3.1.3a Prove the assertion, any prime of the form $3n+1$ is also of the form $6m+1$

Proof. Let p be a prime of the form $3n+1$. Assume n is odd. Then $3n$ is odd, $3n+1$ is even. Also $3n+1>2$, for all $n \geq 1$. Thus, $p=3n+1$ cannot be prime if n is odd. Thus, n must be even. Let $n=2k$ for some $k \in \mathbb{Z}^+$. Notice.

$$p = 3n+1 = 3(2k)+1 = 6k+1$$

Therefore, any prime of the form $3n+1$ is also of the form $6m+1$ □

3.1.3e Prove the assertion, the only prime of the form n^2-4 is 5.

Proof. $5 = 3^2 - 4$ so, 5 is of the form n^2-4.
For $n > 5, n^2-4 > 0$. Since $n > 3 \Rightarrow n^2-4 = (n-2)(n+2)$.
Then, $(n-2) > 1$ and it's a factor of n^2-4, so n^2-4 cannot be prime. Therefore, the only prime of the form n^2-4 is 5. □

3.1.04 If $p \geq 5$ is a prime number, show that p^2+2 is composite.

Proof. Let $p \geq 5$ be a prime number. Then,

$$p^2+2 = (p^2-1)+3 = (p-1)(p+1)+3$$

But p on the form $p=3k+1$ or $p=3k+2$

Case 1: $p = 3k+1 \Rightarrow p-1 = 3k \Rightarrow 3|(p-1) \Rightarrow 3|(p-1)(p+1)+3$. So, $3|(p^2+2) \Rightarrow p^2+2$ is composite.

Case 2: $p = 3k+2 \Rightarrow p+1 = 3(k+1) \Rightarrow 3|(p+1) \Rightarrow 3|(p-1)(p+1)+3$. So, $3|(p^2+2) \Rightarrow p^2+2$ is composite.

Therefore, for $p \geq 5$ that is a prime number, p^2+2 is composite. □

3.1.10 If $p \neq 5$ is an odd prime, prove that either p^2-1 or p^2+1 is divisible by 10.

Proof. Let $p \neq 5$ be an odd prime. Then by division algorithm $p = 10q+r$, for some $q, r \in \mathbb{Z}$. r must be $1, 3, 7$, or 9.

Case 1: $r = 1, p = 10q+1 \Rightarrow p^2-1 = 100q^2+20q = 10(10q^2+2q+0) \Rightarrow 10|(p^2-1)$.

Case 2: $r = 3, p = 10q+3 \Rightarrow p^2+1 = 100q^2+60q+10 = 10(10q^2+6q+1) \Rightarrow 10|(p^2+1)$.

Case 3: $r = 7, p = 10q+7 \Rightarrow p^2+1 = 100q^2+140q+50 = 10(10q^2+14q+5) \Rightarrow 10|(p^2+1)$.

Case 4: $r = 9, p = 10q+9 \Rightarrow p^2-1 = 100q^2+180q+80 = 10(10q^2+18q+8) \Rightarrow 10|(p^2-1)$.

Therefore, in all cases either $10|(p^2-1)$ or $10|(p^2+1)$ □

3.1.15 Prove that a positive integer $a > 1$ is a square if and only if in the canonical form of a all the exponents of the primes are even integers.

Proof. Let a be a square, $a = k^2, k = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}, p_i$ is prime.
Then, $a = k^2 = (p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s})^2 = p_1^{2r_1} p_2^{2r_2} \cdots p_s^{2r_s}$
Thus, all the exponents are even.
Assume all the exponents are even. That is $a = p_1^{2m_1} p_2^{2m_2} \cdots p_s^{2m_s}$
$a = (p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s})^2$
Thus, a is square.
Therefore, a positive integer $a > 1$ is a square if and only if in the canonical form of a all the exponents of the primes are even integers. $\square$