General Topology

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Contents

0 Introduction

These notes were compiled from the lectures of Dr. Jason Parsley and Dr. Hugh Howards at Wake Forest University. The first six chapters are primarily concerned with the development of definitions and canonical results in point-set topology. Chapter 7 details the proof of the Classification Theorem for surfaces without boundary, which is revisited in Chapter 9 for all surfaces. Chapter 8 is an introductory look at homotopy and the fundamental group. The rest of the sections are devoted to several interesting topics in topology, which come from presentations given by various students in Dr. Howards' class. The main texts for these courses are *Introduction to Topology* (Adams and Franzosa) and *Algebraic Topology* (Hatcher).

Definition. Topology *is the study of shapes and how they deform.*

The area of topology has numerous applications, including

- DNA
- Knot theory applications to physics and chemistry
- GIS (geographic information systems)
- Topology of data
- Cosmology and string theory
- Fixed point theory (with applications to game theory)

Sets

Recall: $\mathbb{R}$ has the least upper bound property. For example, given a set $A \subset \mathbb{R}$ which is bounded, A has a least upper bound, or supremum.

Lemma 0.0.1 (Union Lemma)**.** *Let X be a set and let C be a collection of subsets of X. If every element $x \in X$ lies in some set $A_x \in C$ then $\bigcup A_x = X$.*

Proof. Each $A_x \subset X$, so $\bigcup A_x \subset X$. Now let $x \in X$. Then there is some A_x such that $x \in A_x$. Thus $x \in \bigcup A_x$, so $X = \bigcup A_x$. □

Relations

Two types:

- Order relations
- Equivalence relations

Equivalence relations have the following properties:

- Reflexive: $x \sim x$ for all $x \in X$
- Symmetric: If $x \sim y$ then $y \sim x$ for all $x, y \in X$
- Transitive: If $x \sim y$ and $y \sim z$ then $x \sim z$ for all $x, y, z \in X$

Examples:

- Equality (on $\mathbb{R}$ e.g.)
- Set equality (mutual containment)
- $\mathbb{Z}_n$, i.e. equivalence modulo an integer n
- Less mathematical relations, such as shirt color today

Definition. *A* **relation** *R on sets X and Y is a subset $R \subset X \times Y$. We say x* **is related to** *y if $(x, y) \in R$, denoted xRy.*

Definition. *An* **equivalence relation** *is a relation that is reflexive, symmetric and transitive.*

Definition. *The* **equivalence class** *of x is defined by $[x] := \{y \in X \mid x \sim y\}$.*

Definition. *A* **partition** *of X is a collection of mutually disjoint subsets of X whose union is X.*

Definition. *An* **order relation** *C satisfies the following properties:*

1) *Comparability: For all distinct $x, y \in X$, either xCy or yCx*
2) *Nonreflexivity: $x \not{C} x$ for all x*
3) *Transitivity*

Example 0.0.2. $x < y$ is an order relation on $\mathbb{R}$.

Example 0.0.3. xCy if either $x^2 < y^2$, or $x^2 = y^2$ and $x < y$. This order relation folds the real line back on itself.

Without comparability, the remaining properties give you a **partial ordering**.

Example 0.0.4. Divisors in $\mathbb{Z}$: xCy if $x \mid y$ and $x \neq y$.

1 Topological Spaces

1.1 Topology

Definition. *A* **topology** $\mathcal{T}$ *on a set* X *is a collection of subsets such that*

1) $\varnothing, X \in \mathcal{T}$

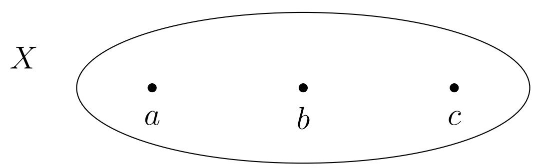
2) The union of arbitrarily many sets in $\mathcal{T}$ *is in* $\mathcal{T}$

3) The intersection of finitely many sets in $\mathcal{T}$ *also lies in* $\mathcal{T}$*.*

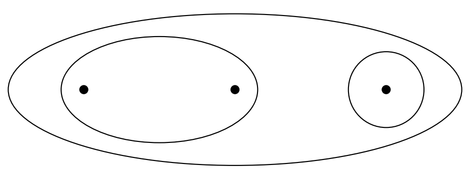
Definition. *The pairing of a set and a topology,* $(X, \mathcal{T})$*, is a* **topological space***. The subsets of* X *which lie in* $\mathcal{T}$ *are called the* **open sets** *in* $(X, \mathcal{T})$*.*

Example 1.1.1. The open sets $\{(a,b) \mid a < b\} \cup \{(-\infty, b)\} \cup \{(a, \infty)\} \cup \mathbb{R}$ form a topology on $\mathbb{R}$. Actually this forms a basis for a topology on $\mathbb{R}$, but taking arbitrary unions and finite intersections, we have the whole topology.

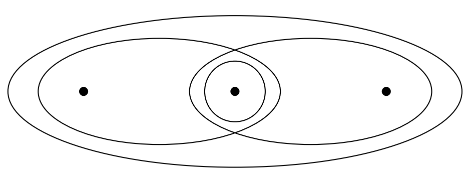
Example 1.1.2. Take a 3-point set $X = \{a, b, c\}$.



