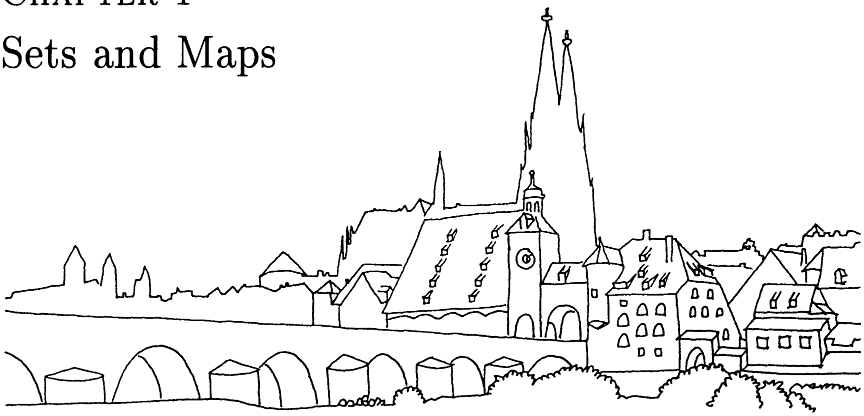
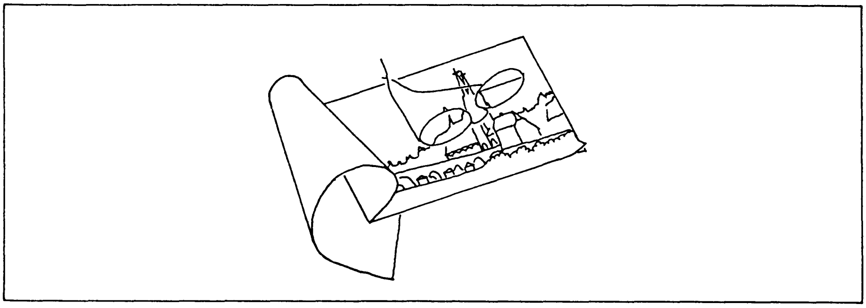
CHAPTER 1

Sets and Maps





1.1 Sets

Throughout your entire mathematical study, and particularly in this work, you will be continuously involved with *sets* and *maps*. In an ordinary mathematical textbook these concepts occur literally thousands of times. The concepts themselves are quite easy to understand; things become more difficult only when we concern ourselves (from Chapter 2 on) with what in mathematics is actually done with sets and maps. First of all, then, let us consider sets. From Georg Cantor, the founder of set theory, comes the following formulation.

> "A ***set*** is a collection into a whole of definite, distinct objects of our intuition or of our thought, which are called the ***elements*** of the set."

A set consists of its elements. If one knows all the elements, then one knows the set. Thus the "collection into a whole" is not to be understood as doing something special with the elements before they can form a set. The elements form, are, and constitute the set — and no more. Consider the following examples.

$\mathbb{N}$ = the set of natural numbers = $\{1,2, \dots \}$,
$\mathbb{N}_0$ = the set of nonnegative integers = $\{0,1,2, \dots \}$,
$\mathbb{Z}$ = the set of integers,
$\mathbb{Q}$ = the set of rational numbers,
$\mathbb{R}$ = the set of real numbers.

The concept of a set consisting of no elements has turned out to be very useful. This is called the *empty set*, for which the notation is as follows.

$\varnothing$ = the empty set.

Next we introduce some signs and symbols, which one uses in connection with sets. Thus we have

The element symbol $\in$
The set brackets $\{\dots\}$
The subset sign $\subset$
The intersection sign $\cap$
The union sign $\cup$
The complementary set sign $\setminus$
The product set sign $\times$

Which among these signs is already known to you? What do they represent, when you simply make a conjecture from the names?

Let's look at the element symbol.

If M is a set and x is an element of M, then one writes $x \in M$. Correspondingly, $y \notin M$ means that y is not an element of M.

For example, $-2 \in \mathbb{Z}$, but $-2 \notin \mathbb{N}$.

For the set brackets, see the following box.

One can describe a set by writing its elements between two curly brackets. This writing out of elements can happen in one of three ways. If the set has only a few elements, then one can simply write them all down, separated by commas. For example, $\{1,2,3\}$ consists of the three numbers one, two, and three. Neither the order of the sequence nor whether some elements are repeated is of importance:

$$\{1,2,3\} = \{3,1,2\} = \{3,3,1,2\}.$$

The second possibility is to use periods to indicate elements that one does not write out. Thus $\{1,2,\dots,10\}$ is immediately understood to be $\{1,2,3,4,5,6,7,8,9,10\}$, and $\{1,2,\dots\}$ to be the set of all natural numbers.

