Math 371 Final Exam (take home part)

1 Guide: read carefully before you start

1. Content: for each numerical method listed, write down a full summary as shown in the template.
2. Format: this exam must be typed in LaTeX, which can be downloaded at `https://www.latex-project.org/`. This guideline will also serve as a template for LaTeX.
3. You are required to keep this document up to date. It will be collected before the final exam.
4. Many quiz and the exam questions will come from this document.
5. It's recommended that you keep a record of this document for the rest of your life.
6. This part worths 50 points and requires an individual submission via google drive.

2 Numerical method list

- done
  1. Bisection method ✓
  2. Newton's method ✓
  3. Secant method ✓
  4. Linear interpolation ✓
  5. Vandermonde matrix method ✓
  6. Newton interpolating polynomial ✓
  7. Lagrange interpolating polynomial ✓
  8. Linear spline function ✓
  9. Natural cubic spline function ✓
  10. Numerical approximation of the first derivative ✓
  11. Numerical integration using left point formula and midpoint formula
  12. Trapezoid rule ✓
  13. Simpson's rule ✓
  14. Adaptive Simpson's rule ✓
  15. Gaussian quadrature formula ✓

16. Naive Gaussian elimination ✓
17. Gaussian elimination with partial and full pivoting ✓

- LU factorization
- Forward and backward Euler's method to solve differential equation

3 Summaries

3.1 Bisection method:

This is a template. Your other summaries should follow the same format and answer all the questions (bold font) as listed in this template if applicable. Most of the content can be find in the lecture slides. Use your own resource to fill in the missing parts.

Area of application: bisection method is used to find the roots of a given equation.

Mathematical theorem: The Intermediate Value Theorem:

If $f(x)$ is continuous on $[a, b]$ with $f(a) \cdot f(b) < 0$, then $f(x) = 0$ for some x in $[a, b]$.

Idea of the method:

1. Find an interval $[a, b]$ with $f(a) \cdot f(b) < 0$.
2. Let c be the middle point of $[a, b]$.
3. Compute $f(c)$,
   - If $f(c) \cdot f(a) < 0$ choose the next interval to be $[a, c]$.
   - If $f(c) \cdot f(b) < 0$ choose the next interval to be $[c, b]$.
4. Repeat this procedure until it converges.

Pseudocode: (you can merge this part with the Matlab code. In the quiz, I will always ask for pseudocode since it's easier compared to Matlab code.)

```
function bisection(f,a,b,tol)

a = a0, b = b0, tol = 1e-05;
while (|a-b| > tol)
c = (a+b)/2;
if f(c) * f(a) < 0
b = c;
else
a = c;
end
x = c;
return x;
```

Matlab code: (compared to pseudocode, your code here should be written in the right grammar of Matlab and guaranteed to run. You also need to consider various invalid cases and display error messages.)

```
function x = bisection(f,a,b,tol)

if (f(a)*(b)>=0)
display('f(a) and f(b) have the same sign. Please use different a,b')
return
end
while(abs(a-b)>tol)
c = 1/2*(a+b);
if (f(c) ==0);
break;
elseif (f(c)*f(a)<0)
b = c;
else
a = c;
end
end

x = c;
```

Numerical analysis

- **When does it converge:** the method will always converge as long as
  - $f(x)$ is continuous on $[a, b]$.
  - $f(a) \cdot f(b) < 0$.
- **Error estimate:** (write down the detailed derivation of the error estimate formula)

$$\begin{aligned} E_0 < b - a, \quad E_{k+1} < \frac{1}{2} E_k, \\ \Rightarrow \quad E_n < \frac{b-a}{2^n}. \end{aligned} \tag{1}$$

- **Pros & Cons:** (you are encouraged to search online for more inputs)
  - Easy to use.
  - Always converge.
  - In the situation of multiple roots available, a carefully chosen starting interval $[a, b]$ is required to find a particular root.
  - Slow.
  - Only works for one dimension.

Numerical experiment: Test bisection method with

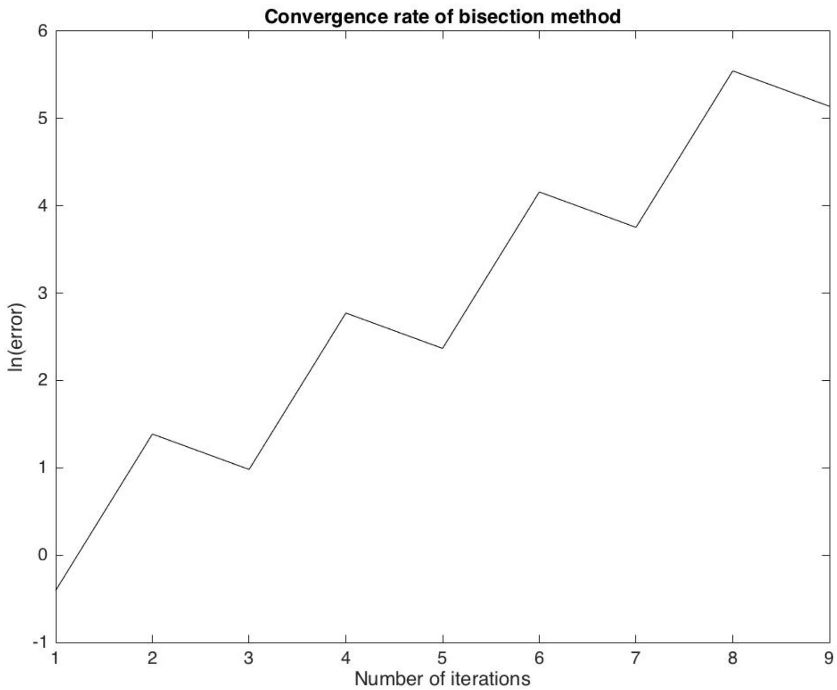
$$f(x) = x^3 + 2x - 3, \quad a = -3, \quad b = 2, \quad r = 1.$$

We obtain the error table

Number of iterations	1	2	3	4	5	6	7
Relative Error	1.500000	0.250000	0.375000	0.062500	0.093750	0.015625	0.023438

The convergence rate graph is (your graph should contain a title, x-label and y-label.)

Figure 1: Convergence rate of bisection method



3.2 Newton Method:

Area of application: Newton method is used to find the roots of a given equation.

Mathematical theorem: The Intermediate Value Theorem:

If $f(x)$ is continuous on $[a, b]$ with $f(a) \cdot f(b) < 0$, then $f(x) = 0$ for some x in $[a, b]$.

Idea of the method: The idea is to start with an initial guess which is reasonably close to the true root, then to approximate the function by its tangent line using calculus, and finally to compute the x-intercept of this tangent line by elementary algebra.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Code:

```
function retval = newton (f,upper,tol)
x = upper;
while (abs(f(x)) > tol)
```

```
x=x-(f(x)/deriv(f,x))
endwhile
retval = x;
endfunction
```

Numerical analysis

- **When does it converge:** When the function is continuous and does not have the following
  - Iteration point is stationary
  - Starting point enters a cycle
  - Derivative does not exist at root
  - Discontinuous derivative
- **Error estimate:**

$$|e_{n+1} \leq \frac{1}{2}\frac{max(|f''(x)|)}{min(|f'(x)|)}(e_n^2) = ce_n^2 \tag{2}$$

- **Pros & Cons:**
  - Fast
  - Accurate
  - Doesn't always converge
  - Needs a derivative

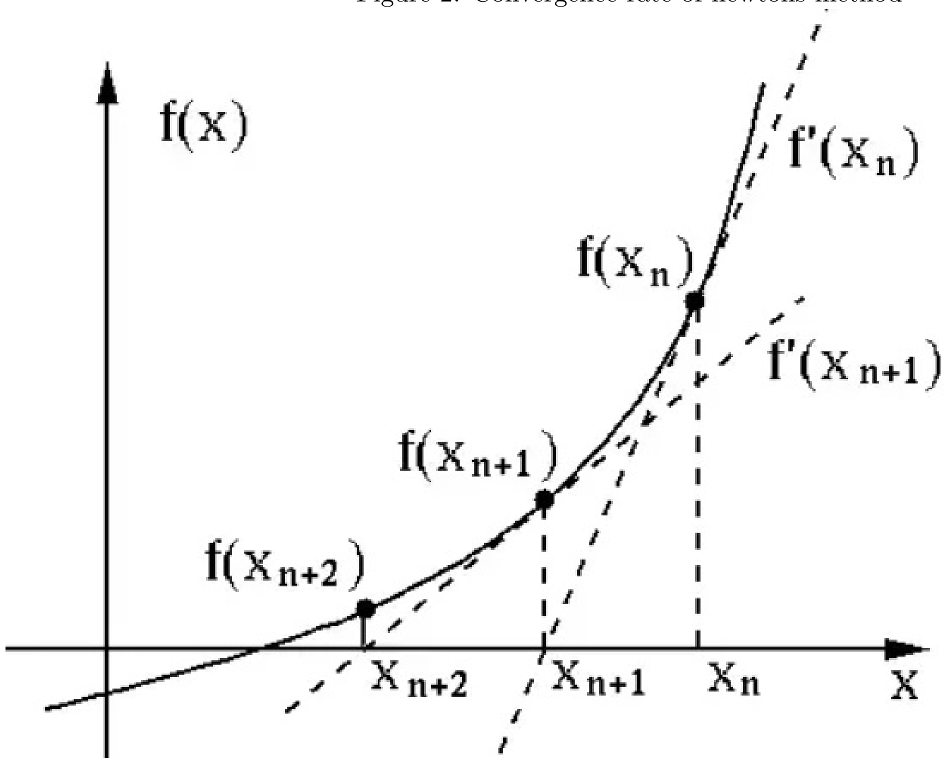
Numerical experiment: Test Newton method with

$$f_1(x) = \sin(x) - x - 1$$
$$f_2(x) = x(1 - \cos(x))$$
$$f_3(x) = e^x - x^2 + 3x - 2$$

Function	Number of Iterations	Approximate Root
$f_1(x)$	5	-1.934563
$f_2(x)$	11	0.010980
$f_3(x)$	3	0.257530

The convergence rate graph is

Figure 2: Convergence rate of newtons method



3.3 Secant Method:

Area of application: Secant method is used to find the roots of a given equation.

