3.01 Let $X = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$, the x -axis in the plane. Describe the topology that X inherits as a subspace of $\mathbb{R}^2$ with the standard topology.
Notice, the basis for the Product Topology is the product of the basis of the individual properties. For $\mathbb{R}^2$, let $a, b, c, d \in \mathbb{R}$, then we have elements of the form $(a, b) \times (c, d)$. When intersected with X, we result in the subspace topology.

3.02 Let $Y = [-1, 1]$ have the standard topology. Which of the following sets are open in Y and which are open in $\mathbb{R}$?

$$A = (-1, -1/2) \cup (1/2, 1) :: \text{Open in } Y \text{ and } \mathbb{R}$$
$$B = (-1, -1/2] \cup [1/2, 1) :: \text{Not open in } Y \text{ and } \mathbb{R}$$
$$C = [-1, -1/2) \cup (1/2, 1] :: \text{Open in } Y \text{ and not open in } \mathbb{R}$$
$$D = [-1, -1/2] \cup [1/2, 1] :: \text{Closed in } Y \text{ and } \mathbb{R}$$
$$E = \bigcup_{n=1}^{\infty} \left(\frac{1}{1+n}, \frac{1}{n} \right) :: \text{Open in } Y \text{ and } \mathbb{R}$$

3.03 **Prove Theorem** 3.4 : Let X be a topological space, and let $Y \subset X$ have the subspace topology. Then $C \subset Y$ is closed in Y if and only if $C = D \cap Y$ for some closed set D in X.
Let $C \subset Y$ be closed in Y. Then, there exists a closed subset D in X with $D \cap Y = C$. Define $U = X - D$ and $V = Y - C$. Note, U is open. Observe,

$$\begin{aligned} U \cap Y &= (X - D) \cap Y \\ &= (X \cap Y) - (D \cap Y) \\ &= Y - C \\ &= V \end{aligned}$$

Thus, V is open. Hence, $C = Y - V$ is closed.
Therefore, there exists a closed set in Y such that C equals the intersection of such a set and Y.
Let D be a closed set in X and $C = D \cap Y$. Then, this implies that $C = D - Y^{\complement}$. Notice, as D is closed and $Y^{\complement}$ is closed, we must have that C is closed by definition of the intersection of closed sets.

3.07 Let X be a Hausdorff topological space, and Y be a subset of X. Prove that the subspace topology on Y is Hausdorff.
Let $a, b \in Y$ with $a \neq b$. Since X is Hausdorff, there exist disjoint neighborhoods U and V in X of a and b, respectively. Notice, a set containing a in Y is $W_1 = U \cap Y$. Then, W_1 must be open in Y by the definition of the subspace topology. Hence, W_1 is a neighborhood of a in Y.

Notice, a set containing b in Y is $W_2 = V \cap Y$. Then, W_2 must be open in Y by definition of the subspace topology. Hence, W_2 is a neighborhood of b in Y.
Observe that as U and V are disjoint and$W_1 \subset U$ and $W_2 \subset V$, we must have that W_1 and W_2 are also disjoint.
Therefore, Y is Hausdorff.

3.08 Let X be a topological space, and let $Y \subset X$ have the subspace topology.

(a) If A is open in Y, and Y is open in X, show that A is open in X.
Let A be open in Y and Y be open in X. Then, $A = Y \cap U$ for some open set U in X. Since, Y and U are both open in X, their intersection must also be open by definition.
Therefore, A is open in X.

(b) If A is closed in Y, and Y is closed in X, show that A is closed in X.
Let A be closed in Y and Y be closed in X. Then, $A = Y \cap U$ for some closed set U in X. Since, Y and U are both closed in X, their intersection must also be closed by definition.
Therefore, A is closed in X.