However, one should only use this procedure when certain that each observer of the formula knows what the periods mean. For example, it would not be clear how to read $\{37, 50, \dots\}$. The third, most frequently used, and always correct method is this: after the initial bracket, $\{$, one first writes a letter or symbol, which one has chosen to denote the elements of the set. One then makes a vertical line, on the other side of which one states in terms of this symbol, verbally or otherwise, what precisely *are* the elements of the set. Thus, instead of $\{1,2,3\}$, one can write: $\{x \mid x \text{ integral and } 1 \leq x \leq 3\}$. If the elements that one wishes to describe already belong to a specific set for which one already has a name, then one writes the property of belonging to the left of the vertical line: $\{1,2,3\} = \{x \in \mathbb{Z} \mid 1 \leq x \leq 3\}$. This reads: "The set of all x from $\mathbb{Z}$ with 1 less than or equal to x less than or equal to 3."

To describe the third and most generally applicable way of using the set brackets, let E be a property that each x in a set X either has or does not have. Then $\{x \in X \mid x \text{ has property } E\}$ denotes the set of all elements of X which have the property E.

We use the subset sign as described below.

If A and B are two sets, and if each element of A is also contained in B, then one says that A is a ***subset*** of B, and writes $A \subset B$.

Thus, in particular, each set is a subset of itself: $M \subset M$. Furthermore, the empty set is a subset of each set: $\varnothing \subset M$. For the sets introduced so far as examples, one has $\varnothing \subset \{1,2,3\} \subset \{1,2,\dots,10\} \subset \mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. In diagrams serving to illustrate the concepts introduced here, a set is often represented by a closed oval shape, labeled by a letter, as in Fig. 1. Then M is meant to be the set of points lying in the region "enclosed" by the oval. Sometimes, for the sake of greater clarity, we shall also shade the region where the points are elements of a set of interest to us. For example, in Fig. 2 we apply shading to indicate intersection, union, and difference (or complement) of two sets A and B. In case you are not yet acquainted with intersection, union, and complement, before reading further it would be a good exercise to try to understand the definitions of $\cap$, $\cup$, and $\setminus$ in terms of the pictures in Figs. 2a, b, and c.

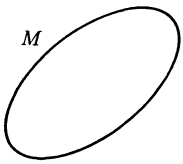


Fig. 1. A set M

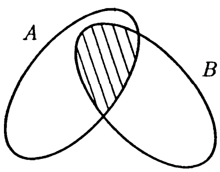


Fig. 2a. $A \cap B$

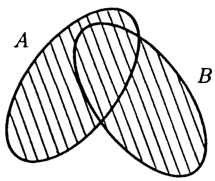


Fig. 2b. $A \cup B$

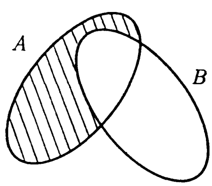


Fig. 2c. $A \setminus B$

Definition: If A and B are two sets, then the ***intersection*** $A \cap B$ (read as "A intersection B") consists of those elements that are contained in both A and B.

Definition: If A and B are two sets, then the ***union*** $A \cup B$ ("A union B") consists of those elements that are contained either in A or in B (or in both).

Definition: If A and B are two sets, then the ***complement*** $A \setminus B$ ("A minus B") consists of those elements that while contained in A are not contained in B.

When there are no elements "contained in both A and B," does it make sense to speak of the intersection $A \cap B$? Certainly! Then we have $A \cap B = \varnothing$, an example of the utility of the empty set. If $\varnothing$ were not admissible as a set, then in defining $A \cap B$ we would have to specify that there must exist some common element. Now think of what $A \setminus B = \varnothing$ means?

Before moving on to maps, we want to discuss Cartesian products of sets. To this end one must first define what is meant by an ordered *pair* of elements.

A ***pair*** consists in giving a first and a second element. If a denotes the first and b the second element, then the pair is denoted by (a, b).

The equality $(a, b) = (a', b')$ therefore denotes that $a = a'$ and $b = b'$. This is the essential difference between a pair and a two-element set: for the pair the sequential order is important, for the set it is not. Thus one always has that $\{a, b\} = \{b, a\}$, but $(a, b) = (b, a)$ only holds when $a = b$. A further distinction is that there exists no two-element set $\{a, a\}$ because $\{a, a\}$ has only one element, a. In contrast, (a, a) is a genuine pair.

