4.3.7 (d) An integer is divisible by 5 if and only if its units digit is 0 or 5.

Let $N \in \mathbb{Z}$ be an integer of the form $N = a_m 10^m + \cdots + a_1 10 + a_0 = a_m 2^m \cdot 5^m + \cdots a_1 2 \cdot 5$ with each $0 \le a_i < 10$. Notice that every term other than a_0 is divisible by 5 as it is some multiple of 5, i.e. $a_i 10^i \equiv 0 \pmod 5$ for $1 \le i \le m$. So, $5 \mid N$ if and only if $5 \mid a_0$. Now a_0 is divisible by 5 if and only if $a_0 = 5$ or 0. Thus an integer N is divisible by 5 if and only if its units digit is 0 or 5.

4.3.11 Assuming that 495 divides $273x49y5$, obtain the digits of x and y.

Suppose that 495 divides $273x49y5$. So $495n = 273x49y5$ for some $n \in \mathbb{Z}$. Then notice that $495 \equiv 0 \pmod 9$ and $495 \equiv 0 \pmod{11}$ since $4+9+5 = 9 \cdot 2$ and $5 - 9 + 4 = 0$. Since $n \in \mathbb{Z}$, we have $495n \equiv 0 \pmod 9$ and $495n \equiv 0 \pmod{11}$, i.e.

$$273x49y5 \equiv 0 \pmod 9 \text{ and } 273x49y5 \equiv 0 \pmod{11}.$$

From this we know that

$$2+7+3+x+4+9+y+5 = 30+x+y \equiv 3+x+y \equiv 0 \pmod 9$$

and

$$5-y+9-4+x-3+7-2 = 12-y+x \equiv 1-y+x \equiv 0 \pmod{11}.$$

Then $x+y \equiv 6 \equiv 15 \pmod 9$ and $y - x \equiv 1 \pmod{11}$. Note: $x,\ y \in \{0,1,2,\cdots,9\}$. Solving the system of linear equations $x+y = 15$ and $y-x = 1$ gives that $x = 7$ and $y = 8$.

4.3.16 Show that 2^n divides an integer N if and only if 2^n divides the number made up of the last n digits of N.

Let $N \in \mathbb{Z}$ so that $N = a_{n+i}10^{n+i} + \cdots a^n 10^n + \cdots a_1 10 + a_0$ where $n \ge 0$ and $i \ge 0$. If 2^n divides the last n digits of N, then

$$2^n \mid a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0.$$

Notice that

$$a_{n+i}10^{n+i} + \cdots + a_n 10^n = 10^n(a_{n+i}10^i + \cdots + a_n) = 2^n 5^n (a_{n+i}10^i + \cdots + a_n).$$

So $2^n \mid (a_{n+i}10^{n+i} + \cdots + a_n 10^n)$. Thus $2^n \mid a_{n+i}10^{n+i} + \cdots a^n 10^n + \cdots + a_1 10 + a_0 = N$.

Now suppose that $2^n \mid N$. Notice that $2^n \mid a_{n+i}10^{n+i} + \cdots + a_n 10^n$ since

$$2^n 5^n (a_{n+i}10^i + \cdots + a_n) = a_{n+i}10^{n+i} + \cdots + a_n 10^n.$$

Then

$$2^n \mid N - (a_{n+i}10^{n+i} + \cdots + a_n 10^n) = a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0.$$

So 2^n divides the last n digits of N.

4.3.28 When printing the ISBN $a_1a_2\dots a_9$, two unequal digits were transposed. Show that the check digits detected this error.

Suppose we have some ISBN number $a_1a_2\cdots a_9$. Then we can write $a_1a_2\cdots a_ja_{i+1}\cdots a_ia_{j+1}\cdots a_9$ where a_i and a_j were transposed and not equal with $1 \le i < j \le 9$. We know that

$$a_1 + 2\cdot a_2 + \cdots + i\cdot a_i + (i+1)\cdot a_{i+1} + \cdots + j\cdot a_j + \cdots + 9\cdot a_9 \equiv a_{10} \pmod{11}.$$

Assume for the sake of contradiction that the check digits did not detect the error in the transposition. Then

$$a_1 + 2\cdot a_2 + \cdots + i\cdot a_j + (i+1)\cdot a_{i+1} + \cdots + i\cdot a_i + \cdots + 9\cdot a_9 \equiv a_{10} \pmod{11}.$$

So $a_1 + 2\cdot a_2 + \cdots + i\cdot a_i + (i+1)\cdot a_{i+1} + \cdots + j\cdot a_j + \cdots + 9\cdot a_9$
$\equiv a_1 + 2\cdot a_2 + \cdots + i\cdot a_j + (i+1)\cdot a_{i+1} + \cdots + j\cdot a_i + \cdots + 9\cdot a_9 \pmod{11}$. This implies that

$$i\cdot a_i + j\cdot a_j \equiv i\cdot a_j + j\cdot a_i \pmod{11}.$$

Simplifying more gives that

$$(j-i)\cdot a_j \equiv (j-i)a_i \pmod{11}.$$

Because $0 < j-i < 9$ and $\gcd(j-i, 11) = 1$, it follows that $a_j \equiv a_i \pmod{11}$. Since both a_i and a_j are less than 11 and not negative, it follows that $a_i = a_j$, a contradiction. Thus the check digits detected the error.