S_α does not have the point p, we have that any arbitrary union will not have p either.
Therefore, $\mathcal{T}$ is a topology on X □

10. #1.9 in Section 1.1
Let $\mathcal{T}$ consist of $\varnothing, \mathbb{R}$, and all intervals $(-\infty, p)$ for $p \in \mathbb{R}$. Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$

Proof. Let $\mathcal{T}$ consist of $\varnothing, \mathbb{R}$, and all intervals $(-\infty, p)$ for $p \in \mathbb{R}$ namely $(-\inf, p_1), (-\inf, p_2), \ldots, (-$
Then, by definition $\mathbb{R}, \varnothing \in \mathcal{T}$. We define $P_{min} := min(\{p_1, p_2, \ldots, p_n\})$. Notice, $\bigcap_{i=1}^n (-\inf, p_i) = (-\inf, P_{min}) \in \mathcal{T}$.
Similarly, $\bigcup_{i=1} (-\inf, p_i) = (-\inf, max\{p_i\}) \in \mathcal{T}$.
Therefore, $\mathcal{T}$ is a topology on $\mathbb{R}$. □

11. #1.10 in Section 1.2
Show that $\mathcal{B} = \{[a,b) \subset \mathbb{R} | a < b\}$ is a basis for a topology on $\mathbb{R}$

Proof. *WTS:* $\forall x \in \mathbb{R}, \exists B \in \mathcal{B}$ *such that* $x \in B$
$\forall B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 \cap B_2, \exists B_3 \in \mathcal{B}$ *such that* $x \in B_3 \subset B_1 \cap B_2$.
Let $\mathcal{B} := \{[a,b) \subset \mathbb{R} | a < b\}$ and $x \in \mathbb{R}$. Let $B \in \mathcal{B}$ such that $B := [x-1, x+1)$. Notice, x is the center of B. So, $x \in B$.
Let $a, b, a', b' \in \mathbb{R}$ such that $a < b$ and $a' < b'$ Let $B_1, B_2 \in \mathcal{B}$ such that $B_1 := [a,b), B_2 := [a', b')$. We define $a_{max} = max\{a, a'\}$ and $b_{min} = min\{b, b'\}$. So, $B_3 := [a_{max}, b_{min})$. Hence, $x \in B_3 \subset B_1 \cap B_2$.
Therefore, $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$ □

12. #1.12 in Section 1.2
Determine which of the following are open sets in $\mathbb{R}_l$. In each case, prove your assertion.
$A = [4,5) \in \mathbb{R}_l$ as $A \in \mathcal{B}$
$B = \{3\}, \notin \mathbb{R}_l$ as open sets of R_l cannot be singleton sets.
$C = [1,2], \notin \mathbb{R}_l$ as $C \notin \mathcal{B}$
$D = (7,8) \in \mathbb{R}_l$ as $(7,8) = \bigcup_{7<a<8} [a,8)$

13. Prove that $\mathbb{R}_\ell$ is strictly finer than the standard topology on $\mathbb{R}$.

Proof. *WTS:* $\mathbb{R} \subset \mathbb{R}_l$
. Let $(a,b) \in \mathbb{R}$ and $\varepsilon < \frac{b-a}{2}$. Notice, $(a,b) = \bigcup_{n \geq 1} [a + \frac{\varepsilon}{n}, b) \in \mathbb{R}_\ell$. Thus, R_ℓ is finer than the standard topology $\mathbb{R}$. As $[0,1)$ is not in the standard topology on $\mathbb{R}$, yet $[0,1)$ is in the lower limit topology, we must have that $\mathbb{R}_\ell$ is strictly finer. □

14. Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a given set X. What do you need to show to prove that the two topologies are not comparable? Prove that the upper limit topology and lower limit topology on $\mathbb{R}$ are not comparable.
In order to show two topologies are not comparable we need show $\mathcal{T}_1 \not\subset \mathcal{T}_2$ and $\mathcal{T}_2 \not\subset \mathcal{T}_1$

15. #1.15 in Section 1.2
An arithmetic progression in $\mathbb{Z}$ is a set

$$A_{a,b} = \{\ldots, a-2b, a-b, a, a+b, a+2b, \ldots\}$$

with $a, b \in \mathbb{Z}$ and $b \neq 0$. Prove that the collection of arithmetic progressions

$$\mathcal{A} = \{A_{a,b} | a, b \in \mathbf{Z} \text{ and } b \neq 0\}$$

is a basis for a topology on $\mathbb{Z}$. The resulting topology is called the arithmetic progression topology on Z.