There are 2^3 subsets of X. To form a topology, $\varnothing$ and X must be in $\mathcal{T}$, and there are 6 other subsets to possibly add in. One topology could be $\mathcal{T} = \{\varnothing, X, \{a, b\}, \{c\}\}$:



Other examples are pictured below. The first is a topology, while the second and third are not.



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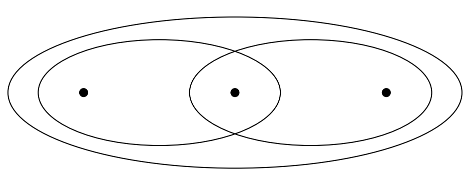

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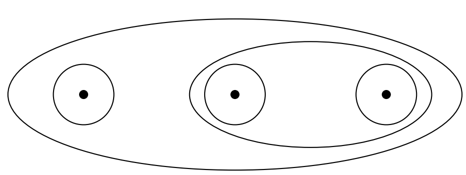

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Example 1.1.3. There are a minimum of two topologies on every set X. The **trivial topology** consists of just X and the empty set. The **discrete topology** consists of all subsets of X, i.e. the power set of X.

Example 1.1.4. The **finite complement topology** is defined as

$$\mathcal{T}_{fc} = \{U \mid X - U \text{ is finite}\} \cup \{\varnothing\}.$$

Note that $\varnothing$ is in $\mathcal{T}_{fc}$ by construction, and $X - X$ is finite so $X \in \mathcal{T}_{fc}$ as well. Now consider $U = \bigcup\limits_{\alpha} U_\alpha$, the arbitrary union of open sets $U \in \mathcal{T}_{fc}$. Then by DeMorgan's Laws, $X - U = \bigcap\limits_{\alpha}(X - U_\alpha)$, and the intersection of finite subsets is finite. So arbitrary unions of open sets are open. Next, let $V = \bigcap\limits_{i=1}^{n} U_i$, the intersection of finitely many open sets. Then $X - V = \bigcup\limits_{i=1}^{n}(X - U_i)$, and the union of finitely many finite sets is finite. Thus intersections hold, and $\mathcal{T}_{fc}$ is indeed a topology. $\square$

Example 1.1.5. The **countable complement topology** is defined as

$$\mathcal{T}_{cc} = \{A \mid X - A \text{ is countable}\} \cup \{\varnothing\}.$$

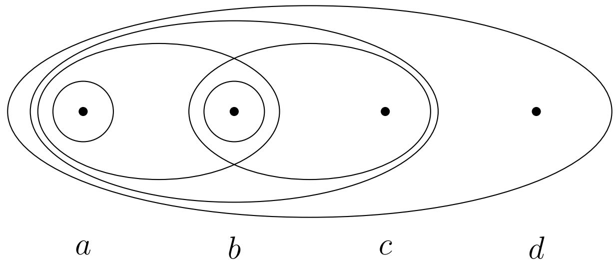
Example 1.1.6. Define $\mathcal{T} = \{A \mid A \text{ is finite}\} \cup \{X\}$.

First, $\varnothing$ and X are in $\mathcal{T}$. For finite intersections, if A_i are all finite, then so is $\bigcap\limits_{i=1}^{n} A_i$. However, an infinite union of finite sets may be infinite. For a counterexample, consider $X = \mathbb{R}$ and take the collection $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. Then arbitrary unions of elements in the collection are almost always infinite. Hence $\mathcal{T}$ is not a topology in general. (However, if X is finite, arbitrary unions will be finite as well.)

Definition. *Given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, we say*

- *$\mathcal{T}'$ is* **finer** *than $\mathcal{T}$ if $\mathcal{T} \subset \mathcal{T}'$*
- *$\mathcal{T}'$ is* **coarser** *than $\mathcal{T}$ if $\mathcal{T}' \subset \mathcal{T}$*
- *The two topologies are* **comparable** *if one is finer than the other. Otherwise they are* **incomparable***.*

Example 1.1.7. $X = \{a, b, c, d\}$, $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$.



Finer sets include:

- The discrete topology
- Adding $\{c\}$ and $\{a, c\}$

Coarser sets include:

- The trivial topology
- $\{\varnothing, X, \{a\}\}$

Incomparable:

- Anything with $\{d\}$, e.g. $\{\varnothing, X, \{d\}, \{b, c, d\}\}$

Remark. In general, the trivial topology is coarser than all topologies on X, and the discrete topology is always finer.

Definition. *A* **neighborhood** *of x is an open set U such that $x \in U$.*

Proposition 1.1.8. *A set A is open in a topological space X if and only if for all points $x \in A$, there is a neighborhood U containing x.*

Proof omitted. □

1.2 Basis

Definition. *For a set X, a* **basis** *for topology $\mathcal{T}$ on X is a collection $\mathcal{B}$ of subsets of X, called* **basis elements***, such that*

1) For all $x \in X$, x lies in some basis element;

2) If $x \in B_1 \cap B_2$ then there exists a B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Example 1.2.1. On $\mathbb{R}$, let $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a < b\}$.

