7.01 Show that every set $A \subset R$ is a compact subset of $\mathbb{R}$ in the finite complement topology on $\mathbb{R}$.

7.02 Prove Theorem 7.6 : Let X be a topological space.

(a) If $C_1, \ldots, C_n$ are each compact in X, then $U_{i=1}^{n} C_j$ is compact in X

(b) If X is Hausdorff, and $\{C_\alpha\}_{\alpha \in A}$ is a collection of sets that are compact in X, then $\cap_{\alpha \in A} C_\alpha$ is compact in X .

7.03 Provide an example demonstrating that an arbitrary union of compact sets in a topological space X is not necessarily compact.

7.07 Recall that the arithmetic progression topology on Z is generated by the basis $\mathcal{B} = \{A_{a,b} | a, b \in \mathbb{Z}, b \neq 0\}$, where each

$$A_{a,b} = \{\ldots, a - 2b, a - b, a, a + b, a + 2b, \ldots\}$$

is an arithmetic progression. Determine whether or not $\mathbb{Z}$ is compact in this topology.

7.12 Show that the Tube Lemma does noes not necessarily hold if we drop the assumption that Y is compact. That is, provide an example of a noncompact space Y and an open set U in $X \times Y$ such that U contains a slice $\{x\} \times Y \subset X \times Y$ but does not contain an open tube $W \times Y$ containing the slice.

7.17 Use compactness to prove that the plane is not homeomorphic to the sphere. (Recall, in Section 6.2 we distinguished between a number of pairs of spaces, including the line and the plane and the line and the sphere, but we indicated that we were not yet in a position to distinguish between the plane and the sphere. With compactness, we can now make that distinction.)

7.18 In this exercise we demonstrate that if we drop the condition that X is Hausdorff in Theorem 7.6 , then the intersection of compact sets in X is not necessarily a compact set. Define the extra-point line as follows. Let $X = \mathbb{R} \cup (p_e)$, where p_e is an extra point, not contained in $\mathbb{R}$. Let $\mathcal{B}$ be the collection of subsets of X consisting of all intervals $(a, b) \subset \mathbb{R}$ and all sets of the form $(c, 0) \cup \{p_e\} \cup (0, d)$ for $c < 0$ and $d > 0$.

(a) Prove that $\mathcal{B}$ is a basis for a topology on X.

(b) Show that the resulting topology on X is not Hausdorff.

(c) Find two compact subsets of X whose intersection is not compact. Prove that the sets are compact and that the intersection is not.

7.19 (a) Let (X, d) be a metric space. Prove that if A is compact in X, then A is closed in X and bounded under the metric d.

(b) Provide an example demonstrating that a subset of a metric space can be closed and bounded but not compact.

Summary