Mathematical theorem: The Intermediate Value Theorem:

If $f(x)$ is continuous on $[a, b]$ with $f(a) \cdot f(b) < 0$, then $f(x) = 0$ for some x in $[a, b]$.

Idea of the method: In numerical analysis, the secant method is a root-finding algorithm that uses a succession of roots of secant lines to better approximate a root of a function f.

$$x_n = x_{n-1} - f\left(x_{n-1}\right) \frac{x_{n-1} - x_{n-2}}{f\left(x_{n-1}\right) - f\left(x_{n-2}\right)} = \frac{x_{n-2} f\left(x_{n-1}\right) - x_{n-1} f\left(x_{n-2}\right)}{f\left(x_{n-1}\right) - f\left(x_{n-2}\right)}$$

Code:

```
    function retval = secant (f,start1,start2,tol)
x = start1;
xPrev = start2;
while (abs(f(x)) > tol)
tmp = x;
x=x-f(x)*((x - xPrev)/(f(x)-f(xPrev)));
xPrev = tmp;
endwhile
retval = x;
endfunction
```

Numerical analysis

- **When does it converge:** When the function is continuous and does not have the following
  - Iteration point is stationary
  - Starting point enters a cycle
  - Derivative does not exist at root
  - Discontinuous derivative
- **Error estimate:**

$$|e_{n+1} \leq \frac{1}{2}\frac{f''(\xi_1)}{f'(\xi_2))}|e_n| \cdot |e_{n-1}) \leq c|e_n| \cdot |e_{n-1}| \tag{3}$$

- **Pros & Cons:**
  - Fast
  - Accurate
  - Doesn't always converge
  - Needs a derivative

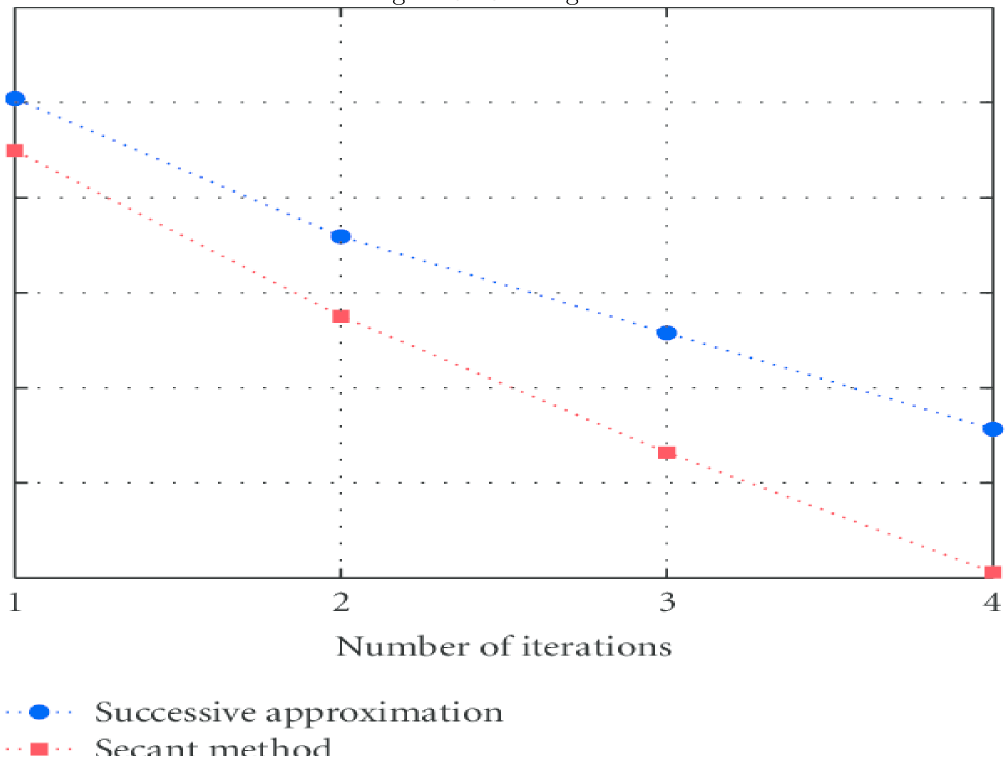
Numerical experiment: Test secant method with

$$\begin{aligned} f_1(x) &= \sin(x) - x - 1 \\ f_2(x) &= x(1 - \cos(x)) \\ f_3(x) &= e^x - x^2 + 3x - 2 \end{aligned}$$

Function	Number of Iterations	Approximate Root
$f_1(x)$	6	-1.934563
$f_2(x)$	15	0.012283
$f_3(x)$	3	0.257530

The convergence rate graph is

Figure 3: Convergence rate of secant method



3.4 Linear Interpolation:

Area of application: In mathematics, linear interpolation is a method of curve fitting using linear polynomials to construct new data points within the range of a discrete set of known data points.
Idea of the method: If the two known points are given by the coordinates $(x_0, y_0)(x_0, y_0)$ and $(x_1, y_1)(x_1, y_1)$, the linear interpolant is the straight line between these points. For a value x in the interval $(x_0, x_1)(x_0, x_1)$, the value y along the straight line is given from the equation of slopes.
This formula can also be understood as a weighted average. The weights are inversely related to the distance from the end points to the unknown point; the closer point has more influence than the farther point. Thus, the weights are $\frac{x-x_0}{x_1-x_0}\frac{x-x_0}{x_1-x_0}$ and $\frac{x_1-x}{x_1-x_0}\frac{x_1-x}{x_1-x_0}$, which are normalized distances between the unknown point and each of the end points

$$y = y_0\left(1-\frac{x-x_0}{x_1-x_0}\right)+y_1\left(1-\frac{x_1-x}{x_1-x_0}\right) = y_0\left(1-\frac{x-x_0}{x_1-x_0}\right)+y_1\left(\frac{x-x_0}{x_1-x_0}\right)$$

Code:

```
function retval = linerInterpolation(f,a,b,tol)
while (abs(a-b) > tol)
slope = (f(b)-f(a))/(b-a)
line = @(x)slope*(x-a)+f(a);
guess = fzero(line,a);
if f(guess) * f(a) > 0
```

```
a = guess
else
b = guess
endif
endwhile
retval = guess;
endfunction
```

Numerical analysis

- **When does it converge:** Provided we have the proper data, linear interpolation always converges.
- **Error estimate:**

$$|f(x)P1(x)| \leq \frac{h^2}{8} max|f''(x)|, x \in [x0, x1]. \tag{4}$$

- **Pros & Cons:**
  - Can fit most data
  - Is not smooth unless taken a high number of times

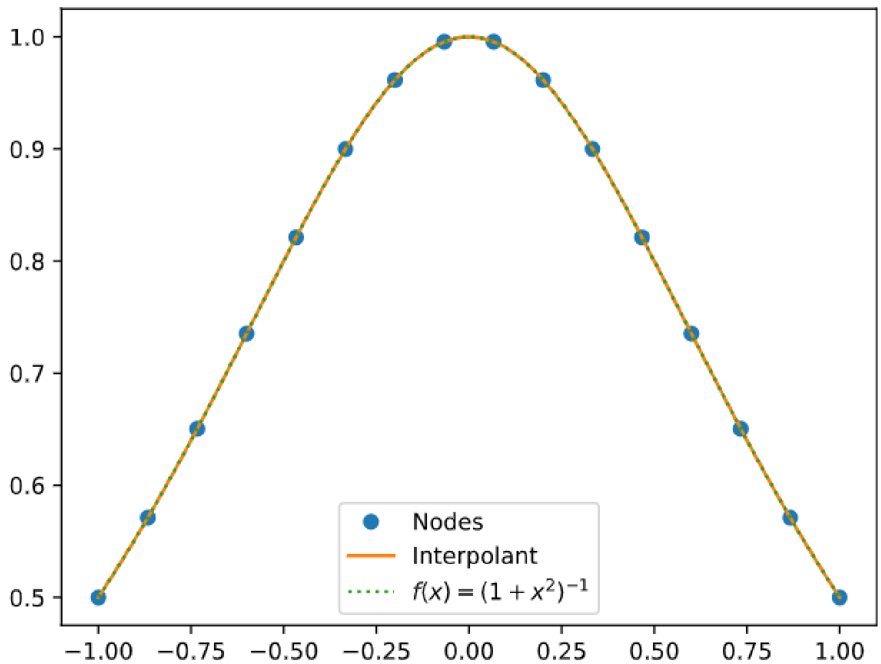
Numerical experiment: Test interpolation method with

$$f_1(x) = \sin(x) - x - 1$$
$$f_2(x) = x(1 - \cos(x))$$
$$f_3(x) = e^x - x^2 + 3x - 2$$

Function	Number of Iterations	Approximate Root
$f_1(x)$	12	-1.934563
$f_2(x)$	500	0.051707
$f_3(x)$	29	0.257530

The convergence rate graph is

Figure 4: Convergence rate of liner interpolation



3.5 Vandermonde Matrix:

Area of application: In polynomial interpolation, since inverting the Vandermonde matrix allows expressing the coefficients of the polynomial in terms of the $\alpha_i$and the values of the polynomial at theα_i.

Idea of the method: If the two known points are given by the coordinates $(x_0, y_0)(x_0, y_0)$ and $(x_1, y_1)(x_1, y_1)$, the linear interpolant is the straight line between these points. For a value x in the interval $(x_0, x_1)(x_0, x_1)$, the value y along the straight line is given from the equation of slopes.

