7 Compactness 7.1

Let A be a subset of a topological space X, and let $\mathcal{O}$ be a collection of subsets of X

7.1 Definition of Cover

The collection $\mathcal{O}$ is said to **cover** A if A is contained in the union of the sets in $\mathcal{O}$.

7.2 Definition of Open Cover

If $\mathcal{O}$ covers A, and each set in $\mathcal{O}$ is open, then we call $\mathcal{O}$ an **open cover** of A.

7.3 Definition of Subcover

If $\mathcal{O}$ covers A, and $\mathcal{O}'$ is a subcollection of $\mathcal{O}$ that also covers A then $\mathcal{O}'$ is called a **subcover** of $\mathcal{O}$.

7.4 Definition of Compact

A topological space X is **compact** if every open cover of X has a finite subcover.

7.5 Definition of Compact In

Let X be a topological space, and assume $A \subset X$. Then A is said to be **compact in** X if A is compact in the subspace topology inherited from X.

7.6 Lemma: Checks weather or not a subspace A is compact

Let X be a topological space, and assume $A \subset X$. Then A is compact in X if and only if every cover of A by sets that are open in X has a finite subcover.

Proof:
Let A be compact in X, and suppose that $\mathcal{O}$ is a cover of A by open sets in X. Then $\mathcal{O}' = \{U \cap A | U \in \mathcal{O}\}$ is a cover of A by open sets in A. Hence, there exists a finite subcover $\{U_1 \cap A, U_2 \cap A, \ldots, U_n \cap A\}$ of $\mathcal{O}'$. But then $\{U_1, U_2, \ldots, U_n\}$ is a finite subcover of $\mathcal{O}$. Therefore every cover of A by open sets in X has a finite subcover.

Conversely, suppose every cover of A by sets that are open in X has a finite subcover. Let $\mathcal{O} = \{V_\beta\}_{\beta \in B}$ be a cover of A by open sets in A. Then, by definition of the subspace topology, for each V_β there vis an open set U_β in X such that $V_\beta = U_\beta \cap A$. It follows that the collection $\mathcal{O}' = \{U_B\}_{B \in R}$ is a cover of A by open sets in X . Since $\mathcal{O}'$ has a finite subcover

$\{U_{\beta_1}, \ldots, U_{\beta_n}\}$, it follows that $\{V_{\beta_1}, \ldots, V_{\beta_n}\}$ is a finite subcover of $\mathcal{O}$. Thus every cover of A by open sets in A has a finite subcover, and therefore A is compact.

7.7 Compactness will be preserved through continuous functions

Let $f : X \to Y$ be continuous, and let A be compact in X. Then $f(A)$ is compact in Y.
Proof:

7.8 Compact sets unioned together are compact

Let X be a topological space. If $C_1, \ldots, C_n$ are each compact in X, then $U_{j=1}^{n} C_j$ is compact in X.

7.9 Intersection of Hausdorff compact sets are compact

If X is Hausdorff, and $\{C_\alpha\}_{\alpha \in A}$ is a collection of sets that are compact in X, then $\bigcap_{\alpha \in A} C_\alpha$ is compact in X.

7.10 If a subset of a compact set is closed, then that subset is also compact

Let X be a topological space and let D be compact in X. If C is closed in X, and $C \subset D$, then C is compact in X.

7.11 All compact subsets of a Hausdorff space are closed

Let X be a Hausdorff topological space and A be compact in X. Then A is closed in X.

7.12 Tube Lemma i.e. You can take a slice of a space and it'll be compact still

Let X and Y be topological spaces, and assume that Y is compact. If $x \in X$, and U is an open set in $X \times Y$ containing $\{x\} \times Y$, then there exists a neighborhood W of x in X such that $W \times Y \subset U$.

7.13 Product topology preserves compactness

THEOREM 7.10. If X and Y are compact topological spaces, then the product $X \times Y$ is compact.

7 Compactness in Metric Spaces 7.2

7.1 Closed Bounded Intervals are compact

Every closed and bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$ with the standard topology.

7.2 Product of closed bounded intervals are compact

Let $[a_1, b_1], \ldots, [a_n, b_n]$ be closed bounded intervals in R. Then $[a_1, b_1] \times \ldots \times [a_n, b_n]$ is a compact subset of $\mathbb{R}^n$

7.3 The standard topology in standard metric in $\mathbb{R}^n$ is compact iff it is closed and bounded

Let $\mathbb{R}^n$ have the standard topology and the standard metric d. A set $A \subset \mathbb{R}^n$ is compact in $\mathbb{R}^n$ if and only if it is closed and bounded.

7.4 In a metric space with compact subset A, if there is a sequence, then there is a subsequence that converges to a limit in A

Let (X, d) be a metric space, and assume that A is compact in X. If (x_n) is a sequence in A, then there exists a subsequence (x_{n_m}) of (x_n) that converges to a limit in A.

7.5 Definition of Cauchy Sequence

Let (X, d) be a metric space. A sequence (x_n) in X is called a **Cauchy Sequence** if for every $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that $d(x_n, x_m) < \varepsilon$ for every $n, m \geq N$

7.6 With everything standard, a Cauchy Sequence converges to a limit

Let (x_n) be a Cauchy sequence in $\mathbb{R}^n$ with the standard metric d. Then (x_n) converges to a limit in $\mathbb{R}^n$.

7.7 Definition of Complete

A metric space X is called **complete** if every Cauchy sequence in X converges to a limit in X .