5.01 Show that the taxicab metric on $\mathbb{R}^2$ satisfies the properties of a metric.

(1) Notice, by the definition of the Taxicab metric we take the addition of two absolute values. Since absolute values are never negative, we must have that for some $x, y \in \mathbb{R}^2, d(x,y) \geq 0$. Note, if $x = y$, we must have that $d(x,y) = 0$ and if $x \neq y, d(x,y) > 0$
Thus, property 1 is satisfied.

(2) Let $x, y \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x,y) &= |x_1 - y_1| + |x_2 - y_2| \\ &= |y_1 - x_1| + |y_2 - x_2| \\ &= d(y,x) \end{aligned}$$

Thus, property 2 is satisfied.

(3) Let $x, y, z \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x,z) &= |x_1 - z_1| + |x_2 - z_2| \\ &= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2| \\ &\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| \\ &= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &= d(x,y) + d(y,z) \end{aligned}$$

Thus, property 3 is satisfied.

Therefore, the taxicab metric is a metric.

5.02 (a) Show that the max metric on $\mathbb{R}^2$ satisfies the properties of a metric.

(1) Notice, we are taking the max value of an absolute value. Since absolute values are never negative, we must have that for some $x, y \in \mathbb{R}^2, d(x,y) \geq 0$. Note, if $x = y$, we must have that $d(x,y) = 0$ and if $x \neq y, d(x,y) > 0$
Thus, property 1 is satisfied.

(2) Let $x, y \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x,y) &= max\{|x_1 - y_1|, |x_2 - y_2|\} \\ &= max\{|y_1 - x_1|, |y_2 - x_2|\} \\ &= d(y,x) \end{aligned}$$

Thus, property 2 is satisfied.

(3) Let $x, y, z \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(x,z) &= max\{|x_1 - z_1|, |x_2 - z_2|\} \\ &= max\{|x_1 - y_1 + y_1 - z_1|, |x_2 - y_2 + y_2 - z_2|\} \\ &\leq \max\{|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|\} \\ &= |x_i - y_i| + |y_i - z_i| \\ &\text{where } i \text{ with value 1 or 2 holds the maximum value} \\ |x_i - y_i| &\leq max\{|x_1 - y_1|, |x_2 - y_2|\} \\ |y_i - z_i| &\leq max\{|y_1 - z_1|, |y_2 - z_2|\} \end{aligned}$$

So,

$$d(x,z) \leq max\{|x_1 - y_1|, |x_2 - y_2|\} + max\{|y_1 - z_1|, |y_2 - z_2|\} = d(x,y) + d(y,z)$$

Thus, property 3 is satisfied.

Therefore, the max metric is a metric.

(b) Explain why $d(p,q) = \min\{|p_1 - q_1|, |p_2 - q_2|\}$ does not define a metric on $\mathbb{R}^2$.
The Triangle inequality does not hold.
(1,0)(2,0) Let $p, q, r \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(p,r) &= min\{|p_1 - r_1|, |p_2 - r_2|\} \\ &\geq min\{|p_1 - q_1| + |q_1 - r_1|, |p_2 - q_2| + |q_2 - r_2|\} \\ &= |p_i - q_i| + |q_i - r_i| \\ &\text{where } i \text{ with value 1 or 2 holds the minimum value} \\ |p_i - q_i| &\leq min\{|p_1 - q_1|, |p_2 - q_2|\} \\ |q_i - r_i| &\leq min\{|q_1 - r_1|, |q_2 - r_2|\} \end{aligned}$$

So,

$$d(p,r) \leq min\{|p_1 - q_1|, |p_2 - q_2|\} + min\{|q_1 - r_1|, |q_2 - r_2|\} \leq d(p,q) + d(q,r)$$

5.03 For points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $\mathbb{R}^2$ define

$$d_V(p,q) = \begin{cases} 1 & \text{if } p_1 \neq q_1 \text{ or } |p_2 - q_2| \geq 1 \\ |p_2 - q_2| & \text{if } p_1 = q_1 \text{ and } |p_2 - q_2| < 1 \end{cases}$$

(a) Show that d_V is a metric.

(1) Notice, by the definition of D_v we are either 1 or the absolute value less than 1. Since, absolute values are never negative, we must have that for some $p, q \in \mathbb{R}^2 d(p,q) \geq 0$. Note if $x = y$, we must have that $d(p,q) = 0$ and if $x \neq y, d(p,q) > 0$. Thus, property 1 is satisfied.

(2) Let $p, q \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(p,q) &= 1 \text{ or } |p_2 - q_2| \\ &= 1 \text{ or } |q_2 - p_2| \\ &= d(q,p) \end{aligned}$$

Thus, property 2 is satisfied.

