4 Continuous Functions and Homeomorphisms

4.1 Definition of Continuous

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if for every $x_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists $a\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$

4.2 Open Set Definition of Continuity

Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous** if $f^{-1}(V)$ is open in X for every open set V in Y.

We call this the **open set definition of continuity**. Paraphrased, it states that f is continuous if the preimage of every open set is open.

4.3 Theorem that a function is continuous if and only if the preimage of the basis elements is open

Let X and Y be topological spaces and $\mathcal{B}$ be a basis for the topology on Y. Then $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

Proof: Suppose $f : X \to Y$ is continuous. Then $f^{-1}(V)$ is open in X for every V open in Y. Since every basis element B is open in Y, it follows that $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$.

Now, suppose $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$. We show that f is continuous. Let V be an open set in Y. Then V is a union of basis elements, say $V = \cup B_\alpha$. Thus,

$$f^{-1}(V) = f^{-1}(\cup B_\alpha) = \cup f^{-1}(B_\alpha)$$

By assumption, each set $f^{-1}(B_\alpha)$ is open in X; therefore so is their union. Thus, $f^{-1}(V)$ is open in X, and it follows that the preimage of every open set in Y is open in X. Hence, f is continuous. $\square$

4.4 Theorem that every polynomial is continuous

Let $\mathbb{R}$ have the standard topology. Then every polynomial function $p : \mathbb{R} \to \mathbb{R}$, with $p(x) = a_n x^n + \ldots + a_1 x + a_0$, is continuous.

4.5 Theorem that says the closure of a subset maps to part of the closure of the superset

Let $f : X \to Y$ be continuous and assume that $A \subset X$. If $x \in Cl(A)$, then $f(x) \in Cl(f(A))$.

Proof: Suppose that $f : X \to Y$ is continuous, $x \in X$, and $A \subset X$. We prove that if $f(x) \notin \mathrm{Cl}(f(A))$, then $x \notin Cl(A)$. Thus suppose that $f(x) \notin \mathrm{Cl}(f(A))$. By Theorem 2.5 there exists an open set U containing $f(x)$, but not intersecting $f(A)$. It follows that $f^{-1}(U)$ is an open set containing x that does not intersect A . Thus $x \notin CI(A)$, and the result follows. □

4.6 Translation of $\varepsilon - \delta$

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if, for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$.

$\forall x \in X$ and every open set U containing $f(x), \exists$ neighborhood V of x, such that $f(V) \subset U$

4.7 Theorem that a function is continuous if and only if every element has a neighborhood containing $f(x)$, there exists a neighbor$V$ of x such that $f(V) \subset U$

A function $f : X \to Y$ is continuous in the open set definition of continuity if and only if for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$

Proof: First, suppose that the open set definition holds for functions $f : X \to Y$. Let $x \in X$ and an open set U C Y containing $f(x)$ be given. Set $V = f^{-1}(U)$. It follows that $x \in V$ and that V is open in X since f is continuous by the open set definition. Clearly $f(V) \subset U$, and therefore we have shown the desired result.

Now assume that for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$. We show that $f^{-1}(W)$ is open in X for every open set W in Y . Thus let W be an arbitrary open set in Y. To show that $f^{-1}(W)$ is open in X, choose an arbitrary $x \in f^{-1}(W)$. It follows that $f(x) \in W$, and therefore there exists a neighborhood V_x of x in X such that $f\left(V_x\right) \subset W$, equivalently, such that $V_x \subset f^{-1}(W)$. Thus, for an arbitrary $x \in f^{-1}(W)$ there exists an open set V_x such that $x \in V_x \subset f^{-1}(W)$. Theorem 1.4 implies that $f^{-1}(W)$ is open in X. □

4.8 Theorem that converges points will converge given a function

Assume that $f : X \to Y$ is continuous. If a sequence $(x_1, x_2, \ldots)$ in X converges to a point x, then the sequence $\left(f\left(x_1\right), f\left(x_2\right), \ldots\right)$ in Y converges to $f(x)$.

