PP: 1.25 Prove that, in a topological space X, if U is open and C is closed, then $U - C$ Is open and $C - U$ is closed.

PP: 1.26 Prove that closed balls are closed sets in the standard topology on $\mathbb{R}^2$.

PP: 1.27 The infinite comb C is the subset of the plane illustrated in Figure 1.17 and defined by

$$C = \{(x, 0) | 0 \leq x \leq 1\} \cup \left\{ \left(\frac{1}{2^2}, y \right) | n = 0, 1, 2, \dots \text{ and } 0 \leq y \leq 1 \right\}$$

(a) Prove that C is not closed in the standard topology on $\mathbb{R}^2$.

(b) Prove that C is closed in the vertical interval topology on $\mathbb{R}^2$.

PP: 1.33 Prove theorem 1.17: Let X be a topological space.

(a) Prove that $\varnothing$ and X are closed sets.

(b) Prove that the intersection of any collection of closed sets in X is a closed set

(c) Prove that the union of finitely many closed sets in X is a closed set.

PP: 1.35 Show that $\mathbb{R}$ in the lower limit topology is Hausdorff.

PP: 1.36 Show that $\mathbb{R}$ in the finite complement topology is not Hausdorff.

PP: 2.02 Prove theorem 2.2: Let X be a topological space and A and B be subsets of X.

(a) If C is a closed set in X and $A \subset C$, then $\mathrm{Cl}(A) \subset C$

(b) If $A \subset B$ then $\mathrm{Cl}(A) \subset \mathrm{Cl}(B)$

(c) A is closed if and only if $A = \mathrm{Cl}(A)$

PP: 2.07 Let $B = \left\{ \frac{a}{2^n} \in \mathbb{R} | a \in \mathbb{Z}, n \in \mathbb{Z}_+ \right\}$. Show that B is dense in $\mathbb{R}$

2.10 Prove Theorem 2.5: Let X be a topological space, A be a subset of X, and y be an element of X. Then $y \in \mathrm{Cl}(A)$ if and only if every open set containing y intersects A.

Proof. Let $(X, \mathcal{T})$ be a topological space, $A \subset X$ and $y \in X$.
($\Rightarrow$) *WTS: If $y \in Cl(A)$ then $\forall U \in \mathcal{T}$ such that $y \in U$ we will have the following $U \cap A \neq \varnothing$*
By way of contradiction, suppose $y \notin Cl(A)$. Then there exists a closed set C such that $y \notin C$. Thus, $X - C$ is open and $y \in X - C \subset X - A$. Notice, $(X - C) \cap A = \varnothing$. This is a contradiction as $X - C$ is an open set containing y, yet $(X - C) \cap = \varnothing$.

Therefore, if $y \in Cl(A)$ then every open set containing y intersects A.

($\Leftarrow$) *WTS: If $\forall U \in \mathcal{T}$ such that $y \in U$ and $U \cap A \neq \varnothing$ then $y \in Cl(A)$*
Let U be open and $U \cap A \neq \varnothing$. By theorem 2.4, We have that $U \in Int(A)$. Recall, by definition $Int(A) \subset A \subset Cl(A)$. Since, $y \in U \in Int(A)$ it follows that $y \in Cl(A)$
Therefore $\forall U \in \mathcal{T}$ such that $y \in U$ and $U \cap A \neq \varnothing$ then $y \in Cl(A)$ □

PP: 2.11 Prove Theorem 2.6: For sets A and B in a topological space X, the following hold:

(a) $\mathrm{Cl}(X - A) = X - \mathrm{Int}(A)$

(b) $\mathrm{Int}(A) \cap \mathrm{Int}(B) = \mathrm{Int}(A \cap B)$

2.13 Determine the set of limit points of A in each case.

(a) $A = (0, 1]$ in the lower limit topology on $\mathbb{R}$.
$A' = [0, 1)$

(b) $A = \{a\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$
$A' = \{b, c\}$

(c) $A = \{a, c\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$
$A' = \{b, c\}$

(d) $A = \{b\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$
$A' = \{a, c\}$

(e) $A = (-1, 1) \cup \{2\}$ in the standard topology on $\mathbb{R}$
$A' = [-1, 2]$

(f) $A = (-1, 1) \cup \{2\}$ in the lower limit topology on $\mathbb{R}$ $A' = [-1, 2)$

(g) $A = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the standard topology.
$A' = \mathbb{R}$

(h) $A = \{(0, x) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the topology generated by the basis in Exercise 1.19

(i) $A = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the topology generated by the basis Exercise $1, 19$
$A' = \mathbb{R}$

2.20 Prove Theorem 2.11 : Let A be a subset of R^n in the standard topology. If x is a limit point of A, then there is a sequence of points in A that converges to x.

Proof. *WTS:* $x \in A' \Rightarrow \exists (x_n)$

□

PP: 2.21 Determine the set of limit points of the set

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \in \mathbb{R}^2 | 0 < x \leq 1 \right\}$$

as a subset of $\mathbb{R}^2$ in the standard topology. (The closure of S in the plane is known as the topologist's sine curve.)