5.2.03 From Fermat's theorem deduce that, for any integer $n \geq 0, 13 | 11^{12n+6} + 1$.

Proof. As $13 \not| 11, 11^{12} \equiv 1 \pmod{13}$, by Fermat's Theorem. Then

$$\begin{aligned} 11^{12n+6} + 1 &\equiv (11^{12})^n \cdot 11^6 + 1 \pmod{13} \\ &\equiv 1^n \cdot (-2)^6 + 1 \pmod{13} \\ &\equiv 65 \pmod{13} \\ &\equiv 0 \pmod{13} \end{aligned}$$

Therefore, $13 | 11^{12n+6} + 1$ □

5.2.4d Derive the following congruence: $a^9 \equiv a \pmod{30}$ for all a.
Note, $30 = 2 \cdot 3 \cdot 5$. Then, using Fermat's Theorem,

$$a^9 \equiv (a^2)^4 \cdot a \equiv a^5 \equiv a^3 \equiv a^2 \equiv a \pmod{2}$$

$$a^9 \equiv (a^3)^3 \equiv a^3 \equiv a \pmod{3}$$
$$a^9 \equiv a^5 \cdot a^4 \equiv a^5 \equiv a \pmod{5}$$

Thus, $a^9 \equiv a \pmod{2 \cdot 3 \cdot 5}$
Therefore, $a^9 \equiv a \pmod{30}$ for all a.

5.2.13 Assume that p and q are distinct odd primes such that $p - 1 | q - 1$. If $\gcd(a, pq) = 1$, show that $a^{q-1} \equiv 1 \pmod{pq}$

Proof. As $\gcd(a, pq) = 1$, then $\gcd(a, p) = 1$ and $\gcd(a, q) = 1$ Then, $a^{p-1} \equiv 1 \pmod{p}$ and $a^{q-1} \equiv \pmod{p}$
As, $p - 1 | q - 1$ then $q - 1 = k(p - 1)$ for some k. Thus,

$$a^{q-1} \equiv (a^{p-1})^k \equiv 1^k \equiv 1 \pmod{p}$$

Thus, $a^{q-1} \equiv 1 \pmod{p}$
Therefore, $a^{q-1} \equiv 1 \pmod{pq}$ □

5.2.20a Show that $561 | 2^{561} - 2$.
Note, $561 = 3 \cdot 11 \cdot 17$.
Then, by Fermat's Theorem,

$$2^{3-1} = 2^2 \equiv 1 \pmod{3}$$

$$2^{11-1} = 2^{10} \equiv 1 \pmod{11}$$
$$2^{17-1} = 2^{16} \equiv 1 \pmod{17}$$

We then have,

$$2^{561} = (2^2)^{280} \cdot 2 \equiv 2 \pmod 3$$
$$2^{561} = (2^{10})^{56} \cdot 2 \equiv 2 \pmod{11}$$
$$2^{561} = (2^{16})^{35} \cdot 2 \equiv 2 \pmod{17}$$

Thus,

$$2^{561} \equiv 2 \pmod{3 \cdot 11 \cdot 17}$$

Therefore, $561 | 2^{561} - 2$

5.2.20b Show that $561 | 3^{561} - 3$.
Note, $561 = 3 \cdot 11 \cdot 17$.
Then, by Fermat's Theorem,

$$3^{11-1} = 3^{10} \equiv 1 \pmod{11}$$

$$3^{17-1} = 3^{16} \equiv 1 \pmod{17}$$

We then have,

$$3^{561} = (3^{10})^{56} \cdot 3 \equiv 3 \pmod{11}$$
$$3^{561} = (3^{16})^{35} \cdot 3 \equiv 3 \pmod{17}$$

Thus,

$$3^{561} \equiv 3 \pmod{3 \cdot 11 \cdot 17}$$

Therefore, $561 | 3^{561} - 3$