9.01 Show that the identity map from the disk D in the plane to itself is homotopic to the map that takes D to the origin.

9.02 Show that if $f_1, f_2 : X \to Y$ are homotopic and $g_1, g_2 : Y \to Z$ are homotopic, then $g_2 \circ f_2$ is homotopic to $g_1 \circ f_1$.
Suppose $f_1, f_2 : X \to Y$ are homotopic and $g_1, g_2 : Y \to Z$ are homotopic. Notice, we must have the homotopies

$$F : X \times I \to Y \text{ with } F(x,0) = f_1(x), F(x,1) = f_2(x)$$
$$G : X \times I \to Y \text{ with } G(x,0) = g_1(x), G(x,1) = g_2(x)$$

We then define $H : X \times I \to Z$ by

$$H(x,t) = \{ \begin{array}{ll} g_1(F(x,2t)) & 0 \leq t \leq \frac{1}{2} \\ G(f_2(x), 2t-1) & \frac{1}{2} \leq t \leq 1 \end{array} .$$

We then check that H gives us a homotopy by checking $t = 0, \frac{1}{2}, 1$.

$$H(x,0) = g_1(F(x,0)) = g_1(f_1(x)) = g_1 \circ f_1(x)$$
$$H(x, \frac{1}{2}) = g_1(F(x,1)) = g_1(f_2(x)) = G(f_2(x), 0)$$
$$H(x,1) = G(f_2(x), 1) = f_2(g_2(x)) = f_2 \circ g_2(x)$$

Therefore, $g_2 \circ f_2$ is homotopic to $g_1 \circ f_1$.

9.03 Let f and g be paths in $\mathbb{R}$. Show that f is homotopic to g. .
Let f, g be paths in $\mathbb{R}$ defined as $f, g : [0,1] \to \mathbb{R}$ with $f(0) = x, f(1) = y, g(0) = a, g(1) = b$ for all $x, y, a, b \in \mathbb{R}$. We define $F : I \times I \to \mathbb{R}$ with

$$F(t) = \{ \begin{array}{ll} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{array} .$$

Checking the points for $t = 0, \frac{1}{2}, 1$

$$F(0) = f(0)$$
$$F(\frac{1}{2}) = f(1)$$
$$F(\frac{1}{2}) = g(0)$$
$$F(1) = g(1)$$

9.04 Let f and g be paths in $\mathbb{R}^2 - \{O\}$. Show that f is homotopic to g . (Hint: Show that every path is homotopic to the constant path that sends the entire interval to the path's starting point. Then show that two constant paths are homotopic using the fact that $\mathbb{R}^2 - \{O\}$ is path connected.)

9.06 Show that if X is a topological space, and D is the disk in the plane, then there is only one homotopy class of continuous functions from X to D .

9.07 Consider the following definition:

DEFINITION 9.4. A topological space X is said to be **contractible** if the identity function on X is homotopic to a constant function.

(a) Prove that contractibility is a topological invariant. That is, prove that if X and Y are homeomorphic, then X is contractible if and only if Y is contractible.

(b) Prove that $\mathbb{R}^n$ is contractible.

(c) Let X be a contractible space. Prove that X is path connected.
Let X be a contractible space. Then the identity function on X is homotopic to a constant function, call it f with $f(x) = c$ where c is a constant in X and $x \in X$. We define this homotopy:

$$F : X \times I \to X \text{ with } F(x, 0) = id_X(x), F(x, 1) = f(x) = c$$

Let $a, b \in X$ be our fixed points for our path. Notice, $F(a, t)$ is a path from a to c and $F(b, t)$ is a path from b to c. So, we define the path as

$$p(t) = \{ \begin{array}{ll} F(a, 2t) & 0 \leq t \leq \frac{1}{2} \\ F(b, 2 - 2t) & \frac{1}{2} \leq t \leq 1 \end{array}$$

We then check the points $t = 0, \frac{1}{2}, 1$.

$$p(1) = F(a, 0) = id_X(a) = a$$
$$p(\frac{1}{2}) = F(a, 1) = c = F(b, 1)$$
$$p(1) = F(b, 0) = id_X(b) = b$$

Summary