5.01 Show that the taxicab metric on $\mathbb{R}^2$ satisfies the properties of a metric.

(1) Notice, by the definition of the Taxicab metric we take the addition of two absolute values. Since absolute values are never negative, we must have that for some $x, y \in \mathbb{R}^2, d(x, y) \geq 0$. Note, if $x = y$, we must have that $d(x, y) = 0$ and if $x \neq y, d(x, y) > 0$
Thus, property 1 is satisfied.

(2) Let $x, y \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x, y) &= |x_1 - y_1| + |x_2 - y_2| \\ &= |y_1 - x_1| + |y_2 - x_2| \\ &= d(y, x) \end{aligned}$$

Thus, property 2 is satisfied.

(3) Let $x, y, z \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x, z) &= |x_1 - z_1| + |x_2 - z_2| \\ &= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2| \\ &\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| \\ &= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &= d(x, y) + d(y, z) \end{aligned}$$

Thus, property 3 is satisfied.

Therefore, the taxicab metric is a metric.

5.02 (a) Show that the max metric on $\mathbb{R}^2$ satisfies the properties of a metric.

(1) Notice, we are taking the max value of an absolute value. Since absolute values are never negative, we must have that for some $x, y \in \mathbb{R}^2, d(x, y) \geq 0$. Note, if $x = y$, we must have that $d(x, y) = 0$ and if $x \neq y, d(x, y) > 0$
Thus, property 1 is satisfied.

(2) Let $x, y \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x, y) &= max\{|x_1 - y_1|, |x_2 - y_2|\} \\ &= max\{|y_1 - x_1|, |y_2 - x_2|\} \\ &= d(y, x) \end{aligned}$$

Thus, property 2 is satisfied.

(3) Let $x, y, z \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x,z) &= max\{|x_1 - z_1|, |x_2 - z_2|\} \\ &= max\{|x_1 - y_1 + y_1 - z_1|, |x_2 - y_2 + y_2 - z_2|\} \\ &\leq \max\{|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|\} \\ &= |x_i - y_i| + |y_i - z_i| \\ &\text{where } i \text{ with value 1 or 2 holds the maximum value} \\ |x_i - y_i| &\leq max\{|x_1 - y_1|, |x_2 - y_2|\} \\ |y_i - z_i| &\leq max\{|y_1 - z_1|, |y_2 - z_2|\} \end{aligned}$$

So,

$$d(x,z) \leq max\{|x_1 - y_1|, |x_2 - y_2|\} + max\{|y_1 - z_1|, |y_2 - z_2|\} = d(x,y) + d(y,z)$$

Thus, property 3 is satisfied.

Therefore, the max metric is a metric.

(b) Explain why $d(p,q) = \min\{|p_1 - q_1|, |p_2 - q_2|\}$ does not define a metric on $\mathbb{R}^2$.
The Triangle inequality does not hold.
Let $p, q, r \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(p,r) &= min\{|p_1 - r_1|, |p_2 - r_2|\} \\ &\geq min\{|p_1 - q_1| + |q_1 - r_1|, |p_2 - q_2| + |q_2 - r_2|\} \\ &= |p_i - q_i| + |q_i - r_i| \\ &\text{where } i \text{ with value 1 or 2 holds the minimum value} \\ |p_i - q_i| &\leq min\{|p_1 - q_1|, |p_2 - q_2|\} \\ |q_i - r_i| &\leq min\{|q_1 - r_1|, |q_2 - r_2|\} \end{aligned}$$

So,

$$d(p,r) \leq min\{|p_1 - q_1|, |p_2 - q_2|\} + min\{|q_1 - r_1|, |q_2 - r_2|\} \leq d(p,q) + d(q,r)$$

5.03 For points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $\mathbb{R}^2$ define

$$d_V(p,q) = \begin{cases} 1 & \text{if } p_1 \neq q_1 \text{ or } |p_2 - q_2| \geq 1 \\ |p_2 - q_2| & \text{if } p_1 = q_1 \text{ and } |p_2 - q_2| < 1 \end{cases}$$

(a) Show that d_V is a metric.

(b) Describe the open balls in the metric d_V.

5.10 (a) Let (X, d) be a metric on a space. For $x, y \in X$, define

$$D(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

Show that D is also a metric on X

(b) Explain why no two points in X are distance one or more apart in the metric D.

5.24 Prove Theorem 5.13 : Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous in the open set definition if and only if for each $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x' \in X$ and $dx\,(x, x') < \delta$ then $d_Y\,(f(x), f\,(x')) < \varepsilon$. (Hint: Consider Exercise 4.3 and the proof of Theorem 4.6.)

5.25 Let (X, d) be a metric space, and assume $p \in X$ and $A \subset X$

(a) Provide an example showing that $d(\{p\}, A) = 0$ need not imply that $p \in A$.

(b) Prove that if A is closed and $d(\{p\}, A) = 0$, then $p \in A$

5.26 Use Theorem 5.15 to prove that the taxicab metric and the max metric induce the same topology on $\mathbb{R}^2$.

5.28 Let (X, d) be a metric space. The function

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a bounded metric on X . (See Exercise 5.10.) Show that the topologies induced by D and d are the same.

5.29 On the set of continuous functions $C[a, b]$ consider the metrics ρ_M and ρ defined by

$$\rho_M(f, g) = \max_{x\in[a,b]}[|f(x) - g(x)\|\},$$

and

$$\rho(f, g) = \int_a^b |f(x) - g(x)|dx$$

These metrics were introduced in Exercise 5.8 and Example 5.5, respectively.

(a) Use Theorem 5.15 to prove that the topology induced by ρ_M on $C[a, b]$ is finer than the topology induced by ρ .

(b) Show that for every $c_1, c_2 > 0$ there exists $f \in C[a, b]$ such that $\max_{x\in[a,b]}\{|f(x)|\} = c_1$ and

$$\int_a^b |f(x)|dx = c_2$$

(c) Let $Z \in C[a, b]$ be the function defined by $Z(x) = 0$ for all $x \in [a, b]$ Given $\varepsilon > 0$, show that no $\delta > 0$ exists such that $B_\rho(Z, \delta) \subset B_{\rho M}(Z, \varepsilon)$ (Hint: Part (b) helps.)

(d) What does Theorem 5.15 allow us to conclude from (c)?

Summary