Given $x \in \mathbb{R}$, there is always some ε such that $x \in (x - \varepsilon, x + \varepsilon)$. And if $x \in B_1 \cap B_2$ where $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$, then let $B_3 = B_1 \cap B_2 = (a_2, b_1)$, which is in $\mathcal{B}$ as well. Thus $\mathcal{B}$ is a basis. □

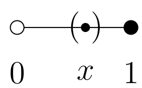
Theorem 1.2.2. *A basis generates a topology $\mathcal{T}$ on X. The open sets are precisely the subsets $U \subset X$ such that for all $x \in U$ there is a basis element $B \subset U$ containing x.*

Proof. (1) $\varnothing$ is satisfied vacuously. $X \in \mathcal{T}$ because every $x \in X$ must lie in a basis element $B \subset X$.

(2) For arbitrary unions, let $U = \bigcup\limits_{\alpha} U_\alpha$. Take $x \in U$. Then x lies in some U_α which is open. By definition there is some basis element $B \subset U_\alpha$ such that $x \in B$ and $B \subset U_\alpha \subset U$. Thus U is open.

(3) Lastly, for finite intersections, let $V = \bigcap\limits_{i=1}^{n} V_i$, where each V_i is open. Take $x \in V$. Then $x \in V_1$, which is open. So there exists a basis element $B_1 \subset V_1$ containing x. This is true for each V_i, $1 \leq i \leq n$; namely there exists a basis element $B_i \subset V_i$ containing x. By definition, there is a basis element $C_2 \subset B_1 \cap B_2$ such that $x \in C_2$. And $x \in B_3 \cap C_2$, so there exists a $C_3 \subset B_3 \cap C_2$ such that $x \in C_3$. And so forth. Eventually there will be some $C_n \subset B_3 \cap C_2 \cap \cdots \cap C_{n-1}$ such that $x \in C_n$. Note that $C_n \subset \bigcap\limits_{i=1}^{n} B_i \subset \bigcap\limits_{i=1}^{n} V_i = V$, so V is open. Hence $\mathcal{T}$ is a topology. $\square$

Example 1.2.3.



$[0,1)$ is not open in $\mathbb{R}$ since there's not an open interval containing $x = 1$. Thus for a set $A \subset \mathbb{R}$ to be open, every point must lie in some open interval contained in A.

Theorem 1.2.4. *The topology generated by $\mathcal{B}$ is the collection of unions of sets in $\mathcal{B}$.*

Proof. Let $\mathcal{T}_1$ be the topology with open sets U where for all $x \in U$, there is some $B \in \mathcal{B}$ such that $x \in B \subset U$. Let $\mathcal{T}_2$ be the collection of unions of elements of $\mathcal{B}$. Take U in $\mathcal{T}_1$. For each $x \in U$, let B_x be the basis element such that $x \in B_x \subset U$. Then $\bigcup\limits_{x \in U} B_x \subset U$. Suppose there is some $x_0 \in U$ such that $x_0 \notin \bigcup\limits_{x \in U} B_x$. Then there is no basis element containing x_0, a contradiction. Thus $\bigcup\limits_{x \in U} B_x = U$. Now let V in $\mathcal{T}_2$, so $V = \bigcup\limits_{\alpha} B_\alpha$. Take $x \in V$. Then x lies in some B_α, so V is in $\mathcal{T}_1$. $\square$

Example 1.2.5. Topologies on $\mathbb{Z}$

- The standard and discrete topologies are equivalent on $\mathbb{Z}$.
- The **digital line topology** is generated by the following basis:

$$\mathcal{B} = \left\{ B_n \,\middle|\, B_n = \left[\begin{matrix} \{n\} & n \text{ is odd} \\ \{n-1, n, n+1\} & n \text{ is even} \end{matrix} \right. \right\}$$

Bases are not unique. For example, on $\mathbb{R}^2$, the standard topology has a basis of open balls:

$$\mathcal{B} = \{\mathring{B}(x, \varepsilon) \mid \bar{x} \in \mathbb{R}^2, \varepsilon > 0\}$$

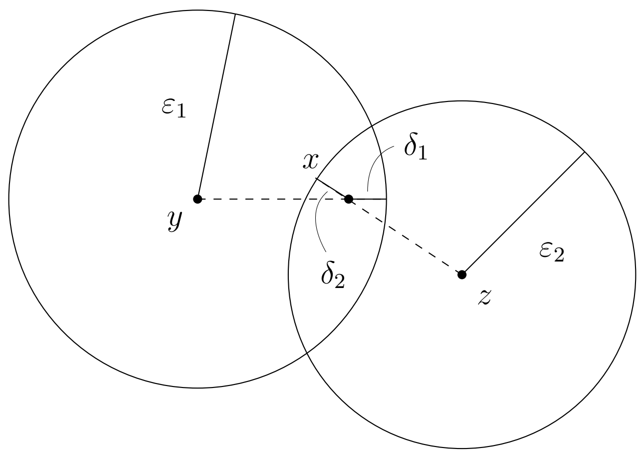
where $\mathring{B}(x, \varepsilon) = \{\bar{y} \mid d(x, y) < \varepsilon\}$. However, another possible topology on $\mathbb{R}^2$ is open rectangles:

$$\mathcal{B}' = \{(a, b) \times (c, d) \subset \mathbb{R}^2 \mid a < b, c < d\}.$$