This formula can also be understood as a weighted average. The weights are inversely related to the distance from the end points to the unknown point; the closer point has more influence than the farther point. Thus, the weights are $\frac{x-x_0}{x_1-x_0}\frac{x-x_0}{x_1-x_0}$ and $\frac{x_1-x}{x_1-x_0}\frac{x_1-x}{x_1-x_0}$, which are normalized distances between the unknown point and each of the end points

$$y = y_0 \left(1 - \frac{x - x_0}{x_1 - x_0}\right) + y_1 \left(1 - \frac{x_1 - x}{x_1 - x_0}\right) = y_0 \left(1 - \frac{x - x_0}{x_1 - x_0}\right) + y_1 \left(\frac{x - x_0}{x_1 - x_0}\right)$$

Code:

```
function A = VanInverse(B)
% This function inverse a Vandermonde Matrix B.
% Matrix B is a n-by-n matrix, its (i,j) entry is i^(j-1),
% where i,j = 1,2,...,n
% for example, n = 4
% B =
```

```
%     1     1     1     1
%     1     2     4     8
%     1     3     9    27
%     1     4    16    64
% This routine uses a Stirling polynomial(the first kind) coefficients
% For fast operation, a C Stirling coefficient function has
% been posted with name: mStirling.c. The C-version of this
% inverse function is also available upon request.
n = size(B,1);
A = zeros(n,n);
% compute the Stirling coeffs (by column), could use mStirling.c
for (i=1:n)
if (i==1)
k = 3;
coeff = [1 -2];
else
k = 2;
coeff = [1 -1];
end
while (k<=n)
if (k == i);
k = k + 1;
continue;
else
coeff = conv(coeff, [1 -k]);
k = k + 1;
end
end
A(:,i) = coeff' .* ((-1).^(n-i)/(factorial(n-i)*factorial(i-1)));
end
A = flipud(A);
```

Numerical analysis

- **When does it converge:** Provided we have the proper data, linear interpolation always converges.
- **Error estimate:**

$$\pi(x) = (xx_1)(xx_2)\cdots(xx_n) \tag{5}$$

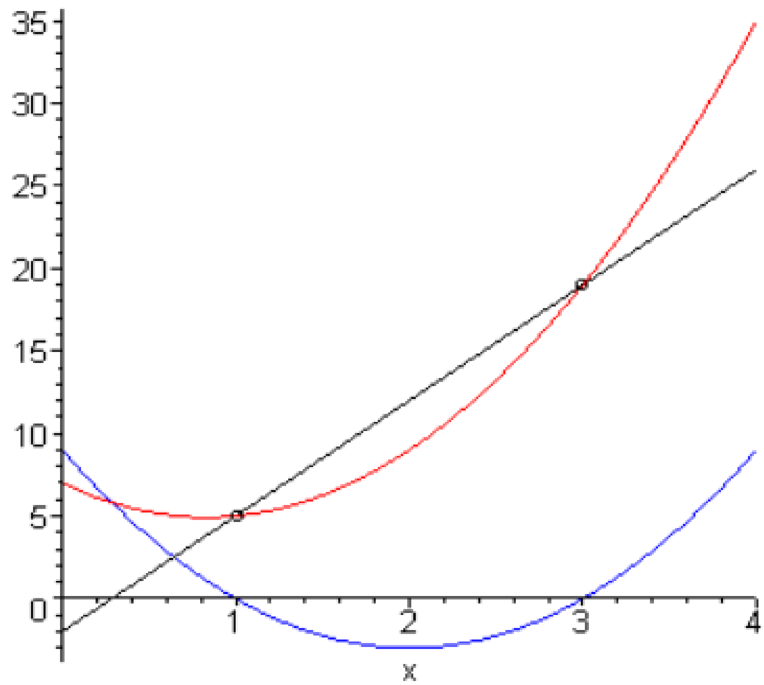
- **Pros & Cons:**
  - Nice Structure
  - Expensive
  - Matrix is ill-conditioned

Numerical experiment: $3x^2 5x + 7$ at the points (1,5) and (3,19)

Function	Number of Iterations	Approximate
$f_1(x)$	23	5.425501

The convergence rate graph is

Figure 5: Convergence rate



3.6 Newton Interpolating Polynomial:

Area of application: An interpolation polynomial for a given set of data points. The Newton polynomial is sometimes called Newton's divided differences interpolation polynomial because the coefficients of the polynomial are calculated using Newton's divided differences method.
Idea of the method: Solving an interpolation problem leads to a problem in linear algebra where we have to solve a system of linear equations. Using a standard monomial basis for our interpolation polynomial we get the very complicated Vandermonde matrix. By choosing another basis, the Newton basis, we get a system of linear equations with a much simpler lower triangular matrix which can be solved faster. **Code:**

```
function fp = newton_interpolation(x,y,p)
% Script for Newton's Interpolation.
% Muhammad Rafiullah Arain
% Mathematics & Basic Sciences Department
% NED University of Engineering & Technology - Karachi
% Pakistan.
% ---------
% x and y are two Row Matrices and p is point of interpolation
```

```
%
% Example
% >> x=[1,2,4,7,8]
% >> y=[-9,-41,-189,9,523]
% >> newton_interpolation(x, y, 5)
% OR
% >> a = newton_interpolation(x, y, 5)
n = length(x);
a(1) = y(1);
for k = 1 : n - 1
d(k, 1) = (y(k+1) - y(k))/(x(k+1) - x(k));
end
for j = 2 : n - 1
for k = 1 : n - j
d(k, j) = (d(k+1, j - 1) - d(k, j - 1))/(x(k+j) - x(k));
end
end
d
for j = 2 : n
a(j) = d(1, j-1);
end
Df(1) = 1;
c(1) = a(1);
for j = 2 : n
Df(j)=(p - x(j-1)) .* Df(j-1);
c(j) = a(j) .* Df(j);
end
fp=sum(c);
```

Numerical analysis

- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

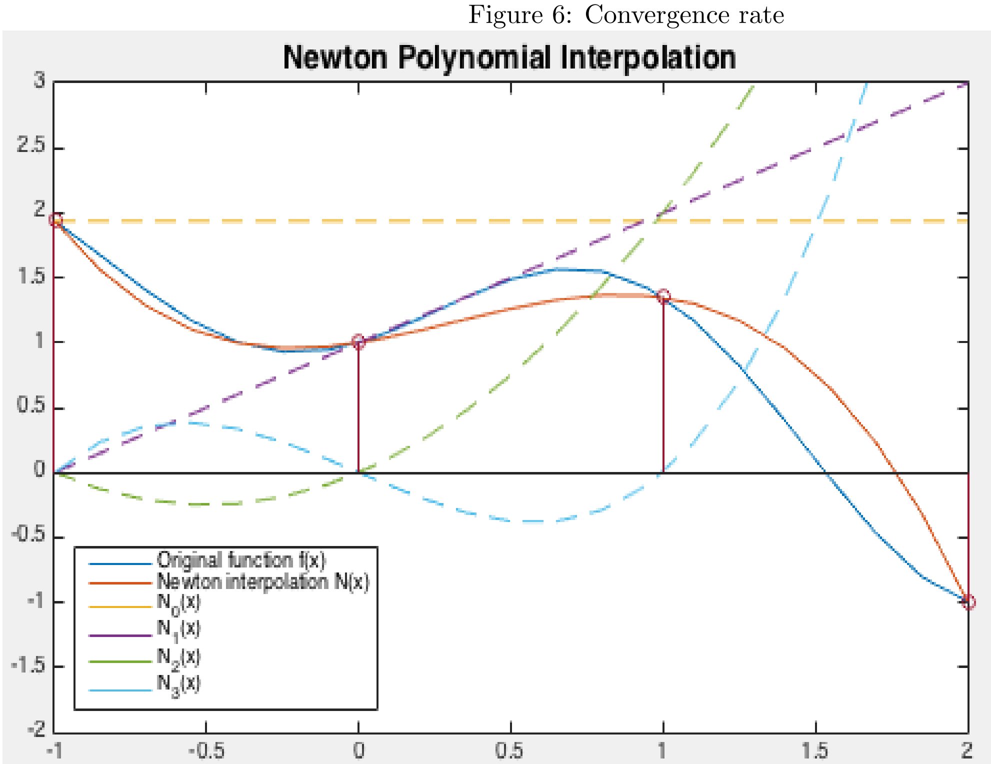
$$p_n(z) = \left(\begin{array}{c} z \\ n \end{array} \right) = \frac{z(z-1)\cdots(z-n+1)}{n!} \tag{6}$$

- **Pros & Cons:**
  - Intuitive
  - fast
  - accurate
  - must be differentiable

$$
\begin{array}{cccccc}
-\frac{3}{2} & -14.1014 & & & & \\
 & & 17.5597 & & & \\
-\frac{3}{4} & -0.931596 & & -10.8784 & & \\
 & & 1.24213 & & 4.83484 & \\
0 & 0 & & 0 & & 0 \\
 & & 1.24213 & & 4.83484 & \\
\frac{3}{4} & 0.931596 & & 10.8784 & & \\
 & & 17.5597 & & & \\
\frac{3}{2} & 14.1014 & & & &
\end{array}
$$

Numerical experiment:
$-14.1014+17.5597\left(x+\frac{3}{2}\right)-10.8784\left(x+\frac{3}{2}\right)\left(x+\frac{3}{4}\right)+4.83484\left(x+\frac{3}{2}\right)\left(x+\frac{3}{4}\right)(x)+0\left(x+\frac{3}{2}\right)\left(x+\frac{3}{4}\right)(x)\left(x\right.$
$= -0.00005 - 1.4775x - 0.00001x^2 + 4.83484x^3$
The convergence rate graph is

