9.02 Find all orbits of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}$
1, 5, 7, 8, 2, 3, 6, 4

9.04 Find all orbits of $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 1$
One orbit, being $\mathbb{Z}$

9.06 Find all orbits of $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n - 3$

$$\{3n|n \in \mathbb{Z}\}, \{3n+1|n \in \mathbb{Z}\}, \{3n+2|n \in \mathbb{Z}\}$$

9.08 Compute the indicated product of cycles $(1, 3, 2, 7)(4, 8, 6)$ that are permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 2 & 8 & 5 & 4 & 1 & 6 \end{pmatrix}$$

9.12 Express the permutation of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disjoint cycles, and then as a product of transpositions. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$
$(1, 3, 4, 7, 8, 6, 5, 2)$ and $(1, 2)(1, 5)(1, 6)(1, 8)(1, 7)(1, 4)(1, 3)$

9.19 Complete figure 9.22 of the Cayley digraph for the alternating group A_4 using the generating set $S = \{(1, 2, 3), (1, 2)(3, 4)\}$

9.20-22 Correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

9.20 For a permutation σ of a set A, an *orbit* of σ is a nonempty subset of A that is mapped onto itself by σ
Correct as stated.

9.21 A *cycle* is a permutation having only one orbit.
A *cycle* is a permutation having at most one orbit containing more than one element.

9.22 The *alternating group* is the group of even permutations.
The *alternating group* A_n is the subgroup of S_n consisting of the even permutations in S_n.

9.24 Which of the permutations in S_3 of Example 8.7 are even permutations? Give the table for the alternating group of A_3. The permutations that even are $\rho_0 = (12)(12), \rho_1 = (1, 2, 3) = (1, 3)(1, 2)$, and $\rho_2 = (1, 3, 2) = (1, 2)(1, 3)$

	ρ_0	ρ_1	ρ_2
ρ_0	ρ_0	ρ_1	ρ_2
ρ_1	ρ_1	ρ_2	ρ_0
ρ_2	ρ_2	ρ_0	ρ_1

9.33 Consider S_n for a fixed $n \geq 2$ and let σ be a fixed odd permutation. Show that every odd permutation in S_n is a product of σ and some permutation in A_n.
Consider S_n for a fixed $n \geq 2$, and let σ be a fixed odd permutation in S_n. Let σ' be a odd permutation in S_n. Then, σ^{-1} is also an odd permutation. Let $\mu = \sigma^{-1}\sigma'$ which must be an even permutation as its the product of two odd permutations. Then,

$$\sigma' = \sigma(\sigma^{-1}\sigma')$$

We see that σ' is in fact a product of σ and a permutation in A_n
Therefore, every odd permutation in S_n is a product of σ and some permutation in A_n

10.12 Find the index of $\langle 3 \rangle$ in the group Z_{24}
$\langle 3 \rangle = \{1, 3, 6, 9, 12, 15, 18, 21\}$. Thus, index is $24/8 = 3$

10.16 Let $\mu = (1, 2, 4, 5)(3, 6)$ in S_6. Find the index of $\langle \mu \rangle$ in S_6.
Notice that μ generates a cyclic subgroup S_6 of order 4. Thus we have for the index $6!/4 = 720/4 = 180$.

10.17 Let G be a group and let $H \subseteq G$. The *left coset of H containing a* is $aH = \{ah | h \in H\}$
Let G be a group and let $H \leq G$. The *left coset of H containing a* is $aH = \{ah | h \in H\}$

10.18 Let G be a group and let $H \leq G$. The *index of H in G* is the number of right cosets of H in G
Correct as stated.

10.20 A subgroup of an abelian group G whose left cosets and right cosets give different partitions of G. Impossible, as an abelian group cannot have a subgroup whose left and right cosets give different partitions.

10.21 A subgroup of a group G whose left cosets give a partition of G into just one cell.
Let G be a group, then use the improper subgroup $H = G$. Then the left cosets give a partition of G into just one cell.

10.22 A subgroup of a group of order 6 whose left cosets give a partition of the group into 6 cells.
Consider the subgroup $H := 0$ of $\mathbb{Z}_6$. Then $0 + H = \{0\}, 1 + H = \{1\}, \cdots 5 + H = \{5\}$

10.23 A subgroup of a group of order 6 whose left cosets give a partition of the group into 12 cells.
Impossible as the order cannot be less than the number of cells when the left cosets partition a subgroup.

10.24 A subgroup of a group of order 6 whose left cosets give a partition of the group into 4 cells. Impossible as 4 does not divide 6. Thus, a group of order 6 cannot be partitioned into 4 cells.

