4.3.7d Establish the following divisibility criteria, an integer is divisible by 5 if and only if its units digit is 0 or 5

Proof. Let $N = a_m 10^m + ...a_1 10 + a_0, x, m \in \mathbb{Z}$. Notice $10 \equiv 0 \pmod 5$. Then, for some $j \in \mathbb{Z}^+ 10^j \equiv 0 \pmod 5$.

$$N \equiv a_m 0^m + ...a_1 x 0 + a_0 \qquad a_0 \pmod 5$$

Therefore, N is divisible by 5 if its last digit is 0 or 5 □

4.3.11 Assuming 495 divides $273x49y5$, obtain the digits x and y. Notice $495 = 11 \cdot 9 \cdot 5$. Thus, $495 \equiv 0 \pmod 9$ and as 495 divides $273x49y, 273x49y \equiv 0 \pmod 9$. Then $(2 + 7 + 3 + x + 4 + 9 + y + 5) = 30 + x + y \equiv 3 + x + y \pmod 9$. Thus, $x + y \equiv 0 \pmod 9$ or $x + y \equiv 15 \pmod 9$.
Notice $495 \equiv 0 \pmod{11}$. As 495 divides $273x49y, 273x49y \equiv 0 \pmod{11}$.

$$5 - y + 9 - 4 + x - 3 + 7 - 2 = x - y + 12 \equiv 0 \pmod{11}$$

$$x - y \equiv -1 \pmod{11} \text{ or } y - x \equiv 1 \pmod{11}$$

Then as $x + y = 15$ and $y - x = 1$, we result in $2y = 16$.
Therefore, $y = 8$ and $x = 7$

4.3.16 Show that 2^n divides an integer N if and only if 2^n divides the number made of the last n digits of N.

Proof. Let $N = a_{n+j}10^{n+j} + \cdots + a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0$ be the decimal representation for $N, n \geq 1, j \geq 0$ Assume 2^n divides the last n digits of N. Then, $2^n | (a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0)$ Thus,

$$a_{n+j}10^{n+j} + \cdots + a_n 10^n = 10^n(a_{n+j}10^j + \cdots + a_n) = 2^n 5^n (a_{n+j}10^j + \cdots + a_n)$$

So, $2^n | (a_{n+j} + 10^{n+j} + \cdots + a_n 10^n)$. From the previous statement and that fact of $2^n | (a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0)$
$2^n | (a_{n+j}10^{n+j} + \cdots + a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_o)$.
Therefore $2^n | N$
Suppose $2^n | N$ As $2^n | 2^n 5^n$, then $2^n | 10^n (a_{n+j}10^j + \cdots + a_n)$.
Thus, $2^n | (a_{n+j}10^{n+j} + \cdots + a_n 10^n)$
Then, $2^n | N - (a_{n+j}10^{n+j} + \cdots + a_n 10^n) \Rightarrow 2^n | (a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0)$
Therefore, 2^n divides the last n digits of N. □

4.3.28 When printing the ISBN $a_1 a_2 \cdots a_9$, two unequal digits were transposed. Show that the check digits detected this errors.

$N = a_1 a_2 \cdots a_k a_{k+1} \cdots a_9, M = a_1 a_2 \cdots a_{k+1} a_k \cdots a_9$
Check digits of N is

$$N[a_{10}] \equiv a_1 + 2a_2 + \cdots + ka_k + (k+1)a_{k+1} + \cdots + 9a_9 \pmod{11}$$

Check digits of M is

$$M[a_{10}] \equiv a_1 + 2a_2 + \cdots + ka_{k+1} + (k+1)a_k + \cdots + 9a_9 \pmod{11}$$

Let $N[a_{10}] = M[a_{10}]$, then

$$a_1+2a_2+\cdots+ka_k+(k+1)a_{k+1}+\cdots+9a_9 \equiv a_1+2a_2+\cdots+ka_{k+1}+(k+1)a_k+\cdots+0a_9 \pmod{11}$$

$$\Rightarrow ka_k + (k+1)a_{k+1} \equiv ka_{k+1} + (k+1)a_k \pmod{11}$$
$$\Rightarrow ka_k + ka_{k+1} + a_{k+1} \equiv ka_{k+1} + ka_k + a_k \pmod{11}$$
$$\Rightarrow a_{k+1} \equiv a_k \pmod{11}$$

Therefore, $11 | a_{k+1} - a_k \Rightarrow a_{k+1} - a_k = 0 \Rightarrow a_{k+1} = a_k$ since $a_k, a_{k+1} \leq 9$