(3) Let $p, q, r \in \mathbb{R}^2$. Observe.

$$\begin{aligned} d(p,r) &= 1 \text{ or } |p_2 - r_2| \\ &= 1 \text{ or } |p_2 - q_2 + q_2 - r_2| \\ * &= \end{aligned}$$

Thus, property 3 is satisfied.

Therefore, D_v is a metric.

(b) Describe the open balls in the metric d_V . If $\epsilon > 1$, then $d_V(p, \epsilon) = \mathbb{R}^2$ since $d_V(p,q) \leq 1$
If $\epsilon < 1$, then $d_v(p, \epsilon) = |p_2 - q_2|$

5.10 (a) Let (X, d) be a metric on a space. For $x, y \in X$, define

$$D(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

Show that D is also a metric on X

(1) Notice, that $d(x,y) \geq 0$. Since, we always get a non-negative value back from $d(x,y)$ we know that our definition for $D(x,y)$ must also return a non-negative values Thus, property 1 is satisfied.

(2) Let $x, y \in \mathbb{R}^2$. Observe.

$$\begin{aligned} D(x,y)+ &= \frac{d(x,y)}{1 + d(x,y)} \\ &= \frac{d(y,x)}{1 + d(y,x)} \\ &= D(y,x) \end{aligned}$$

Thus, property 2 is satisfied.

(3) Let $x, y, z \in \mathbb{R}^2$. Observe.

$$\begin{aligned}
D(x,y)+D(y,z) &= \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \\
&= \frac{d(x,y)(1+d(y,z))}{(1+d(x,y))(1+d(y,z))} + \frac{d(y,z)(1+d(x,y))}{(1+d(x,y))(1+d(y,z))} \\
&\geq \frac{d(x,y)+d(y,z)}{(1+d(x,y))(1+d(y,z))} \\
&\geq \frac{d(x,z)}{(1+d(x,y))(1+d(y,z))} \\
&\geq \frac{d(x,z)}{1+d(x,z)} \\
&\text{As } d(x,z)=d(x,z) \text{ and } (1+d(x,y))(1+d(y,z)) \geq 1+d(x,z)
\end{aligned}$$

Thus, property 3 is satisfied.

Therefore, D is a metric.

(b) Explain why no two points in X are distance one or more apart in the metric D. The numerator is always smaller than the denominator, so the distance will always be less than 1 apart.

5.24 Prove Theorem 5.13 : Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \rightarrow Y$ is continuous in the open set definition if and only if for each $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x' \in X$ and $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$. (Hint: Consider Exercise 4.3 and the proof of Theorem 4.6.)
(WTS: $\forall x \in X \exists \delta > 0$ such that $x' \in X$ and $d_x(x, x') < \delta$, we have $d_Y(f(x), f(x')) < \epsilon$)
Suppose f is continuous. Let $x \in X, \epsilon > 0$, and $\delta > 0$ such that $x' \in X$ and $d_x(x, x') < \delta$. Notice, as f is continuous, $f(x), f(x') \in Y$. Since, x is bound by ϵ, $d(x, x')$ is bounded by δ, and $f(x), f(x') \in Y$, we must have that $d_Y(f(x), f(x') < \epsilon$.

Let U be an open set in Y. Let $x \in f^{-1}(U)$ and define $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq U$. Define δ such that$x' \in X$ satisfies $d(x, x') < \delta$. Which implies $x' \in B(x, \delta)$. Notice, we must have $d(f(x), f(x')) < \epsilon$. From this result, we have $f(x') \in B(f(x), \epsilon) \subseteq U$. So, $x' \in B(x, \delta)$ as $x' \in f^{-1}(U)$.
Thus, $B(x, \delta) \subseteq f^{-1}(U)$.
Thus, f is continuous.

Therefore, A function $f : X \rightarrow Y$ is continuous in the open set definition if and only if for each $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x' \in X$ and $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$.

5.25 Let (X, d) be a metric space, and assume $p \in X$ and $A \subset X$

(a) Provide an example showing that $d(\{p\}, A) = 0$ need not imply that $p \in A$.
Suppose p is a limit point of A. Then, $d(\{p\}, A)$ would equal 0, but $p \notin A$.

(b) Prove that if A is closed and $d(\{p\}, A) = 0$, then $p \in A$
Notice, if $d(\{p\}, A) = 0, p$ must be in A or a limit point of A.
Suppose p is a limit point of A. That is $p \in A'$ We know that as A is closed, we must have that A contains all of its limit points. That is $A' \subset A$.
Thus, as $p \in A'$ we must have $p \in A$.
Therefore, if A is closed and $d(\{p\}, A) = 0$, then $p \in A$

