2.3.3 Prove or Disprove: if $a|(b+c)$, then either $a|b$ or $a|c$.
Let $a=2, b=1$, and $c=1$. Then,

$$a|(b+c) \text{ as } 2|(1+1)=2$$

But $2 \nmid 1$. Therefore by counterexample the original statement must be false.

2.3.5 Prove that for any integer a, one of the integers $a, a+2, a+4$ is divisible by 3.

Proof. From the division algorithm we have $a=3n+0, a=3n+1, a=3n+2$ for some $n \in \mathbb{Z}$.

Case 1: Let $a=3n+0$. Then, a is divisible by 3 by definition.

Case 2: Let $a=3n+1$. Then,

$$a+2=3n+3=3(n+1)$$

Thus, $a+2$ is divisible by 3.

Case 3: Let $a=3n+2$. Then,

$$a+4=3n+6=3(n+2)$$

Thus, $a+4$ is divisible by 3.

Therefore, for any integer a, one of the integers of the form $a, a+2, a+4$ is divisible by 3. □

2.3.9 Establish that the difference of two consecutive cubes is never divisible by 2.

Proof. Let the difference of two consecutive cubes be represented by $(n+1)^3-(n)^3$ for some $n \in \mathbb{Z}$. As we know the two cubes are consecutive, one of the cubes must be even.

Case 1: The smaller cube is even. Then, n is even. That is to say, there exists a $k \in \mathbb{Z}$ such that $n=2k$ and $n+1=2k+1$. Notice.

$$\begin{aligned}
(n+1)^3-(n)^3 &= (2k+1)^3-(2k)^3 \\
&= 12k^2+6k+1 \\
&= 2(6k^2+3k)+1 \\
\text{Let } s := 6k^2+3k & \\
&= 2s+1 \quad \text{which is odd}
\end{aligned}$$

Case 2: The smaller cube is odd. Then, n is odd. That is to say, there exists a $k \in \mathbb{Z}$ such that $n = 2k + 1$ and $n + 1 = 2k + 2$. Notice.

$$\begin{aligned} (n+1)^3 - (n)^3 &= (2k+2)^3 - (2k+1)^3 \\ &= 12k^2 + 18k + 7 \\ &= 2(6k^2 + 9k + 3) + 1 \\ \text{Let } s := 6k^2 + 9k + 3 & \\ &= 2s + 1 \quad \text{which is odd} \end{aligned}$$

As we have shown that in both cases the result from the difference of two consecutive cubes is odd. Therefore the difference of two consecutive cubes is never divisible by 2. □

2.3.15 If a and b are integers, not both of which are zero, prove that $\gcd(2a - 3b, 4a - 5b)$ divides b; hence, $\gcd(2a + 3, 4a + 5) = 1$.

Proof. $\gcd(2a - 3b, 4a - 5b) = x(2a - 3b) + y(4a - 5b)$ for some $x, y \in \mathbb{Z}$. Let $x = -2$ and $y = 1$. Then,

$$-2(2a - 3b) + 1(4a - 5b) = -4a + 6b + 4a - 5b = b$$

Thus, $\gcd(2a - 3b, 4a - 5b)|b$.

Continuing,
$\gcd(2a + 3, 4a + 5) = x(2a + 3) + y(4a + 5) = 1$ for some $x, y \in \mathbb{Z}$. Let $x = 2$ and $y = -1$. Then,

$$2(2a + 3) - 1(4a + 5) = 4a + 6 - 4a + 5 = 1$$

As $\gcd(2a + 3, 4a + 5)|1 \Rightarrow \gcd(2a + 3, 4a + 5) = 1$

Therefore, both $\gcd(2a - 3b, 4a - 5b)$ divides b; hence, $\gcd(2a + 3, 4a + 5) = 1$ are true. □

2.3.17 Prove that the expression $(3n)!/(3!)^n$ is an integer for all $n \geq 0$.

Proof. In the case of $n = 1$, We have $3!/3!^1 = 6/6 = 1$. So, the expression hold true for $n = 1$. Assume that the expression holds true for some $k \geq 0$, that is $n = k$,

$(3k)!/(3!)^k$. We will now show that the expression holds true for $k+1$. We see that

$$\begin{aligned}\frac{[3(k+1)]!}{(3!)^{k+1}} &= \frac{(3k+3)!}{(3!)^k 3!}\\ &= \frac{(3k)!}{3!^k}\cdot\frac{(3k+3)(3k+2)(3k+1)}{6}\\ \text{Let } q := \frac{(3k)!}{3!^k}\\ &= q\cdot\frac{(k+1)(3k+2)(3k+1)}{2}\end{aligned}$$

As q must be an integer, we need to prove that $\frac{(k+1)(3k+2)(3k+1)}{2}$ is also an integer.

Case 1: k is even. Then, for some $r \in \mathbb{Z}, k = 2r \Rightarrow 3k+2 = 6r+2 = 2(3r+1)$. Thus, $2|3k+2$.

Case 2: k is odd. Then, for some $r \in \mathbb{Z}, k = 2r+1 \Rightarrow k+1 = 2r+2 = 2(r+1)$. Thus, $2|k+1$.

Thus, $\frac{(k+1)(3k+2)(3k+1)}{2}$ is an integer. Which means $\frac{[3(k+1)]!}{(3!)^{k+1}}$ must also be an integer. Therefore, by mathematical induction, $(3n)!/(3!)^n$ is an integer for all $n \geq 0$. □

2.3.23 If $a|bc$ show that $a|\gcd(a,b)\gcd(a,c)$

Proof. Assume $a|bc$. Then, $bc = aq$ for some $q \in \mathbb{Z}$. By definition, $\gcd(a,b) = am+bn$ for some $m,n \in \mathbb{Z}$ and $\gcd(a,c) = ar+bs$ for some $r,s \in \mathbb{Z}$. Then,

$$\begin{aligned}\gcd(a,b)\gcd(a,c) &= (am+bn)(ar+bs)\\ &= a^2mr + acms + abrn + bcns\\ &= a^2mr + acms + abrn + aqns, \text{ as } bc = aq\\ &= a(amr + cms + brn + qns)\end{aligned}$$

Therefore, $a|\gcd(a,b)\gcd(a,c)$ □