7.3.2 Use Euler's theorem to confirm that , for any integer $n \geq 0$,

$$51 \mid 10^{32n+9} - 7.$$

Let $n \geq 0$ be an integer. We want to show $51 \mid 10^{32n+9} - 7$ or $10^{32n+9} - 7 \equiv 0 \pmod{51}$. Notice $\phi(51) = \phi(17 \cdot 3) = \phi(17)\phi(3) = 16 \cdot 2 = 32$. Since $\gcd(51, 10) = 1$, Euler's theorem gives

$$10^{32} \equiv 1 \pmod{51}.$$

Then

$$10^{32n+9} = (10^{32})^n \cdot 10^9 \equiv 1^n \cdot 10^9 \equiv (1000)^3 \equiv (-20)^3 \equiv -8000 \equiv 7 \pmod{51}.$$

Thus

$$10^{32n+9} - 7 \equiv 7 - 7 \equiv 0 \pmod{51}$$

hence

$$51 \mid 10^{32n+9} - 7.$$

7.3.5 If m and n are relatively prime positive integers, prove that

$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}.$$

Suppose that m and n are relatively prime integers. Euler's theorem gives

$$n^{\phi(m)} \equiv 1 \pmod{m} \text{ and } m^{\phi(n)} \equiv 1 \pmod{n}.$$

Note that

$$m^{\phi(n)} \equiv 0 \pmod{m} \text{ and } n^{\phi(m)} \equiv 0 \pmod{n}.$$

Then

$$n^{\phi(m)} + m^{\phi(n)} \equiv 1 + 0 \equiv 1 \pmod{m} \text{ and } m^{\phi(n)} + n^{\phi(m)} \equiv 1 + 0 \equiv 1 \pmod{n}.$$

So $m \mid n^{\phi(m)} + m^{\phi(n)} - 1$ and $n \mid m^{\phi(n)} + n^{\phi(m)} - 1$. Since $\gcd(m, n) = 1$, we have

$$mn \mid n^{\phi(m)} + m^{\phi(n)} - 1$$

or

$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}.$$

7.3.8(a) If $\gcd(a, n) = 1$, show that the linear congruence $ax \equiv b \pmod{n}$ has the solution $x \equiv ba^{\phi(n)-1} \pmod{n}$.

Suppose that $\gcd(a, n) = 1$. Euler's theorem gives $a^{\phi(n)} \equiv 1 \pmod{n}$. So

$$ax \equiv b \equiv b \cdot 1 \equiv b \cdot a^{\phi(n)} \pmod{n}.$$

Since $\gcd(a, n) = 1$, we can divide both sides of the congruence by a giving

$$x \equiv ba^{\phi(n)-1} \pmod{n}$$

as desired.

7.3.10 For any integer a, show that a and a^{4n+1} have the same last digit.

Let $a \in \mathbb{Z}$. To show that a and a^{4n+1} have the same last digit, we want

$$a \equiv a^{4n+1} \pmod{10}.$$

It is enough to show that $a^{4n+1} \equiv a \pmod 5$ and $a^{4n+1} \equiv a \pmod 2$. Notice that if $a \equiv 0 \pmod 2$, then $a^{4n+1} \equiv 0 \pmod 2$. If $a \equiv 1 \pmod 2$, then $a^{4n+1} \equiv 1^{4n+1} \equiv 1 \pmod 2$. So

$$a^{4n+1} \equiv a \pmod 2$$

for all a.

By Fermat's theorem, if $\gcd(a, 5) = 1$, then $a^4 \equiv 1 \pmod 5$. In this case,

$$a^{4n+1} = (a^4)^n a \equiv 1^n a \equiv a \pmod 5.$$

The only other case is when $\gcd(a, 5) = 5$. Then $a \equiv 0 \pmod 5$ and $a^{4n+1} \equiv 0 \pmod 5$ as a is a multiple of 5. So

$$a^{4n+1} \equiv a \pmod 5.$$

Then $a^{4n+1} \equiv a \pmod 5$ and $a^{4n+1} \equiv a \pmod 2$ for all a. Putting these together with the fact that $\gcd(2, 5) = 1$ gives

$$a^{4n+1} \equiv a \pmod{10}.$$