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0.1 DeMorgan's Laws

1. $A - (B \cup C) = (A - B) \cap (A - C)$
2. $A - (B \cap C) = (A - B) \cup (A - C)$

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1.1 Definition of a Topology

Let X be a set. A **topology** $\mathcal{T}$ on X is a collection of subsets of X, each called an **open set** such that:

(i) $\varnothing$ and X are open sets;
(ii) The intersection of finitely many open sets is an open set;
(iii) The union of any collection of open sets is an open set.

The set X together with a topology $\mathcal{T}$ on X is called a topological space.

1.2 Trivial Topology

Define $\mathcal{T} = \{\varnothing, X\}$. Notice, $\mathcal{T}$ satisfies all three conditions for being a topology. For obvious reasons, it is called the **Trivial Topology** define on X.

1.3 Discrete Topology

Let X be a nonempty set and let $\mathcal{T}$ be the collection of all subsets of X. This is called the **discrete topology** on X. This is the largest topology that we can define on X.

1.4 Finite Complement Topology

On the real line, $\mathbb{R}$, define a topology whose open sets are the empty set and every set in $\mathbb{R}$ with a finite complement. We call this topology the **finite complement topology** on $\mathbb{R}$ and denote it by $\mathbb{R}_{fc}$

1.5 Definition of a Neighborhood

Let X be a topological space and $x \in X$. An open set U containing x is said to be a **neighborhood** of x.

1.6 Theorem for using Neighborhood to determine if sets are open

Let X be a topological space and let A be a subset of X. Then A is open in X if and only if for each $x \in A$, there is a neighborhood U of x such that $x \in U \subset A$

1.7 Definition of Basis for a Topology

Let X be a set and $\mathcal{B}$ be a collection of subsets of X. We say $\mathcal{B}$ is a **basis (for a topology on X)** if the following statements hold:

(i) For each x in X, there is aB in $\mathcal{B}$ such that $x \in B$
(ii) If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists B_3 in $\mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

We call the sets in B **basis elements**.

1.8 Definition of a Topology Generated by a Basis

Let $\mathcal{B}$ be a basis on a set X. The **topology $\mathcal{T}$generated by $\mathcal{B}$** is obtained by defining the open sets to be the empty set and every set that is equal to a union of basis elements.

1.9 Standard Topology

On the real line $\mathbb{R}$, let $\mathcal{B} = \{(a, b) \subset \mathbb{R} | a < b\}$ The topology generated by $\mathcal{B}$ is called the **standard topology** on $\mathbb{R}$

Open sets in the standard topology on $\mathbb{R}$ are unions of open intervals.

1.10 Theorem: Bases Generate Topologies

The topology $\mathcal{T}$ generated by a basis $\mathcal{B}$ is a topology.

1.11 Lower/Upper Limit Topology

On $\mathbb{R}$, let $B = \{[a, b) \subset \mathbb{R} | a < b\}$. The collection $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$ We call the topology generated by this basis the **lower limit topology** since each basis element contains its lower limit. We denote $\mathbb{R}$ with this topology by $\mathbb{R}_l$.

We can similarly define the **upper limit topology** on $\mathbb{R}$ via the basis $\mathcal{B} = \{(a, b] \subset \mathbb{R} | a < b\}$

1.12 Digital Line Topology

For each $n \in \mathbb{Z}$, define

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even} \end{cases}$$

The collection $\mathcal{B} = \{B(n) | n \in \mathbb{Z}\}$ is a basis for a topology on $\mathbb{Z}$. The resulting topology is called the **digital line topology**

1.13 Theorem: A set is open if and only if every element is contained in a basis element

Let X be a set and $\mathcal{B}$ be a basis for a topology on X. Then U is open in the topology generated by $\mathcal{B}$ if and only if for each $x \in U$ there exists a basis element $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$.

1.14 Open Balls and Standard Topology on $\mathbb{R}^2$

For each x in $\mathbb{R}^2$ and $\varepsilon > 0$, define

$$B(x, \varepsilon) = \left\{p \in \mathbb{R}^2 | d(x, p) < \varepsilon\right\}$$

The set $B(x, \varepsilon)$ is called the **open ball of radius ε centered at x** . Let

$$\mathcal{B} = \{B(x, \varepsilon) | x \in \mathbb{R}^2, \varepsilon > 0\}$$

So $\mathcal{B}$ is the collection of all open balls associated with the Euclidean distance d. We call the topology generated by $\mathcal{B}$ the **standard topology on $\mathbb{R}^2$** .