10.28 Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left cosets gH is the same as the right coset Hg.
Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$.

Let $g \in G$ and $x \in gH$. Then $\exists h \in H$ such that $x = gh$. Notice.

$$gh = ghe = ghg^{-1}g = (ghg^{-1}g) = [(g^{-1})^{-1}hg^{-1}]$$

We then have, $ghg^{-1} \in H$
Thus, $x \in Hg$
Therefore, $gH \subset Hg$

Let $x \in Hg$ and $h \in H$ such that $x = hg$. Notice.

$$hg = ehg = gg^{-1}hg = g(g^{-1}hg)$$

Thus, $g^{-1}hg \in H$ and $x \in gH$
Therefore, $Hg \subset gH$ for all $g \in G$

Therefore, as the two are subsets of one another,every left cosets gH is the same as the right coset Hg, $gH = hg$

10.29 Let H be a subgroup of a group G. Prove that if the partition of G into left cosets of H is the same as the partition into right cosets of H, then $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$

Let $g \in G$ and $h \in H$ such that $hg \in Hg$. Since $H \leq G$, $e \in H$. Notice. $g = eg \in Hg$ and $g = ge \in gH$. Thus, $g \in gH \cap Hg$. Then as the left and right cosets are the same partition, we have $gH = Hg$. From this there exists $h' \in H$ such that $hg = gh' \Rightarrow g^{-1}hg = g^{-1}gh' = eh' = h' \in H$.
Therefore, we have $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$

10.37 Show that a group with at least two elements but with no proper nontrivial subgroups must be finite and of prime order.
Let G be a group with order ≥ 2 and with no proper nontrivial subgroups. Let $a \in G$ and $a \neq e$. Then $\langle a \rangle$ is a nontrivial subgroup of G. Thus, $\langle a \rangle$ must be G. As we've seen every cyclic group of not of prime order has proper subgroups, we must have that G is finite of prime order.

10.40 Show that if a group G with identity e has finite order n, then $a^n = e$ for all $a \in G$ Let G be a group with identity e with finite order n. Let $a \in G$. Let $\langle a \rangle$ have order d and must divide the order of G. i.e. $n = dq$ for some $q \in \mathbb{Z}$. Then $a^d = e$. Thus by the theorem of Lagrange, $a^n = (a^d)^q = e^q = e$

11.01 List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each of the elements. Is the group cyclic?
$\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)\}$
Orders are $1, 4, 2, 4, 2, 4, 2, 4$, respectively.
Not cyclic.

11.02 Repeat for the group $\mathbb{Z}_3 \times \mathbb{Z}_4$ $\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,$
Order are $1, 4, 2, 4, 2, 12, 6, 12, 3, 12, 6, 12$, respectively
Cyclic as there are elements of order 12.

11.14 Fill in the blanks.

(a) The Cyclic subgroup of Z_{24} generated by 18 has order ___.
4

(b) $\mathbb{Z}_3 \times \mathbb{Z}_4$ is of order ___.
12

(c) The element $(4, 2)$ of $\mathbb{Z}_{12} \times \mathbb{Z}_8$ has order ___.
12

(d) The Klein 4-group is isomorphic to $\mathbb{Z}_{\_\_} \times \mathbb{Z}_{\_\_}$.
2, 2

(e) $\mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_4$ has ___ elements of finite order.
8

11.15 Find the maximum possible order for some element of $\mathbb{Z}_4 \times \mathbb{Z}_6$. As 4 and 6 are not relatively prime, $\mathbb{Z}_4 \times \mathbb{Z}_6$ is not cyclic and has no element of order 24. Thus, the maximum possible order is lcm(4,6) = 12.

11.16 Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ isomorphic? Why or why not? Yes, both are isomorphic. As $\mathbb{Z}_2 \times \mathbb{Z}_{12} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_6 \simeq \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2$
Thus, $\mathbb{Z}_2 \times \mathbb{Z}_{12} \simeq \mathbb{Z}_4 \times \mathbb{Z}_6$

11.46 Prove the direct product of abelian groups is abelian. Let each G_i be an abelian group

$$G_1 \times \cdots \times G_n$$

$$\Rightarrow (a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (b_1, \ldots, b_n) \cdot (a_1, \ldots, a_n)$$
$$\Rightarrow (a_1 b_1, \ldots, a_n b_n) = (b_1 a_1, \ldots, b_n a_n)$$
$$\Rightarrow \forall i, a_i b_i = b_i a_i$$

Thus, as the components are abelian the groups are abelian as well.