MTH 411, Fall 2018, Quiz 3 (Tuesday Short Quiz/6 pts)

1. **(2 points)** Given a binary algebraic structure $\langle U, \star \rangle$, define what it means for an element $x \in U$ to be an **idempotent** for $\star$.

   - **Solution.** (p. 28[1]) An element $x \in U$ is an **idempotent** for $\star$ if $x \star x = x$.

2. **(2 points)** Given a binary algebraic structure $\langle U, \star \rangle$, define what it means for an element $z \in U$ to be an **identity element** for $\star$.

   - **Solution.** (p. 32) An element $z \in U$ is an **identity element** for $\star$ if for every $x \in U$ it follows that

$$z \star x = x \star z = x.$$

3. **(2 points)** Define what it means for two binary algebraic structures $\langle U, \star \rangle$ and $\langle V, \smiley \rangle$ to be **isomorphic**.

   - **Solution.** (p. 29) The binary algebraic structures $\langle U, \star \rangle$ and $\langle V, \smiley \rangle$ are **isomorphic** if there exists a bijective map $\phi : U \to V$ such that for every $x, y \in U$ we have that

$$\phi(x \ \star \ y) = \phi(x) \ \smiley \ \phi(y).$$

4. **(Bonus!)** Define what it means for a binary structure $\langle U, \star \rangle$ to be a **group**.

   - **Solution.** (p. 37) A binary structure $\langle U, \star \rangle$ is a **group** if the following axioms are satisfied:

     $\boxed{\mathcal{G}_1 \textbf{ (associativity of } \star\textbf{)}}$ For every $a, b, c \in U$ we have

$$(a \star b) \star c = a \star (b \star c).$$

     $\boxed{\mathcal{G}_2 \textbf{ (existence of an identity element for } \star\textbf{)}}$ There exists an element $e \in U$ such that for all $x \in G$

$$e \star x = x \star e = x.$$

     $\boxed{\mathcal{G}_3 \textbf{ (existence of inverse elements)}}$ For each $x \in G$ there exists $x' \in G$ such that

$$x \star x' = x' \star x = e.$$

[1] "To avoid an overwhelming quantity of such labels and numberings, we define many terms within the body of the text and exercises using boldface type." – Page 2 in Fraleigh.