3.15 **Prove Theorem** 3.9 : Let X and Y be topological spaces, and assume that $A \subset X$ and $B \subset Y$. Then the topology on $A \times B$ as a subspace of the product $X \times Y$ is the same as the product topology on $A \times B$, where A has the subspace topology inherited from X, and B has the subspace topology inherited from Y.
Notice, the subspace topology on $A \times B$ is with $\mathcal{T}_{X \times Y}$ with $A \times B \subset X \times Y$

$$S_1 = \mathcal{T}_{A\times B} = \{\bigcup_{i,j}(U_i \cap A) \times (V_j \cap B) | U_i \in \mathcal{T}_X, V_j \in \mathcal{T}_Y$$

Notice, the subspace topology on $A \times B$ in the context of $\mathcal{T}_x$ and T_Y

$$S_2 = \mathcal{T}_{A\times B} = \{(\bigcup_{i,j} W_i \times V_j) \cap (A \times B) | W_i \in \mathcal{T}_X, V_j \in \mathcal{T}_Y\}$$

Let $(x, y) \in S_1$. We then have $x \in U_i \cap A$ for some i and $y \in V_j \cap B$ for some j. So, $x \in U_i$ and $x \in A$. In addition, $y \in V_j$ and $y \in B$.
Hence, $(x, y) \in U_i \times V_j$ and $(x, y) \in A \times B$.
Thus, $(x, y) \in (U_i \times V_j) \cap (A \times B)$. Notice, this is the definition for S_2.
Thus, $(x, y) \in S_2$
Therefore, $S_1 \subset S_2$

Let $(x, y) \in S_2$. We then have $(x, y) \in W_i \times V_j) \cap (A \times B)$ for some i, j. So, $(x, y) \in W_i \times V_j$ and $(x, y) \in A \times B$. Notice, $x \in W_i$ and $x \in A$. Also note $y \in V_j$ and $y \in B$.
Hence, $x \in W_i \cap A$ and $y \in V_j \cap B$.

Thus, $(x,y) \in U_i \cap A \times V_j \cap B$. Notice, this is the definition for S_1.
Thus, $(x,y) \in S_1$
Therefore, $S_2 \subset S_1$

Therefore, as each set is a subset of the other we have that the sets must be equal.

3.16 Let S^2 be the sphere, D be the disk, T be the torus, S^1 be the circle, and $I = [0,1]$ with the standard topology. Draw pictures of the product spaces $S^2 \times I, T \times I, S^1 \times I \times I$, and $S^1 \times D$

3.18 Show that if X and Y are Hausdorff spaces, then so is the product space $X \times Y$
Let X and Y be Hausdorff spaces and $X \times Y$ be the product space. Suppose $(x_1, y_1), (x_2, y_2)$ are two distinct point in $X \times Y$. Then, $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality, assume $x_1 \neq x_2$. Let $V_1, V_2 \subset_{open} X$ with $x_1 \in V_1$, $x_2 \in V_2$ and since X is a a Hausdorff Space, also assume $V_1 \cap V_2 = \varnothing$.
Similarly, let $U_1, U_2 \subset_{open} Y$ with $y_1 \in U_1$ and $y_2 \in U_2$. Notice, $(x_1, y_1) \in V_1 \times U_1$ and $(x_2, y_2) \in V_2 \times U_2$. Observe.

$$(V_1 \times U_1) \cap (V_2 \times U_2) = (V_1 \cap V_2) \times (U_1 \cap U_2) = \varnothing$$

Thus, for two distinct point in $X \times Y$, we can find two disjoint open sets containing the two points respectively.
Therefore, $X \times Y$ is Hausdorff.