Definition: The set $A \times B := \{(a, b) \mid a \in A,\ b \in B\}$ of pairs is called the ***Cartesian product*** of the sets A and B.

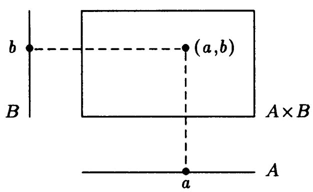


Fig. 3. Cartesian product $A \times B$

The symbol ":=" (analogously, "=: ") means that the expression on the side of the colon is first *defined* by the equation. Hence one does not have to search through one's memory to decide if one already knows it or to what the equation refers. Of course this should be clear from the context, but the notation eases its reading.

In order to illustrate the Cartesian product, one usually uses a rectangle and indicates A and B by intervals below and to the left of this rectangle, as in Fig. 3. For each $a \in A$ and $b \in B$, onethen "sees" the pair (a, b) as a point in $A \times B$. These pictures have only a symbolic significance: they illustrate the situation in a very simplified way, since in general A and B are not intervals. Nonetheless, as aids to thought and visualization, such diagrams should not to be discounted. One proceeds slightly differently when it is not a matter of consideringtwo sets A and B, but rather the special case $A = B = \mathbb{R}$. Here one "draws" $\mathbb{R}^2 := \mathbb{R}\times\mathbb{R}$ by sketching two mutually perpendicular copies of the real line. The horizontal line plays the role of $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}$, the vertical line that of $\{0\} \times \mathbb{R}$. An arbitrary element $(x, y) \in \mathbb{R}^2$ is then formed from $(x, 0)$ and $(0, y)$ as the diagram in Fig. 4 shows.

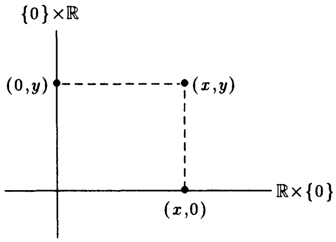


Fig. 4. Cartesian product ℝ×ℝ

Analogous to the definition of pairs, one also has ***triples*** (a, b, c) and ***n-tuples*** $(a_1, \ldots, a_n)$. If $A_1, \ldots, A_n$ are sets, the set

$$A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}$$

is called the Cartesian product of the sets $A_1, \ldots, A_n$. Particularly often in this book we shall have to do with the so-called $\mathbb{R}^n$; this is the Cartesian product of n factors $\mathbb{R}$:

$$\mathbb{R}^n := \mathbb{R} \times \cdots \times \mathbb{R}\,.$$

$\mathbb{R}^n$ is thus the set of all n-tuples of real numbers. Of course, between $\mathbb{R}^1$ and $\mathbb{R}$ there is only a formal distinction, if indeed one wants to make one. For the illustration of $\mathbb{R}^3$, as for $\mathbb{R}^2$, one uses the "axes" $\mathbb{R} \times \{0\} \times \{0\}, \{0\} \times \mathbb{R} \times \{0\}$, and $\{0\} \times \{0\} \times \mathbb{R}$, but we only half draw them; otherwise, the picture becomes difficult to read (see Fig. 5).

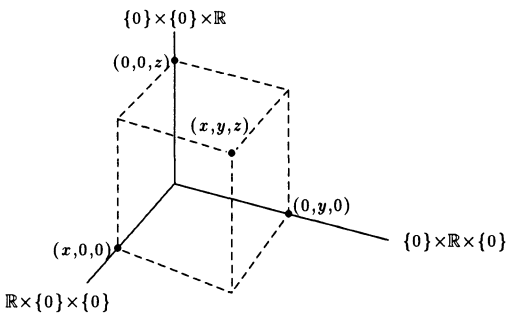


Fig. 5. Cartesian product ℝ×ℝ×ℝ

Such pictures are not meant to imply that $\mathbb{R}^3$ is "the space" (physical or geometric): $\mathbb{R}^3$ is and remains the set of all real number triples.

1.2 Maps

Definition: Let X and Y be sets. A ***map*** f from X to Y is a rule which to each $x \in X$ assigns precisely one element $f(x) \in Y$. Instead of "f is a map from X to Y," one writes $f : X \to Y$ as an abbreviation. Frequently it is practical also to describe the association of a single element x with its "image point" $f(x)$ by means of an arrow, but in this case, in order to avoid confusion, one uses another arrow, namely $x \mapsto f(x)$.