Figure 6: Convergence rate



3.7 Lagrange interpolating polynomial:

Area of application: In numerical analysis, Lagrange polynomials are used for polynomial interpolation. For a given set of points $(x_j, y_j)(x_j, y_j)$with no two $x_j x_j$values equal, the Lagrange polynomial is the polynomial of lowest degree that assumes at each value$x_j x_j$ the corresponding value$y_j y_j$ (i.e. the functions coincide at each point). The interpolating polynomial of the least degree is unique,
Idea of the method: Given a set of k + 1 data points
$(x_0, y_0), \ldots, (x_j, y_j), \ldots, (x_k, y_k)(x_0, y_0), \ldots, (x_j, y_j), \ldots, (x_k, y_k)$ where no two $x_j x_j$ are the same, the interpolation polynomial in the Lagrange form is a linear combination. Thus,

$$\ell_j(x) := \prod_{\substack{0 \le m \le k \\ m \neq j}} \frac{x - x_m}{x_j - x_m} = \frac{(x - x_0)}{(x_j - x_0)} \cdots \frac{(x - x_{j-1})}{(x_j - x_{j-1})} \frac{(x - x_{j+1})}{(x_j - x_{j+1})} \cdots \frac{(x - x_k)}{(x_j - x_k)}$$

Code:

```
function y=lagrange(x,pointx,pointy)
%
%LAGRANGE   approx a point-defined function using the Lagrange polynomial interpolation
%
%      LAGRANGE(X,POINTX,POINTY) approx the function definited by the points:
%      P1=(POINTX(1),POINTY(1)), P2=(POINTX(2),POINTY(2)), ..., PN(POINTX(N),POINTY(N))
%      and calculate it in each elements of X
%
%      If POINTX and POINTY have different number of elements the function will return t
%
%      function wrote by: Calzino
%      7-oct-2001
%
n=size(pointx,2);
L=ones(n,size(x,2));
if (size(pointx,2)~=size(pointy,2))
fprintf(1,'\nERROR!\nPOINTX and POINTY must have the same number of elements\n');
y=NaN;
else
for i=1:n
for j=1:n
if (i~=j)
L(i,:)=L(i,:).*(x-pointx(j))/(pointx(i)-pointx(j));
end
end
end
y=0;
for i=1:n
y=y+pointy(i)*L(i,:);
end
end
```

Numerical analysis

- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

$$R(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n) f^{(n+1)}(\xi)}{(n+1)!} \tag{7}$$

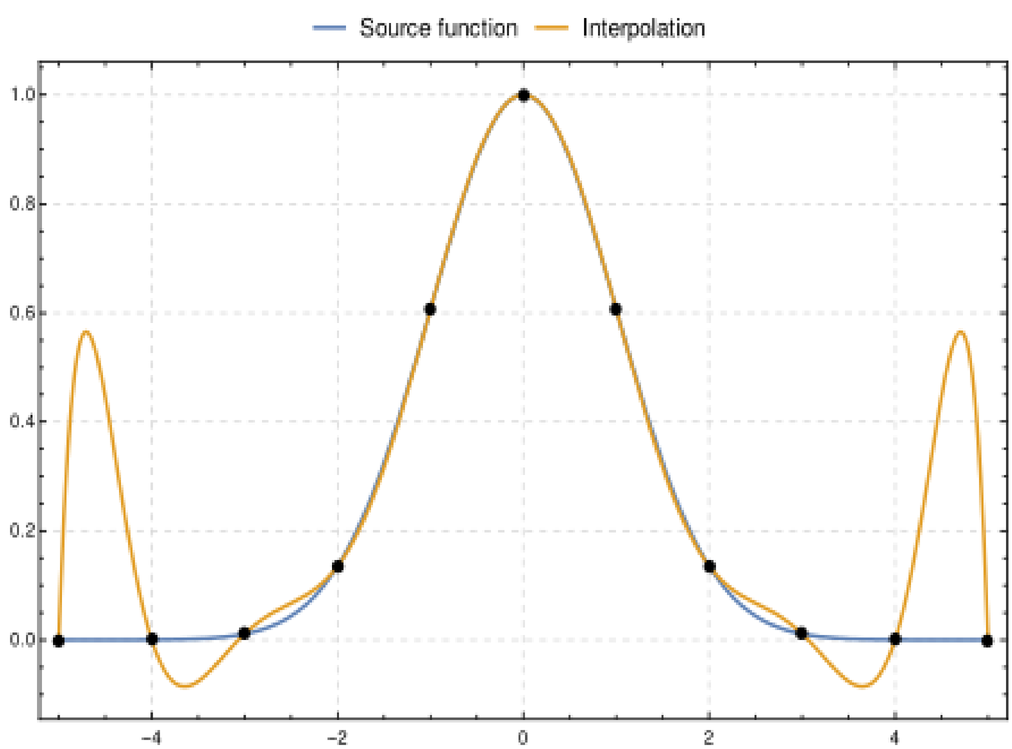
- **Pros & Cons:**
  - Intuitive
  - Fast
  - Accurate

Numerical experiment: Approximating $f(x) = x^2, 1 \leq x \leq 3$ $\begin{array}{ll} x_0 = 1 & f(x_0) = 1 \\ x_1 = 2 & f(x_1) = 4 \\ x_2 = 3 & f(x_2) = 9 \end{array}$ We get the polynomial

$$L(x) = 1 \cdot \frac{x-2}{1-2} \cdot \frac{x-3}{1-3} + 4 \cdot \frac{x-1}{2-1} \cdot \frac{x-3}{2-3} + 9 \cdot \frac{x-1}{3-1} \cdot \frac{x-2}{3-2}$$

The convergence rate graph is

Figure 7: Convergence rate



3.8 Linear spline function:

Area of application: Spline interpolation is a form of interpolation where the interpolant is a special type of piecewise polynomial called a spline. Spline interpolation is often preferred over polynomial interpolation because the interpolation error can be made small even when using low degree polynomials for the spline[1]. Spline interpolation avoids the problem of Runge's phenomenon, in which oscillation can occur between points when interpolating using high degree polynomials.
Idea of the method: Taking a third-order polynomial $q(x)$
$q(x) = (1 - t(x))y_1 + t(x)y_2 + t(x)(1 - t(x))(a(1 - t(x)) + bt(x))$
Where
$t(x) = \frac{x-x_1}{x_2-x_1}$ $a = k_1 (x_2 - x_1) - (y_2 - y_1)$ $b = -k_2 (x_2 - x_1) + (y_2 - y_1)$ We will take the derivatives and use those to build up linear equations. From that we will repeat as needed.
$q_n'' (x_n) = -2\frac{3(y_n-y_{n-1})-(2k_n+k_{n-1})(x_n-x_{n-1})}{(x_n-x_{n-1})^2} = 0$ **Code:**

```
    function s = LinearSpline (x, y)
s = zeros(length(x)-1, 4)
n = length(x);
for i = 1:n-1
m = (y(i+1)-y(i))./(x(i+1)-x(i));
b = -m*x(i)+y(i);
s(i,:) = [m b x(i) x(i+1)]
endfor
endfunction
```

Numerical analysis

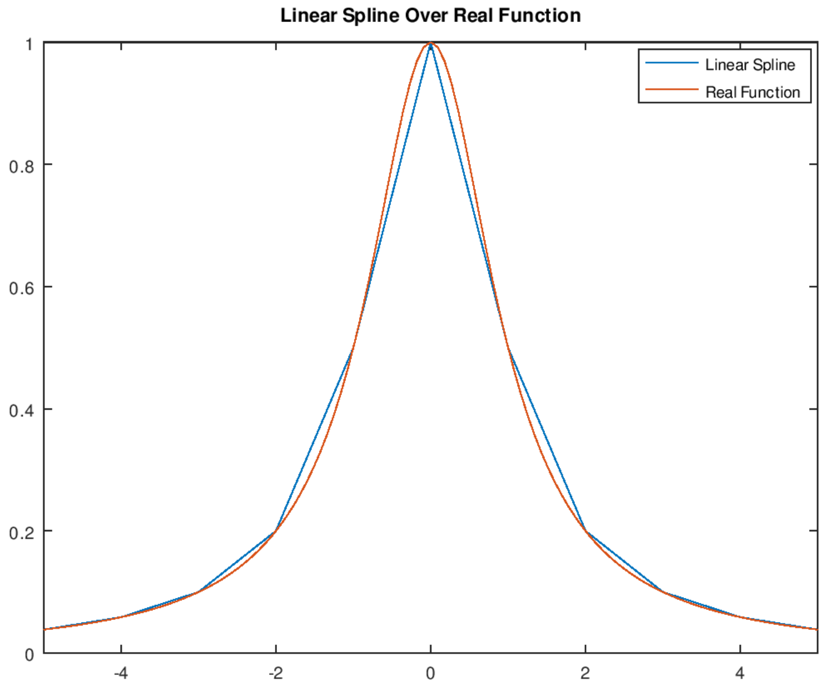
- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

$$\frac{1}{(n+1)!} f^{(n+1)}(\xi)\Pi_{i=0}^{n}(x - x_i) \tag{8}$$

- **Pros & Cons:**
  - Intuitive
  - More Points $\Rightarrow$ More Better
  - Avoids Runge

Numerical experiment: Approximating $f(x) = \frac{1}{1+x^2}$

Figure 8: Convergence rate



3.9 Natural Cubic spline function:

3.10 Numerical approximation of the first derivative

Area of application: In numerical analysis, Richardson extrapolation is a sequence acceleration method, used to improve the rate of convergence of a sequence.
Idea of the method: Suppose that we wish to approximateA^*A^*, and we have a method $A(h)$that depends on a small parameter h in such a way that
$A(h) = A^* + Ch^n + O\left(h^{n+1}\right)$

