2.14 For each $n \in \mathbb{Z}_+$, let $B_n = \{n, n+1, n+2, \ldots\}$, and consider the collection $\mathcal{B} = \{B_n | n \in \mathbb{Z}_+\}$

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}_+$
Let $x \in \mathbb{Z}_+$. Notice, $x \in B_x := \{x, x+1, x+2, \cdots\}$
Thus, every point in $\mathbb{Z}_+$ is contained in a basis element. Let $a, b \in \mathbb{Z}, B_a := \{a, a+1, a+2, \cdots\}$ and $B_b := \{b, b+1, b+2, \cdots\}$ with $y \in B_a \cup B_b$. Suppose $m = max(a, b)$. Then, $y \in B_m \subset B_a \cup B_b$
Thus, every point in the intersection of two basis elements is contained in a basis element contained in that intersection.
Therefore, $\mathcal{B}$ is a basis on $\mathbb{Z}_+$

(b) Show that the topology on X generated by $\mathcal{B}$ is not Hausdorff.
Let X be a set with $\mathcal{B}$ as a basis for X and let $x, y \in X$. Without loss of generality, assume $x < y$ with basis elements of the form $B_x := \{x, \cdots, y, y+1, y+2, \cdots\}$ and $B_y := \{y, y+1, y+2, \cdots\}$ Notice, $B_x \cap B_y = B_y$.
Thus, the basis are not disjoint.
Therefore, the topology generated by $\mathcal{B}$ is not Hausdorff.

(c) Show that the sequence $(2, 4, 6, 8, \ldots)$ converges to every point in $\mathbb{Z}_+$ with the topology generated by $\mathcal{B}$ Let $j \in \mathbb{Z}_+$. Suppose U is a neighborhood of j. Suppose $k = 2j$. Then for all elements of $(2, 4, 6, c \ldots) \geq 2j$, are in U.
Therefore, the sequence $(2, 4, 6, \cdots)$ converges to every point in $\mathbb{Z}_+$ with the topology generate by $\mathcal{B}$.

(d) Prove that every injective sequence converges to every point in $\mathbb{Z}_+$ with the topology generated by $\mathcal{B}$
Let s be an injective sequence and $z \in \mathbb{Z}_+$. Notice, that $s = B_z$. Thus, the every injective sequence converges to every point in $\mathbb{Z}_+$.
[To be honest, I have no idea what this is asking of me.]

2.15 Determine the set of limit points of $[0, 1]$ in the finite complement topology on $\mathbb{R}$
Notice, $[0, 1]$ is an infinite subset of $\mathbb{R}$. Let $x \in \mathbb{R}$ and U be a neighborhood of x. Then $[0, 1] \cap U \neq \varnothing$ and is infinite. Thus, the limit points of $[0, 1]$ is every point.

2.17 (a) Let $\mathcal{B} = \{[a, b) \subset \mathbb{R} | a, b \in \mathbb{Q} \text{ and } a < b\}$. Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. The resulting topology is called the rational lower limit topology and is denoted $\mathbb{R}_{rl}$
Let $x \in \mathbb{R}$. Suppose $B \in \mathcal{B}$ such that $x \in B := [x - \epsilon, x + \epsilon)$ for some $\epsilon > 0$.
Thus, every point in $\mathbb{Z}_+$ is contained in a basis element.

Let $B_1 = [a, b)$ and $B_2 = [c, d)$ such that $x \in B_1 \cap B_2$. Let $x = max(a, c)$ and $y = min(b, d)$. Notice, $x \in B = [x, y) \subset B_1 \cap B_2$.
Thus, every point in the intersection of two basis elements is contained in a basis element contained in that intersection.

Therefore, $\mathcal{B}$ is a basis

(b) Determine the closures of $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in $\mathbb{R}_l$ and in $\mathbb{R}_{rl}$
<u>Lower Limit:</u>

$$\begin{aligned} Cl(A) &= [0, \sqrt{2}) \\ Cl(B) &= [\sqrt{2}, 3) \end{aligned}$$

<u>Rational Lower Limit:</u>

$$\begin{aligned} Cl(A) &= [0, \sqrt{2}] \\ Cl(B) &= [\sqrt{2}, 3) \end{aligned}$$

2.21 Determine the set of limit points of the set

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \in \mathbb{R}^2 | 0 < x \leq 1 \right\}$$

as a subset of $\mathbb{R}^2$ in the standard topology. (The closure of S in the plane is known as the topologist's sine curve.
Let $y \in [-1.1]$ and $p = (0, y)$. Notice, for every neighborhood $U - \{p\}$ contain points in S . Thus, every point in S is a limit point.

