7.22 Prove Corollary 7.24 : Let $[a, b]$ be a closed and bounded interval in $\mathbb{R}$, and assume that $f : [a, b] \to \mathbb{R}$ is continuous. Then the image of f is a closed and bounded interval in $\mathbb{R}$.

Let $[a, b]$ be a closed bounded interval in $\mathbb{R}$ and suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Notice, that $[a, b]$ is compact and so by the extreme value theorem f has a maximum and minimum, c at a point $y \in [a, b]$ and d at a point $x \in [a, b]$ respectively. So, $f([a, b]) \subset [c, d]$. Without loss of generality, suppose $x < y$. Let $z \in [c, d]$. By the intermediate value theorem, there exists a z' such that $x < z' < y$ and $f(z') = z$. So, $f([a, b]) = [c, d]$.

Therefore, the image of f is a closed and bounded interval in $\mathbb{R}$.

7.23 Provide an example of closed sets, A and B, in a metric space (X, d) such that A and B are disjoint and $d(A, B) = 0$

Let $A = \mathbb{N}$ and $B = \{n + \frac{1}{2^n} | n \in \mathbb{N}\}$. By definition the two sets are closed subsets of $\mathbb{R}$ and for any $\varepsilon > 0$ there is some $n \in \mathbb{N}$ such that $1/n < \varepsilon$. So, $d(A, B) = 0$.

7.24 Prove Lemma 7.26 : Let (X, d) be a metric space, and let A be a subset of X . The function $f_A : X \to \mathbb{R}$, defined by $f_A(x) = d(\{x\}, A)$, is continuous.

(WTS: $f_A^{-1}(U)$ is open in X for every open U in R)

Let U be an open set in $\mathbb{R}$ with $U = [a, b]$ for $a, b \in \mathbb{R}$

$f_A^{-1} = \{x \in X | f(x) \in \mathbb{R}\}$

7.38 Prove Theorem 7.40 : Let X be a Hausdorff space, and let $Y = X \cup \{\infty\}$ be its one-point compactification. Then the subspace topology that X inherits from Y is equal to the original topology on X .

Let X be a Hausdorff space and let $Y = X \cup \{\infty\}$ be its one point compactification. Let Y' be the subspace topology of Y. We know that the open sets in Y are the open sets in X and $Y - C$ where C is compact in X. Thus, $X \subset Y$. Since, every open set that is $Y - C$

7.39 Show that the one-point compactification of $(0, 1)$ is homeomorphic to the circle.

Consider bijective $f : (0, 1) \to S^1 - \{(0, 1)\}$ defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Since, trigonometric functions are continuous we must have f is continuous. Let f^{-1} be defined by:

$$f^{-1}(x, y) = \{ \begin{array}{ll} \frac{1}{2\pi}\cos^{-1}(x) & \text{if } y > 0 \\ \frac{1}{2} - \frac{1}{2\pi}\sin^{-1}(y) & \text{if } x < 0 \\ 1 - \frac{1}{2\pi}\cos^{-1}(x) & \text{if } y < 0 \end{array}$$

Notice, by the pasting lemma since each piece of the function is comprised of continuous trig functions we must have that f^{-1} is continuous.

Thus, f is a bijection, continuous, and f^{-1} is continuous.

Thus, f is a homeomorphism.

Therefore, the one-point compactification of $(0, 1)$ is homeomorphic to the circle, S^1.

7.40 Show that the one-point compactification of Q is not Hausdorff.
By way of contradiction, suppose that the one point compactification of $\mathbb{Q}$, $\mathbb{Q}'$, is Hausdorff. Let $x, \infty \in \mathbb{Q}'$ with $x \neq \infty$ and let U, V be open disjoint sets such that $x \in U$ and $\infty \in V$. Notice, U is on open neighborhood of $x \in \mathbb{Q}$, thus it contains an open neighborhood , W, of x with the form $(a, b) \cap \mathbb{Q}$ for some irrational $a, b \in \mathbb{R}$. We must have that W and V are disjoint by definition. So, $\infty \notin Cl_{\mathbb{Q}'}(W)$. Observe.

$$Cl_{\mathbb{Q}'}(W) = Cl_{\mathbb{Q}}(W) = [a, b] \cap \mathbb{Q} = W$$

This is a contradiction, since W is not compact.
Therefore, the one-point compactification of Q is not Hausdorff

7.41 (a) Describe and illustrate the result of taking the one-point compactification of the open annulus $S^1 \times (0, 1)$.

(b) An open Mobius band is the space obtained from $[0, 1] \times (0, 1)$ by gluing the ends as we do with the usual Mobius band. Describe and illustrate the result of taking the one-point compactification of the open Mobius band. (Hint: The resulting space is one that we have previously encountered.)

7.42 Let X be Hausdorff and assume $Y = X \cup \{\infty\}$ is the one-point compactification of X.

(a) Show that if X is not compact, then $Cl(X) = Y$

(b) Show that if X is compact, then $Cl(X) = X$, and Y is disconnected with $\{\infty\}$ being one of its components. (This shows that not much interesting happens when taking the one-point compactification of a space that is already compact.)
Let X be compact. Since, X is compact and Hausdorff we have that the compact sets, $C = X$. Thus, since Y is comprised of open sets of X and $Y - C$, we must have that $Y - C$ is open and must be $\{\infty\}$.
Hence, the $Cl(X) = X$ and Y is disconnected with $\{\infty\}$

Summary