MATH 411 HOMEWORK 3 SOLUTIONS

ADAM LEVINE

2.17.6. *Let A, B, and A_α denote subsets of a space X.*

(a) *Show that if $A \subset B$, then $\bar{A} \subset \bar{B}$.*

By definition, $\bar{A}$ is the intersection of all closed sets containing A. Since $\bar{B}$ is a closed set that contains B and hence A, it must therefore contain $\bar{A}$.

(b) *Show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.*

The set $\overline{A \cup B}$ is a closed set that contains $A \cup B$, so it contains both A and B, and therefore it contains both $\bar{A}$ and $\bar{B}$. Therefore, $\overline{A \cup B} \supset \bar{A} \cup \bar{B}$.

Conversely, $\bar{A} \cup \bar{B}$ is a closed set (since it is the union of two closed sets) that contains $A \cup B$, so it contains $\overline{A \cup B}$. (Note that we had to use the fact that finite intersections of closed sets are closed!)

(c) *Show that $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.*

Just as in (b), the set $\overline{\bigcup A_\alpha}$ is a closed set that contains each A_α, so it contains each $\bar{A}_\alpha$, and thus it contains $\bigcup \bar{A}_\alpha$. (But we can't do the converse because an arbitrary union of closures isn't necessarily closed!)

As a counterexample, let A_i be the closed set $[\frac{1}{i}, 1] \subset \mathbb{R}$. Then $\bigcup_{i \in \mathbb{N}} \bar{A}_i = \bigcup_{i \in \mathbb{N}} A_i = (0,1]$, while $\overline{\bigcup_{i \in \mathbb{N}} A_i} = [0,1]$.

2.17.11. *Show that the product of two Hausdorff spaces is Hausdorff.*

Let X and Y be Hausdorff spaces. For any two distinct points (x,y) and (x',y') in $X \times Y$, we may assume that they differ in at least one coordinate. If $x \neq x'$, choose disjoint open sets $U, U' \subset X$ with $x \in U$ and $x' \in U'$. Then $U \times Y$ and $U' \times Y$ are disjoint open sets in $X \times Y$ containing (x,y) and (x',y') respectively. Likewise, if $y \neq y'$, choose disjoint open sets $V, V' \subset Y$ with $y \in V$ and $y' \in V'$; then $X \times V$ and $X \times V'$ are the needed open sets.

2.17.12. *Show that a subspace of a Hausdorff space is Hausdorff.*

Let X be Hausdorff, and let $A \subset X$ be a subspace. For any $x, x' \in A$, we may find disjoint open sets U, U' with $x \in U$ and $x' \in U'$. Then $U \cap A$ and $U' \cap A$ are disjoint open sets in A containing x and x' respectively.

2.17.13. *Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.*

We will show an equivalent condition: that $X \times X - \Delta$ is open.

If X is Hausdorff, then for any $x \times x' \in X \times X - \Delta$, we have $x \neq x'$. Choose disjoint open sets U, U' with $x \in U$ and $x' \in U'$. Then $U \times U'$ is open in $X \times X$, and it does not contain any points of Δ. Therefore, $X \times X - \Delta$ is open.

Conversely, if $X \times X \smallsetminus \Delta$ is open, then for any point $x \times x' \in X \times X - \Delta$, there is a basic open set $U \times U'$ containing $x \times x'$ and disjoint from Δ. Therefore, U and U' are disjoint open sets containing x and x' respectively.

2.17.17. *Consider the lower limit topology and the topology given by the basis $\mathcal{C} = \{[a,b) \mid a,b \in \mathbb{Q},\ a < b\}$. Determine the closures of the intervals $A = (0,\sqrt{2})$ and $B = (\sqrt{2},3)$ in these two topologies.*

In the lower limit topology, we claim that the closure of any interval (a,b) is $[a,b)$. If $x < a$ or $x \geq b$, we can easily find an open set $[x, x+\epsilon)$ disjoint from (a,b). On the other hand, a is a limit point: any open set containing a must contain an interval $[a, a+\epsilon)$, which intersects (a,b). Thus, $\bar{A} = [0,\sqrt{2})$ and $\bar{B} = [\sqrt{2},3)$.

In the topology given by $\mathcal{C}$, the only difference is $\sqrt{2}$ is a limit point of A. Indeed, if $[a,b)$ is an element of $\mathcal{C}$ containing $\sqrt{2}$, we must have $a < \sqrt{2}$. Thus, $\bar{A} = [0,\sqrt{2}]$ and $\bar{B} = [\sqrt{2},3)$.

2.17.18. *Determine the closures of the following subsets of the ordered square:*

(a) $A = \{\frac{1}{n} \times 0 \mid n \in \mathbb{Z}_+\}$

We claim that $\bar{A} = A \cup \{0 \times 1\}$. The point 0×1 is a limit point because any open set containing 0×1 must contain $(0,\epsilon) \times [0,1]$ for some $\epsilon > 0$, and therefore meets A. Any other point $x \times y \in I \times I - A$ can be seen to have a neighborhood (specifically, an interval in the dictionary ordering) that is disjoint from A. Namely, if $x = 0$ and $y < 1$, then we can use $[0 \times 0, 0 \times 1)$. If $0 < x < 1$, then choose $n \in \mathbb{Z}_+$ with $\frac{1}{n+1} \leq x < \frac{1}{n}$, and take the interval $(\frac{1}{n+1} \times 0, \frac{1}{n} \times 0)$. If $x = 1$ and $y > 0$, then take $(1 \times 0, 1 \times 1]$.

