7.2.4 (c) $\phi(3n) = 3\phi(n)$ if and only if $3 \mid n$.

Suppose that $\phi(3n) = 3\phi(n)$. Assume for the sake of contradiction that $3 \nmid n$. Since 3 is a prime number, $\gcd(3, n) = 1$. Since ϕ is multiplicative,

$$\phi(3n) = \phi(3)\phi(n) = 2\phi(n).$$

This contradicts the assumption that $\phi(3n) = 3\phi(n)$. Thus $3 \mid n$.

Conversely suppose that $3 \mid n$. Then $n = 3^k m$ where $\gcd(3^k, m) = 1$. So

$$3\phi(n) = 3\phi(3^k m) = 3\phi(3^k)\phi(m) = (3^{k+1} - 3^k)\phi(m).$$

Also

$$\phi(3n) = \phi(3^{k+1}m) = \phi(3^{k+1})\phi(m) = (3^{k+1} - 3^k)\phi(m).$$

Thus

$$\phi(3n) = 3\phi(n).$$

7.2.5 Prove that the equation $\phi(n) = \phi(n+2)$ is satisfied by $n = 2(2p-1)$ whenever p and $2p-1$ are odd primes.

Suppose that p and $2p-1$ are odd primes and $n = 2(2p-1)$. Then $n + 2 = 2^2 p$. So

$$\phi(n) = \phi(2(2p-1)) = (2-1)((2p-1)-1) = 2p-2$$

since $2p - 1 \neq 2$ as it is an odd prime. Also notice $\gcd(4, p) = 1$

$$\phi(n+2) = \phi(2^2 p) = (2^2 - 2)(p-1) = 2p-2.$$

Therefore

$$\phi(n) = \phi(n+2)$$

for $n = 2(2p-1)$.

7.2.10 If every prime that divides n also divides m, establish that $\phi(nm) = n\phi(m)$; in particular, $\phi(n^2) = n\phi(n)$ for every positive integer n.

Since every prime that divides n also divides m, we can suppose that $n = p_1^{k_1} \cdots p_r^{k_r}$ and $m = p_1^{t_1} \cdots p_r^{t_r} \cdot p_{r+1}^{t_{r+1}} \cdots p_q^{t_q}$, where each p_j is a prime. So

$$\begin{aligned}
\phi(nm) &= (p_1^{k_1+t_1} - p_1^{k_1+t_1-1}) \cdots (p_r^{k_r+t_r} - p_r^{k_r+t_r-1})(p_{r+1}^{t_{r+1}} - p_{r+1}^{t_{r+1}-1}) \cdots (p_q^{t_q} - p_q^{t_q-1}) \\
&= p_1^{k_1}(p_1^{t_1} - p_1^{t_1-1}) \cdots p_r^{k_r}(p_r^{t_r} - p_r^{t_r-1})(p_{r+1}^{t_{r+1}} - p_{r+1}^{t_{r+1}-1}) \cdots (p_q^{t_q} - p_q^{t_q-1}) \\
&= n(p_1^{t_1} - p_1^{t_1-1}) \cdots (p_r^{t_r} - p_r^{t_r-1})(p_{r+1}^{t_{r+1}} - p_{r+1}^{t_{r+1}-1}) \cdots (p_q^{t_q} - p_q^{t_q-1}) \\
&= n\phi(m).
\end{aligned}$$

Since n contains every prime divisor of itself, it follows by the same argument that $\phi(n^2) = n\phi(n)$.

7.2.20 If p is a prime and $k \geq 2$, show that $\phi(\phi(p^k)) = p^{k-2}\phi((p-1)^2)$.

Suppose that p is a prime and $k \geq 2$. Since $\gcd(p, p-1) = 1$ implies $\gcd(p^{k-1}, p-1) = 1$,

$$\begin{aligned}
\phi(\phi(p^k)) &= \phi(p^k - p^{k-1}) \\
&= \phi(p^{k-1}(p-1)) \\
&= \phi(p^{k-1})\phi(p-1) \\
&= (p^{k-1} - p^{k-2})\phi(p-1) \\
&= p^{k-2}(p-1)\phi(p-1).
\end{aligned}$$

Because $\phi(n^2) = n\phi(n)$, we know

$$p^{k-2}(p-1)\phi(p-1) = p^{k-2}\phi((p-1)^2).$$