What does one write when defining a map? Here there is a choice of formulation. By way of example we use the map from $\mathbb{Z}$ to $\mathbb{N}_0$ that associates its square to each integer. Then one can either write

Let $f : \mathbb{Z} \to \mathbb{N}_0$ be the map given by
$f(x) := x^2$ for all $x \in \mathbb{Z}$,

or, somewhat shorter,

Let $f : \mathbb{Z} \to \mathbb{N}_0$ be the map given by $x \mapsto x^2$,

or, even shorter,

Consider $f : \mathbb{Z} \to \mathbb{N}_0$, $x \mapsto x^2$.

Finally, it is sometimes unnecessary to give the map a label; then one simply writes

$\mathbb{Z} \to \mathbb{N}_0$, $x \mapsto x^2$,

a very suggestive and practical notation.

One cannot avoid specifying which sets X and Y are involved (in our example, $\mathbb{Z}$ and $\mathbb{N}_0$), and it is also not permissible to call our map simply x^2. This is the value of our map at the point x, or as one also says, the *image* of x under the map, but not the mapping itself, for which we must choose some other notation.

Addition of real numbers is also a map, namely

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x + y. \end{aligned}$$

One can (and should) describe all arithmetic operations to oneself in this way.

A mapping does not need to be given by a formula; one can also describe the association in words. In order to distinguish between cases, one often uses a large bracket. For example, the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$x \longmapsto \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

is occasionally mentioned in analysis for one or another reason.

In a sequence of definitions, we shall now label some special maps as well as concepts and constructions referring to maps.

Definition: Let M be a set. Then one calls the map

$$\mathrm{Id}_M : M \longrightarrow M,\ x \mapsto x,$$

the ***identity*** on M. Sometimes one sloppily omits the subscript M and simply writes Id, if it is clear which M is involved.

Definition: Let A and B be sets. Then one calls the map

$$\begin{aligned} \pi_1 : A \times B &\longrightarrow A \\ (a,b) &\longmapsto a \end{aligned}$$

the ***projection on the first factor*** (see Fig. 6).

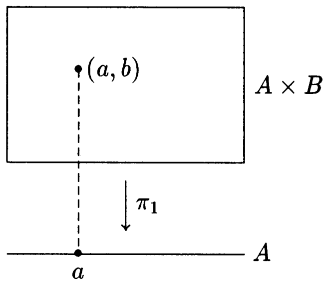


Fig. 6. Visualization of the projection on the first factor

Definition: Let X and Y be sets and $y_0 \in Y$. Then one calls the map

$$\begin{aligned} X &\longrightarrow Y \\ x &\longmapsto y_0 \end{aligned}$$

a ***constant*** map.

Definition: Let $f : X \to Y$ be a map and $A \subset X$, $B \subset Y$. Then one calls the set

$$f(A) := \{f(x) \mid x \in A\}$$

the ***image set*** of A or "the image of A," and the set

$$f^{-1}(B) := \{x \mid f(x) \in B\}$$

the ***preimage set*** of B or simply the preimage of B.

$f^{-1}(B)$ is read as "f minus 1 of B". It is important to observe that in no way have we defined $f^{-1}(B)$ using an "inverse map" f^{-1}. In this connection, the symbol f^{-1} alone, without an adjoining (B), has no meaning.

One can picture the concepts of image set and preimage set, shown in Figs. 7a and b, through the example of the projection onto the first factor of a Cartesian product given in Fig. 6.

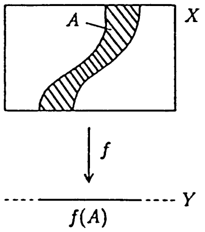


Fig. 7a. Image $f(A)$ of A

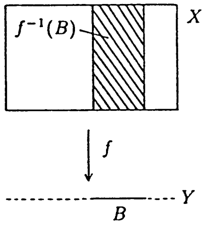


Fig. 7b. Preimage $f^{-1}(B)$ of B

The elements of $f(A)$ are precisely the $f(x)$ for $x \in A$. However, it can also happen that $f(z)$ belongs to $f(A)$ for some $z \notin A$, namely when by chance there exists $x \in A$ with $f(x) = f(z)$, as in Fig. 8.

