6.1.06 For any integer $n \geq 1$, establish the inequality $\tau(n) \leq 2\sqrt{n}$

Proof. Assume $d|n$. Then, d or $n/d \leq 2\sqrt{n}$. Note, there can be at most $\sqrt{n}$ divisor pairs $(d, n/d)$.
Thus, $\sqrt{2} \leq 2\sqrt{2}$.
Therefore, $\tau(n) \leq 2\sqrt{n}$ □

6.1.08 Show that $\sum_{d|n} 1/d = \sigma(n)/n$ for every positive integer n

Proof. Note, d is a divisor of $n \Rightarrow n/d$ is a divisor of n, as $d \cdot \frac{n}{d} = n$. Let the set of divisors of $n = \{d_1, d_2, \cdots, d_k\} \Rightarrow \{n/d_1, n/d_2, \cdots, n/d_k\}$ Then,

$$\tau(n) = d_1 + d_2 + \cdots + d_k = \frac{n}{d_1} + \frac{n}{d_2} + \cdots + \frac{n}{d_k} = n(\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k})$$

Thus, $\frac{\tau(n)}{n} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = \sum_{d|n} 1/d$
Therefore, $\sum_{d|n} 1/d = \sigma(n)/n$ for every positive integer n □

6.1.14a For $k \geq 2$, $n = 2^{k-1}$ satisfies the equation $\sigma(n) = 2n - 1$

Proof. Notice,

$$\sigma(n) = \sigma(2^{k-1}) = \frac{2^{k-1+1} - 1}{2 - 1} = 2^k - 1 = 2 \cdot 2^{k-1} - 1 = 2n - 1$$

□

6.1.14b For $k \geq 2$, if $2^k - 1$ is prime, then $n = 2^{k-1}(2^k - 1)$ satisfies the equation $\sigma(n) = 2n$

Proof. Assume $2^k - 1$ is prime. Notice,

$$\begin{aligned} \sigma(n) &= \frac{2^{k-1+1} - 1}{2 - 1} \cdot \frac{(2^k - 1)^2 - 1}{2^k - 1 - 1} \\ &= (2^k - 1)(2^k - 1 + 1) = (2^k - 1)(2^k) \\ &= 2(2^{k-1})(2^{k-1}) \\ &= 2n \end{aligned}$$

Therefore, $n = 2^{k-1}(2^k - 1)$ satisfies the equation $\sigma(n) = 2n$ □

6.1.14c For $k \geq 2$, if $2^k - 3$ is prime, then $n = 2^{k-1}(2^k - 3)$ satisfies $\sigma(n) = 2n + 2$

Proof. Assume $2^k - 3$ is prime. Notice,

$$\begin{aligned}
\sigma(n) &= \frac{2^{k-1+1}-1}{2-1} \cdot \frac{(2^k-3)^2-1}{2^k-3-1} \\
&= (2^k-1)(2^k-3+1) = (2^k-1)(2^k-3+1) \\
&= (2^k-1)(2^k-2) \\
&= 2^{2k} - 3 \cdot 2^k + 2 \\
&= 2^k(2^k-3) + 2 \\
&= 2(2^{k-1})(2^k-3) + 2 \\
&= 2n + 2
\end{aligned}$$

Therefore, $n = 2^{k-1}(2^k - 3)$ satisfies $\sigma(n) = 2n + 2$ $\square$