Determine weather the given map ϕ is a homomorphism.

13.07 Let $\phi_i : G_i \to G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_R$ be given by $\phi_i(g_i) = (e_1, e_2, \ldots, g_i, \ldots, e_R)$, where $g_i \in G_i$ and e_j is the identity element of G_j
Let $a, b \in G_i$. Observe.

$$\begin{aligned}\phi(ab) &= (e_1, e_2, \cdots, ab, \cdots, e_r)\\ &= (e_1, e_2, \cdots, a, \cdots, e_r)(e_1, e_2, \cdots, b, \cdots, e_r)\\ &= \phi(a)\phi(b)\end{aligned}$$

Thus, a homomorphism

13.08 Let G be any group and let $\phi : G \to G$ be given by $\phi(g) = g^{-1}$ for $g \in G$
If G is abelian, let $a, b \in G$. Notice.

$$\phi(ab) = (ab)^{-1} = b^{-1}a^{-1} = \phi(b)\phi(a)$$

Thus, a homomorphism if G is abelian.

13.12 Let M_n be the additive group of all $n \times n$ matrices with real entries, and let $\mathbb{R}$ be the additive group of real numbers. Let $\phi(A) = \det(A)$, the determinant of A, for $A \in M_n$
Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Then,

$$\phi(A + B) = \det(A + B) = 4$$

but

$$\phi(A) + \phi(B) = \det(A) + \det(B) = 1 + 1 = 2$$

Thus, not a homomorphism.

13.13 Let M_n and $\mathbb{R}$ be as in Exercise 12 . Let $\phi(A) = \mathrm{tr}(A)$ for $A \in M_n$, where the trace $\mathrm{tr}(A)$ is the sum of the elements on the main diagonal of A, from the upper-left to the lower-right corner.
Let $A = (a_{ij})$ and $B = (b_{ij})$. Observe.

$$\begin{aligned}\phi(A + B) &= \mathrm{tr}(A + B)\\ &= \sum_{i=1}^{n}(a_{ii} + b_{ii})\\ &= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}\\ &= \mathrm{tr}(A) + \mathrm{tr}(B)\\ &= \phi(A) + \phi(B)\end{aligned}$$

Thus, a homomorphism.

13.14 Let $GL(n, \mathbb{R})$ be the multiplicative group of invertible $n \times n$ matrices, and let $\mathbb{R}$ be the additive group of real numbers. Let $\phi : GL(n, \mathbb{R}) \to \mathbb{R}$ be given by$\phi(A) = \text{tr}(A)$, where tr (A) is defined in Exercise 13.
Notice, $\phi(I_n I_n) = \phi(I_n) = \text{tr}(I_n) = n$ and $\phi(I_n) + \phi(I_n) = \text{tr}(I_n) + \text{tr}(I_n) = n + n = 2n$
Thus, not a homomorphism.

13.30 A *homomorphism* is a map such that $\phi(xy) = \phi(x)\phi(y)$
A *homomorphism* is a map from a group G into a group G' such that $\phi(xy) = \phi(x)\phi(y)$

13.31 Let $\phi : G \to G'$ be a homomorphism of groups. The *kernel* of ϕ is $\{x \in G | \phi(x) = e'\}$ where e' is the identity in G'
Correct as stated.

13.44 Let $\phi : G \to G'$ be a group homomorphism. Show that if $|G|$ is finite, then, $|\phi[G]|$ is finite and is a divisor of $|G|$
Note that $\phi[G] := \{\phi(x) | x \in G\}$ and we have $|\phi[G]| \leq |G|$, thus $\phi[G]$ must also be finite.
By Theorem 13.15, we have $|\phi[G]| = |G| / \ker(\phi)|$ and thus $\phi[G]$ is a divisor of $|G|$

13.45 Let $\phi : G \to G'$ be a group homomorphism. Show that if $|G'|$ is finite, then, $|\phi[G]|$ is finite and is a divisor of $|G'|$
Note that $\phi[G] := \{\phi(x) | x \in G\}$ and we have $\phi[G] \subseteq G'$, thus $\phi[G]$ must also be finite.
By Lagrange's Theorem we have that $|\phi[G]|$ is a divisor of $|G'|$

