Section 2 For 2.9 - 2.11, determine whether the binary operation $*$ defined is commutative and associative.

2.9 $*$ defined on $\mathbb{Q}$ by letting $a * b = ab/2$.

2.10 $*$ defined on $\mathbb{Z}^+$ by letting $a * b = 2^{ab}$

Communative: Let $a, b \in \mathbb{Z}^+$. Then,

$$a * b = 2^{ab} = 2^{ba} = b * a$$

Therefore, $*$ is commutative.

Associative: Let $a, b, c \in \mathbb{Z}^+$ Then,

$$(a * b) * c = 2^{ab} * c = 2^{abc}$$

$$a * (b * c) = a * 2^{bc} = 2^{abc}$$

Therefore, $*$ is associative.

2.11 $*$ defined on $\mathbb{Z}^+$ by letting $a * b = a^b$

For 2.14 - 2.16, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form for publication.

2.14 A binary operation $*$ is *commutative* if and onlt if $a * b = b * a$.

2.15 A binary operation $*$ on a set S is *associative* if and only if, for all $a, b, c \in S$, we have $(b * c) * a = b * (c * a)$.

2.16 A subset H of a set S is *closed* under a binary operation $*$ on S if $(a * b) \in H$ for all $a, b \in S$.

For 2.18, 2.20, and 2.22, determine whether the definition of $*$ does five a binary operation on the set. In the event that $*$ is not a binary operation, state whether Condition 1, Condition 2, or both of these conditions on page 24 are violated.

Condition 1: Exactly one element is assigned to each possible ordered pair of elements of S.
Condition 2: For each ordered pair of elements of S, the element assigned to it is again in S.

2.18 On $\mathbb{Z}^+$, define $*$ by letting $a * b = a^b$

2.20 On $\mathbb{Z}^+$, define $*$ by letting $a * b = c$ where c is the smallest integer greater than a and b.

2.22 On $\mathbb{Z}^+$, define $*$ by letting $a * b = c$ where c is the largest integer less than the product of a and b.

2.26 Prove that if $*$ is an associative and commutative binary operation on a set S, then

$$(a * b) * (c * d) = [(d * c) * a)] * b$$

for all $a, b, c, d \in S$. Assume the associative law only for triples as in the definition, that is, assume only

$$(x * y) * z = x * (y * z)$$

for all $x, y, z \in S$.

2.36 Suppose that $*$ us an *associative binary* operation on a set S. Show that $H = \{a \in \S | a * x = x * a \text{ for all } x \in S\}$. Show that H is closed under $*$.

2.37 Suppose that $*$ is an associative and commutative binary operation on a set S. Show that $H = \{a \in \S | a * a = a\}$ is closed under $*$. (check sheet for hint).

Section 3 For 3.6-3.10, determine whether the given map ϕ is an isomorphism of the first binary structure with the second. If not an isomorphism, why not?

3.6 $< \mathbb{Q}, \cdot >$ with $< \mathbb{Q}, \cdot >$ where $\phi(x) = x^2$ for $x \in \mathbb{Q}$. Not bijective thus, not isomorphic. (not onto) proof this more.

3.7 $< \mathbb{R}, \cdot >$ with $< \mathbb{R}, \cdot >$ where $\phi(x) = x^3$ for $x \in \mathbb{R}$.

Injective: Let $a, b \in \mathbb{R}$. Notice, $\phi(a) = \phi(b)$ which implies $a^3 = b^3 \to a = b$.

Surjective: Let $b \in \mathbb{R}$. Then. there must exist an arbitrary a such that $\phi(a) = b$. Let $a = \sqrt[3]{b}$. Then,

Let $a, b \in \mathbb{R}$. $\phi(a * b) = (ab)^3$ and conversely $\phi(a) * \phi(b) = a^3 * b^3 = (ab)^3$
Therefore, ϕ is a homomorphism. As ϕ is a bijective homomorphism, ϕ must be isomorphic.

3.8 $< M_2(\mathbb{R}), \cdot >$ with $< \mathbb{R}, \cdot >$ where $\phi(A)$ is is the determinate of matrix A.

3.9 $< M_1(\mathbb{R}), \cdot >$ with $< \mathbb{R}, \cdot >$ where $\phi(A)$ is is the determinate of matrix A.

3.10 $< \mathbb{R}, + >$ with $< \mathbb{R}^+, \cdot >$ where $\phi(r) = .5^r$ for $r \in \mathbb{R}$.

For 3.11 - 3.13 let F be the set of all functions f mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. Determine if they are isomorphism. Why or why not?

3.11 $< F, + >$ with $< F, + >$ where $\phi(f) = f'$, the derivative of f.

3.12 $< F, + >$ with $< \mathbb{R}, + >$ where $\phi(f) = f'(0)$

3.13 $< F, + >$ with $< F, + >$ where $\phi(f)(x) = \int_0^x f(t)dt$

3.16 The map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(n) = n + 1$ for $n \in \mathbb{Z}$ is one to one and onto $\mathbb{Z}$. Give the definition of a binary operation $*$ on $\mathbb{Z}$ such that ϕ is an isomorphic mapping.

3.16(a) $< \mathbb{Z}, + >$ onto $< \mathbb{Z}, * >$

Define $*$ have the operation $m * n = m + n - 1$. Let $a, b \in \mathbb{Z}$. Observe.

$$\phi(a + b) = (a + b) + 1 = (a + 1) + (b + 1) - 1 = \phi(a) + \phi(b) - 1 = \phi(a) * \phi(b)$$

Thus, there exists a homomorphism

3.16(b) $< \mathbb{Z}, * >$ onto $< \mathbb{Z}, + >$

3.17 The map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(n) = n + 1$ for $n \in \mathbb{Z}$ is one to one and onto $\mathbb{Z}$. Give the definition of a binary operation $*$ on $\mathbb{Z}$ such that ϕ is an isomorphic mapping.

3.17(a) $< \mathbb{Z}, \cdot >$ onto $< \mathbb{Z}, * >$

3.17(b) $< \mathbb{Z}, * >$ onto $< \mathbb{Z}, \cdot >$

3.18 The map $\phi : \mathbb{Q} \to \mathbb{Q}$ defined by $\phi(x) = 3x - 1$ for $x \in \mathbb{Q}$ is one to one and onto $\mathbb{Q}$. Give the definition of a binary operation $*$ on $\mathbb{Q}$ such that ϕ is an isomorphic mapping. Give the identity element.

3.18(a) $< \mathbb{Q}, + >$ onto $< \mathbb{Q}, * >$

3.18(b) $< \mathbb{Q}, * >$ onto $< \mathbb{Q}, + >$

For 2.21,2.22 correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

3.21 A function $\phi : S \to S'$ is an *isomorphism* if and only if $\phi(a * b) = \phi(a) * \phi(b)$

3.22 Let $*$ be a binary operation on a set S. An element e of S with the property $s*e = s = e*s$ is an *identity element for* $*$ for all $s \in S$.

3.31 Give a careful proof for a skeptic that the indicated property of a binary structure $< S, * >$ is indeed a structural property. (In Theorem 3.14 we did this for the property, "There is an identity element for $*$."). For each $c \in S$, the equation $x * x = c$ has a solution x in S.

3.33 Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form for $a, b \in \mathbb{R}$. Exercise 23 of Section 2 shows that H is closed under both matrix addition and multiplication.

3.33(a) Show that $< \mathbb{C}, + >$ is isomorphic to $< H, + >$

3.33(b) Show that $< \mathbb{C}, \cdot >$ is isomorphic to $< H, \cdot >$