From this we can get the function:
$R(h,t) = \frac{t^n\left(A^*+C\left(\frac{h}{t}\right)^n+O\left(h^{n+1}\right)\right)-\left(A^*+Ch^n+O\left(h^{n+1}\right)\right)}{t^n-1} = A^* + O\left(h^{n+1}\right)$
Code:

```
tStart = 0           %Starting time
tEnd = 5             %Ending time
f = -y^2             %The derivative of y, so y' = f(t, y(t)) = -y^2
% The solution to this ODE is y = 1/(1 + t)
y0 = 1               %The initial position (i.e. y0 = y(tStart) = y(0) = 1)
tolerance = 10^-11   %10 digit accuracy is desired

maxRows = 20                 %Don't allow the iteration to continue indefinitely
initialH = tStart - tEnd     %Pick an initial step size
haveWeFoundSolution = false %Were we able to find the solution to within the desired tol
```

```
h = initialH

%Create a 2D matrix of size maxRows by maxRows to hold the Richardson extrapolates
%Note that this will be a lower triangular matrix and that at most two rows are actually
% needed at any time in the computation.
A = zeroMatrix(maxRows, maxRows)

%Compute the top left element of the matrix
A(1, 1) = Trapezoidal(f, tStart, tEnd, h, y0)

%Each row of the matrix requires one call to Trapezoidal
%This loops starts by filling the second row of the matrix, since the first row was comp
for i = 1 : maxRows - 1 %Starting at i = 1, iterate at most maxRows - 1 times
h = h/2             %Half the previous value of h since this is the start of a new row

%Call the Trapezoidal function with this new smaller step size
A(i + 1, 1) = Trapezoidal(f, tStart, tEnd, h, y0)

for j = 1 : i         %Go across the row until the diagonal is reached
%Use the value just computed (i.e. A(i + 1, j)) and the element from the
% row above it (i.e. A(i, j)) to compute the next Richardson extrapolate

A(i + 1, j + 1) = ((4^j).*A(i + 1, j) - A(i, j))/(4^j - 1);
end

%After leaving the above inner loop, the diagonal element of row i + 1 has been computed
% This diagonal element is the latest Richardson extrapolate to be computed
%The difference between this extrapolate and the last extrapolate of row i is a good
% indication of the error
if(absoluteValue(A(i + 1, i + 1) - A(i, i)) < tolerance)   %If the result is within tole
print("y(5) = ", A(i + 1, i + 1))                          %Display the result of the Richar
haveWeFoundSolution = true
break                                                      %Done, so leave the loop
end
end

if(haveWeFoundSolution == false)   %If we weren't able to find a solution to within the
print("Warning: Not able to find solution to within the desired tolerance of ", toleranc
print("The last computed extrapolate was ", A(maxRows, maxRows))
end
```

Numerical analysis

- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

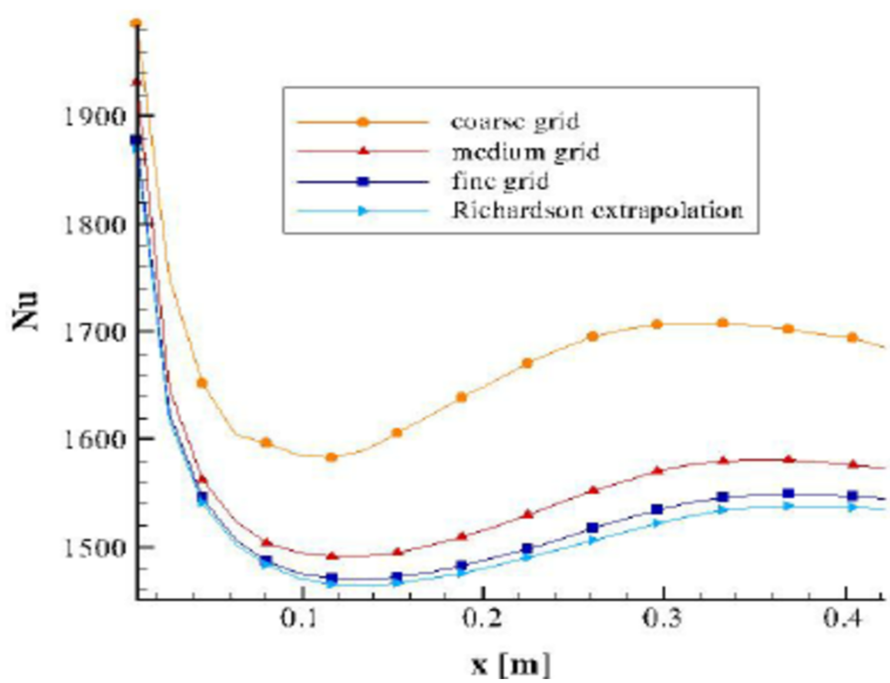
$$Kh^2 + O\left(h^3\right) \tag{9}$$

- **Pros & Cons:**
  - Intuitive
  - Accurate
  - Powerful Tool

Numerical experiment: $f'(x) \approx \frac{1}{4h}(f(x+2h) - f(x-2h))$
We then get the following:
$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x)$
$f(x-2h) = f(x) - 2hf'(x) + \frac{4h^2}{2!}f''(x) - \frac{8h^3}{3!}f'''(x)$
$f(x+2h) - f(x-2h) = 4hf'(x) + \frac{8h^3}{3!}f'''(\zeta) + \frac{8h^3}{3!}f'''(\xi)$
$f'(x) - \frac{f(x+2h)-f(x-2h)}{4h} = \frac{8h^2}{4!}f'''(\zeta) + \frac{8h^2}{4!}f'''(\xi)$

$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f''''(x)$
$f(x-2h) = f(x) - 2hf'(x) + \frac{4h^2}{2!}f''(x) - \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f''''(x)$
$f(x+2h) + f(x-2h) = 2f(x) + \frac{8h^2}{2!}f''(x) + \frac{16h^4}{4!}f''''(\zeta) + \frac{16h^4}{4!}f''''(\xi)$
$f(x+2h) + f(x-2h) - 2f(x) = \frac{8h^2}{2!}f''(x) + \frac{16h^4}{4!}f''''(\zeta) + \frac{16h^4}{4!}f''''(\xi)$
$\frac{f(x+2h)+f(x-2h)-2f(x)}{4h^2} = f''(x) + \frac{4h^2}{4!}f''''(\zeta) + \frac{4h^2}{4!}f''''(\xi)$
$\frac{f(x+2h)+f(x-2h)-2f(x)}{4h^2} - f''(x) = \frac{4h^2}{4!}f''''(\zeta) + \frac{4h^2}{4!}f''''(\xi)$

$x - 2h \leq \zeta \leq x$
$x \leq \xi \leq x + 2h$
The graph is



3.11 Numerical integration using left point formula and midpoint formula

Area of application: Used to find a numerical approximation for a definite integral
Idea of the method: Take the sum of the area under the curve by splitting the region into shapes. Take the shapes that together form a region that is similar to the region being measured, then calculating the area for each of these shapes, and finally adding all of these small areas together.
Code:

```
x1 = 1;
x2 = 3;
f=@(x)  2*x.^5 - 3*x.^2 - 5;
n=500;
dx=(x2-x1)/n;
summe=0.0;
for i=1:500
summe=summe+f(x1+dx*(i-1));
end
summe=summe*dx;
z=integral(f,x1,x2);
```

```
  dx = (b-a)/N;
sum_mdp = 0;
for p = 1:N
X = a + p*(dx)-(dx/2);
Y = sqrt(((X^2)-(c^2)))/X;
sum_mdp = sum_mdp + Y*dx;
end
Int_mdp(N) = sum_mdp;
```

Numerical analysis

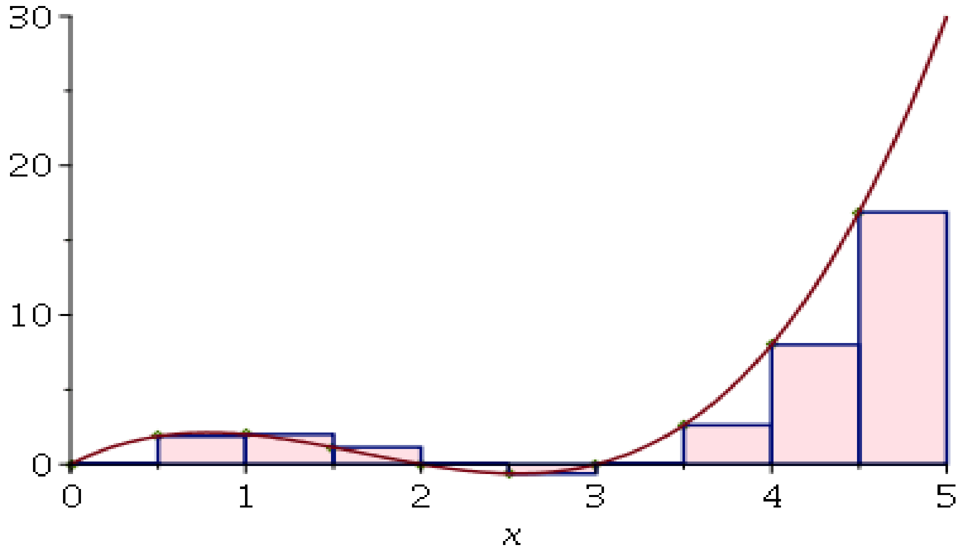
- **When does it converge:** Provided we have the proper data and a continuous function with no gaps; always converges.
- **Error estimate:**

$$\Delta x|f(b) - f(a)|$$

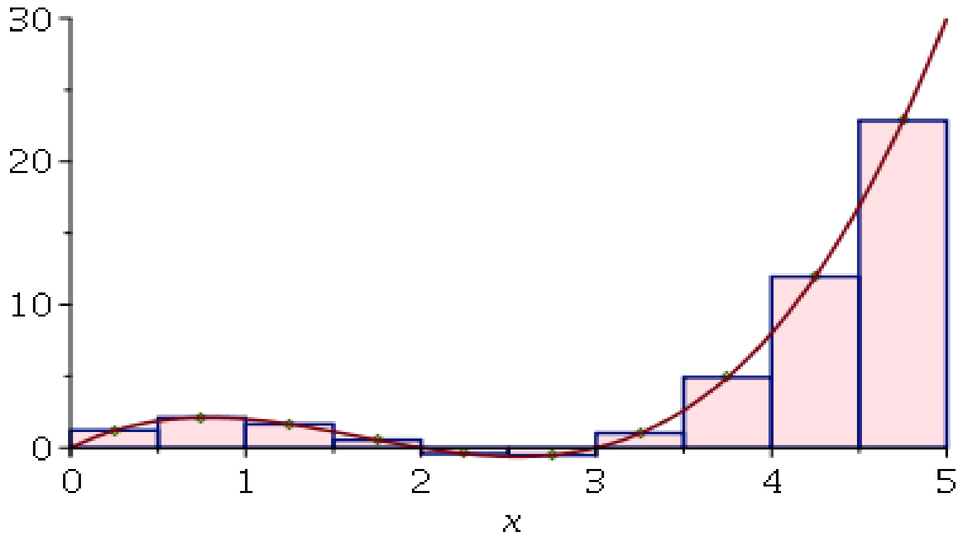
$$\frac{(h/2)^{\xi}}{3!} f'''(M) \tag{10}$$

