7.2.4c $\phi(3n) = 3\phi(n)$ if and only if $3|n$

Proof. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$
Assume $3|n$. Thus, one of the $p_i = 3$ and we can write $n = 3^k m$, where $m \in \mathbb{Z}^+$ and $\gcd(3, m) = 1$. Observe.

$$\begin{aligned}\phi(3n) &= \phi(3^{k+1}m)\\ &= \phi(3^{k+1})\phi(m)\\ &= (3^{k+1} - 3^k)\phi(m)\\ &= 3(3^k - 3^{k-1})\phi(m)\\ &= 3\phi(3^k m)\\ &= 3\phi(n)\end{aligned}$$

Thus, if $3|n$ then $\phi(3n) = 3\phi(n)$
Going the other way, assume $\phi(3n) = 3\phi(n)$. Suppose $3 \not| n$. Then $\gcd(3, n) = 1$. Notice.

$$\phi(3n) = \phi(3)\phi(n) = 2\phi(n)$$

This contradicts $\phi(3n) = 3\phi(n)$.
Thus, if $\phi(3n) = 3\phi(n)$ then $3|n$
Therefore, $\phi(3n) = 3\phi(n)$ if and only if $3|n$ □

7.2.05 Prove that the equation $\phi(n) = \phi(n + 2)$ is satisfied by $n = 2(2p - 1)$ whenever p and $2p - 1$ are both odd primes.

Proof. Assume p and $2p - 1$ are both odd primes. Notice $\gcd(2, 2p - 1) = 1$. Then,

$$\phi(n) = \phi(2)\phi(2p - 1) = (2p - 1)(1 - \frac{1}{2p - 1}) = 2p - 2$$

Notice,

$$n + 2 = 2(2p - 1) + 2 = 4p$$

As p is a odd prime, $\gcd(4, p) = 1$

$$\phi(n + 2) = \phi(4)\phi(p) = 2p(1 - \frac{1}{p}) = 2p - 2$$

Thus, $\phi(n) = \phi(n + 2)$
Therefore, the equation $\phi(n) = \phi(n + 2)$ is satisfied by $n = 2(2p - 1)$ whenever p and $2p - 1$ are both odd primes. □

7.2.10 If every prime that divides n also divides m, establish that $\phi(nm) = n\phi(m)$; in particular $\phi(n^2) = n\phi(n)$ for every positive integer n.

Proof. Let $p_1, p_2, \cdots, p_r$ be the primes of n that divide m. Let $n = p_1^{k_1} \cdots p_r^{k_r}$. Then $m = p_1^{j_1} \cdots p_r^{j_r} q_1^{m_1} \cdots q_s^{m_s}$ where q_i is prime and $q_i \not| p_j$. We then have, $nm = p_1^{k_1+j_1} \cdots p_r^{k_r+j_r} q_1^{m_1} \cdots q_s^{m_s}$. Observe.

$$\begin{aligned}
\phi(nm) &= p_1^{k_1+j_1} \cdots p_r^{k_r+j_r} q_1^{m_1} \cdots q_s^{m_s} (1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_r})(1 - \frac{1}{q_1}) \cdots (1 - \frac{1}{q_s}) \\
&= p_1^{j_1} \cdots p_r^{j_r} (1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{q_1}) \cdots (1 - \frac{1}{q_s}) p_1^{k_1} \cdots p_r^{k_r} \\
&= \phi(m) \cdot p_1^{k_1} \cdots p_r^{k_r} \\
&= n\phi(m)
\end{aligned}$$

Therefore, If every prime that divides n also divides m, then $\phi(nm) = n\phi(m)$ ☐

7.2.20 If p is a prime and $k \geq 2$, show that $\phi(\phi(p^k)) = p^{k-2}\phi((p-1)^2)$

Proof. Let p be a prime and $k \geq 2$. Notice, $\phi(\phi(p^k)) = p^{k-1}(p-1)$. Since $\gcd(p, p-1) = 1 \Rightarrow \gcd(p^{k_1}, p-1) = 1$. As phi is multiplicative, notice.

$$\begin{aligned}
\phi(\phi(p^k)) &= \phi(p^{k-1}(p-1)) \\
&= \phi(p^{k-1})\phi(p-1) \\
&= p^{k-2}(p-1)\phi(p-1) \\
&\text{By the previous problem. We have } \phi(n^2) = n\phi(n) \text{ i.e } (p-1)\phi(p-1) = \phi((p-1)^2) \\
&= p^{k-2}\phi((p-1)^2)
\end{aligned}$$

Therefore, if p is a prime and $k \geq 2$, $\phi(\phi(p^k)) = p^{k-2}\phi((p-1)^2)$ ☐