4 Continuous Functions and Homeomorphisms

4.1 Definition of Continuous

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if for every $x_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists $a\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$

4.2 Open Set Definition of Continuity

Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous** if $f^{-1}(V)$ is open in X for every open set V in Y.

We call this the **open set definition of continuity**. Paraphrased, it states that f is continuous if the preimage of every open set is open.

4.3 Theorem that a function is continuous if and only if the preimage of the basis elements is open

Let X and Y be topological spaces and $\mathcal{B}$ be a basis for the topology on Y . Then $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

Proof: Suppose $f : X \to Y$ is continuous. Then $f^{-1}(V)$ is open in X for every V open in Y. Since every basis element B is open in Y, it follows that $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$.

Now, suppose $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$. We show that f is continuous. Let V be an open set in Y . Then V is a union of basis elements, say $V = \cup B_\alpha$. Thus,

$$f^{-1}(V) = f^{-1}(\cup B_\alpha) = \cup f^{-1}(B_\alpha)$$

By assumption, each set $f^{-1}(B_\alpha)$ is open in X; therefore so is their union. Thus, $f^{-1}(V)$ is open in X , and it follows that the preimage of every open set in Y is open in X . Hence, f is continuous. $\square$

4.4 Theorem that every polynomial is continuous

Let $\mathbb{R}$ have the standard topology. Then every polynomial function $p : \mathbb{R} \to \mathbb{R}$, with $p(x) = a_n x^n + \ldots + a_1 x + a_0$, is continuous.

4.5 Theorem that says the closure of a subset maps to part of the closure of the superset

Let $f : X \to Y$ be continuous and assume that $A \subset X$. If $x \in Cl(A)$, then $f(x) \in Cl(f(A))$.

Proof: Suppose that $f : X \to Y$ is continuous, $x \in X$, and $A \subset X$. We prove that if $f(x) \notin \mathrm{Cl}(f(A))$, then $x \notin Cl(A)$. Thus suppose that $f(x) \notin \mathrm{Cl}(f(A))$. By Theorem 2.5 there exists an open set U containing $f(x)$, but not intersecting $f(A)$. It follows that $f^{-1}(U)$ is an open set containing x that does not intersect A . Thus $x \notin CI(A)$, and the result follows. □

4.6 Translation of $\varepsilon - \delta$

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if, for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$.

$\forall x \in X$ and every open set U containing $f(x), \exists$ neighborhood V of x, such that $f(V) \subset U$

4.7 Theorem that a function is continuous if and only if every element has a neighborhood containing $f(x)$, there exists a neighbor$V$ of x such that $f(V) \subset U$

A function $f : X \to Y$ is continuous in the open set definition of continuity if and only if for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$

Proof: First, suppose that the open set definition holds for functions $f : X \to Y$. Let $x \in X$ and an open set U C Y containing $f(x)$ be given. Set $V = f^{-1}(U)$. It follows that $x \in V$ and that V is open in X since f is continuous by the open set definition. Clearly $f(V) \subset U$, and therefore we have shown the desired result.

Now assume that for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$. We show that $f^{-1}(W)$ is open in X for every open set W in Y . Thus let W be an arbitrary open set in Y. To show that $f^{-1}(W)$ is open in X, choose an arbitrary $x \in f^{-1}(W)$. It follows that $f(x) \in W$, and therefore there exists a neighborhood V_x of x in X such that $f(V_x) \subset W$, equivalently, such that $V_x \subset f^{-1}(W)$. Thus, for an arbitrary $x \in f^{-1}(W)$ there exists an open set V_x such that $x \in V_x \subset f^{-1}(W)$. Theorem 1.4 implies that $f^{-1}(W)$ is open in X. □

4.8 Theorem that converges points will converge given a function

Assume that $f : X \to Y$ is continuous. If a sequence $(x_1, x_2, \ldots)$ in X converges to a point x, then the sequence $(f(x_1), f(x_2), \ldots)$ in Y converges to $f(x)$.

Proof: Let U be an arbitrary neighborhood of $f(x)$ in Y. Since f is continuous, $f^{-1}(U)$ is open in X . Furthermore, $f(x) \in U$ implies that $x \in f^{-1}(U)$. The sequence $(x_1, x_2, \ldots)$ converges to x; thus, there exists $N \in Z_+$ such that $x_n \in f^{-1}(U)$ for all $n \geq N$. It follows that $f(x_n) \in U$ for all $n \geq N$, and therefore the sequence $(f(x_1), f(x_2), \ldots)$ converges to $f(x)$ $\square$

4.9 Theorem that we can map closed sets between each other

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$.

My Proof: Let C be a closed set in Y. Notice, $Y - C$ is open in Y and so $f^{-1}(Y - C)$ must also be open in X.
We claim that $f^{-1}(Y - C) = f^{-1}(Y) - f^{-1}(C)$ and is open.
We define $f^{-1}(Y - C) = \{x \in X | f(x) \in Y - C\}$. This implies that $f(x) \in Y$ and $f(x) \notin C$.
We also define $f^{-1}(Y) = \{x \in X | f(x) \in Y\}$ and $f^{-1}(C) = \{x \in X | f(x) \in C\}$ Thus, $f^{-1}(Y) - f^{-1}(C)$ implies $f(x) \in Y$ and $f(x) \notin C$. Notice, this is our definition of $f^{-1}(Y - C)$.
Thus, $f^{-1}(Y - C) \subseteq f^{-1}(Y) - f^{-1}(C)$
Going the other direction, we have the definition of $f^{-1}(Y) - f^{-1}(C)$ from our implication of $f^{-1}(Y - C)$
Thus, $f^{-1}(Y) - f^{-1}(C) \subseteq f^{-1}(Y - C)$
Therefore, $f^{-1}(Y - C) = f^{-1}(Y) - f^{-1}(C)$ (Thank you for recognizing this needed to be proved.)
Since, $f^{-1}(Y)$ is defined to be $\{x \in X | f(x) \in Y\}$ we know that $f^{-1}(Y) \subseteq X$. But since all of X is mapped in the preimage we also have $X \subseteq f^{-1}(Y)$. Thus, $f^{-1}(X) = Y$(You mean $f^{-1}(Y) = X$)
Taking $X - C$ will result in an open set as $X - C = X \cap C^{\complement}$ and since $C^{\complement}$ is open and the intersection of open sets are open.
Observe.