- **Pros & Cons:**
  - Intuitive
  - Simple to Use
  - Requires very small shapes to remain accurate

Numerical experiment: $f(x) = x^2 - x$ from -1 to 3 with $n = 4$.
Left:
$= f(-1) \cdot 1 + f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1$
$= -2 + 0 + 0 + 6$
$= 4$
Mid:
$f\left(\frac{x_0 + x_1}{2}\right) = f\left(\frac{(-1) + (0)}{2}\right) = f\left(-\frac{1}{2}\right) = \frac{3}{4} = 0.75$
$f\left(\frac{x_1 + x_2}{2}\right) = f\left(\frac{(0) + (1)}{2}\right) = f\left(\frac{1}{2}\right) = -\frac{1}{4} = -0.25$
$f\left(\frac{x_2 + x_3}{2}\right) = f\left(\frac{(1) + (2)}{2}\right) = f\left(\frac{3}{2}\right) = \frac{3}{4} = 0.75$
$f\left(\frac{x_3 + x_4}{2}\right) = f\left(\frac{(2) + (3)}{2}\right) = f\left(\frac{5}{2}\right) = \frac{15}{4} = 3.75$
$1(0.75 - 0.25 + 0.75 + 3.75)$
$= 5$
The graph is



A left Riemann sum approximation of $\int_0^5 f(x)\,dx$, where $f(x) = x\,(x-2)\,(x-3)$ and the partition is



A midpoint Riemann sum approximation of $\int_0^5 f(x)\,dx$, where $f(x) = x\,(x-2)\,(x-3)$

3.12 Trapezoid Rule

Area of application: Used to find a numerical approximation for a definite integral
Idea of the method: Take the sum of the area under the curve by splitting the region into shapes. Take the shapes that together form a region that is similar to the region being

measured, then calculating the area for each of these shapes, and finally adding all of these small areas together.
Code:

```
function y = Trap (f, a, b, n)
h = abs((b-a)/n);
y = 0;
for i = a:h:b
if (i==a || i==b)
y+=(h/2)*f(i);
elseif
y+=(h.*f(i));
endif
endfor
endfunction
```

Numerical analysis

- **When does it converge:** Provided we have the proper data and a continuous function with no gaps; always converges.
- **Error estimate:**

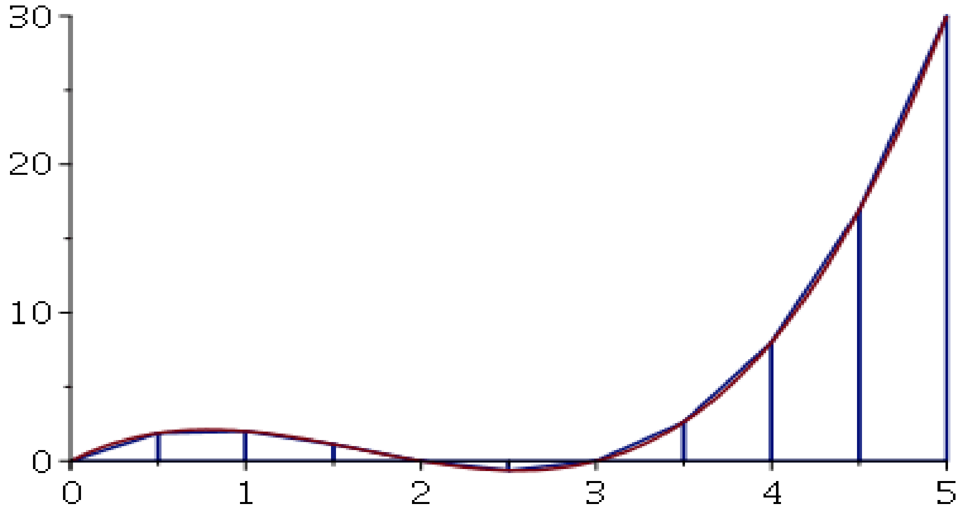
$$\begin{gathered} \Delta x|f(b)-f(a)| \\ (h/2)f(x_i)+(h/2)f(x_{i+1}) \end{gathered} \tag{11}$$

- **Pros & Cons:**
  - Intuitive
  - Simple to Use
  - Requires very small shapes to remain accurate

Numerical experiment: $f(x) = x^2 - x$ from -1 to 3 with $n = 4$.

$f(x_0) = f(a) = f(-1) = 2 = 2$ $2f(x_1) = 2f(0) = 0 = 0$ $2f(x_2) = 2f(1) = 0 = 0$ $2f(x_3) = 2f(2) = 4 = 4$ $f(x_4) = f(b) = f(3) = 6 = 6$

$\frac{1}{2}(2 + 0 + 0 + 4 + 6) = 6$ The graph is



An approximation of $\int_0^5 f(x)\,dx$ using trapezoid rule,
where $f(x) = x\,(x-2)\,(x-3)$

3.13 Simpson's Rule

Area of application: Used to find a numerical approximation for a definite integral
Idea of the method: Take the sum of the area under the curve by splitting the region into shapes. Take the shapes that together form a region that is similar to the region being measured, then calculating the area for each of these shapes, and finally adding all of these small areas together.
$\frac{\Delta x}{3}\left(f\left(x_0\right)+4f\left(x_1\right)+2f\left(x_2\right)+4f\left(x_3\right)+2f\left(x_4\right)+\cdots+4f\left(x_{n-1}\right)+f\left(x_n\right)\right)$
Code:

```
function y = Simpson (f, a, b, n)
hBar = abs((b-a)/n);
h = hBar/2;
y=0;
for i = a:hBar:b-hBar
y+=(h/3)*(f(i)+4*f(i+h)+f(i+(2*h)));
endfor
endfunction
```

Numerical analysis

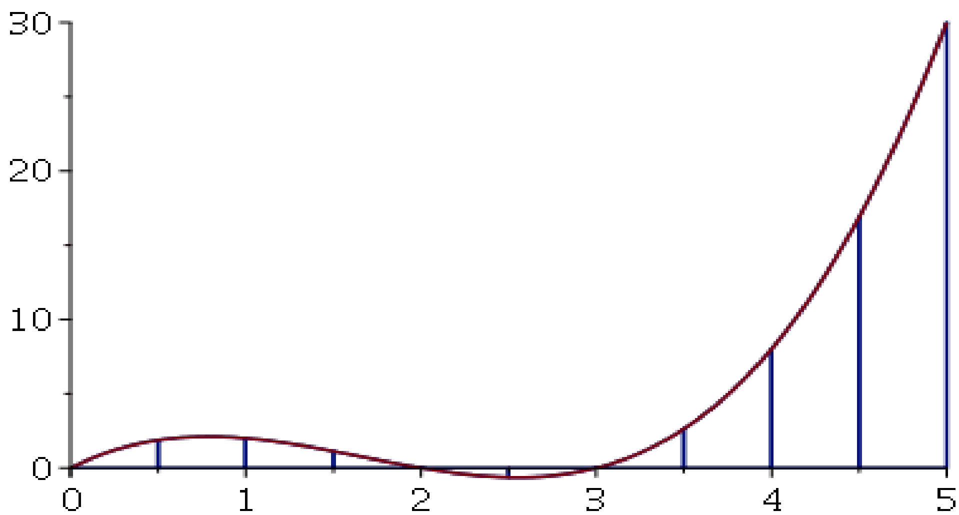
- **When does it converge:** Provided we have the proper data and a continuous function with no gaps; always converges.
- **Error estimate:**

$$-\frac{1}{90}\left(\frac{b-a}{n}\right)^5 f^{(4)}(\xi) \tag{12}$$

- **Pros & Cons:**
  - Intuitive
  - Simple to Use
  - Requires very small shapes to remain accurate

Numerical experiment: $f(x) = x^2 - x$ from -1 to 3 with $n = 4$.