Proof: Let U be an arbitrary neighborhood of $f(x)$ in Y. Since f is continuous, $f^{-1}(U)$ is open in X . Furthermore, $f(x) \in U$ implies that $x \in f^{-1}(U)$. The sequence $(x_1, x_2, \ldots)$ converges to x; thus, there exists $N \in Z_+$ such that $x_n \in f^{-1}(U)$ for all $n \geq N$. It follows that $f(x_n) \in U$ for all $n \geq N$, and therefore the sequence $(f(x_1), f(x_2), \ldots)$ converges to $f(x)$ $\square$

4.9 Theorem that we can map closed sets between each other

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$.

My Proof: Let C be a closed set in Y. Notice, $Y - C$ is open in Y and so $f^{-1}(Y - C)$ must also be open in X.
We claim that $f^{-1}(Y - C) = f^{-1}(Y) - f^{-1}(C)$ and is open.
We define $f^{-1}(Y - C) = \{x \in X | f(x) \in Y - C\}$. This implies that $f(x) \in Y$ and $f(x) \notin C$.
We also define $f^{-1}(Y) = \{x \in X | f(x) \in Y\}$ and $f^{-1}(C) = \{x \in X | f(x) \in C\}$ Thus, $f^{-1}(Y) - f^{-1}(C)$ implies $f(x) \in Y$ and $f(x) \notin C$. Notice, this is our definition of $f^{-1}(Y - C)$.
Thus, $f^{-1}(Y - C) \subseteq f^{-1}(Y) - f^{-1}(C)$
Going the other direction, we have the definition of $f^{-1}(Y) - f^{-1}(C)$ from our implication of $f^{-1}(Y - C)$
Thus, $f^{-1}(Y) - f^{-1}(C) \subseteq f^{-1}(Y - C)$
Therefore, $f^{-1}(Y - C) = f^{-1}(Y) - f^{-1}(C)$ (Thank you for recognizing this needed to be proved.)
Since, $f^{-1}(Y)$ is defined to be $\{x \in X | f(x) \in Y\}$ we know that $f^{-1}(Y) \subseteq X$. But since all of X is mapped in the preimage we also have $X \subseteq f^{-1}(Y)$. Thus, $f^{-1}(X) = Y$(You mean $f^{-1}(Y) = X$)
Taking $X - C$ will result in an open set as $X - C = X \cap C^{\complement}$ and since $C^{\complement}$ is open and the intersection of open sets are open.
Observe.

$$f^{-1}(Y - C) = f^{-1}(Y) - f^{-1}(C) = X - f^{-1}(C)$$

Thus, $f^{-1}(C)$ must be closed by our previous result.

Suppose, $f^{-1}(C)$ is closed. We then know that $X - f^{-1}(C)$ must be open. Notice, that $X = f^{-1}(Y)$. Observe.

$$X - f^{-1}(C) = f^{-1}(Y) - f^{-1}(C) = f^{-1}(Y - C)$$

As, $Y - C$ is open and f^{-1} is defined with an arbitrary open set U as $f^{-1}(U) = \{x \in X | f(x) = U\}$, we then have that f^{-1} maps open sets to open sets.
Thus, f must be continuous.5/5 $\square$

4.10 Theorem that function composition works for continuity

Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then the composition function, $g \circ f : X \to Z$, is continuous.

Proof: Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous, and let U be an open set in Z . Then $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right)$, since g is continuous, $g^{-1}(U)$ is open in Y , and since f is continuous, $f^{-1}\left(g^{-1}(U)\right)$ is open in X. Thus, $(g \circ f)^{-1}(U)$ is open in X for an arbitrary U open in Z, implying that $g \circ f$ is continuous. $\square$

4.11 The Pasting Lemma

Let X be a topological space and let A and B be closed subsets of X such that $A \cup B = X$. Assume that $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for all x in $A \cap B$. Then $h : X \to Y$, defined by

$$h(x) = \begin{cases} f(x) \text{ if } x \in A \\ g(x) \text{ if } x \in B \end{cases}$$

is a continuous function.