$$f^{-1}(Y - C) = f^{-1}(Y) - f^{-1}(C) = X - f^{-1}(C)$$

Thus, $f^{-1}(C)$ must be closed by our previous result.

Suppose, $f^{-1}(C)$ is closed. We then know that $X - f^{-1}(C)$ must be open. Notice, that $X = f^{-1}(Y)$. Observe.

$$X - f^{-1}(C) = f^{-1}(Y) - f^{-1}(C) = f^{-1}(Y - C)$$

As, $Y - C$ is open and f^{-1} is defined with an arbitrary open set U as $f^{-1}(U) = \{x \in X | f(x) = U\}$, we then have that f^{-1} maps open sets to open sets.
Thus, f must be continuous.5/5 $\square$

4.10 Theorem that function composition works for continuity

Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then the composition function, $g \circ f : X \to Z$, is continuous.

Proof: Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous, and let U be an open set in Z . Then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$, since g is continuous, $g^{-1}(U)$ is open in Y , and since f is continuous, $f^{-1}(g^{-1}(U))$ is open in X. Thus, $(g \circ f)^{-1}(U)$ is open in X for an arbitrary U open in Z, implying that $g \circ f$ is continuous. $\square$

4.11 The Pasting Lemma

Let X be a topological space and let A and B be closed subsets of X such that $A \cup B = X$. Assume that $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for all x in $A \cap B$. Then $h : X \to Y$, defined by

$$h(x) = \begin{cases} f(x) \text{ if } x \in A \\ g(x) \text{ if } x \in B \end{cases}$$

is a continuous function.

Proof: Proof. By Theorem 4.8 , it suffices to show that if C is closed in Y , then $h^{-1}(C)$ is closed in X. Thus suppose that C is closed in Y. Note that $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. since f is continuous, it follows by Theorem 4.8 that $f^{-1}(C)$ is closed in A . Theorem 3.4 then implies that $f^{-1}(C) = D \cap A$ where D is closed in X . Now, D and A are both closed in X, and $f^{-1}(C) = D \cap A$; therefore , $f^{-1}(C)$ is closed in X. Similarly, $g^{-1}(C)$ is closed in X. Thus, $h^{-1}(C)$ is the union of two closed sets in X and therefore is closed in X as well. It follows that h is continuous. $\square$

4.12 Definition of a Homeomorphism

Let X and Y be topological spaces, and let $f : X \to Y$ be a bijection with inverse $f^{-1} : Y \to X$. If both f and f^{-1} are continuous functions, then f is said to be a **homeomorphism**. If there exists a homeomorphism between X and Y, we say that X and Y are **homeomorphic** or **topologically equivalent**, and we denote this by $X \cong Y$.

4.13 Facts about Homeomorphisms

(i) The function $id : X \to X,$ defined by $id(x) = x,$ is a homeomorphism.
(ii) If $f : X \to Y$ is a homeomorphism, then so is $f^{-1} : Y \to X$.
(iii) If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then so is $g \circ f : X \to Z$

4.15 Definition of Embedding

An **embedding** of X in Y is a function $f : X \to Y$ that maps X homeomorphically to the subspace $f(X)$ in Y.

4.16 Definition of Arc & Simple Closed Curve

Let X be a topological space. If $f : [-1, 1] \to X$ is an embedding, then the image of f is called an **arc** in X , and if $f : S^1 \to X$ is an embedding, then the image of f is called a **simple closed curve** in X .

4.17 Theorem about Hausdorffness being a topological property

If $f : X \to Y$ is a homeomorphism and X is Hausdorff, then Y is Hausdorff.

Proof: Suppose that X is Hausdorff and $f : X \to Y$ is a homeomorphism. Let x and y be distinct points in Y. Then $f^{-1}(x)$ and $f^{-1}(y)$ are distinct points in X. Thus, there exist disjoint open sets U and V containing $f^{-1}(x)$ and $f^{-1}(y)$, respectively. It follows that $f(U)$ and $f(V)$ are disjoint open sets containing x and y , respectively. Therefore Y is Hausdorff. $\square$

Definition of Topological Property

A property of topological spaces that is preserved by homeomorphism is said to be a **topological property.**

5 Metric Spaces

5.1 Definition of Metric

A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ with the following properties:

(O) $d(x, y) = 0$ for some $x, y \in X$; if and only if $x = y$

(i) $d(x, y) \geq 0$ for all $x, y \in X$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

Definition of Metric Space

We call $d(x, y)$ the distance between x and y, and we call the pair (X, d) , consisting of the set X and the metric d , a **metric space**.

Definition of Standard Metric

Given points $p = (p_1, p_2)$ and $q = (q_1, q_2)$

$$d(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$$

We call d the **standard metric** on $\mathbb{R}^2$. This metric measures the straight-line distance between points in the plane.