$f(x_0) = f(a) = f(-1) = 2 = 2$ $4f(x_1) = 4f(0) = 0 = 0$ $2f(x_2) = 2f(1) = 0 = 0$ $4f(x_3) = 4f(2) = 8 = 8$ $f(x_4) = f(b) = f(3) = 6 = 6$
$\frac{1}{3}(2 + 0 + 0 + 8 + 6) = 5.33333333333333$ The graph is



An approximation of $\int_0^5 f(x)\,dx$ using Simpson's rule, where $f(x) = x\,(x-2)\,(x-3)$

3.14 Adaptive Simpson's rule

Area of application: Used to find a numerical approximation for a definite integral
Idea of the method: Adaptive Simpson's method uses an estimate of the error we get from calculating a definite integral using Simpson's rule. If the error exceeds a user-specified tolerance, the algorithm calls for subdividing the interval of integration in two and applying adaptive Simpson's method to each subinterval in a recursive manner
Code: Python.

```
from __future__ import division # python 2 compat
# "structured" adaptive version, translated from Racket
def _quad_simpsons_mem(f, a, fa, b, fb):
"""Evaluates the Simpson's Rule, also returning m and f(m) to reuse"""
m = (a+b) / 2
```

```
fm = f(m)
return (m, fm, abs(b-a) / 6 * (fa + 4 * fm + fb))

def _quad_asr(f, a, fa, b, fb, eps, whole, m, fm):
"""
Efficient recursive implementation of adaptive Simpson's rule.
Function values at the start, middle, end of the intervals are retained.
"""
lm, flm, left  = _quad_simpsons_mem(f, a, fa, m, fm)
rm, frm, right = _quad_simpsons_mem(f, m, fm, b, fb)
delta = left + right - whole
if abs(delta) <= 15 * eps:
return left + right + delta / 15
return _quad_asr(f, a, fa, m, fm, eps/2, left , lm, flm) +\
_quad_asr(f, m, fm, b, fb, eps/2, right, rm, frm)

def quad_asr(f, a, b, eps):
"""Integrate f from a to b using Adaptive Simpson's Rule with max error of eps."""
fa, fb = f(a), f(b)
m, fm, whole = _quad_simpsons_mem(f, a, fa, b, fb)
return _quad_asr(f, a, fa, b, fb, eps, whole, m, fm)

from math import sin
print(quad_asr(sin, 0, 1, 1e-09))
```

Numerical analysis

- **When does it converge:** Provided we have the proper data and a continuous function with no gaps; always converges.
- **Error estimate:**

$$-\frac{1}{90}\left(\frac{b-a}{n}\right)^5 f^{(4)}(\xi) \tag{13}$$

- **Pros & Cons:**
  - Intuitive
  - Simple to Use
  - Fewer functions than before

Numerical experiment:
Consider the integral $\int_1^3 e^{2x}\sin 3x dx$
$S(1,3) = 35.42697658812284$
$S(1,2) = -15.45828245392933$
$S(2,3) = 117.9751755250024$
Continuing:

$S(1,2) = -15.45828245392933$
$S(1,1.5) = -3.87030357255464$
$S(1.5,2) = -12.38881686458909$
The final approx for the problem is
$S(1,1.5)+S(1.5,2)+S(2,2.25)+S(2.25,2.5)+S(2.5,2.75)+S(2.75,3)$ Thus, 108.5722885413671

3.15 Gaussian Quadrature Formula

Area of application: Quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration
Idea of the method: $Pn(x)$. With the n-th polynomial normalized to give $Pn(1) = 1$, the i-th Gauss node, xi, is the i-th root of Pn and the weights are given by the formula
$w_i = \frac{2}{(1-x_i^2)\,[P'_n(x_i)]^2}.w_i = \frac{2}{(1-x_i^2)[P'_n(x_i)]^2}.$
Code:

```
function [x,w]=lgwt(N,a,b)
% lgwt.m
%
% This script is for computing definite integrals using Legendre-Gauss
% Quadrature. Computes the Legendre-Gauss nodes and weights  on an interval
% [a,b] with truncation order N
%
% Suppose you have a continuous function f(x) which is defined on [a,b]
% which you can evaluate at any x in [a,b]. Simply evaluate it at all of
% the values contained in the x vector to obtain a vector f. Then compute
% the definite integral using sum(f.*w);
%
% Written by Greg von Winckel - 02/25/2004
N=N-1;
N1=N+1; N2=N+2;
xu=linspace(-1,1,N1)';
% Initial guess
y=cos((2*(0:N)'+1)*pi/(2*N+2))+(0.27/N1)*sin(pi*xu*N/N2);
% Legendre-Gauss Vandermonde Matrix
L=zeros(N1,N2);
% Derivative of LGVM
Lp=zeros(N1,N2);
% Compute the zeros of the N+1 Legendre Polynomial
% using the recursion relation and the Newton-Raphson method
y0=2;
% Iterate until new points are uniformly within epsilon of old points
while max(abs(y-y0))>eps


L(:,1)=1;
```

```
Lp(:,1)=0;

L(:,2)=y;
Lp(:,2)=1;

for k=2:N1
L(:,k+1)=( (2*k-1)*y.*L(:,k)-(k-1)*L(:,k-1) )/k;
end

Lp=(N2)*( L(:,N1)-y.*L(:,N2) )./(1-y.^2);

y0=y;
y=y0-L(:,N2)./Lp;

end
% Linear map from[-1,1] to [a,b]
x=(a*(1-y)+b*(1+y))/2;
% Compute the weights
w=(b-a)./((1-y.^2).*Lp.^2)*(N2/N1)^2;
```

Numerical analysis

- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

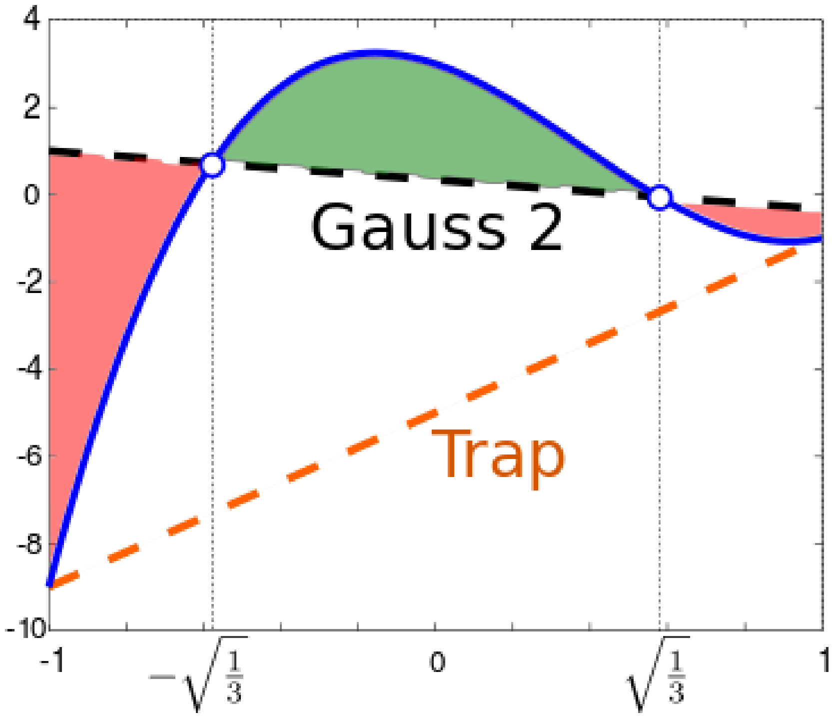
$$\frac{f^{(2n)}(\xi)}{(2n)!}\,(p_n, p_n) \tag{14}$$

- **Pros & Cons:**
  - Intuitive
  - Super Accurate
  - Fast

Numerical experiment: Gaussian Quadrature of $\sin(x), 0, \pi$

Gaussian Quadrature
0.064180425349
0.001388913608
0.000015771542
0.000000110284
0.000000000523

The graph is



Area of application: Quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration

Idea of the method: $Pn(x)$. With the n-th polynomial normalized to give $Pn(1) = 1$, the i-th Gauss node, xi, is the i-th root of Pn and the weights are given by the formula

$w_i = \frac{2}{(1-x_i^2)\,[P'_n(x_i)]^2}.w_i = \frac{2}{(1-x_i^2)[P'_n(x_i)]^2}.$

Code:

```
function [x,w]=lgwt(N,a,b)
% lgwt.m
%
% This script is for computing definite integrals using Legendre-Gauss
% Quadrature. Computes the Legendre-Gauss nodes and weights  on an interval
% [a,b] with truncation order N
%
% Suppose you have a continuous function f(x) which is defined on [a,b]
% which you can evaluate at any x in [a,b]. Simply evaluate it at all of
% the values contained in the x vector to obtain a vector f. Then compute
% the definite integral using sum(f.*w);
%
% Written by Greg von Winckel - 02/25/2004
N=N-1;
N1=N+1; N2=N+2;
xu=linspace(-1,1,N1)';
% Initial guess
```

```
y=cos((2*(0:N)'+1)*pi/(2*N+2))+(0.27/N1)*sin(pi*xu*N/N2);
% Legendre-Gauss Vandermonde Matrix
L=zeros(N1,N2);
% Derivative of LGVM
Lp=zeros(N1,N2);
% Compute the zeros of the N+1 Legendre Polynomial
% using the recursion relation and the Newton-Raphson method
y0=2;
% Iterate until new points are uniformly within epsilon of old points
while max(abs(y-y0))>eps


L(:,1)=1;
Lp(:,1)=0;

L(:,2)=y;
Lp(:,2)=1;

for k=2:N1
L(:,k+1)=( (2*k-1)*y.*L(:,k)-(k-1)*L(:,k-1) )/k;
end

Lp=(N2)*( L(:,N1)-y.*L(:,N2) )./(1-y.^2);

y0=y;
y=y0-L(:,N2)./Lp;

end
% Linear map from[-1,1] to [a,b]
x=(a*(1-y)+b*(1+y))/2;
% Compute the weights
w=(b-a)./((1-y.^2).*Lp.^2)*(N2/N1)^2;
```