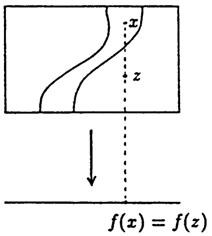


Fig. 8. It can happen that $f(z) \in f(A)$ for some $z \notin A$.

The elements of $f^{-1}(B)$ are precisely *the* elements of X that under the map f land in B. With maps it can also happen that *no* element lands in B. Well, then one has that $f^{-1}(B) = \varnothing$.

Definition: A map $f : X \to Y$ is called ***injective*** if no two elements of X are mapped onto the same element of Y. It is called ***surjective***, or a map ***onto*** Y, if each element $y \in Y$ is an $f(x)$. Finally, it is called ***bijective*** if it is both injective *and* surjective.

Let X, Y, Z be sets and f, g maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then in an obvious way one can form a map from X to Z, which one writes as $g \circ f$, or in short form as gf:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ x & \longmapsto & f(x) & \longmapsto & (gf)(x). \end{array}$$

The reason why one writes g first in gf (read "g following f"), even though one has first to apply f, is that the image of x under the composition of maps is $g(f(x))$. We formulate this as described in the following box.

Definition: If $f: X \to Y$ and $g: Y \to Z$ are maps, then the composition gf is defined by $X \to Z$, $x \mapsto g(f(x))$. If one has to deal with several maps between different sets, it is often clearer to arrange them in a diagram; for example, one can write the maps $f: X \to Y$, $g: Y \to Z$, $h: X \to Z$ in the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

If maps $f: X \to Y$, $g: Y \to B$, $h: X \to A$, and $i: A \to B$ are given, the corresponding diagram looks like this:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ A \arrow[r, "i"'] & B \end{tikzcd}$$

Of course, more sets and maps can occur in such a diagram; it is certainly clear enough what is meant by "diagram" without having to formalize this concept.

Definition: If in a diagram all maps between any two sets (including compositions and possibly multiple compositions) agree, then one calls the diagram ***commutative***.

The above diagram, for example, is commutative if and only if $gf = ih$.

If $f: X \to Y$ is a map and one would like to construct an "inverse map" from Y to X, which so to speak *reverses* f, then in general this fails for two reasons. First, the map f does not need to be surjective, therefore for some $y \in Y$ there possibly exists no $x \in X$ with $f(x) = y$, and hence one does not know which x to associate with y. Second, the map does not need to be injective, and therefore for some $y \in Y$ there may be *several* $x \in X$ with $f(x) = y$. But for a map $Y \to X$, only *one* x is allowed to be associated to each y. If, however, f is bijective, then there exists a natural inverse map that we can define as follows.

Definition: If $f: X \to Y$ is bijective, the ***inverse map*** to f is defined by

$$\begin{aligned} f^{-1}: Y &\longrightarrow X, \\ f(x) &\longmapsto x. \end{aligned}$$

One reads f^{-1} either as "f minus one" or as "f inverse."

Bijective maps will usually be denoted by the "isomorphism sign" $\cong$, thus

$$f : X \xrightarrow{\cong} Y.$$

Just as a precaution, let me add one further remark about the concept of an inverse map. Let $f : X \to Y$ be a map and $B \subset Y$ (see Fig. 9). You have just heard that only bijective maps have an inverse. However, experience shows that beginners are tempted to assume that *every* map f ought "somehow" *still* to have an inverse, and that $f^{-1}(B)$ has something to do with this inverse. I agree that the notation suggests this, but it should still be possible to distinguish between the bijective and nonbijective cases. When f is indeed bijective, then $f^{-1}(B)$ certainly has something to do with the inverse map, since you can describe it either as the f-preimage of B or as the f^{-1}-image of B. Clearly, one has (f assumed to be bijective):

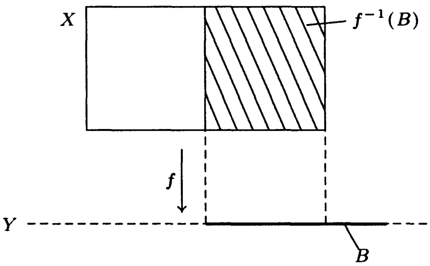


Fig. 9. Let $f{:}X\to Y$ be a map and $B\subset Y$

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} = \{f^{-1}(y) \mid y \in B\}.$$

One final definition: the restriction of a map to a subset of the domain of definition, shown in Fig. 10.