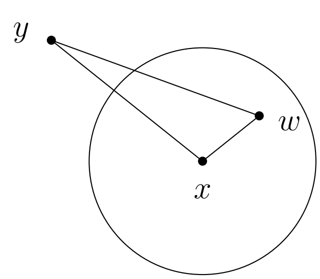
Claim. *The collection $\mathcal{B}$ of open balls in $\mathbb{R}^2$ is a basis for the standard topology.*

Proof. (1) For each $\bar{x} \in \mathbb{R}^2$, $\mathring{B}(x, 1)$ is a basis element containing $\bar{x}$.

(2) Suppose $x \in B_1 \cap B_2$, where $B_1 = \mathring{B}(y, \varepsilon_1)$ and $B_2 = \mathring{B}(z, \varepsilon_2)$, then let δ_1 be the distance from x to B_1, and δ_2 be the distance from x to B_2.



Choose $\delta = \min(\delta_1, \delta_2)$. Then $B_3 = \mathring{B}(x, \delta)$ contains x and lies in $B_1 \cap B_2$. We can verify this using the Triangle Inequality. Take some $\bar{w}$ in B_3:



Then $d(x, w) < \delta$. Without loss of generality, say $\delta = \delta_1$. Then by the Triangle Inequality,

$$d(y, w) \le d(y, x) + d(x, w) < \delta_1 + e - \delta_1 = \varepsilon.$$

Hence $\mathcal{B}$ is a basis. □

Lemma 1.2.6. *Given a topological space $(X, \mathcal{T})$, let $\mathcal{B}$ be a collection of open sets such that for each open set U and for all $x \in U$, there is some set $B_x \in \mathcal{B}$ with $B_x \subset U$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$.*

Proof. (1) X is an open set so for all $x \in X$ there is some $B_x \in \mathcal{B}$ with $x \in B_x \subset X$.

(2) Since $B_1 \cap B_2$ is the intersection of open sets, it too is open. Therefore for all $x \in B_1 \cap B_2$ there exists a $B_x \subset B_1 \cap B_2$. Hence $\mathcal{B}$ is a basis.

Now let $\mathcal{T}'$ be the topology generated by $\mathcal{B}$ (we will show $\mathcal{T}' = \mathcal{T}$). Take U' open in $\mathcal{T}'$. Then U' is a union of sets in $\mathcal{B}$, which are all open in $\mathcal{T}$. Thus U' is open in $\mathcal{T}$, implying $\mathcal{T}' \subset \mathcal{T}$. On the other hand, take V open in $\mathcal{T}$. For all $x \in V$ there is some basis element $B_x \subset V$ such that $x \in B_x$ and B_x is open in $\mathcal{T}$. By the Union Lemma (0.0.1), $V = \bigcup\limits_{x \in V} B_x$ so V is the union of open sets in $\mathcal{T}'$, implying V is itself open in $\mathcal{T}'$. Therefore we conclude that $\mathcal{B}$ generates the topology $\mathcal{T}$ on X. □

Theorem 1.2.7. *Let $\mathcal{B}, \mathcal{B}'$ be bases for topologies $\mathcal{T}, \mathcal{T}'$, respectively, on the set X. Then the following are equivalent:*

1) *$\mathcal{T}'$ is finer than $\mathcal{T}$*
2) *For all $x \in X$ and for each basis element $B \in \mathcal{B}$ containing x, there is some $B' \in \mathcal{B}'$ with $x \in B' \subset B$.*

Proof. ($2 \Rightarrow 1$) Take U open in $\mathcal{T}$ and $x \in U$. Then there exists a basis element $B \subset U$ such that $x \in B$. By hypothesis, there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$, by which U is open in $\mathcal{T}'$. Hence $\mathcal{T} \subset \mathcal{T}'$.

($1 \Rightarrow 2$) Given $x \in B$, since $\mathcal{T} \subset \mathcal{T}'$ we know that B is open in $\mathcal{T}'$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, there is some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. □

1.3 The Continuum Hypothesis

Definition. *A* **countable set** *A is either finite or in bijection with the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Otherwise, A is said to be* **uncountable**.

Example 1.3.1.

Coutably infinite		Uncountable
$\mathbb{N}$	$\mathbb{N} \times \mathbb{N}$	$\mathbb{R}$
$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$[0, 1]$
$\mathbb{Q}$	$\mathbb{Q} \times \mathbb{Q}$	

Facts:

- Finite unions of countable sets are countable.
- Countable unions of countable sets are countable.
- Finite products of countable sets are countable (e.g. $\mathbb{N} \times \mathbb{N}$).

Claim. *X^ω is uncountable.*

This is proven using Cantor's Diagonalization Argument, with $X = \{0, 1\}$, which can easily be generalized to any space X.