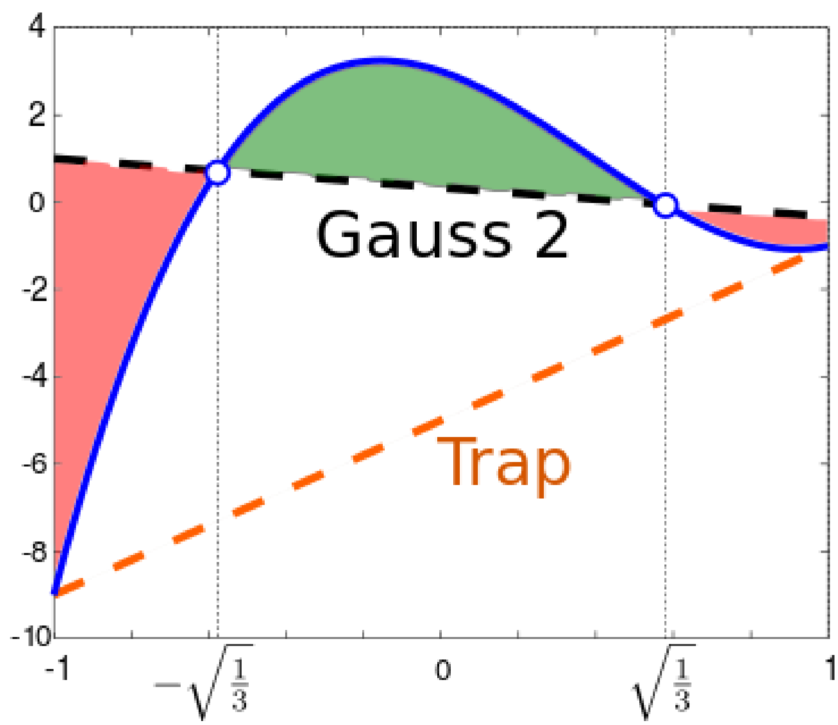
Numerical analysis

- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

$$\frac{f^{(2n)}(\xi)}{(2n)!}\,(p_n, p_n) \tag{15}$$

- **Pros & Cons:**
  - Intuitive
  - Super Accurate
  - Fast

Numerical experiment: Gaussian Quadrature of $\sin(x), 0, \pi$
Gaussian Quadrature
0.064180425349
0.001388913608
0.000015771542
0.000000110284
0.000000000523
The graph is



3.16 Naive Gaussian Elimination

Area of application: An algorithm for solving systems of linear equations. It is usually understood as a sequence of operations performed on the corresponding matrix of coefficients. This method can also be used to find the rank of a matrix, to calculate the determinant of a matrix, and to calculate the inverse of an invertible square matrix.
Idea of the method: To perform row reduction on a matrix, one uses a sequence of elementary row operations to modify the matrix until the lower left-hand corner of the matrix is filled with zeros, as much as possible.
Code:

```
function Aug = GaussianElimination (A, b)
ACol = columns(A);
ARows = rows(A);
Aug = [A b];
AugCol = columns(Aug);
for i = 1:ACol
```

```
pivotVal = Aug(i,i);
for j = i+1:ARows
underVal = Aug(j, i);
for k = i:AugCol
Aug(j, k) = Aug(j, k) - (underVal/pivotVal)*Aug(i, k);
endfor
endfor
endfor
endfunction
```

Numerical analysis

- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

$$\frac{f^{(2n)}(\xi)}{(2n)!}(p_n, p_n) \tag{16}$$

- **Pros & Cons:**
  - Intuitive
  - Slow
  - Easy to implement

Numerical experiment: $\left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 1 & -14 & -18 \end{array}\right] \rightarrow$
$\left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right]$
$\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

3.17 Gaussian Elimination partial and full pivoting

Area of application: An algorithm for solving systems of linear equations. It is usually understood as a sequence of operations performed on the corresponding matrix of coefficients. This method can also be used to find the rank of a matrix, to calculate the determinant of a matrix, and to calculate the inverse of an invertible square matrix.
Idea of the method: To perform row reduction on a matrix, one uses a sequence of elementary row operations to modify the matrix until the lower left-hand corner of the matrix is filled with zeros, as much as possible.
Code:

```
function [Aug swaps]= GaussianEliminationPartial (A, b)
ACol = columns(A);
ARows = rows(A);
Aug = [A b];
AugCol = columns(Aug);
swaps = zeros(0, 2);
for i = 1:ACol
maxIdx = MaxColValIdx(Aug(:, i), i);
swaps = [swaps ; i maxIdx];
tmpCurrPivot = Aug(i, :);
Aug
Aug(i, :) = Aug(maxIdx, :);
Aug(maxIdx, :) = tmpCurrPivot;
Aug
pivotVal = Aug(i,i)
for j = i+1:ARows
underVal = Aug(j, i);
for k = i:AugCol
Aug(j, k) = Aug(j, k) - (underVal/pivotVal)*Aug(i, k);
endfor
endfor
endfor
swaps
endfunction

function idx = MaxColValIdx (currCol, start)
idx = start;
maxVal = abs(currCol(idx));
for i = start:length(currCol)
if(abs(currCol(i))>maxVal)
maxVal = abs(currCol(i));
idx = i;
endif
endfor
endfunction
```

Numerical analysis

- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

$$e_n + f(\xi) \tag{17}$$

- **Pros & Cons:**
  - Intuitive
  - Faster than naive elimination

Numerical experiment: $\left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 1 & -14 & -18 \end{array}\right] \rightarrow$
$\left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right]$
$\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

3.18 LU factorization

Area of application: Factors a matrix as the product of a lower triangular matrix and an upper triangular matrix. The product sometimes includes a permutation matrix as well. LU decomposition can be viewed as the matrix form of Gaussian elimination.
Idea of the method: Let $A = LU$ where L is a lower triangular matrix and U is an upper triangular matrix. Use Gaussian elimination as normal.
Code:

```
function [L U] = LUDecomposition (A)
ACol = columns(A);
ARows = rows(A);
U = A;
L = eye(columns(A));
UCol = columns(U);
for i = 1:ACol
pivotVal = U(i,i);
for j = i+1:ARows
underVal = U(j, i);
for k = i:UCol
U(j, k) = U(j, k) - (underVal/pivotVal)*U(i, k);
L(j, k) = L(j, k) + (underVal/pivotVal)*L(i, k);
endfor
endfor
endfor
endfunction
```

Numerical analysis

- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

$$e_n + f(\xi) \tag{18}$$

- **Pros & Cons:**
  - Intuitive
  - Nice Structure
  - Hard to setup

Numerical experiment: We start by reducing A to echelon form to produce U.

$$A = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} = U$$

Now we will do the inverse of the row reduction operations to the identity matrix I.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & .5 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ .5 & 3 & 1 & 0 \\ -1 & -.5 & 2 & 1 \end{bmatrix}$$

Now we need to solve these two equations using forward and backward substitution.
$L\vec{z} = \vec{b}$
$U\vec{x} = \vec{z}$

$z_1 = 16$
$z_2 = 26 - 2(16) = -6$
$z_3 = -19 - 0.5(16) - 3(-6) = -9$
$z_4 = -34 + 16 - 0.5(-6) - 2(-9) = -3$

$$\vec{z} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

$U\vec{x} = \vec{z}$ $x_4 = -3/-3 = 1$
$x_3 = (-9 + 5(1))/2 = -2$
$x_2 = (-6 - 2(1) - 2(-2))/-4 = 1$
$x_1 = (16 - 4(1) - 2(-2) + 2(1))/6 = 3$

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

3.19 Forward and backward Euler's

Area of application:a numerical method to solve first order first degree differential equation with a given initial value
Idea of the method: The Forward Euler scheme is as follows. For $1 \leq n \leq N$ $y(0) = y0$,
$\frac{dy}{dt} = F(t, y)$,
$\frac{y_{n+1} y_n}{\Delta t} = F(t_n, y_n)$.
The backward Euler method computes the approximations using

$y_{k+1} = y_k + hf(t_{k+1}, y_{k+1}).y_{k+1} = y_k + hf(t_{k+1}, y_{k+1})$
Code:

```
y0 = 1;
t0 = 0;
h = 1; % try: h = 0.01
tn = 4; % equal to: t0 + h*n, with n the number of steps
[t, y] = Euler(t0, y0, h, tn);
plot(t, y, 'b');

% exact solution (y = e^t):
tt = (t0:0.001:tn)';
yy = exp(tt);
hold('on');
plot(tt, yy, 'r');
hold('off');
legend('Euler', 'Exact');

function [t, y] = Euler(t0, y0, h, tn)
fprintf('%10s%10s%10s%15s\n', 'i', 'yi', 'ti', 'f(yi,ti)');
fprintf('%10d%+10.2f%+10.2f%+15.2f\n', 0, y0, t0, f(y0,t0));
t = (t0:h:tn)';
y = zeros(size(t));
y(1) = y0;
for i = 1:1:length(t)-1
y(i+1) = y(i) + h*f(y(i),t(i));
fprintf('%10d%+10.2f%+10.2f%+15.2f\n', i, y(i+1), t(i+1), f(y(i+1),t(i+1)));
end
end

% in this case, f(y,t) = f(y)
function dydt = f(y,t)
dydt = y;
end
```

Numerical analysis

- **When does it converge:** Provided we have the proper data; always converges.
- **Error estimate:**

$$e_n + f(\xi) \tag{19}$$

- **Pros & Cons:**
  - Intuitive
  - Backwards avoids the problems forward has

Numerical experiment: Here $F(t, y) = y+2te^{2t}$ and $y_0 = 1$. Since the time step is$\Delta t = .1$, we need to iterate the solution twice. That is, we need to find y_2. The first iterate gives

$$
\begin{aligned}
y_1 &= y_0 + \Delta t F\left(t_0, y_0\right) \\
&= 1 + 0.1\left(1 + 2 \cdot 0 \cdot e^{2-0}\right) \text{ and} \\
&= 1.1 \\
y_2 &= y_1 + \Delta t F\left(t_1, y_1\right) \\
&= 1.1 + 0.1\left(1.1 + 2 \cdot 0.1 \cdot e^{2 \cdot 0.1}\right) \\
&\simeq 1.2344
\end{aligned}
$$