Proof: Proof. By Theorem 4.8 , it suffices to show that if C is closed in Y , then $h^{-1}(C)$ is closed in X. Thus suppose that C is closed in Y. Note that $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. since f is continuous, it follows by Theorem 4.8 that $f^{-1}(C)$ is closed in A . Theorem 3.4 then implies that $f^{-1}(C) = D \cap A$ where D is closed in X . Now, D and A are both closed in X, and $f^{-1}(C) = D \cap A$; therefore , $f^{-1}(C)$ is closed in X. Similarly, $g^{-1}(C)$ is closed in X. Thus, $h^{-1}(C)$ is the union of two closed sets in X and therefore is closed in X as well. It follows that h is continuous. $\square$

4.12 Definition of a Homeomorphism

Let X and Y be topological spaces, and let $f : X \to Y$ be a bijection with inverse $f^{-1} : Y \to X$. If both f and f^{-1} are continuous functions, then f is said to be a **homeomorphism**. If there exists a homeomorphism between X and Y, we say that X and Y are **homeomorphic** or **topologically equivalent**, and we denote this by $X \cong Y$.

4.13 Facts about Homeomorphisms

(i) The function $id : X \to X,$ defined by $id(x) = x,$ is a homeomorphism.
(ii) If $f : X \to Y$ is a homeomorphism, then so is $f^{-1} : Y \to X$.
(iii) If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then so is $g \circ f : X \to Z$

4.15 Definition of Embedding

An **embedding** of X in Y is a function $f : X \to Y$ that maps X homeomorphically to the subspace $f(X)$ in Y.

4.16 Definition of Arc & Simple Closed Curve

Let X be a topological space. If $f : [-1, 1] \to X$ is an embedding, then the image of f is called an **arc** in X , and if $f : S^1 \to X$ is an embedding, then the image of f is called a **simple closed curve** in X .

4.17 Theorem about Hausdorffness being a topological property

If $f : X \to Y$ is a homeomorphism and X is Hausdorff, then Y is Hausdorff.

Proof: Suppose that X is Hausdorff and $f : X \to Y$ is a homeomorphism. Let x and y be distinct points in Y. Then $f^{-1}(x)$ and $f^{-1}(y)$ are distinct points in X. Thus, there exist disjoint open sets U and V containing $f^{-1}(x)$ and $f^{-1}(y)$, respectively. It follows that $f(U)$ and $f(V)$ are disjoint open sets containing x and y , respectively. Therefore Y is Hausdorff. □

Definition of Topological Property

A property of topological spaces that is preserved by homeomorphism is said to be a **topological property.**

5 Metric Spaces

5.1 Definition of Metric

A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ with the following properties:

(O) $d(x, y) = 0$ for some $x, y \in X$; if and only if $x = y$

(i) $d(x, y) \geq 0$ for all $x, y \in X$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

Definition of Metric Space

We call $d(x, y)$ the distance between x and y, and we call the pair (X, d) , consisting of the set X and the metric d , a **metric space**.

Definition of Standard Metric

Given points $p = (p_1, p_2)$ and $q = (q_1, q_2)$

$$d(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$$

We call d the **standard metric** on $\mathbb{R}^2$. This metric measures the straight-line distance between points in the plane.

Definition of Taxicab Metric

Given points $p = (p_1, p_2)$ and $q = (q_1, q_2)$

$$d_M(p,q) = \max\{|p_1 - q_1|, |p_2 - q_2|\}$$

We call d the **max metric** on $\mathbb{R}^2$. The distance in this metric is the maximum of the differences between their coordinates.

Definition of Max Metric

Given points $p = (p_1, p_2)$ and $q = (q_1, q_2)$

$$d_M(p,q) = \max\{|p_1 - q_1|, |p_2 - q_2|\}$$

We call d the **max metric** on $\mathbb{R}^2$. This metric measures the distance between two points is the maximum of the differences between their coordinates.

5.5 Definition of Metric Topology

Let (X, d) be a metric space. The topology generated by the basis of open balls $\mathcal{B} = \{B_d(x, \varepsilon) | x \in X, \varepsilon > 0\}$ is called the topology induced by d and is referred to as a **metric topology.**

5.6 Theorem about A set U is open iff for each $y \in U$ there is an open ball centered at y and contained in U

Let (X, d) be a metric space. A set $U \subset X$ is open in the topology induced by d if and only if for each $y \in U$, there is $a\delta > 0$ such that $B_d(y, \delta) \subset U$.

