

3.01 Let  $X = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ , the  $x$ -axis in the plane. Describe the topology that  $X$  inherits as a subspace of  $\mathbb{R}^2$  with the standard topology.

Notice, the basis for the Product Topology is the product of the basis of the individual properties. For  $\mathbb{R}^2$ , let  $a, b, c, d \in \mathbb{R}$ , then we have elements of the form  $(a, b) \times (c, d)$ . When intersected with  $X$ , we result in the subspace topology.

3.02 Let  $Y = [-1, 1]$  have the standard topology. Which of the following sets are open in  $Y$  and which are open in  $\mathbb{R}$ ?

$$A = (-1, -1/2) \cup (1/2, 1) :: \text{Open in } Y \text{ and } \mathbb{R}$$

$$B = (-1, -1/2] \cup [1/2, 1) :: \text{Not open in } Y \text{ and } \mathbb{R}$$

$$C = [-1, -1/2) \cup (1/2, 1] :: \text{Open in } Y \text{ and not open in } \mathbb{R}$$

$$D = [-1, -1/2] \cup [1/2, 1] :: \text{Closed in } Y \text{ and } \mathbb{R}$$

$$E = \bigcup_{n=1}^{\infty} \left( \frac{1}{1+n}, \frac{1}{n} \right) :: \text{Open in } Y \text{ and } \mathbb{R}$$

3.03 **Prove Theorem 3.4** : Let  $X$  be a topological space, and let  $Y \subset X$  have the subspace topology. Then  $C \subset Y$  is closed in  $Y$  if and only if  $C = D \cap Y$  for some closed set  $D$  in  $X$ .

Let  $C \subset Y$  be closed in  $Y$ . Then, there exists a closed subset  $D$  in  $X$  with  $D \cap Y = C$ . Define  $U = X - D$  and  $V = Y - C$ . Note,  $U$  is open. Observe,

$$\begin{aligned} U \cap Y &= (X - D) \cap Y \\ &= (X \cap Y) - (D \cap Y) \\ &= Y - C \\ &= V \end{aligned}$$

Thus,  $V$  is open. Hence,  $C = Y - V$  is closed.

Therefore, there exists a closed set in  $Y$  such that  $C$  equals the intersection of such a set and  $Y$ .

Let  $D$  be a closed set in  $X$  and  $C = D \cap Y$ . Then, this implies that  $C = D - Y^c$ . Notice, as  $D$  is closed and  $Y^c$  is closed, we must have that  $C$  is closed by definition of the intersection of closed sets.

3.07 Let  $X$  be a Hausdorff topological space, and  $Y$  be a subset of  $X$ . Prove that the subspace topology on  $Y$  is Hausdorff.

Let  $a, b \in Y$  with  $a \neq b$ . Since  $X$  is Hausdorff, there exist disjoint neighborhoods  $U$  and  $V$  in  $X$  of  $a$  and  $b$ , respectively. Notice, a set containing  $a$  in  $Y$  is  $W_1 = U \cap Y$ . Then,  $W_1$  must be open in  $Y$  by the definition of the subspace topology. Hence,  $W_1$  is a neighborhood of  $a$  in  $Y$ .

Notice, a set containing  $b$  in  $Y$  is  $W_2 = V \cap Y$ . Then,  $W_2$  must be open in  $Y$  by definition of the subspace topology. Hence,  $W_2$  is a neighborhood of  $b$  in  $Y$ .

Observe that as  $U$  and  $V$  are disjoint and  $W_1 \subset U$  and  $W_2 \subset V$ , we must have that  $W_1$  and  $W_2$  are also disjoint.

Therefore,  $Y$  is Hausdorff.

3.08 Let  $X$  be a topological space, and let  $Y \subset X$  have the subspace topology.

- (a) If  $A$  is open in  $Y$ , and  $Y$  is open in  $X$ , show that  $A$  is open in  $X$ .
- (b) If  $A$  is closed in  $Y$ , and  $Y$  is closed in  $X$ , show that  $A$  is closed in  $X$ .

3.15 **Prove Theorem 3.9** : Let  $X$  and  $Y$  be topological spaces, and assume that  $A \subset X$  and  $B \subset Y$ . Then the topology on  $A \times B$  as a subspace of the product  $X \times Y$  is the same as the product topology on  $A \times B$ , where  $A$  has the subspace topology inherited from  $X$ , and  $B$  has the subspace topology inherited from  $Y$ .

3.16 Let  $S^2$  be the sphere,  $D$  be the disk,  $T$  be the torus,  $S^1$  be the circle, and  $I = [0, 1]$  with the standard topology. Draw pictures of the product spaces  $S^2 \times I$ ,  $T \times I$ ,  $S^1 \times I \times I$ , and  $S^1 \times D$

3.18 Show that if  $X$  and  $Y$  are Hausdorff spaces, then so is the product space  $X \times Y$

3.19 Show that if  $A$  is closed in  $X$  and  $B$  is closed in  $Y$ , then  $A \times B$  is closed in  $X \times Y$ .

3.20 Show that if  $A \subset X$  and  $B \subset Y$ , then  $\text{Cl}(A \times B) = \text{Cl}(A) \times \text{Cl}(B)$

3.23 If  $\mathbb{R}$  has the standard topology, define

$$p : \mathbb{R} \rightarrow \{a, b, c, d, e\} \text{ by } p(x) = \begin{cases} a & \text{if } x > 2 \\ b & \text{if } x = 2 \\ c & \text{if } 0 \leq x < 2 \\ d & \text{if } -1 < x < 0 \\ e & \text{if } x \leq -1 \end{cases}$$

- (a) List the open sets in the quotient topology on  $\{a, b, c, d, e\}$
- (b) Now assume that  $\mathbb{R}$  has the lower limit topology. What are the open sets in the resulting quotient topology on  $\{a, b, c, d, e\}$ ?

3.24 Let  $X = \mathbb{R}$  in the standard topology. Take the partition

$$X^* = \{\dots, (-1, 0], (0, 1], (1, 2], \dots\}$$

Describe the open sets in the resulting quotient topology on  $X^*$ .

- 3.25 Define a partition of  $X = \mathbb{R}^2 - \{O\}$  by taking each ray emanating from the origin as an element in the partition. (See Figure 3.25. Which topological space that we have previously encountered appears to be topologically equivalent to the quotient space that results from this partition?
- 3.27 Provide an example showing that a quotient space of a Hausdorff space need not be a Hausdorff space.
- 3.29 Consider the equivalence relation on  $\mathbb{R}^2$  defined by  $(x_1, x_2) \sim (w_1, w_2)$  if  $x_1 + x_2 = w_1 + w_2$ . Describe the quotient space that results from the partition of  $\mathbb{R}^2$  into the equivalence classes in this equivalence relation.
- 3.30 Consider the equivalence relation on  $\mathbb{R}^2$  defined by  $(x_1, x_2) \sim (w_1, w_2)$  if  $x_1^2 + x_2^2 = w_1^2 + w_2^2$ . Describe the quotient space that results from the partition of  $\mathbb{R}^2$  into the equivalence classes in this equivalence relation.
- 3.35 On a sketch of the surface  $T \# T$ , illustrate where the glued edges of the octagon in Figure 3.33 appear.
- 3.36 (a) Show that a hexagon with opposite edges glued together straight across yields a torus.
- (b) Show that a hexagon with opposite edges glued together with a flip yields a projective plane.
- 3.38 Show that the quotient space in Example 3.27 is topologically equivalent to  $S^1 \times P$ , the product of a circle and a projective plane.