Proof. Suppose X^ω is countable. Then we should be able to list the elements:

$$\begin{aligned} x_1 &= (0, 1, 1, 1, 0, 1, \ldots) \\ x_2 &= (1, 1, 0, 1, 0, 1, \ldots) \\ x_3 &= (1, 0, 1, 1, 0, 1, \ldots) \\ &\text{etc.} \end{aligned}$$

Let $y = (1, 0, 0, \ldots)$, i.e. y is the sequence where each component y_i is the *opposite* of the ith term of x_i. Thus $y \neq x_i$ for any i. We were unable to list all the elements, hence X^ω is uncountable. □

This shows that countable products of countable sets, such as X^ω, are uncountable. Other examples of uncountable sets, as listed above, are $\mathbb{R}$ and $[0, 1]$.

Definition. *The* **cardinality** *of a set A is the size of A.*

Note that for any set A, $\text{card}(A) < \text{card}(\mathbb{P}(A))$, where $\mathbb{P}(A)$ is the power set of A. For example, $\mathbb{N}$ is countable, so $\mathbb{P}(\mathbb{N})$ is uncountable. Below is a list of possible cardinalities, ordered from smallest to "largest", although this concept becomes murky as we move from the countable to the uncountable (and beyond).

COUNTABLE	UNCOUNTABLE
$0, 1, 2, 3, \ldots, n, \ldots, \aleph_0,$	$\aleph_1, \aleph_2, \ldots, \aleph_n, \ldots, \aleph_\omega, \ldots$
$0 \to \text{card}(\varnothing)$, $\aleph_0 \to \text{card}(\mathbb{N})$	

The Continuum Hypothesis: The cardinality of the real numbers is $\aleph_1$. In other words, there are no cardinalities which lie between $\text{card}(\mathbb{N}) = \aleph_0$ and $\text{card}(\mathbb{R}) = \aleph_1$.

This hypothesis is undecidable. It is independent of the rest of the ZFT axioms which describe standard set theory. Gödel showed that the Continuum Hypothesis cannot be proven false. However, Paul Cohen later showed that it cannot be proven true either.

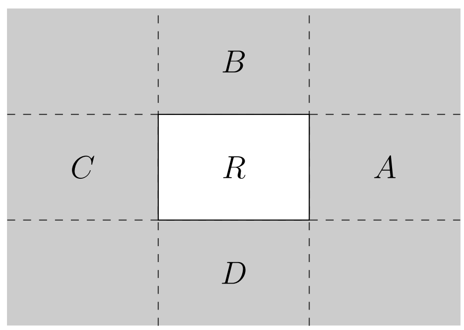
1.4 Closed Sets

Definition. *A* **closed set** *A is the complement of an open set U: $A = X - U$.*

Example 1.4.1.

- Closed intervals $[a, b]$
- Closed disks
- $\mathbb{R}$
- X and $\varnothing$ in any space X

Example 1.4.2. Since open half-planes are open, closed rectangles are closed:



$X - R = A \cup B \cup C \cup D$ which is open.

Proposition 1.4.3. *The complement of an open set is closed.*

Proof. Let C be closed in X. Then by definition, $U = X - C$ is open. The complement of U is $A = X - U$, and $A = X - (X - C) = C$, which is closed. □

Proposition 1.4.4. *Every set in the discrete topology on X is closed.*

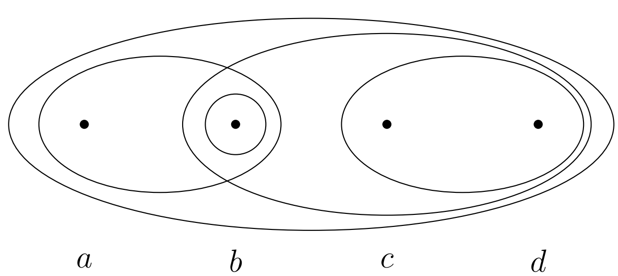
Proof. Take $A \in \mathcal{T}_{dis}$. Then $X - A \in \mathcal{T}_{dis}$ so $X - A$ is open. Hence A is closed. □

Example 1.4.5. In any topological space X, $\varnothing$ and X are both closed and open. In $\mathbb{R}$, the only sets that are both open and closed are $\varnothing$ and $\mathbb{R}$.

Example 1.4.6. $(-\infty, a)$ and $[a, \infty)$ are both open and closed in $\mathbb{R}_l$, but $[a, \infty)$ is not open in $\mathbb{R}_{std}$.

Example 1.4.7. The only closed sets in X_{triv} are $\varnothing$ and X.

Example 1.4.8.



Open only: $\{b\}, \{b, c, d\}$
Closed only: $\{a\}, \{a, c, d\}$
Open and closed: $\{a, b\}, \{c, d\}, X, \varnothing$
Neither: $\{c\}, \{d\}, \{a, b, c\}, \{a, b, c\}, \{a, c\}, \{b, d\}, \{a, d\}, \{b, c\}$

Theorem 1.4.9. *Let X be a topological space. Then*

1) $\varnothing$ and X are closed

2) Arbitrary intersections of closed sets are closed

3) Finite unions of closed sets are closed.