5.26 Use Theorem 5.15 to prove that the taxicab metric and the max metric induce the same topology on $\mathbb{R}^2$.
Without loss of generality, assume we are centered at the origin. Let $\epsilon > 0$ and $B_T(0, \epsilon) = \{q \in \mathbb{R}^2 | d_T(0, q) < \epsilon\} = \{q \in \mathbb{R}^2 || q_1| + |q_2| < \epsilon\}$. Where B_T is the open ball in our taxicab topology, T_T
Define $B_M(0, \delta) = \{q \in \mathbb{R}^2 | d_m(0, q) < \delta\} = \{q \in \mathbb{R}^2 | max\{|q_1|, |q_2| < \delta\}\}$. Where B_M is the open ball in our max topology, T_M

Assume, $\epsilon > \delta$. Notice, the boundary of B_M contained inside B_T is simply $a + b = \epsilon$. Since, $a = b$ we then have $2a = \epsilon \Rightarrow a = \epsilon/2$. So, we can define $\delta = \epsilon/2$
Thus, $B_M(0, \delta) \subset B_T(0, \epsilon)$
Thus, T_M is finer than T_T

Going the other way, assume $\epsilon < \delta$ (WTS: $B_T \subset B_M$)
Notice, the boundary of B_T contained inside B_M is limited by $a = \delta$. So, $\epsilon = \delta$.
Thus, $B_T(0, \epsilon) \subset B_M(0, \delta)$
Thus, T_T is finer than T_M

Therefore, the taxicab metric and max metric induce the same topology on $\mathbb{R}^2$

5.28 Let (X, d) be a metric space. The function

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a bounded metric on X . (See Exercise 5.10.) Show that the topologies induced by D and d are the same.
(WTS: The topologies are finer than each other.)
Without loss of generality, assume we are centered at the origin Let $\epsilon > 0$ and $B_D(0, \epsilon) = \{q \in \mathbb{R}^2 | D(0, q) < \epsilon\} = \{q \in \mathbb{R}^2 | \frac{d(0,q)}{1+d(0,q)} < \epsilon\}$. Where B_D is the open ball in our $D(x, y)$ topology, T_D
Define $B_d(0, \delta) = \{q \in \mathbb{R}^2 | d(0, q) < \delta\} = \{q \in \mathbb{R}^2 | d(0, q) < \delta\}\}$. Where B_d is the open ball in our $d(x, y)$ topology, T_d

Assume, $\epsilon > \delta$. Notice, the boundary of B_D contained inside B_d is simply $\delta = \frac{\epsilon}{1+\epsilon}$.
Thus, $B_d(0, \delta) \subset B_D(0, \epsilon)$
Thus, T_d is finer than T_D

Going the other way, assume $\epsilon < \delta$ (WTS: $B_D \subset B_d$)
Notice, the boundary of B_D contained inside B_d is limited by δ. So, $\epsilon = \delta$.
Thus, $B_D(0, \epsilon) \subset B_d(0, \delta)$
Thus, T_D is finer than T_d

Therefore, the topologies induced by D and d are the same.

5.29 ρ_M and ρ defined by

$$\rho_M(f, g) = \max_{x \in [a,b]} [|f(x) - g(x)\|\},$$

and

$$\rho(f, g) = \int_a^b |f(x) - g(x)| dx$$

These metrics were introduced in Exercise 5.8 and Example 5.5, respectively.

(a) Use Theorem 5.15 to prove that the topology induced by ρ_M on $C[a, b]$ is finer than the topology induced by ρ .

Without loss of generality, assume we are centered at the origin. Let $\epsilon > 0$ and $B_\rho(0, \epsilon) = \{f, g \in C[a, b] | \rho(f, g) < \epsilon\} = \{f, g \in C[a, b] | \int_a^b |f(x) - g(x)| dx < \epsilon\}$. Where B_ρ is the open ball in our ρ topology, T_ρ
Define $B_{\rho_M}(0, \delta) = \{f, g \in C[a, b] | \rho_M(f, g) < \delta\} = \{f, g \in C[a, b] | max_{x \in [a,b]}\{|f(x) - g(x)| < \delta\}\}$. Where B_{ρ_M} is the open ball in our ρ_M topology, T_{ρ_M}

Notice, the boundary of B_{ρ_M} contained inside B_ρ is simply $M(b - a)$. Where $M(b-a)$ is the maximum value for $|f-g|$ over $[a, b]$. So, we can define $\delta = M(b-a)$
Thus, $B_{\rho_M}(0, \delta) \subset B_\rho(0, \epsilon)$
Therefore, T_{ρ_M} is finer than T_ρ

Summary

It's going I guess. I've been more stuck lately. But now that I am making more time to go to office hours often and discussing with a few other classmates, I feel like I have a better understanding of some of the material. I'm noticing that I will occasionally have some intuition on where to go next in a problem. So, when I've been getting stuck, I've been getting stuck big as in I have no idea how to continue, but now I have some where I'm not super stuck and only have to struggle through them. I plan on continuing to work on my homeworks earlier and come to office hours more regularly.