MTH 411, Fall 2018, Quiz 5 (Thursday Long Quiz/12 pts) sample solutions

1. **(10 points)** Indicate whether the statement is true or false, with a brief justification, counterexample or sketch of a proof.

   (a) The empty set can be considered a group.

   #4.25(i)

   ☐ True

   ☒ False   A group must contain an id. element (axiom $\mathcal{G}_2$).

   #4.25(h) **(b)** For given elements a, b, c in a group G there exists a unique solution $x \in G$ to the equation $axb = c$.

   ☒ True   Let $a, b, c \in G$. Then $x := a^{-1}cb^{-1} \in G$ is a solution to $axb = c$, since $a(a^{-1}cb^{-1})b = aa^{-1}c\,b^{-1}b = ece = c$. So we've shown there exists a solution in G. Next, let's show it's unique. Suppose $y \in G$ is a solution to $axb = c$, i.e. $ayb = c$. Then $a^{-1}(ayb)b^{-1} = a^{-1}cb^{-1}$, which implies $y = a^{-1}cb^{-1}$. Thus the solution to $axb = c$ is unique.

   ☐ False

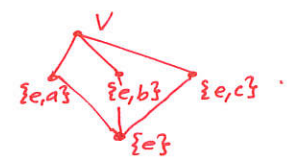
   #5.39(f) **(c)** A cyclic group has a unique generator.

   ☐ True

   ☒ False $\mathbb{Z}_6 = \langle 1 \rangle = \langle 5 \rangle$

   (d) If H and K are subgroups of a group G, then $H \cup K$ is also a subgroup of G.

   ☐ True   p.52 The Klein-4-group V has a subgroup diagram given by



   ☒ False   Then $H := \{e, a\} \leq V$, $K := \{e, b\} \leq V$, but $H \cup K = \{e, a, b\} \not\leq V$.

   alt. solution: $\langle 2 \rangle = \{0, 2, 4\} \leq \mathbb{Z}_6$, $\langle 3 \rangle = \{0, 3\} \leq \mathbb{Z}_6$. But $\langle 2 \rangle \cup \langle 3 \rangle \not\leq \mathbb{Z}_6$ (why?)

   #5.54 **(e)** If H and K are subgroups of a group G, then $H \cap K$ is also a subgroup of G.

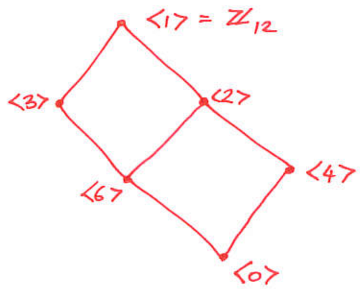
   ☒ True   (see HW solutions)

   ☐ False   (also try and prove the generalization of this statement, i.e. Theorem

#5.37 (variation) #6.22

2. **(2 points)** Draw the subgroup diagram for $\mathbb{Z}_{12}$.

(see lecture notes)



we generalized this to $\mathbb{Z}_{p^2q}$ where p and q are arbitrary primes.