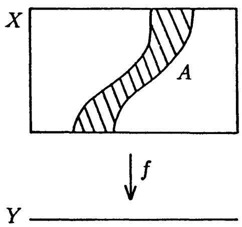


Fig. 10a. Map $f{:}X\to Y$

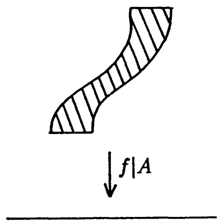


Fig. 10b. Restricted map $f|A{:}A\to Y$

Definition: Let $f : X \to Y$ be a map and $A \subset X$. Then the map

$$\begin{aligned} f|A : A &\longrightarrow Y \\ a &\longmapsto f(a) \end{aligned}$$

is called the ***restriction*** of f to A. One reads $f|A$ as "f restricted to A."

1.3 Test

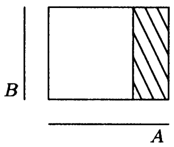
(1) If for each $a \in A$ we have $a \in B$, one writes

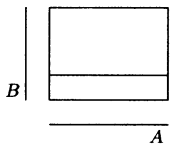
☐ $A \subset B$ ☐ $A = B$ ☐ $A \cup B$

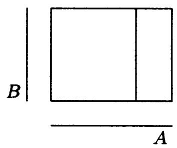
(2) For each set M, which of the following sets is empty?

☐ $M \cup M$ ☐ $M \cap M$ ☐ $M \setminus M$

(3) As usual, represent $A \times B$ by a rectangle. How would one picture $\{a\} \times B$?

☐ 
Fig. 11a.

☐ 
Fig. 11b.

☐ 
Fig. 11c.

(4) Which of the following statements is false? The map

$$\begin{aligned} \mathrm{Id}_M : M &\longrightarrow M \\ x &\longmapsto x \end{aligned}$$

is always

☐ surjective ☐ bijective ☐ constant

(5) Let A, B be sets and $A \times B$ the Cartesian product. By projection onto the second factor, one understands the map π_2 as:

☐ $\begin{aligned} A \times B &\longrightarrow A \\ (a,b) &\longmapsto b \end{aligned}$ ☐ $\begin{aligned} A \times B &\longrightarrow B \\ (a,b) &\longmapsto b \end{aligned}$ ☐ $\begin{aligned} B &\longrightarrow A \times B \\ b &\longmapsto (a,b) \end{aligned}$

(6) Let $f : X \to Y$ be a map. Which of the following statements implies that f is surjective?

☐ $f^{-1}(Y) = X$ ☐ $f(X) = Y$ ☐ $f^{-1}(X) = Y$

(7) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps. Then the map $gf : X \to Z$ is defined by

☐ $x \longmapsto g(f(x))$ ☐ $x \longmapsto f(g(x))$ ☐ $x \longmapsto g(x)(f)$

(8) Let

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & Z \arrow[uu, "h"'] \end{tikzcd}$$

be a commutative diagram. Then we have

☐ $h = gf$ ☐ $f = hg$ ☐ $g = fh$

(9) The map $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$, $x \mapsto \frac{1}{x}$ is bijective. The inverse map $f^{-1} : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ is defined by

☐ $x \longmapsto \frac{1}{x}$ ☐ $x \longmapsto x$ ☐ $x \longmapsto -\frac{1}{x}$

(10) $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is

☐ surjective but not injective
☐ injective but not surjective
☐ neither surjective nor injective

1.4 Remarks on the Literature

I imagine that the reader of the beginning of a first-year text is just starting her or his studies and therefore might be interested in what a lecturer — in this case, I — thinks of the relation between books and lectures.

Many years ago, as I was preparing the notes for my students, from which this book has emerged, the books and manuscripts on linear algebra occupied four feet of shelf space in the departmental library; today, they occupy more than fifteen. According to mood one can find this reassuring or terrifying, but one thing has certainly not changed: a beginning student in mathematics actually needs no textbook. The lectures are autonomous, and the most important work for the student is her or his *own* lecture notes. This perhaps strikes you as a task from mediaeval times. Take notes? Somewhere in the fifteen feet there must be some book containing the material of the lectures. And if I don't have to write with the lecturer, then I can *think* much better with him — so you say. Besides, you say to yourself: write? And what if I can't decipher what is written on the board? Or what if the lecturer writes so fast that I can't follow him?[1] And what if I'm ill and can't come to the lecture? Then I'm stuck with my fragmentary notes.

