3.2.03 Given that $p \not| n$ for all primes $p \leq \sqrt[3]{n}$, show that $n > 1$ is either prime or the product of two primes.

Proof. By way of contradiction, let us assume $n = p_1 p_2 \cdots p_k, k \geq 3$ The first three prime factors are p_1, p_2, p_3. Then, $p_1|n, p_2|n$, and $p_3|n \Rightarrow p_1p_2p_3|n \Rightarrow p \geq p_1p_2p_3$. Since $p \not| n$ for all $p \leq \sqrt[3]{n}$. Then, $p_1 > \sqrt[3]{n}, p_2 > \sqrt[3]{n}$, and $p_3 > \sqrt[3]{n}$. Thus, $p_1p_2p_3 > n$ is a contradiction.
Therefore $n > 1$ is either a prime of the product of two primes. □

3.2.05 Show that any composite three-digit number must have a prime factor less than or equal to 31.

Proof. Let n be a composite three-digit number. Then $n \leq 999$. As n is composite, there must exist a p where $p \neq n$, $p|n$ and $p \leq \sqrt{n} \Rightarrow p \leq \sqrt{999} \Rightarrow p \leq 31$
Therefore, any composite three-digit number must have a prime factor less than or equal to 31. □

3.2.9a Prove that if $n > 2$, then there exists a prime p satisfying $n < p < n!$.

Proof. Let $n > 2$. Using the hint given, if $(n! - 1)$ is prime, then the statement is satisfied.
Else, $(n! - 1)$ is composite and thus has a prime divisor p with $p \leq n \Rightarrow p|n!$ and $p|(n! - 1) \Rightarrow p|(n! - (n! - 1)) = 1 \Rightarrow p > n$. But $p > n$ is a contradiction.
Therefore, if $n > 2$, then there exists a prime p satisfying $n < p < n!$ □

3.2.9b For $n > 1$, show that every prime divisor of $n! + 1$ is an odd integer that is greater than n.

Proof. Let $n > 1$. Then,

$$
\begin{aligned}
n! + 1 &= n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 + 1 \\
&= 2n(n-1)(n-2)\cdots 3 + 1 \\
&\text{let } m := n(n-1)(n-2)\cdots 3 \\
&= 2m + 1
\end{aligned}
$$

Thus, $2m + 1$ is odd and greater than n. Therefore, For $n > 1$, every prime divisor of $n! + 1$ is an odd integer that is greater than n □