6.1.6 For any integer $n \geq 1$, establish the inequality $\tau(n) \leq 2\sqrt{n}$.

Let d be a positive divisor of n. Then any integer $m = n/d$ is also a positive divisor of n. Then one of d and m is less than or equal to $\sqrt{n}$, i.e. without loss of generality suppose $d \leq \sqrt{n}$ and $m \geq \sqrt{n}$. We can separate the positive divisors into pairs such that their product $dm = n$. Then the number of these pairs is at most $\sqrt{n}$ and so there can be no more than $2\sqrt{n}$ divisors. Thus $\tau(n) \leq 2\sqrt{n}$.

6.1.8 Show that $\sum_{d|n} 1/d = \sigma(n)/n$ for all positive integers n.

If $d \mid n$ then $n/d \mid n$ since $d(n/d) = n$. Then we know that the set of divisors, call them $d_1, \ldots, d_k$, can be written as $n/d_1, \ldots, n/d_k$. So

$$\sigma(n) = \sum_{d|n} d = n \sum_{d|n} \frac{1}{d}.$$

Thus

$$\sigma(n)/n = \sum_{d|n} \frac{1}{d}.$$

6.1.14 For $k \geq 2$, show

(a) $n = 2^{k-1}$ satisfies the equation $\sigma(n) = 2n - 1$.
Let $n = 2^{k-1}$. Then

$$\sigma(n) = 2^k - 1.$$

Notice that $2n - 1 = 2(2^k - 1) - 1 = 2^k - 1$. Thus $\sigma(n) = 2n - 1$.

(b) If $2^k - 1$ is prime, then $n = 2^{k-1}(2^k - 1)$ satisfies the equation $\sigma(n) = 2n$.
Suppose that $2^k - 1$ is prime. Let $n = 2^{k-1}(2^k - 1)$. Then

$$\sigma(n) = \sigma(2^{k-1}) \cdot \sigma(2^k - 1) = (2^k - 1)(2^k).$$

Notice that

$$2n = 2(2^{k-1})(2^k - 1) = 2^k(2^k + 1).$$

Therefore

$$\sigma(n) = 2n.$$

(c) If $2^k - 3$ is prime, then $n = 2^{k-1}(2^k - 3)$ satisfies $\sigma(n) = 2n + 1$.
Suppose that $2^k - 3$ is prime. Let $n = 2^{k-1}(2^k - 3)$. Then

$$\sigma(n) = \sigma(2^{k-1}) \cdot \sigma(2^k - 3) = (2^k - 1)(2^k - 2) = 2^{2k} - 3 \cdot 2^k + 2.$$

Notice that

$$2n + 1 = 2(2^{k-1})(2^k - 3) + 1 = 2^{2k} - 3 \cdot 2^k + 2.$$

Therefore

$$\sigma(n) = 2n + 2.$$