Proof omitted. □

1.5 The Separation Axioms

The T1 Property: All finite point sets are closed.

- In Example 1.4.8 on $X = \{a, b, c, d\}$ is not T1: $\{b\}, \{c\}$ and $\{d\}$ are not closed sets.
- $\mathbb{R}$ with the standard topology is T1
- X with the discrete topology is T1
- X with the trivial topology is not T1
- $\mathbb{R}$ with the finite complement topology ($\mathbb{R}_{fc}$) is T1

Definition. *A set X is* **Hausdorff** *if for all $x, y \in X$ there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.*

Example 1.5.1.

- $\mathbb{R}$, $\mathbb{R}_l$ and $\mathbb{R}_U$ are all Hausdorff
- X with the discrete topology is always Hausdorff
- $\mathbb{R}_{fc}$ is not Hausdorff:

  Take $x \in U$ and $y \in V$, where U, V are open. Then $U \cap V = X - [(X - U) \cup (X - V)]$, where $X - U$ and $X - V$ are each finite. Thus $U \cap V$ only excludes finitely many points, so it is nonempty. Hence there do not exist disjoint, open sets U and V, so $\mathbb{R}_{fc}$ is not Hausdorff.

- The digital line is not Hausdorff:

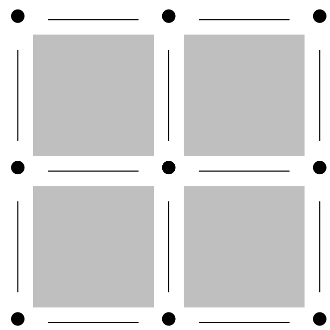
  As a counterexample, 1 is in every open set containing 0.

Theorem 1.5.2. *If a set is Hausdorff, it is T1.*

Proof omitted. □

Example 1.5.3. The digital plane

Consider a set of pixels in the xy-plane:

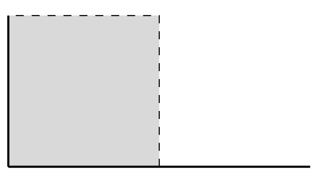


The **digital plane topology** is a product of two digital lines: $\mathbb{R}_{DL} \times \mathbb{R}_{DL}$. Open sets in the digital plane arise from the following basis:

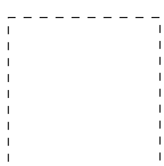
$$B_{m,n} = \begin{cases} \{(m,n)\} & \text{if } m,n \text{ are odd} \\ \{m\} \times \{n-1,n,n+1\} & \text{if } m \text{ is odd and } n \text{ is even} \\ \{m-1,m,m+1\} \times \{n\} & \text{if } m \text{ is even and } n \text{ is odd} \\ \{m-1,m,m+1\} \times \{n-1,n,n+1\} & \text{if } m,n \text{ are even.} \end{cases}$$

1.6 Interior and Closure

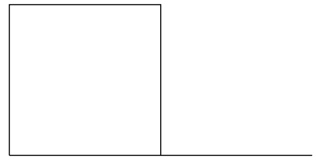
Given a set A,



what is the largest open set contained in A? What is the smallest closed set containing A? In the standard topology on $\mathbb{R}^2$, the largest open set in A is



and the smallest closed set containing A is



(The latter case is not obvious.)

Definition. *For $A \subset X$, the* **interior** *of A, denoted $\mathring{A}$ of* $\mathrm{Int}(A)$*, is the union of all open sets contained in A.*

Definition. *The* **closure** *of A, denoted $\bar{A}$ or* $\mathrm{Cl}(A)$*, is the intersection of all closed sets containing A.*

Facts:

1) $\mathrm{Int}(A)$ is open
2) $\mathrm{Cl}(A)$ is closed
3) $\mathrm{Int}(A) \subset A \subset \mathrm{Cl}(A)$

Proposition 1.6.1. *For any topological space X,*

i) If U is open and $U \subset A$ then $U \subset \mathrm{Int}(A)$.

ii) If C is closed and $C \supset A$ then $C \supset \mathrm{Cl}(A)$.

iii) If $A \subset B$ then $\mathrm{Int}(A) \subset \mathrm{Int}(B)$ and $\mathrm{Cl}(A) \subset \mathrm{Cl}(B)$.

iv) A is open $\iff A = \mathrm{Int}(A)$.

v) A is closed $\iff A = \mathrm{Cl}(A)$.

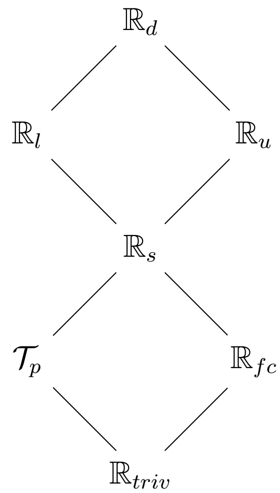
Proof. (i) and (ii) follow directly from the definitions of interior and closure.