1.15 Theorem: The collection $\mathcal{B} = \{B(x, \varepsilon) | x \in \mathbb{R}^2, \varepsilon > 0\}$ is a basis for a topology on $\mathbb{R}^2$

1.16 Theorem: If all sets have an open set with element, we have a basis

Let X be a set with topology $\mathcal{T}$, and let $\mathcal{C}$be a collection of open sets in X. If, for each open set U in X and for each $x \in U$, there is an open set V in $\mathcal{C}$ such that $x \in V \subset U$, then $\mathcal{C}$ is a basis that generates the topology T.

1.17 Definition: Closed Sets

A subset A of a topological space X is **closed** if the set $X - A$ is open. (i.e. if a set is open then it's complement is closed)

1.18 Closed Ball

For each x in $\mathbb{R}^2$ and $\varepsilon > 0$, define the **closed ball of radius** ε **centered** at x to be the set

$$\overline{B}(x, \varepsilon) = \{y \in \mathbb{R}^2 | d(x, y) \leq \varepsilon\}$$

where $d(x, y)$ is the Euclidean distance between x and y

If $[a, b]$ and $[c, d]$ are closed bounded intervals in $\mathbb{R}$, then the product $[a, b] \times [c, d] \subset \mathbb{R}^2$ is called a **closed rectangle**.

1.19 Theorem: Closed balls and closed rectangles are closed sets in the standard topology on $\mathbb{R}^2$

1.20 Warning: A set can be open, closed, both, or neither. If a set is not opened $\not\Rightarrow$ a set is closed

1.21 Theorem: Closed Sets have the same properties of open sets

Let X be a topological space. The following statements about the collection of closed sets in X hold:

(i) $\varnothing$ and X are closed.
(ii) The intersection of any collection of closed sets is a closed set.
(iii) The union of finitely many closed sets is a closed set.

1.22 Definition: Hausdorff

A topological space X is **Hausdorff** if for every pair of distinct points$x$ and y in X, there exist disjoint neighborhoods U and V of x and y, respectively.

Points are "housed off" from other points by disjoint neighborhoods.

1.23 Theorem: If X is a Hausdorff space, then every single-point subset of X is closed.

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2.1 Interior

Let A be a subset of a topological space X. The **interior of** A, denoted $\dot{A}$ or Int(A), is the union of all open sets contained in A.

The interior of A is open and a subset of A.

2.2 Closure

Let A be a subset of a topological space X. The **closure** of A, denoted A or $Cl(A)$, is the intersection of all closed sets containing A.

The closure of A is closed and contains A

2.3 Theorem: Facts on Previous Definitions

Let X be a topological space and A and B be subsets of X.
Notice, $Int(A) \subset A \subset Cl(A)$

(i) If U is an open set in X and $U \subset A$, then $U \subset \ln t(A)$
(ii) If C is a closed set in X and $A \subset C$, then $Cl(A) \subset C$
(iii) If $A \subset B$ then Int $(A) \subset Int(B)$
(iv) If $A \subset B$ then $Cl(A) \subset Cl(B)$
(v) A is open if and only if $A = \text{Int}(A)$
(vi) A is closed if and only if $A = Cl(A)$

2.4 Dense

A subset B of a topological space X is called dense if $Cl(B) = X$.

2.5 Theorem: An element is in the Interior, if the element is contained in an open set

Let X be a topological space, A be a subset of X, and y be an element of X. Then $y \in \text{Int}(A)$ if and only if there exists an open set U such that $y \in U \subset$ A.

2.6 Theorem: An element is in the Closure, if the element is contained in the intersection of every open set

Let X be a topological space, A be a subset of X, and y be an element of X. Then $y \in Cl(A)$ if and only if every open set containing y intersects A.

2.7 Theorem: Facts on Int and Cl with sets

For sets A and B in a topological space X, the following statements hold:
(i) $\ln t(X - A) = X - Cl(A)$

(ii) $\mathrm{Cl}(X-A)=X-\mathrm{Int}(A)$
(iii) $\ln t(A)\cup \mathrm{Int}(B)\subset \mathrm{Int}(A\cup B)$, and in general equality does not hold.
(iv) $\mathrm{lnt}(A)\cap Int(B)=\mathrm{Int}(A\cap B)$

2.8 Definition: Limit Point

Let A be a subset of a topological space X. A point x in X is a **limit point of** A if every neighborhood of x intersects A in a point other than x.

2.9 Theorem: Limit points provide an easy way to find the closure of a set

Let A be a subset of a topological space X, and let $A' be$ the set of limit points of A. Then $Cl(A)=A\cup A'$.

> Corollary: A subset A of a topological space is closed if and only if it contains all of its limit points.

