2.4.2b Use the Euclidean Algorithm to obtain integers x and y that satisfy $\gcd(24, 138) = 24x + 138y$

Using the Euclidean Algorithm, observe.

$$\begin{aligned} 138 &= 5(24) + 18 \\ 24 &= 1(18) + 6 \\ 18 &= 3(6) + 0 \end{aligned}$$

Working these results in reverse, notice.

$$\begin{aligned} 6 &= 24 - 18(1) \\ &= 24 - 24(138 - 15) \\ &= 6(24) - 1(138) \end{aligned}$$

Thus, $x = 6$ and $y = -1$.

2.4.2c Use the Euclidean Algorithm to obtain integers x and y that satisfy $\gcd(119, 272) = 119x + 272y$

Using the Euclidean Algorithm, observe.

$$\begin{aligned} 272 &= 2(119) + 34 \\ 119 &= 3(34) + 17 \\ 34 &= 2(17) + 0 \end{aligned}$$

Working these results in reverse, notice.

$$\begin{aligned} 17 &= 119 - 3(34) \\ &= 119 - 3(272 - 2(119)) \\ &= 7(119) - 3(272) \end{aligned}$$

Thus, $x = 7$ and $y = -3$.

2.4.4c Assuming that $\gcd(a, b) = 1$, prove the following: $\gcd(a + b, a^2 + b^2) = 1$ or 2.

Proof. Let $d = \gcd(a, b) = 1$. Then, $\gcd(a, b) = 1$ can be written as $ax + by = 1$.

$$(2a)x + (2b)y = 2$$

Then, $d|2 \Rightarrow d = 1$ or 2. Therefore, $\gcd(a + b, a^2 + b^2) = 1$ or 2. □

2.4.6 Prove that if $\gcd(a, b) = 1$, then $\gcd(a + b, ab) = 1$

Proof. Let $\gcd(a, b) = 1$. Then, $\gcd(a, b) = 1$ can be written as $ax + by = 1$. Squaring both sides yields in

$$a^2x^2 + 2abxy + b^2y^2 = 1$$

. Notice.

$$a^2x^2 + 2abxy + b^2y^2 = ab(2xy - x^2 - y^2) + (a + b)(ax^2 + by^2)$$

This can be rewritten as,

$$ab(x) + (a + b)(y)$$

Therefore, $\gcd(a + b, ab) = 1$ □