(iii) Note that $\mathrm{Int}(A) \subset A \subset B$, so $\mathrm{Int}(A)$ is an open set contained in B. Then by (i), $\mathrm{Int}(A) \subset \mathrm{Int}(B)$. Likewise, $A \subset B \subset \mathrm{Cl}(B)$. Thus $\bar{B}$ contains A and is closed. So $\bar{B} \supset \bar{A}$.

(iv) If A is open then $A \subset A$. So by (i), $A \subset \mathrm{Int}(A)$. And clearly $\mathrm{Int}(A) \subset A$, so we see that $\mathrm{Int}(A) = A$. For the other direction, if $A = \mathrm{Int}(A)$ then $\mathrm{Int}(A)$ is an open set, which means A is as well. (v) is proved in a similar fashion. □

Example 1.6.2. $A = [0, 1)$

Find its closure and interior in the following topologies, where $\mathcal{T}_p = \{(-\infty, p) \mid p \in \mathbb{R}\}$:



Topology	$\mathring{A}$	$\bar{A}$
$\mathbb{R}_d$	A	A
$\mathbb{R}_l$	A	A
$\mathbb{R}_u$	$(0,1)$	$[0,1]$
$\mathbb{R}_s$	$(0,1)$	$[0,1]$
$\mathcal{T}_p$	$\varnothing$	$[0,\infty)$
$\mathbb{R}_{fc}$	$\varnothing$	$\mathbb{R}$
$\mathbb{R}_{triv}$	$\varnothing$	$\mathbb{R}$

Proposition 1.6.3. *If $\mathcal{T}_1 \subset \mathcal{T}_2$ on X, then $\mathrm{Int}_{\mathcal{T}_1}(A) \subset \mathrm{Int}_{\mathcal{T}_2}(A)$ and $\mathrm{Cl}_{\mathcal{T}_1}(A) \supset \mathrm{Cl}_{\mathcal{T}_2}(A)$ for all $A \subset X$.*

Proof omitted. □

Properties of Interior and Closure

1) $\mathrm{Int}(A) \neq \mathrm{Int}(\bar{A})$ in general

   For example, $A = (0,1) \cup (1,2)$ is open in $\mathbb{R}$, so $\mathrm{Int}(A) = A$. But $\bar{A} = [0,2]$, and $\mathrm{Int}(\bar{A}) = (0,2) \neq A$.

2) Consider $\mathbb{Q} \subset \mathbb{R}$. Then $\bar{\mathbb{Q}} = \mathbb{R}$ and $\mathrm{Int}(\mathbb{Q}) = \varnothing$.

Definition. *A set $A \subset X$ is* **dense** *if $\bar{A} = X$.*

Example 1.6.4. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example 1.6.5. What sets are dense in $\mathbb{R}_{fc}$?

All finite sets are closed, so they are their own closure, thus not dense. All infinite sets are not closed, and $\mathbb{R}$ is the only closed set that contains all infinite sets. Therefore all infinite sets are dense.

Theorem 1.6.6. *Let $y \in A \subset X$. Then*

1) $y \in \mathrm{Int}(A) \iff$ there is an open set U with $y \in U \subset A$

2) $y \in \bar{A} \iff$ every open set containing y intersects A.

Proof omitted. □

Theorem 1.6.7. *Let $A, B \subset X$. Then*

1) $\mathrm{Int}(X - A) = X - \mathrm{Cl}(A)$

2) $\mathrm{Cl}(X - A) = X - \mathrm{Int}(A)$

3) $\mathrm{Int}(A) \cup \mathrm{Int}(B) \subset \mathrm{Int}(A \cup B)$ *but these are not equal in general*

4) $\mathrm{Int}(A) \cap \mathrm{Int}(B) = \mathrm{Int}(A \cap B)$.

Proof. (1) Take $x \in \mathrm{Int}(X - A)$, which is open. By definition, $\mathrm{Int}(X - A) \subset X - A$, so $x \notin A$, and $X - A$ is an open set that does not intersect A. Thus by Theorem 1.6.6, $x \in X - \mathrm{Cl}(A) \implies \mathrm{Int}(X - A) \subset X - \mathrm{Cl}(A)$. Now take $x \in X - \mathrm{Cl}(A)$, so $x \notin \mathrm{Cl}(A)$. Then there is some open U containing x that is disjoint from A. In other words, $U \subset X - A$. Since $\mathrm{Int}(X - A)$ is the union of all open sets in $X - A$, $U \subset \mathrm{Int}(X - A)$. Hence $x \in \mathrm{Int}(X - A) \implies X - \mathrm{Cl}(A) = \mathrm{Int}(X - A)$.

(2) is proven similarly.

(3) Without loss of generality, take $x \in \mathrm{Int}(A)$. Then by Theorem 1.6.6, there is some open set U such that $x \in U \subset A$. Thus $x \in U \subset A \cup B$ so by the same theorem, $x \in \mathrm{Int}(A \cup B)$. This proves that $\mathrm{Int}(A) \cup \mathrm{Int}(B) \subset \mathrm{Int}(A \cup B)$. However, note that equality doesnt hold. For example, if $A = (0, 1)$ and $B = (1, 2)$, then $\mathrm{Int}(A \cup B) = (0, 2) \neq \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

(4) is proven similarly. □

1.7 Limit Points

In analysis, the limit points of A are the limits of convergent sequences in A. In topology, limit points are defined using open neighborhoods.