(b) $B = \{(1 - \frac{1}{n}) \times \frac{1}{2} \mid n \in \mathbb{Z}_+\}$

We claim that $\bar{A} = B \cup \{1 \times 0\}$. The argument is similar to the previous example.

(c) $C = \{x \times 0 \mid 0 < x < 1\}$

Any point $x \times y$ with $0 < y < 1$ has a neighborhood distinct from C, as do the points 0×0 and 1×1. On the other hand, any neighborhood of a point $x \times 0$ (with $0 < x \leq 1$) or $x \times 1$ (with $0 \leq x < 1$) must include an entire vertical strip (as in (a)), and thus intersect C. Thus,

$$\bar{C} = (0,1] \times \{0\} \cup [0,1) \times \{1\}$$

(d) $D = \{x \times \frac{1}{2} \mid 0 < x < 1\}$

Any point $x \times y$ with $0 < y < \frac{1}{2}$ or $\frac{1}{2} < y < 1$, or with $x = 0$ and $0 \leq y < 1$, or with $x = 1$ and $0 < y \leq 1$, has a neighborhood disjoint from D. On the other hand, any neighborhood of $x \times 0$ (with $0 < x \leq 1$) or of $x \times 1$ (with $0 \leq x < 1$) must include entire vertical segments, and thus intersects D. Thus,

$$\bar{D} = D \cup \{x \times 0 \mid 0 < x \leq 1\} \cup \{x \times 1 \mid 0 \leq x < 1\}.$$

(e) $E = \{\frac{1}{2} \times y \mid 0 < y < 1\}$

Any point $x \times y$ with $0 \leq x < \frac{1}{2}$ or $\frac{1}{2} < x <\leq 1$ has a neighborhood disjoint from E. The points $\frac{1}{2} \times 0$ and $\frac{1}{2} \times 1$ are limit points. Therefore, $\bar{E} = \{\frac{1}{2} \times y \mid 0 \leq y \leq 1\}$.

2.18.2. *Suppose that $f\colon X \to Y$ is continuous. If x is a limit point of $A \subset X$, is it necessarily true that $f(x)$ is a limit point of $f(A)$?*

Suppose f is a constant function: $f(x) = y_0$ for all $x \in X$. Then for any limit point x of A, we have $f(x) = y_0$. Since $f(A)$ has only one point, that point cannot be a limit point.

2.18.7(a). *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is "continuous from the right," i.e.,* $\lim_{x\to a^+} f(x) = f(a)$ *for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.*

As a review from calculus, we say that $\lim_{x\to a^+} f(x) = b$ if for all $\epsilon > 0$, there is a $\delta > 0$ such that for all x with $a < x < x + \delta$, we have $|f(x) - b| < \epsilon$.

Now, assume that $\lim_{x\to a^+} f(x) = f(a)$ for each $a \in \mathbb{R}$. Let $U \subset \mathbb{R}$ be an open set. For any $a \in f^{-1}(U)$, choose some $\epsilon > 0$ such that $(f(a) - \epsilon, f(a) + \epsilon) \subset U$. By assumption, there is a $\delta > 0$ such that $f([a, a + \delta)) \subset (f(a) - \epsilon, f(a) + \epsilon) \subset U$, and therefore $[a, a + \delta) \subset f^{-1}(U)$. Thus, $f^{-1}(U)$ is open in $\mathbb{R}_\ell$.

Note: We will see in Chapter 3 that the only continuous maps from $\mathbb{R}$ to $\mathbb{R}_\ell$ are constant maps.

2.18.8. *Let Y be an ordered set in the order topology. Let $f, g\colon X \to Y$ be continuous.*

(a) *Show that the set $\{x \mid f(x) \le g(x)\}$ is closed in X.*

In class we proved the following: Given an ordered set Y, for any two distinct elements $a, b \in Y$ with $a < b$, there are open sets U, V such that $a \in U$, $b \in V$, and for all $c \in U$ and $d \in V$, $c < d$.

Now, let us show that the set $A = \{x \mid f(x) > g(x)\}$ (which is the complement of the one above) is open in X. Suppose $x \in A$. We may find open sets U and V as above with $g(x) \in U$ and $f(x) \in V$. Then $g^{-1}(U)$ and $f^{-1}(V)$ are open in X, and $g^{-1}(U) \cap f^{-1}(V)$ contains x and is contained in A. Thus, A is open as required.

(b) *Let $h\colon X \to Y$ be the function $h(x) = \min\{f(x), g(x)\}$. Show that h is continuous.*

By the preceding discussion, the sets $C_1 = \{x \mid f(x) \le g(x)\}$ and $C_2 = \{x \mid f(x) \ge g(x)\}$ are closed, and their union equals X. The function h equals f on C_1 and g on C_2, and these agree on the overlap. Hence, by the pasting lemma, h is continuous.