Properties of Metric Spaces

5.12 Theorem about Every Metric Space is Hausdorff

Proof: Let (X, d) be a metric space. Suppose x and y are distinct points in X with $d(x, y) = \varepsilon$. Consider the sets $U = B_d(x, \varepsilon/2)$ and $V = B_d(y, \varepsilon/2)$. It follows that $x \in U, y \in V$, and U and V are open sets. We claim that U and V are disjoint. Suppose $U \cap V \neq \varnothing$, and z is in the intersection. Then $d(x, z) < \varepsilon/2$ and $d(y, z) < \varepsilon/2$. Therefore, by the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

that is, $d(x, y) < \varepsilon$. This contradicts $d(x, y) = \varepsilon$. Thus $U \cap V = \varnothing$. Hence, there exist disjoint open sets U and V containing x and y respectively, implying that X is Hausdorff. $\square$

5.13 Theorem about

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous in the open set definition if and only if for each $x \in X$ and $\varepsilon > 0$, there exists $a\delta > 0$ such that if $x' \in X$ and $d_X(x, x') < \delta$ then $dy(f(x), f(x')) < \varepsilon$

My Proof: (WTS: $\forall x \in X \exists \delta > 0$ such that $x' \in X$ and $d_x(x, x') < \delta$, we have $d_Y(f(x), f(x')) < \epsilon$)
Suppose f is continuous. Let $x \in X, \epsilon > 0$, and $\delta > 0$ such that $x' \in X$ and $d_x(x, x') < \delta$. Notice, as f is continuous, $f(x), f(x') \in Y$. Since, x is bound by ϵ, $d(x, x')$ is bounded by δ, and $f(x), f(x') \in Y$, we must have that $d_Y(f(x), f(x') < \epsilon$.

Let U be an open set in Y. Let $x \in f^{-1}(U)$ and define $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq U$. Define δ such that$x' \in X$ satisfies $d(x, x') < \delta$. Which implies $x' \in B(x, \delta)$. Notice, we must have $d(f(x), f(x')) < \epsilon$. From this result, we have $f(x') \in B(f(x), \epsilon) \subseteq U$. So, $x' \in B(x, \delta)$ as $x' \in f^{-1}(U)$.
Thus, $B(x, \delta) \subseteq f^{-1}(U)$.
Thus, f is continuous.

Therefore, A function $f : X \to Y$ is continuous in the open set definition if and only if for each $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x' \in X$ and $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$. $\square$

5.14 Definition of Distance between

Let (X, d) be a metric space. For sets $A, B \subset X$ define the distance between A and B by

$$d(A, B) = glb\{d(a, b) | a \in A, b \in B\}$$

The greatest lower bound in Definition 5.14 exists for every pair of sets A and B since the set of values $\{d(a, b) | a \in A, b \in B\}$ is bounded below by 0.

5.15 Theorem about Proving one topology is finer than other with metrics

THEOREM 5.15. Let d and d' be metrics on a set X, and let $\mathcal{T}$ and T' , respectively, be the topologies that they induce. Then T' is finer than $\mathcal{T}$ if and only if for each $x \in X$ and $\varepsilon > 0$, there exists $a\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$.

Proof: Proof. Suppose that T' is finer than $\mathcal{T}$. Then every open set in $\mathcal{T}$ is open in $\mathcal{T}'$. In particular, for every $x \in X$ and $\varepsilon > 0, B_d(x, \varepsilon)$ is open in $\mathcal{T}$ and hence is open in T'. since $B_d(x, \varepsilon)$ is open in T' and contains x Theorem 5.6 implies that there is a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$ as we wished to show.

Suppose now that for each $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$. We prove that T' is finer than $\mathcal{T}$. Let U be an open set in $\mathcal{T}$. We show that

U is open in $\mathcal{T}'$. Let x be an arbitrary point in U . Since U is open in T, Theorem 5.6 implies that there is an $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$. By assumption there is a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon) \subset U$. It follows that for each $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subset U$. Theorem 5.6 implies that U is open in $\mathcal{T}'$, as we wished to show. $\square$

5.17 Definition of Bounded Metric

Let (X, d) be a metric space. A subset A of X is said to be bounded under d if there exists a $\mu > 0$ such that $d(x, y) \leq \mu$ for all $x, y \in A$. If X itself is bounded under d, then we say that d is a **bounded metric.**

5.18 Theorem about A topology is bounded if it is induced by a bounded metric

THEOREM 5.18. Let (X, d) be a metric space, and define $d' : X \times X \to \mathbb{R}$ by $d'(x, y) = \min[d(x, y), 1\}$. Then d' is a bounded metric that induces the same topology as d .

