3.3.02a If 1 is added to a product of twin primes, prove that a perfect square is always obtained.

Proof. Let p and $p+2$ be twin primes. Then,

$$p(p+2)+1 = p^2+2p+1 = (p+1)^2$$

Which is a perfect square.
Therefore, if 1 is added to a product of twin primes, a perfect square is always obtained $\square$

3.3.02b Show that the sum of twin primes p and $p+2$ is divisible by 12, provided that $p > 3$.

Proof. Let p and $p+2, p > 3$ be twin primes. Then we can rewrite p as $6k+1$ and $6k-1$ for some $k \in \mathbb{Z}$. Then,

$$p+p+2 = 6k+1+6k-1 = 12k|12$$

Therefore, the sum of twin primes p and $p+2$ is divisible by 12, provided that $p > 3$. $\square$

3.3.06 Prove that the Goldbach conjecture that every integer greater than 2 is the sum of two primes is equivalent to the statement that every integer greater than 5 is the sum of three primes.

Proof. Let p_1 and p_2 be primes. Let $n \in \mathbb{Z} > 2$ then $2n-2 > 2$. Then,

$$2n-2 = p_1+p_2 \Rightarrow 2n = p_1+p+2+2$$

But $n > 2 \Rightarrow 2n > 4 \Rightarrow 2n+1 > 5$. So, $2n+1 = p_1+p_2+3$.
Therefore, every integer greater than 2 is the sum of two primes is equivalent to the statement that every integer greater than 5 is the sum of three primes $\square$

3.3.24 Determine all twin primes p and $q = p+2$ for which $pq-2$ is also prime.

Proof. Let p and q be twin primes. If $p=3$ and $q=5$ then $pq-2=13$ which is prime. Suppose that $p>3$. Then $p=6k+1$ or $p=6k+5$. If $p=6k+1 \Rightarrow p+2=6k+3$ which is composite. Thus $p=6k+5 \Rightarrow p+2=6k+7$. Notice.

$$pq-2 = (6k+5)(6k+7)-2 = 36k^2+72k+33 = 3(12k^2+24k+11)$$

which is divisible by 3.
Thus, 3 and 5 are the only twin primes for which $pq-2$ is prime. $\square$

3.3.28a If $n > 1$, show that $n!$ is never a perfect square.

Proof. Let $n > 1$.

Case 1: n is prime. Then for $n!$ to be a perfect square one of $n-1, n-2, \cdots, 2$ must contain n as a factor. But this means one of $n-1, n-2, \cdots, 2 \geq n$ which is impossible.

Case 2: n is not prime. Then the first prime less than n is for all $p, k \in \mathbb{Z}, p = n-k, 0 < k < n-1, 2 \leq p < n$ No number less than p will contain p as a factor. Thus, for $n!$ to be a perfect square there exists a multiple of p, called $bp, 1 < b < n$, such that $p < bpn$. There must exists a prime number between p and $2p$. Then if $r < n < 2r$ and also $p < n$, so such an$n!$ would never be a perfect square.

□

3.3.28b Find the values of $n \geq 1$ for which

$$n! + (n+1)! + (n+2)!$$

is a perfect square.

Proof. Let $n \geq 1$. Notice.

$$n! + (n+1)! + (n+2)! = n!(1 + (n+1) + (n+1)(n+2)) = n!(n+2)^2$$

As $n!$ is never a perfect square, the only n for which

$$n! + (n+1)! + (n+2)!$$

is a perfect square is when $n = 1$. □