13.48 The sign of an even permutation is $+1$ and the sign of an odd permutation is -1. Observe that the map $\text{sgn}_n : S_n \to \{1, -1\}$ defined by $\text{sgn}_n(\sigma) =$ sign of σ is a homomorphism of S_n onto the multiplicative group$\{1, -1\}$
What is the Kernel? The $\ker(sgn_n) = \{\sigma \in S_n | \sigma \text{is an even permutation.}\}$

13.50 Let $\phi : G \to H$ be a group homomorphism. Show that$\phi[G]$ is abelian if and only if for all $x, y \in G$, we have $xyx^{-1}y^{-1} \in \text{Ker}(\phi)$
Assume $\phi[G]$ is abelian. Let $x, y \in G$. Observe.

$$\begin{aligned}
\phi(xyx^{-1}y^{-1}) &= \phi(x)\phi(y)\phi(x^{-1})\phi(y^{-1}) \\
&= \phi(y)\phi(x)\phi(x^{-1})\phi(y^{-1}) \\
&= \phi(y)\phi(xx^{-1})\phi(y^{-1}) \\
&= \phi(y)\phi(y^{-1}) \\
&= e
\end{aligned}$$

Therefore, $xyx^{-1}y^{-1} \in \ker(\phi)$

Assume $\forall x, y \in G, xyx^{-1}y^{-1} \in \ker(\phi)$. Then $\phi(xyx^{-1}y^{-1}) = e$ and $\phi(x)\phi(y)\phi(x^{-1})\phi(y^{-1}) = e$.
Note, $\phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1} = e$.
We can rewrite $\phi(x)^{-1}\phi(y)^{-1}$ as $(\phi(y)\phi(x))^{-1}$.
Multiplying on the right by $(\phi(y)\phi(x))$ we have, $\phi(x)\phi(y) = \phi(y)\phi(x)$.
Therefore, $\phi[G]$ is abelian

14.06 Find the order of the factor group, $(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle(4,3)\rangle$
As $|\langle(4,3)\rangle| = 6$ and $|(\mathbb{Z}_{12} \times \mathbb{Z}_{18})| = 216$ we have $216/6 = 36$

14.16 Compute $i_{\rho_1}[H]$ for the subgroup $H = \{\rho_0, \mu_1\}$ of the group S_3 of Example 8.7 .
$i_{\rho_1}(H) = \{\rho_0, \mu_2\}$

14.17 A *normal subgroup* H of G is one satisfying $hG = Gh$ for all $h \in H$
A *normal subgroup* H of a group G is a subgroup satisfying $gH = Hg$ for all $g \in G$

14.18 A *normal subgroup* H of G is one satisfying $g^{-1}hg \in H$ for all $h \in H$ and all $g \in G$
Correct as stated.

14.19 An *automorphism* of a group G is a homomorphism mapping G into G
An *automorphism* of a group G is a isomorphism mapping G onto G

14.24 Show that A_n is a normal subgroup of S_n and compute S_n/A_n;that is, find a known group to which S_n/A_n is isomorphic.
If $n = 1$ we have $S_1 = A_1$ which gives us that A_1 is a normal subgroup of S_1
If $n \geq 2$, we know that $|A_n| = |S_n|/2$. Thus, there are only 2 cosets of A_n, being A_n itself and the odd permutations of S_n. Then, the left and right cosets must be the same, hence A_n is a normal subgroup of S_n.
Notice, S_n/A_n has order 2, and thus is isomorphic to $\mathbb{Z}_2$

14.37a Show that all automorphisms of a group G form a group under function composition.
Let G, G' and G'' be groups, $a, b \in G$, and let $\phi : G \to G'$ and $\gamma : G' \to G''$ be homomorphisms. Then,

$$\gamma\phi(ab) = \gamma(\phi(ab)) = \gamma(\phi(a)\phi(b)) = \gamma(\phi(a))\gamma(\phi(b)) = \gamma\phi(a)\gamma\phi(b)$$

