7.3.02 Use Euler's theorem to confirm that, for any integer $n \geq 0$,

$$51 | 10^{32n+9} - 7$$

Since, $51 = 17 \cdot 3$ we have $\phi(51) = \phi(17)\phi(3) = 16 \cdot 2 = 32$. By Euler's theorem we have $10^{32} \equiv 1 \pmod{51}$. Notice,

$$10^{32n+9} - 7 = 10^{32n} \cdot 10^9 - 7 = 1^n \cdot 10^9 - 7 = 10^9 - 7 \equiv 0 \pmod{51}$$

Then observe the following.

$$\begin{aligned} 10^9 = (10^3)^3 &= 1000^3 \\ &\equiv 31^3 \pmod{51} \\ &\equiv 29791 \pmod{51} \\ &\equiv 7 \pmod{51} \end{aligned}$$

Thus, we have $10^9 \equiv 7 \pmod{51}$.
Therefore, $51 | 10^{32n+9} - 7$

7.3.05 If m and n are relatively prime positive integers, prove that

$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}$$

By Euler's Theorem, we have $m^{\phi(n)} \equiv 1 \pmod{n}$ and $n^{\phi(m)} \equiv 1 \pmod{m}$. Also, by Euler's Theorem we get $m^{\phi(n)} \equiv 0 \pmod{m}$ and $n^{\phi(m)} \equiv 0 \pmod{n}$. Notice,

$$m^{\phi(n)} + n^{\phi(m)} \equiv (1 + 0) \equiv 1 \pmod{m}$$

and

$$m^{\phi(n)} + n^{\phi(m)} \equiv (1 + 0) \equiv 1 \pmod{m}$$

Therefore, $m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}$

7.3.8a If $\gcd(a, n) = 1$, show that the linear congruence $ax \equiv b \pmod{n}$ has the solution $x \equiv ba^{\phi(n)-1} \pmod{n}$.
Let $\gcd(a, n) = 1$. Then by Euler's Theorem we have $a^{\phi(n)} \equiv 1 \pmod{n}$. Observe.

$$\begin{aligned} &a^{\phi(n)} \equiv 1 \pmod{n} \\ &\Rightarrow ba^{\phi(n)} \equiv b \pmod{n} \\ &\Rightarrow aa^{\phi(n)-1}b \equiv b \pmod{n} \\ &\Rightarrow a(^{\phi(n)-1}b) \equiv b \pmod{n} \end{aligned}$$

Thus, the linear congruence $ax \equiv b$ (mod n) has the solution $x = ba^{\phi(n)-1}$ (mod n)

7.3.10 For any integer a, show that a and a^{4n+1} have the same last digit.

If $\gcd(a, 10) = 1$, then $a^{\phi(10)} \equiv 1 \pmod{10}$. Thus, as $\phi(10) = 4$, we have $a^4 \equiv 1 \pmod{10}$. Then we have $a^{4n} \equiv 1 \pmod{10}$ and $a^{4n+1} \equiv a \pmod{10}$

Notice, then

$$a^{4n+1} \equiv a \pmod{2}$$

is true , because it is obviously true for $a \equiv 0 \pmod{2}$ and $a \equiv 1 \pmod{2}$

Suppose $\gcd(a, 5) = 1$. Then by Fermat's Theorem, $a^4 \equiv 1 \pmod{5}$. Then $a^{4n+1} = (a^4)^n \cdot a \equiv a \pmod{5}$. Thus, $a^{4n+1} \equiv a \pmod{5}$

Therefore, for any integer a, a and a^{4n+1} have the same last digit