

7.3.02 Use Euler's theorem to confirm that, for any integer $n \geq 0$,

$$51 | 10^{32n+9} - 7$$

Since, $51 = 17 \cdot 3$ we have $\phi(51) = \phi(17)\phi(3) = 16 \cdot 2 = 32$. By Euler's theorem we have $10^{32} \equiv 1 \pmod{51}$. Notice,

$$10^{32n+9} - 7 = 10^{32n} \cdot 10^9 - 7 = 1^n \cdot 10^9 - 7 = 10^9 - 7 \equiv 0 \pmod{51}$$

Then observe the following.

$$\begin{aligned} 10^9 &= (10^3)^3 = 1000^3 \\ &\equiv 31^3 \pmod{51} \\ &\equiv 29791 \pmod{51} \\ &\equiv 7 \pmod{51} \end{aligned}$$

Thus, we have $10^9 \equiv 7 \pmod{51}$.

Therefore, $51 | 10^{32n+9} - 7$

7.3.05 If m and n are relatively prime positive integers, prove that

$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}$$

By Euler's Theorem, we have $m^{\phi(n)} \equiv 1 \pmod{n}$ and $n^{\phi(m)} \equiv 1 \pmod{m}$. Also, by Euler's Theorem we get $m^{\phi(n)} \equiv 0 \pmod{m}$ and $n^{\phi(m)} \equiv 0 \pmod{n}$. Notice,

$$m^{\phi(n)} + n^{\phi(m)} \equiv (1 + 0) \equiv 1 \pmod{m}$$

and

$$m^{\phi(n)} + n^{\phi(m)} \equiv (1 + 0) \equiv 1 \pmod{n}$$

Therefore, $m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}$

7.3.8a If $\gcd(a, n) = 1$, show that the linear congruence $ax \equiv b \pmod{n}$ has the solution $x \equiv ba^{\phi(n)-1} \pmod{n}$.

Let $\gcd(a, n) = 1$. Then by Euler's Theorem we have $a^{\phi(n)} \equiv 1 \pmod{n}$. Observe.

$$\begin{aligned} a^{\phi(n)} &\equiv 1 \pmod{n} \\ \Rightarrow ba^{\phi(n)} &\equiv b \pmod{n} \\ \Rightarrow aa^{\phi(n)-1}b &\equiv b \pmod{n} \\ \Rightarrow a^{\phi(n)-1}b &\equiv b \pmod{n} \end{aligned}$$

Thus, the linear congruence $ax \equiv b \pmod{n}$ has the solution $x \equiv ba^{\phi(n)-1} \pmod{n}$

7.3.10 For any integer a , show that a and a^{4n+1} have the same last digit.