3.19 Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.
Let A be closed in X and B be closed in Y. Notice, $X - A$ is open in X and $Y - B$ is open in Y. Then, $(X - A) \times (Y - B)$ is open in $X \times Y$ by the definition of the product topology. And so, $(X \times Y) - (A \times B)$ is also open in$X \times Y$.
Therefore, $A \times B$ is closed in $X \times Y$

3.20 Show that if $A \subset X$ and $B \subset Y$, then $\mathrm{Cl}(A \times B) = Cl(A) \times Cl(B)$
Let $A \subset X$ and $B \subset Y$. Notice, $A \subset Cl(A)$ and $B \subset Cl(B)$. So, $A \times B \subset Cl(A) \times Cl(B)$. Hence, $Cl(A \times B) = Cl(A) \times Cl(B)$

3.23 If $\mathbb{R}$ has the standard topology, define

$$p : \mathbb{R} \to \{a,b,c,d,e\} \text{ by } p(x) = \begin{cases} a \text{ if } x > 2 \\ b \text{ if } x = 2 \\ c \text{ if } 0 \leq x < 2 \\ d \text{ if } -1 < x < 0 \\ e \text{ if } x \leq -1 \end{cases}$$

(a) List the open sets in the quotient topology on $\{a,b,c,d,e\}$
Notice, p is surjective and must be injective so that $p^{-1}(U)$ is open in $\mathbb{R}$. Satisfying this we have, $\{\{a\}, \{d\}, \{a,d\}\}$.

(b) Now assume that $\mathbb{R}$ has the lower limit topology. What are the open sets in the resulting quotient topology on $\{a, b, c, d, e\}$?
The lower limit topology gives us the interval of $[a, b)$ for open sets. Thus, $\{c\}$

3.24 Let $X = \mathbb{R}$ in the standard topology. Take the partition

$$X^* = \{\ldots, (-1, 0], (0, 1], (1, 2], \ldots\}$$

Describe the open sets in the resulting quotient topology on X^* .
In quotient topology on X^* we have $U \subset X^*$ is open if the union of equivalence classes in U is open in $X = \mathbb{R}$. Thus, for $n \in \mathbb{Z}$ $(n, n+1] \in X^*$. Hence, $(n, n+1] \cup (n+2, n+2] \cup (n+2, n+3] \cdots$ is open in $\mathbb{R}$.
Therefore, $U \subseteq X^*$ is open in X^*
Resulting open sets: $U = \{(n, n+1] \cup (n+2, n+2] \cup (n+2, n+3] \cdots\}$

3.25 Define a partition of $X = \mathbb{R}^2 - \{O\}$ by taking each ray emanating from the origin as an element in the partition. (See Figure 3.25.) Which topological space that we have previously encountered appears to be topologically equivalent to the quotient space that results from this partition?
The equivalent topological space appears to be the projective line, which is a set of points of a line in $\mathbb{R}^2$. Then, $X = R^2 - \{0\}$, results in an equivalent topological space of S^1.

3.27 Provide an example showing that a quotient space of a Hausdorff space need not be a Hausdorff space.
Take $X = \mathbb{R}$ and let R be the equivalence relation with the classes Q and R Q.

Then $Y = X/R$ in the quotient topology is not Hausdorff. Since, Y has 2 points and no singleton is open.

3.29 Consider the equivalence relation on $\mathbb{R}^2$ defined by $(x_1, x_2) \sim (w_1, w_2)$ if $x_1 + x_2 = w_1 + w_2$. Describe the quotient space that results from the partition of $\mathbb{R}^2$ into the equivalence classes in this equivalence relation.
$X^* = [(a, b)] = \{(x, y)|y = -x + (a + b)\}$

3.30 Consider the equivalence relation on $\mathbb{R}^2$ defined by $(x_1, x_2) \sim (w_1, w_2)$ if $x_1^2 + x_2^2 = w_1^2 + w_2^2$. Describe the quotient space that results from the partition of $\mathbb{R}^2$ into the equivalence classes in this equivalence relation.
$X^* = \{[x_1, x_2]| x_1^2 + x_2^2 = r^2, r \geq 0\}$

3.35 On a sketch of the surface $T\#T$, illustrate where the glued edges of the octagon in Figure 3.33 appear.

3.36 (a) Show that a hexagon with opposite edges glued together straight across yields a torus.

(b) Show that a hexagon with opposite edges glued together with a flip yields a projective plane.

3.38 Show that the quotient space in Example 3.27 is topologically equivalent to $S^1 \times P$, the product of a circle and a projective plane.