5.19 Definition of Isometry

Let (X, d_X) and (Y, d_Y) be metric spaces. A bijective function $f : X \to Y$ is called an **isometry** if $d_X(x, x') = d_Y(f(x), f(x'))$ for every pair of points x and x' in X . If $f : X \to Y$ is an isometry, then we say that the metric spaces X and Y are **isometric**.

6 Connectedness

6.1 Definition of Connected

Let X be a topological space.

(i) We call X **connected** if there does not exist a pair of disjoint nonempty open sets whose union is X.

(ii) We call X **disconnected** if X is not connected.

(iii) If X is disconnected, then a pair of disjoint nonempty open sets whose union is X is called **a separation of** X.

6.2 Theorem about A topological space X is connected if and only if there are no nonempty proper subsets of X that are both open and closed in X .

My Proof: Suppose that X is not connected. Let U, V be a separation of X. That is $U \cup V = X$ and U, V are disjoint nonempty open sets. Notice, $V = X - U$ as U and V are disjoint. Then, we have that U is closed as it is the complement of an open set. We then have $V = X - U \neq \varnothing$. This results says that U is a proper subset. Thus, U is a nonempty, proper, closed, and open set.

Suppose U is a nonempty proper set that is closed and open. Let $V = X - U$. Since, U is proper we have V is nonempty and open. Notice.

$$X = U \cup (X - U) = U \cup V$$

Thus, U, V form a separation of X.
Thus, X is not connected.

Therefore, X is connected if and only if there are no nonempty proper subsets of X that are both open and closed in X.
□

6.4 Theorem about Showing disconnected

A set A is disconnected in X if and only if there exist open sets U and V in X such that $A \subset U \cup V, U \cap A \neq \varnothing, V \cap A \neq \varnothing$, and $U \cap V \cap A = \varnothing$.

Proof: Suppose that A is disconnected in X. Then there exist nonempty y . Then there exist nonempty sets P and Q that are open in A , disjoint, and such that $P \cup Q = A$. since P and Q are open in A there exist sets U and V that are open in X and such that $U \cap A = P$ and $V \cap A = Q$. Clearly, $A \subset U \cup V$, $U \cap A \neq \varnothing, V \cap A \neq \varnothing$, and $U \cap V \cap A = \varnothing$.

Now suppose that U and V are open sets in X such that $A \subset U \cup V$, $U \cap A \neq \varnothing, V \cap A \neq \varnothing$, and $U \cap V \cap A = \varnothing$. If we let $P = U \cap A$ and $Q = V \cap A$, then it follows that the pair of sets, P and Q, is a separation of A in the subspace topology, and therefore A is disconnected in X . □

6.5 Definition of Separation

Let A be a subspace of a topological space X. If U and V are open sets in X such that $A \subset U \cup V, U \cap A \neq \varnothing, V \cap A \neq \varnothing$, and $U \cap V \cap A = \varnothing$, then we say that the pair of sets, U and V, is a separation of A in X.

There's a whole lot more that goes in here. I just don't have the time to be spending on this document.

6.27 Definition of Path Connected

A topological space X is **path connected** if for every $x, y \in X$ there is a path in X from x to $y.A$ subset of a topological space X is path connected in X is path connected in the subspace topology that A inherits from X.

6.28 Theorem about If X is path connected space, than it is connected.

Proof: Proof. Let X be a path connected space. We prove that X is connected by showing that it has only one component, or equivalently that every pair of points $x, y \in X$ is contained in some connected subset of X. Thus, let x and y be arbitrary points in X. since X is path connected, there is a path in X from x to y. The image of such a path is a connected subset of X containing both x and y. Therefore every pair of points in X is contained in a connected subset of X, and it follows that X is connected. $\square$

7 Compactness 7.1

Let A be a subset of a topological space X, and let $\mathcal{O}$ be a collection of subsets of X

7.1 Definition of Cover

The collection $\mathcal{O}$ is said to **cover** A if A is contained in the union of the sets in $\mathcal{O}$.

7.2 Definition of Open Cover

If $\mathcal{O}$ covers A, and each set in $\mathcal{O}$ is open, then we call $\mathcal{O}$ an **open cover** of A.

7.3 Definition of Subcover

If $\mathcal{O}$ covers A, and $\mathcal{O}'$ is a subcollection of $\mathcal{O}$ that also covers A then $\mathcal{O}'$ is called a **subcover** of $\mathcal{O}$.

