7.01 Show that every set $A \subset \mathbb{R}$ is a compact subset of $\mathbb{R}$ in the finite complement topology on $\mathbb{R}$.
Let $A \subset \mathbb{R}$ and $\{U_\alpha\}$ be an open cover. Notice, that any set in the cover it's complement has finitely many elements, namely $x_1, \cdots x_n$ are not in this set. Then, $\{U_\{\alpha_i\}_i^n = 1$ is a finite subcover.
Therefore, A is a compact subset of $\mathbb{R}$ in the finite complement topology on $\mathbb{R}$.

7.02 Prove Theorem 7.6 : Let X be a topological space.

(a) If $C_1, \ldots, C_n$ are each compact in X, then $U_{j=1}^n C_j$ is compact in X
Let $\{C_1, \ldots, C_n\}$ be a collection of compact subspaces of X. We define $C = \cup_{j=1}^n C_j$. Suppose O is a cover for C. Then, notice each C_j is compact and so has a finite subcover O_j. We then will have $O' = \cup_{j=1}^n O_j$.
Thus, O' is an open cover for C.
Therefore, C is compact.

(b) If X is Hausdorff, and $\{C_\alpha\}_{\alpha \in A}$ is a collection of sets that are compact in X, then $\cap_{\alpha \in A} C_\alpha$ is compact in X .
Notice, that each C_j in the collection is closed since it's in a Hausdorff space. Thus, the finite intersection of the collection is also closed. Since every C_α lives inside A for some $\alpha \in A$, we also have that the collection is bounded. Since the collection is both closed and bounded, we must have the collection is compact.

7.03 Provide an example demonstrating that an arbitrary union of compact sets in a topological space X is not necessarily compact.
Let X be an infinite set with the discrete topology. Notice, the collection of singletons gives an open cover with no subcover. Thus, an arbitrary union of compact sets in a topological space X is not necessarily compact.

7.12 Show that the Tube Lemma does noes not necessarily hold if we drop the assumption that Y is compact. That is, provide an example of a noncompact space Y and an open set U in $X \times Y$ such that U contains a slice $\{x\} \times Y \subset X \times Y$ but does not contain an open tube $W \times Y$ containing the slice.
Consider the space $\mathbb{R}$ and open set defined as $U = \{(x, y) | |x \cdot y| < 1\}$. Notice, U contains $\{0\} \times \mathbb{R}$, but cannot contain the a tube. But for U to contain $\{0\}$, we would have $U = \{0\}$ which is not open.
Thus, we must have Y to compact for the Tube Lemma to hold true.

7.17 Use compactness to prove that the plane is not homeomorphic to the sphere. (Recall, in Section 6.2 we distinguished between a number of pairs of spaces, including the line and the plane and the line and the sphere, but we indicated that we were not yet in a position to distinguish between the plane and the sphere. With compactness, we can now make that distinction.)
Notice the sphere is compact. Suppose that there exists a continuous bijection from

the sphere to the plane. Thus, the plane would have to be compact since we've supposed there exists a continuous function mapping the sphere to the plane. This is a contradiction as the plane is not compact. Since, the sphere is compact and the plane is not, we cannot have a homeomorphism.

7.18 In this exercise we demonstrate that if we drop the condition that X is Hausdorff in Theorem 7.6 , then the intersection of compact sets in X is not necessarily a compact set. Define the extra-point line as follows. Let $X = \mathbb{R} \cup (p_e)$, where p_e is an extra point, not contained in $\mathbb{R}$. Let $\mathcal{B}$ be the collection of subsets of X consisting of all intervals $(a, b) \subset \mathbb{R}$ and all sets of the form $(c, 0) \cup \{p_e\} \cup (0, d)$ for $c < 0$ and $d > 0$.

(a) Prove that $\mathcal{B}$ is a basis for a topology on X.
Let $x \in X$ with $B = \{(x-\varepsilon, x+\varepsilon)|\varepsilon > 0\}$. Thus, every point, including the extra point, is contained in a basis element.

Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Notice, as B_1 and B_2 are open intervals their intersection would also be open. Thus, there exists a B_3 such that $x \in B_3 \subset B_1 \cap B_2$. Therefore, $\mathcal{B}$ is a basis for a topology X.

(b) Show that the resulting topology on X is not Hausdorff. Let $x, y \in X$ with $U = \{(x - \varepsilon, x + \varepsilon)|\varepsilon > 0\}$ and $V = \{(y - \delta, y + \delta)|\delta > 0\}$. Without loss of generality suppose $x < y$, then we can define $\varepsilon = \delta = \frac{y-x}{2}$. Thus, $U \cap V = \varnothing$. But, the extra point is not part of $\mathbb{R}$. That is $U \cap V \neq \varnothing$.
Therefore, X is not Hausdorff.

(c) Find two compact subsets of X whose intersection is not compact. Prove that the sets are compact and that the intersection is not.
Let $A = \{\{pe\} \cup 0\}$ and $B = \{\{pe\} \cup 0\}$. Notice, $[-1, 1] - \{0\} \cup A$ is compact and $[-1, 1] - \{0\} \cup B$ is compact. But $([-1, 1] - \{0\} \cup A) \cap ([-1, 1] - \{0\} \cup B)$ is missing the extra point and origin.
Therefore, their intersection is not compact.

7.19 (a) Let (X, d) be a metric space. Prove that if A is compact in X, then A is closed in X and bounded under the metric d.
Suppose A is compact in X. Consider $\{B(0, n)|n \in \mathbb{N}\}$. Notice this is an open cover for X. This must also be an open cover for A since A is a subset of X. Thus, A is bounded.
Let $x \in A^{\complement}$. For every $y \in A$, there are open neighborhoods of y, U_y and V_y of x such that $U_y \cap V_y \varnothing$. Then, $\{U_y|y \in A\}$ is an open cover of A.. Since, A is compact we then have that U and Y are open and $U \cap V = \varnothing$. We then have that $A^{\complement}$ is open. Thus, A is closed.
Therefore, if A is compact in X, then A is closed and bounded.

(b) Provide an example demonstrating that a subset of a metric space can be closed and bounded but not compact.

Let X be the integers and let our metric be defined as such:

$$d(x,y) = \{ \begin{array}{ll} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{array}$$

Notice, $d(x,y)$ is bounded since each point is within a distance 1 of some other point. Notice, every subset of X is open and thus also closed. Thus, we are bounded and closed. It is not compact as there are no finite subcovers, since X is infinite.

Summary

I'm super worried about his upcoming exam. Going to office hours a ton has helped, but I'm not sure if I'll be ready for the exam. There were so many topics we covered, I'm not sure if the 4 questions on the exam will be able to accurately reflect my understanding. There's so many different areas to study and I'm concerned if I study the bigger topics a lot since they're the "most important", I will not have as great of an of understanding on the "less important" topics which of course will be on the exam and frustrate me even more since "I know I've seen it...". The quizzes have also not been going well for me. I like the idea as it allows for a greater understating of the problem stated.