

**Reading:** Sections 6.

**Section 6 Problems:** You should be able to work out all problems

---

The following problems are **due on 11:59pm Friday 10/12/2018**. Submit both LaTeX and pdf files to the appropriate D2L Dropbox.

Please name the files using the following format:

LastName\_FirstName\_MTH411\_Fall2018\_HW\_3

You may discuss the problems with your classmates, but your write-up must be your own. Any problems marked with an asterisk (\*) denote problems you can not discuss with anyone except for me.

Please include the statements of the problems in your HW submissions. For any Extra problems you can copy the statements from the LaTeX file that generated this pdf. However, you will have to transcribe the remaining problems from Fraleigh.

**Section 6:** 9, 10, 14, 16, 18, 22, 24, 26, 27, 30, 33-37, 40, 49

Proofs: 45, 47, 55

6.09 Find the number of generators of a cyclic group of order 8

As all relatively prime solutions to  $\gcd(8, n)$  for  $0 \leq n < 8$  will result in the numbers that generate a cyclic group of order 8. Thus, there are 4 generators of a cyclic group of order 8, namely 1, 2, 5, 7.

6.10 Find the number of generators of a cyclic group of order 12

As all relatively prime solutions to  $\gcd(12, n)$  for  $0 \leq n < 12$  will result in the numbers that generate a cyclic group of order 12. Thus, there are 4 generators of a cyclic group of order 12, namely 1, 5, 7, 11.

6.14 Find the number of automorphisms of  $\mathbb{Z}_8$

There are 4 automorphisms of  $\mathbb{Z}_8$  as 1 can be taken to 1, 3, 5 and 7.

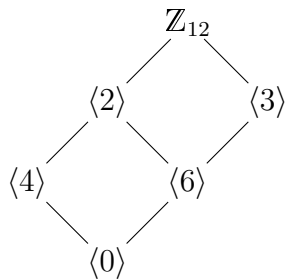
6.16 Find the number of automorphisms of  $\mathbb{Z}_{12}$

There are 4 automorphisms of  $\mathbb{Z}_{12}$  as 1 can be taken to 1, 5, 7 and 11.

6.18 Find the number of elements of the cyclic subgroup of  $\mathbb{Z}_{42}$  generated by 30.

As  $\gcd(42, 30) = 6$  and  $\frac{42}{6} = 7$ ,  $\langle 30 \rangle$  has 7 elements.

6.22 Find all subgroups of  $\mathbb{Z}_{12}$  and draw the subgroup diagram for the subgroup.



6.24 Find all subgroups of  $\mathbb{Z}_8$  and draw the subgroup diagram for the subgroup.



6.26 Find all orders of subgroups of  $\mathbb{Z}_8$ .

By the previous exercise we know the subgroups of  $\mathbb{Z}_8$ . Thus,  $\mathbb{Z}_8$  has subgroups of order 1, 2, 4, and 8.

6.27 Find all orders of subgroups of  $\mathbb{Z}_{12}$ .

By the previous exercise we know the subgroups of  $\mathbb{Z}_{12}$ . Thus,  $\mathbb{Z}_{12}$  has subgroups of order 1, 2, 3, 4, 6 and 12.

6.30 Correct the definition if needed. An element  $a$  of a group  $G$  has *order*  $n \in \mathbb{Z}^+$  if and only if  $a^n = e$ .

An element  $a$  of a group  $G$  has *order*  $n \in \mathbb{Z}^+$  if and only if  $a^n = e$  and  $n$  is the smallest positive integer for which this equation is true.

6.33-6.37 Either give an example, or show why there exists no example.

6.33 A finite group that is not cyclic.

The Klein 4-group

6.34 An infinite group that is not cyclic.

$\langle \mathbb{R}, + \rangle$

6.35 A cyclic group having only one generator.

$\mathbb{Z}_2$

6.36 An infinite cyclic group having four generators.

No example exists. As every infinite cyclic group is isomorphic to  $\langle \mathbb{Z}, + \rangle$  which only has two generators, 1 and -1.

6.37 A finite group having four generators.

$\mathbb{Z}_8$

6.40 Find the primitive  $n$ th roots of unity for  $n = 8$ .

As  $U_8$  is isomorphic to  $\mathbb{Z}_8$ , we know the generators of  $\mathbb{Z}_8$  are isomorphic to the generators of  $U_8$ . Thus, the primitive 8th roots of unity are  $\zeta, \zeta^3, \zeta^5, \zeta^7$

6.49 Show by a counterexample that the following "converse" of Theorem 6.6 is not a theorem: "If a group  $G$  is such that every proper subgroup is cyclic, then  $G$  is cyclic."

The Klein 4 group is a counter example as every proper subgroup is cyclic, but the group itself is not cyclic.

6.45 Let  $r$  and  $s$  be positive integers. Show that  $H = \{nr + ms | n, m \in \mathbb{Z}\}$  is a subgroup of  $\mathbb{Z}$ . Let  $a, b \in H$ . Then, for some  $n_1, m_1 \in \mathbb{Z}$ ,  $a = n_1r + m_1s$  and for some  $n_2, m_2 \in \mathbb{Z}$ ,  $b = n_2r + m_2s$ . Notice.

$$n_1r + m_1s + n_2r + m_2s = (n_1 + n_2)r + (m_1 + m_2)s$$

shows that the set is closed under addition.

As  $0r + 0s = 0$ , we see that 0 is in the set.

Let  $a = nr + ms \in H$  we have that

$$-a = -(nr + ms) = -nr - ms = (-n)r + (-m)s$$

and therefore  $-a \in H$

Therefore,  $H \leq \mathbb{Z}$

6.47 Let  $r$  and  $s$  be positive integers.

- (a) Define the **least common multiple** of  $r$  and  $s$  as a generator of a certain cyclic group.  
As a subgroup of the cyclic group  $\langle \mathbb{Z}, + \rangle$ , the subgroup  $G = r\mathbb{Z} \cap s\mathbb{Z}$  is cyclic. The positive generator of  $G$  is the least common multiple of  $r$  and  $s$ .
- (b) Under what condition is the least common multiple of  $r$  and  $s$  their product,  $rs$ ?  
When  $r$  and  $s$  are relatively prime, the lcm of  $r$  and  $s$  is their product,  $rs$ .
- (c) Generalizing part (b), show that the product of the greatest common divisor and of the least common multiple of  $r$  and  $s$  is  $rs$ .

Let  $i, j, k, q \in \mathbb{Z}$ ,  $\gcd(r, s) = d = ir + js$ , and  $\text{lcm}(r, s) = m = kr = qs$ . Notice.

$$md = mir + mjs = qsir + krjs = (qi + kj)rs$$

, thus  $rs$  is a divisor of  $md$ . Now let  $u, v \in \mathbb{Z}$ . Let  $r = ud$  and  $s = vd$ . Then  $rs = uvdd = (uvd)d$ , and  $uvd = rv = su$  is a multiple of  $r$  and  $s$ . Then,  $uvd = mt$ . Thus,  $rs = mtd = (md)t$ , so  $md$  is divisor of  $rs$ . Therefore,  $md = rs$ .

6.55 Let  $G$  be an abelian group and let  $H$  and  $K$  be finite cyclic subgroups with  $|H| = r$  and  $|K| = s$ .

- (a) Show that if  $r$  and  $s$  are relatively prime, then  $G$  contains a cyclic group of order  $rs$ .
- (b) Generalizing part (a), show that  $G$  contains a cyclic subgroup of order the least common multiple of  $r$  and  $s$ .