sample solutions

MTH 411, Fall 2018, Quiz 4 (Thursday Long Quiz/12 pts)

Choose **any one** of the following two results to prove. Make your proof as precise as possible.

1. Let G be a group. Fix two elements a and b in G. Prove that $(ab)^2 = a^2b^2$ if and only if $ab = ba$.
2. Let $G := GL(2,\mathbb{R})$, and define $H := \{A \in G : \det(A) = 1\}$ Prove that $H \leq G$.

Precisely state and number any theorems/results that you need to call (or, refer to) in your proof.

	Theorem 1: Let G be a group with id elt $e \in G$ and $H \subseteq G$. Then $H \leq G \iff$ ①, H is closed under mult.; ②, $e \in H$; and ③, H is closed under inverses.
	Theorem 2: If $A, B \in M(2,\mathbb{R})$, then $\det(AB) = \det(A)\det(B)$
	Theorem 3: If $A \in GL(2,\mathbb{R})$, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

① Let G be a group, and let $a, b \in G$.

(⇒) Suppose $(ab)^2 = a^2b^2$.

Then $abab = aabb$

Multiplying on left by a^{-1} and on the right by b^{-1} we get

$$a^{-1}ababb^{-1} = a^{-1}aabbb^{-1}$$

axiom G_3

$$ebae = eabe$$

axiom G_2

$$ba = ab$$

Thus we proved $(ab)^2 = a^2b^2 \Rightarrow ba = ab$.

(⇐) Suppose $ab = ba$.

Then multiplying on the left by a and on the right by b gives

$aabb = abab$. Notice this may rewritten as $a^2b^2 = (ab)^2$.

Thus we proved $ab = ba \Rightarrow a^2b^2 = (ab)^2$.

Therefore we have shown that

$$ab = ba \iff a^2b^2 = (ab)^2.$$

② Let $G := GL(2,\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \det(A) \neq 0\}$

$H := \{A \in G \mid \det(A) = 1\}$

WTS: $H \leq G$

proof: We will prove that $H \leq G$ using Theorem 1.

Notice by def. of H we know $H \subseteq G$.

① WTS: $\forall\, x, y \in H$, $xy \in H$.

Let $x, y \in H$. Then $\det(xy) \overset{\text{Theo 2.}}{=} \det(x)\det(y)$ since $x,y \in H$

$= 1 \cdot 1$

$= 1$

Thus $xy \in H$; and we proved that H is closed under mult.

② WTS: $e \in H$.

$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ and $\det(e) = 1$, and so $e \in H$.

③ WTS: $\forall\, x \in H$, $x^{-1} \in H$.

Let $x \in H$. Then $\det(x^{-1}) \overset{\text{Theo 3.}}{=} \frac{1}{\det(x)} \overset{\text{since } x \in H}{=} \frac{1}{1} = 1$,

and so $x^{-1} \in H$. We proved that H is closed under inverses.

Therefore, using Theorem 1 we can conclude that $H \leq G$.