4.01 (a) Let X have the discrete topology and Y be an arbitrary topological space. Show that every function $f : X \to Y$ is continuous.

(b) Let Y have the trivial topology and X be an arbitrary topological space. Show that every function $f : X \to Y$ is continuous.

4.02 Prove Theorem 4.8 : Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$.

4.09 Let $f, g : X \to Y$ be continuous functions. Assume that Y is Hausdorff and that there exists a dense subset D of X such that $f(x) = g(x)$ for all $x \in D$. Prove that $f(x) = g(x)$ for all $x \in X$.

4.13 (a) Let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ be continuous functions. Show that $h : X \to Y_1 \times Y_2$, defined by $h(x) = (f_1(x), f_2(x))$, is continuous as well.

(b) Extend the result of (a) to n functions, for $n > 2$

4.14 Show that the addition function, $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = x+y$, is a continuous function.

4.16 Use Example 4.6, Exercises 4.13 and 4.14, and Theorem 4.9 to show that the sum and product of a finite number of continuous functions are also continuous functions. That is, assuming that $f_1, \ldots, f_m : \mathbb{R} \to \mathbb{R}$ are continuous, prove that $S : \mathbb{R} \to \mathbb{R}$ and $P : \mathbb{R} \to \mathbb{R}$, defined by $S(x) = f_1(x) + \ldots + f_m(x)$ and $P(x) = f_1(x)f_2(x)\ldots f_m(x)$, are continuous.

4.17 Use Exercise 4.16 to show that every polynomial function $p : \mathbb{R} \to \mathbb{R}$, given by $p(x) = a_nx^n + \ldots + a_1x + a_0$, is continuous.

4.22 Consider all of the possible topologies on the two-point set $X = \{a, b\}$. Indicate which ones are homeomorphic.

4.23 Find three different topologies on the three-point set $X = \{a, b, c\}$, each consisting of five open sets (including X and $\varnothing$), such that two of the topologies are homeomorphic to each other, but the third is not homeomorphic to the other two.

4.24 Prove that a bijection $f : X \to Y$ is a homeomorphism if and only if f and f^{-1} map closed sets to closed sets.

4.28 Prove each of the following statements, and then use them to show that topological equivalence is an equivalence relation on the collection of all topological spaces:

(a) The function $id : X \to X$, defined by $id(x) = x$, is a homeomorphism.

(b) If $f : X \to Y$ is a homeomorphism, then so is $f^{-1} : Y \to X$

(c) If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then so is the composition $g \circ f : X \to Z$

4.29 Show that $\mathbb{R}^2 - \{O\}$ in the standard topology is homeomorphic to $S^1 \times \mathbb{R}$.

4.32 Show that homeomorphism preserves interior, closure, and boundary as indicated in the following implications:

(a) If $f : X \to Y$ is a homeomorphism, then $f(\mathrm{Int}(A)) = \mathrm{Int}(f(A))$ for every $A \subset X$.

(b) If $f : X \to Y$ is a homeomorphism, then $f(\mathrm{Cl}(A)) = \mathrm{Cl}(f(A))$ for every $A \subset X$.

(c) If $f : X \to Y$ is a homeomorphism, then $f(\partial(A)) = \partial(f(A))$ for every $A \subset X$.

4.33 Let $X \times Y$ be partitioned into subsets of the form $X \times \{y\}$ for all y in Y. If we let $(X \times Y)^*$ denote the collection of sets in the partition, show that $(X \times Y)^*$ with the resulting quotient topology is homeomorphic to Y.

Summary