7.4 Definition of Compact

A topological space X is **compact** if every open cover of X has a finite subcover.

7.5 Definition of Compact In

Let X be a topological space, and assume $A \subset X$. Then A is said to be **compact in** X if A is compact in the subspace topology inherited from X.

7.6 Lemma: Checks weather or not a subspace A is compact

Let X be a topological space, and assume $A \subset X$. Then A is compact in X if and only if every cover of A by sets that are open in X has a finite subcover.

Proof:
Let A be compact in X, and suppose that $\mathcal{O}$ is a cover of A by open sets in X. Then $\mathcal{O}' = \{U \cap A | U \in \mathcal{O}\}$ is a cover of A by open sets in A. Hence, there exists a finite subcover $\{U_1 \cap A, U_2 \cap A, \ldots, U_n \cap A\}$ of $\mathcal{O}'$. But then $\{U_1, U_2, \ldots, U_n\}$ is a finite subcover of $\mathcal{O}$. Therefore every cover of A by open sets in X has a finite subcover.

Conversely, suppose every cover of A by sets that are open in X has a finite subcover. Let $\mathcal{O} = \{V_\beta\}_{\beta \in B}$ be a cover of A by open sets in A. Then, by definition of the subspace topology, for each V_β there vis an open set U_β in X such that $V_\beta = U_\beta \cap A$. It follows that the collection $\mathcal{O}' = \{U_B\}_{B \in R}$ is a cover of A by open sets in X . Since $\mathcal{O}'$ has a finite subcover $\{U_{\beta_1}, \ldots, U_{\beta_n}\}$, it follows that $\{V_{\beta_1}, \ldots, V_{\beta_n}\}$ is a finite subcover of $\mathcal{O}$. Thus every cover of A by open sets in A has a finite subcover, and therefore A is compact.

7.7 Compactness will be preserved through continuous functions

Let $f : X \to Y$ be continuous, and let A be compact in X. Then $f(A)$ is compact in Y.
Proof:

7.8 Compact sets unioned together are compact

Let X be a topological space. If $C_1, \ldots, C_n$ are each compact in X, then $U_{j=1}^{n} C_j$ is compact in X.

7.9 Intersection of Hausdorff compact sets are compact

If X is Hausdorff, and $\{C_\alpha\}_{\alpha \in A}$ is a collection of sets that are compact in X, then $\bigcap_{\alpha \in A} C_\alpha$ is compact in X .

7.10 If a subset of a compact set is closed, then that subset is also compact

Let X be a topological space and let D be compact in X. If C is closed in X, and $C \subset D$, then C is compact in X .

7.11 All compact subsets of a Hausdorff space are closed

Let X be a Hausdorff topological space and A be compact in X . Then A is closed in X.

7.12 Tube Lemma i.e. You can take a slice of a space and it'll be compact still

Let X and Y be topological spaces, and assume that Y is compact. If $x \in X$, and U is an open set in $X \times Y$ containing $\{x\} \times Y$, then there exists a neighborhood W of x in X such that $W \times Y \subset U$.

7.13 Product topology preserves compactness

THEOREM 7.10. If X and Y are compact topological spaces, then the product $X \times Y$ is compact.

7 Compactness in Metric Spaces 7.2

7.1 Closed Bounded Intervals are compact

Every closed and bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$ with the standard topology.

7.2 Product of closed bounded intervals are compact

Let $[a_1, b_1], \ldots, [a_n, b_n]$ be closed bounded intervals in R. Then $[a_1, b_1] \times \ldots \times [a_n, b_n]$ is a compact subset of $\mathbb{R}^n$

7.3 The standard topology in standard metric in $\mathbb{R}^n$ is compact iff it is closed and bounded

Let $\mathbb{R}^n$ have the standard topology and the standard metric d. A set $A \subset \mathbb{R}^n$ is compact in $\mathbb{R}^n$ if and only if it is closed and bounded.

7.4 In a metric space with compact subset A, if there is a sequence, then there is a subsequence that converges to a limit in A

Let (X, d) be a metric space, and assume that A is compact in X. If (x_n) is a sequence in A, then there exists a subsequence (x_{n_m}) of (x_n) that converges to a limit in A.