[1] "I can't even *speak* as fast as Jänich writes" was one student's comment passed on to me.

So plausible appear these arguments, and yet they do not hold water. First of all, on average no book in those fifteen feet will contain the "material of the lectures." Indeed, the large number of books and manuscripts on linear algebra is more a sign that each lecturer prefers to go his own way. Of course, many a lecture course is rooted in a given book or manuscript, and then you should have the book, if only because the lecturer may leave gaps to be filled from it. But even then you should take notes, and as soon as he makes use of two books, you can be certain that he will follow neither of them exactly. If you can't write fast enough, you must train yourself to do so; if you can't read the board from the back of the room, you must look for a seat nearer the front; and if you are ill, you must copy a colleague's notes. Why this effort? If not, you will lose touch with the material, fall behind, and soon understand nothing else being taught. Ask any older student if he has ever learned anything in a lecture course in which he has not taken notes. It is as if information presented to the eye and ear must first pass through the hand in order really to enter the brain. Perhaps this is linked to the fact that in practicing mathematics, you again have to write. But whatever the reason, experience proves it.

When you are really in the swing of a course of lectures, books will be very useful to you, and for more senior year studies books are essential. You must therefore learn to work with books, but as a novice you must not lightly let a book tempt you away from direct contact with the course.

1.5 Exercises

1.1: If $f : X \to Y$ is a map, one calls the set $\{(x, f(x)) \mid x \in X\}$ the ***graph*** Γ_f of f. The graph is a subset of the Cartesian product $X \times Y$. In diagram (a) it is indicated by a line. But the graph of a map cannot be an arbitrary subset of $X \times Y$ since, for example, to each x there can only correspond *one* $f(x)$, and thus the line drawn in diagram (b) is not a graph. The exercise is to draw graphs of functions with preassigned properties. For example, the graph of a nonsurjective map is illustrated in (c).

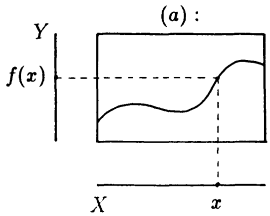


Fig. 12a. Graph

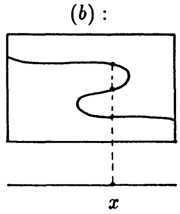


Fig. 12b. Nongraph

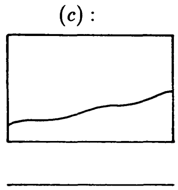


Fig. 12c. Graph of nonsurjective map

Give, in this fashion, examples of graphs of maps f with the following properties:

(i) f surjective, but not injective
(ii) f injective, but not surjective
(iii) f bijective
(iv) f constant
(v) f neither injective nor surjective
(vi) $X = Y$ and $f = \mathrm{Id}_X$
(vii) $f(X)$ consists of only two elements

1.2: The inverse map f^{-1} of a bijective map $f : X \to Y$ clearly has the properties $f \circ f^{-1} = \mathrm{Id}_Y$ and $f^{-1} \circ f = \mathrm{Id}_X$, since in the first case each element $f(x) \in Y$ is mapped by $f(x) \mapsto x \mapsto f(x)$ onto $f(x)$, and in the second case each $x \in X$ is mapped by $x \mapsto f(x) \mapsto x$ onto x. Conversely, one has the following (and the proof is the point of the exercise):

Let $f : X \to Y$ and $g : Y \to X$ be maps such that $fg = \mathrm{Id}_Y$ and $gf = \mathrm{Id}_X$, then f is bijective and $f^{-1} = g$.

(An injectivity proof runs like this: "Let $x, x' \in X$ and $f(x) = f(x')$, then ... . Therefore $x = x'$, and f is proved to be injective." On the other hand, the pattern for a surjectivity proof is: "Let $y \in Y$. Choose $x = \dots$. Then we have ... , therefore $f(x) = y$, and f is proved to be surjective.")

1.3: Let

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, "\alpha", "\cong"'] \arrow[r, "g"'] & B \arrow[u, "\beta", "\cong"'] \end{tikzcd}$$

be a commutative diagram of maps with α, β bijective. Show that g is injective if and only if f is injective.

(We shall frequently meet this kind of diagram in the text. The situation is then mostly: f is the object of our interest, α and β are subsidiary constructions, means to an end, and we already know something about g. This information about g then tells us something about f. In solving this exercise you will see into the mechanism of this information transfer.)