2.27 Determine $\partial([0, 1])$ in $\mathbb{R}$ with the finite complement topology. Justify your result.
Let $A = [0, 1]$. We then have $Cl(A) = \mathbb{R}$ and $Int(A) = \varnothing$. Hence, $\partial A = Cl(A) - Int(A) = \mathbb{R}$.
Therefore, $\partial([0, 1])$ in $\mathbb{R}$ with the finite complement topology is $\mathbb{R}$

2.28 Prove Theorem 2.15 : Let A be a subset of a topological space X.

(a) ∂A is closed.
Observe,

$$\begin{aligned} \partial A &= Cl(A) - Int(A) \\ &= Cl(A) \cap (X - Int(A)) \end{aligned}$$

Notice, $Cl(A)$ is closed and the complement of $Int(A)$ is closed.
Thus, as intersections of closed sets are closed, we have ∂A is closed.

(b) $\partial A = \mathrm{Cl}(A) \cap \mathrm{Cl}(X - A)$ Observe,

$$\begin{aligned} \partial A &= Cl(A) - Int(A) \\ &= Cl(A) \cap (X - Int(A)) \\ &= Cl(A) \cap Cl(X - A) \end{aligned}$$

Thus, $\partial A = \mathrm{Cl}(A) \cap \mathrm{Cl}(X - A)$

(c) $\partial A \cap \operatorname{In} t(A) = \varnothing$
As $\partial A = Cl(A) - Int(A)$, we have already removed all elements of $Int(A)$.
Therefore, $\partial A \cap \operatorname{In} t(A) = \varnothing$

(d) $\partial A \cup \mathrm{Int}(A) = \mathrm{Cl}(A)$
Notice,

$$\begin{aligned} \partial A \cup Int(A) &= (Cl(A) - Int(A))U Int(A) \\ &= Cl(A) \end{aligned}$$

Therefore, $\partial A \cup \mathrm{Int}(A) = \mathrm{Cl}(A)$

(e) $\partial A \subset A$ if and only if A is closed.
Let $\partial A \subset A$. Then, A must be closed as $\partial A = Cl(A) - Int(A)$

Let A be closed. Then, we have that $Cl(A)$ is closed. Thus, $\partial A = Cl(A) - Int(A)$. Hence, $\partial A \subset A$.

Therefore, $\partial A \subset A$ if and only if A is closed.

(f) $\partial A \cap A = \varnothing$ if and only if A is open.
Let $\partial A \cap A = \varnothing$. By way of contradiction, assume A is not open. Then, there exists a $x \in A$ such that no open set containing x is a subset of A. This is a contradiction as $Int(A)$ is open and $Int(A) \subset A$.
Thus, A must be open.
Let A be open. Then,

$$\begin{aligned} \partial A \cap A &= (Cl(A) - Int(A)) \cap A \\ &= (Cl(A) \cap A^{\complement}) \cap A \\ &= Cl(A) \cap (A^{\complement} \cap A) \\ &= Cl(A) \cap \varnothing \\ &= \varnothing \end{aligned}$$

(g) $\partial A = \varnothing$ if and only if A is both open and closed.
Let $\partial A = \varnothing$. Notice, $Int(A) \subset A \subset Cl(A)$. From this we have $Int(A) = A = Cl(A)$. Which shows that A is both opened and closed by each part of the equality, respectively.

Let A be opened and closed. Then, we have $A = Int(A)$ and $A = Cl(A)$. Notice, $Int(A) = Int(A) \cup \partial A \Rightarrow Int(A) = A \cup \partial A$
Notice, $A = A \cup \partial A$ and $Int(A) \cap \partial A = \varnothing$. So, $Int(A) = \partial A = \varnothing$.
Thus, $A = A \cup \partial A$ and $A \cap \partial A = \varnothing$.
Therefore, $\partial A = \varnothing$