Thus, the composition of two automorphisms of G is a homomorphism of G into G
As each automorphism is a bijection, their composition also is a bijection, and then must be automorphism of G. From this, we have that compostion gives a binary operation on the set of all automorphisms of G.
Consider $\mathrm{id}_G : G \to G$. Let *phi* be in the set of all automorphisms of G. Then $\phi \circ \mathrm{id}_G = \phi = \mathrm{id}_G \circ \phi$.
Thus, id_G is an automorphism.
Also, notice $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = \mathrm{id}_G$
Thus, the automorphisms form a group under function composition

14.37b Show that the inner automorphisms of a group G form a normal subgroup of the group of all automorphisms of G under function composition.
For $a, b, x \in \mathbb{G}$, we have

$$i_a(i_b(x)) = i_a(bxb^{-1}) = a(bxb^{-1})a^{-1} = (ab)x(b^{-1}a^{-1}) = (ab)x(ab)^{-1} = i_{ab}(x)$$

Thus, the composition of two inner automorphisms is still an inner automorphism.
Notice i_e is the identity
Notice, $i_a i_{a^{-1}} = i_e$, thus $i_{a^{-1}}$ is the inverse of i_a

Thus, under function composition the set of inner automorphisms is a group.
Let $a, x \in G$ and let ϕ be an automorphism of G. Observe.

$$(\phi i_a \phi^{-1})(x) = \phi(i_a(\phi^{-1}(x))) = \phi(a\phi^{-1}(x)a^{-1}) = \phi(a)\phi(\phi^{-1}(x))\phi(a^{-1}) = \phi(a)x(\phi(a))^{-1} = i_{\phi(a)}(x)$$

Then, $\phi i_a \phi^{-1} = i_{\phi(a)}$
Thus, the inner automorphisms are a normal subgroup of the automorphism group of G.

14.40a The $n \times n$ matrices with determinant 1 form a normal subgroup of $GL(n, \mathbb{R})$
Let H be the subset of $GL(n, \mathbb{R})$ consisting of $n \times n$ matrices with determinant 1. Let $A, B \in H$. Notice, $\det(AB) = \det(A)\det(B)$ and thus must be closed under matrix multiplication. Observe that $\det(I_n) = 1$ and thus is the identity element in H. The inverse of A is A^{-1}. Notice, $\det(A^{-1}) = 1/\det(A) = 1/1 = 1$. Thus $A^{-1} \in H$
Therefore, $H \leq GL(n, \mathbb{R})$

Let $A \in H$ and $B \in GL(n, \mathbb{R})$. Note, $\det(B) \neq 0$. Then, $\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \det(B)\det(A)(1/\det(B)) = \det(A) = 1$. Thus, $BAB^{-1} \in H$.
Thus H is a normal subgroup of $GL(n, \mathbb{R})$.

14.40b The $n \times n$ matrices with determinant $\pm$ 1 form a normal subgroup of $GL(n, \mathbb{R})$
Let H be the subset of $GL(n, \mathbb{R})$ consisting of $n \times n$ matrices with determinant ± 1. Let $A, B \in H$. Notice, $\det(AB) = \det(A)\det(B)$ and thus must be closed under matrix multiplication. Observe that $\det(I_n) = 1$ and thus is the identity element in H. The inverse of A is A^{-1}. Notice, $\det(A^{-1}) = 1/\det(A)$. Which must be either 1 or -1. Thus $A^{-1} \in H$
Therefore, $H \leq GL(n, \mathbb{R})$

Let $A \in H$ and $B \in GL(n, \mathbb{R})$. Note, $\det(B) \neq 0$. Then, $\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \det(B)\det(A)(1/\det(B)) = \det(A)$. Which must be either 1 or -1 Thus, $BAB^{-1} \in H$.
Thus H is a normal subgroup of $GL(n, \mathbb{R})$.