2.5.2b Determine all solutions in the integers of the following Diophantine equations. $24x + 138y = 18$

$$\begin{aligned} 138 &= 5(24) + 18 \\ 24 &= 1(18) + 6 \\ 18 &= 3(6) + 0 \\ \gcd(138, 24) &= 6 \end{aligned}$$

Then,

$$\begin{aligned} 6 &= 24 - 18 \cdot 1 \\ &= 24 - 1(138 - 5(24)) \\ &= 6(24) - 1(138) \\ 18 &= 18(24) - 3(138) \end{aligned}$$

So $x_0 = 18$ and $y_0 = -3$. Therefore, $x = 18 + 23t$ and $y = -3 - 4t, t \in \mathbb{Z}$

2.5.3b Determine all solutions in the positive integers of the following Diophantine. $54x + 21y = 906$

$$\begin{aligned} 54 &= 2(21) + 12 \\ 21 &= 1(12) + 9 \\ 12 &= 1(9) + 3 \\ 9 &= 3(3) + 0 \\ \gcd(54, 21) &= 3 \end{aligned}$$

Then,

$$\begin{aligned} 3 &= 12 - 9 \\ &= 12 - (21 - 12) \\ &= 12 - (21 - (54 - 2(21))) \\ &= 2(54) - 3(21) \\ 906 &= 604(54) - 1510(21) \end{aligned}$$

As we are looking for only positive solutions, $x_0 = 16$ and $y_0 = 2$. Therefore, $x = 16+7t$ and $y = 2 - 18t, t \in \mathbb{Z}$

2.5.5b The neighborhood theater charges \$1.80 for adult admissions and \$.75 for children. On a particular evening the total receipts were \$90. Assuming that more adults than children were present, how many people attended?

The problem can be simplified to $1.80x + .75y = 90$. Then multiplying by 10,

$$\begin{aligned} 180x + 75y &= 9000 \\ 12x + 5y &= 600 \end{aligned}$$

Finding the $\gcd(12, 5)$

$$\begin{aligned} 12 &= 2(5) + 2 \\ 5 &= 2(2) + 1 \\ 2 &= 2(1) + 0 \\ \gcd(12, 5) &= 1 \end{aligned}$$

Then,

$$\begin{aligned} 1 &= 5 - 2(2) \\ &= 5 - 2(12 - 2(5)) \\ &= 5 - 2(12) + 4(5) \\ &= 5(5) - 2(12) \\ 600 &= 3000(5) - 1200(12) \end{aligned}$$

So, $x_0 = 30$ and $y_0 = 12$. Therefore, $x = 30 + 5t$ and $y = 12 - 2t, t \in \mathbb{Z}$

2.5.8e Euler, 1770. Divide 100 into two summands such that one is divisible by 7 and the other by 11.
Suppose that $x + y = 100$ with $\frac{x}{7} = k$ and $\frac{y}{11} = j$. Then, $7j + 11k = 100$.

$$\begin{aligned} 11 &= 1(7) + 4 \\ 7 &= 1(4) + 3 \\ 4 &= 1(3) + 1 \\ 3 &= 3(1) + 0 \end{aligned}$$

Then,

$$\begin{aligned} 1 &= 4 - (7 - 4) \\ &= 4 - (7 - 11 - 1(7)) \\ &= 7(-3) + 11(2) \\ 100 &= 7(-300) + 11(200) \end{aligned}$$

So, $x_0 = -300$ and $y_0 = 200$. Thus, $x = -300 - 11t$ and $y = 200 + 7t, t \in \mathbb{Z}$. As we're only looking for positive solutions, let $t = 28$. Therefore, $100 = 7(8) + 11(4) = 56 + 44$