2.10 Definition: Converge

In a topological space X, a sequence $(x_1, x_2, \ldots)$ **converges to** $x\in X$ if for every neighborhood U of x, there is a positive integer N such that $x_n\in U$ for all $n\geq N$. We say that x is the **limit** of the sequence $(x_1, x_2, \ldots)$, and we write

$$\lim_{n\to\infty} x_n = x$$

The idea behind a sequence converging to a point x is that, given any neighborhood U of x, the sequence eventually enters and stays in U .

2.11 Theorem: If x is a limit point, there is a sequence that converges to x

Let A be a subset of $\mathbb{R}^n$ in the standard topology. If x is a limit point of A , then there is a sequence of points in A that converges to x.

2.12 Theorem: If X is a Hausdorff space, then every convergent sequence of points in X converges to a unique point in X.

2.13 Definition: Boundary

Let A be a subset of a topological space X. The **boundary** of A, denoted ∂A, is the set $\partial A = Cl(A)-\ln t(A)$

2.14 Theorem: An element is in the boundary if and only if all neighborhoods of that element are in the subset and the complement of our topology space

Let Abe a subset of a topological space X and let x be a point in X. Then $x \in \partial A$ if and only if every neighborhood of x intersects both A and$X - A$.

2.15 Theorem: Facts on boundaries

Let A be a subset of a topological space X. Then the following statements about the boundary of A hold:

(i) ∂A is closed.
(ii) $\partial A = Cl(A) \cap Cl(X - A)$
(iii) $\partial A \cap Int(A) = \varnothing$
(iv) $\partial A \cup Int(A) = Cl(A)$
(v) $\partial A \subset A$ if and only if A is closed.
(vi) $\partial A \cap A = \varnothing$ if and only if A is open.
(vii) $\partial A = \varnothing$ if and only if A is both open and closed.

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3.1 Definition: Subspace Topology

Let X be a topological space and let Y be a subset of X. Define $T_Y = \{U \cap Y | U \text{ is open in } X\}$. This is called the **subspace topology** on Y and, with topology, Y is called a **subspace** of X. We say that $V \subset Y$ is **open in** Y if V is an open set in the subspace topology on Y.

Thus, a set is open in the subspace topology on Y if it is the intersection of an open set in X with Y.

3.2 Definition: Standard Topology on $Y \subset \mathbb{R}^n$

Let Y be a subset of $\mathbb{R}^n$. The **standard topology** on Y is the topology that Y inherits as a subspace of $\mathbb{R}^n$ with the standard topology.

3.3 Definition: Closed In $Y \subset X$

Let X be a topological space, and let $Y \subset X$ have the subspace topology. We say that a set $C \subset Y$ is **closed in** Y if C is closed in the subspace topology on Y.

3.4 Theorem: Closed sets are in a subspace Y if there exists a closed set D in the superspace

Let X be a topological space, and let $Y \subset X$ have the subspace topology. Then $C \subset Y$ is closed in Y if and only if $C = D \cap Y$ for some closed set D in X .

3.5 Theorem: Bases for Subspaces

Let X be a topological space and $\mathcal{B}$ be a basis for the topology on X. If $Y \subset X$, then the collection

$$\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y

3.6 Definition: Product Topology

Let X and Y be topological spaces and $X \times Y$ be their product. The product topology on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V | U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

3.7 Theorem: The collection $\mathcal{B}$ is a basis for a topology on $X \times Y$.

3.8 Theorem: The cross product of bases, result in a basis that generates a product topology

If $\mathcal{C}$ is a basis for X and $\mathcal{D}$ is a basis for Y, then

$$\mathcal{E} = \{C \times D | C \in \mathcal{C} \text{ and } D \in \mathcal{D}\}$$

is a basis that generates the product topology on $X \times Y$.

3.9 Theorem: Subspace of a product topology

Let X and Y be topological spaces, and assume that $A \subset X$ and $B \subset Y$. Then the topology on $A \times B$ as a subspace of the product $X \times Y$ is the same as the product topology on $A \times B$, where A has the subspace topology inherited from X , and B has the subspace topology inherited from Y.

3.10 Theorem: Cross product of interior is the same as the interior of the cross product

Let A and B be subsets of topological spaces X and Y , respectively. Then Int $(A \times B) = \text{Int}(A) \times \text{Int}(B)$.

3.11 Definition: Quotient Topology

Let X be a topological space and A be a set that is not necessarily a subset of X). Let $p : X \rightarrow$ A be a surjective map. Define a subset U of A to be open in A if and only if $p^{-1}(U)$ is open in X. The resultant collection of open sets in A is called the **quotient topology induced by** p , and the function p is called the **quotient map.** The topological space A is called a **quotient space**.

3.12 Theorem: Let $p : X \rightarrow A$ be a quotient map. The quotient topology on A is induced by p is a topology