Definition. *A* **limit point** *x of $A \subset X$ has the property that every neighborhood U of x intersects A somewhere other than x. The set of limit points of a set A will be denoted A'.*

Proposition 1.7.1. *Every limit point of A is in the closure of A.*

Proof. This follows from $y \in \bar{A} \iff$ every open set containing y intersects A. □

However, A and A' are incomparable, as shown below.

Example 1.7.2. $A = [0, 1) \cup \{2\}$

Then $\bar{A} = [0, 1] \cup \{2\}$ and $A' = [0, 1]$ since every point in between and including 0,1 is a limit point, but 2 is not. Thus $A' \subset \bar{A}$ but A' and A are incomparable.

Example 1.7.3. $B = \left\{ \dfrac{1}{n} \mid n \in \mathbb{N} \right\}$

0 is a limit point (even though $0 \notin B$) since every neighborhood of 0 contains an interval $(-\varepsilon, \varepsilon)$—by the Archimedes Principle, there is some $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ so every neighborhood of 0 intersects B at a point other than 0. However, no point in B is in B'. So $B' = \{0\}$ which is disjoint from B. Moreover, $\bar{B} = B \cup B'$.

Theorem 1.7.4. $\bar{B} = B \cup B'$.

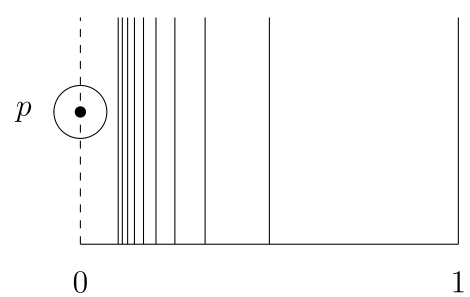
Proof. $B \subset \bar{B}$ for any set B, and $B' \subset \bar{B}$ by Proposition 1.7, so $B \cup B' \subset \bar{B}$. Now take $x \in \bar{B}$. Then every open neighborhood U of x must intersect B at some point y. If $x \in B$, x is in the union already, so assume $x \notin B$. Since $y \in B$, $x \neq y$ and thus U intersects B at a point other than x. So x is a limit point. Hence $x \in B \cup B' \implies \bar{B} = B \cup B'$. □

Corollary 1.7.5. *B is closed if and only if B contains all of its limit points.*

Proof. Assume B is closed, by which $B = \bar{B}$. Then by Theorem 1.7.4, $B = B \cup B'$, which means $B' \subset B$. Now assume $B' \subset B$. Then $B \cup B' = B$ and by the same theorem, $\bar{B} = B \cup B' = B$, so B is closed. □

Example 1.7.6. The Infinite Comb

Let C be the infinite comb $C = \{(x, 0) \mid 0 \leq x \leq 1\} \cup \left\{ \left(\frac{1}{2^n}, y\right) \mid 0 \leq y \leq 1 \right\}$.



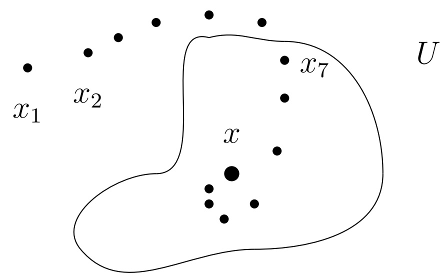
Let Y be the missing "tooth" at $x = 0$. Take $p = (0, y) \in Y$. Then for any neighborhood of p, say $\mathring{B}(p, \varepsilon)$, the Archimedes Principle guarantees there exists an n such that $\frac{1}{n} < \varepsilon$. Further, this gives us $\frac{1}{2^n} < \frac{1}{n} < \varepsilon$. So every neighborhood of p intersects C, thus p is a limit point of C. And $p \notin C$ so C is not closed by Corollary 1.7.5. Let C' be the set of limit points of C. We showed that $Y \subset C'$. Also, any point on C is also in C'. Thus $C \cup Y \subset C'$. There are no other limit points of C, so $\overline{C} = C \cup Y$.

1.8 Sequences

In topology, the notion of a convergent sequence must not rely on distance.

Definition. *In a topological space X, a sequence (x_n) is* **convergent** *if there exists a limit $x \in X$ such that for all open neighborhoods U containing x, there exists an $N > 0$ such that if $n \geq N$ then $x_n \in U$.*

Example 1.8.1.



Let $N = 7$. Then $(x_n) \to x$.

Example 1.8.2. $x_n = \dfrac{(-1)^n}{n}$

In $\mathbb{R}$, $\lim_{n\to\infty} x_n = 0$ (as in calculus class). Any open neighborhood of 0 contains some neighborhood $(-\varepsilon, \varepsilon)$. Does (x_n) converge in $\mathbb{R}_l$? In this topology, $[0, \varepsilon)$ is an open neighborhood of 0 (for some $\varepsilon > 0$). And (x_n) alternates between $+$ and $-$, so for all odd n, $x_n \notin [0, \varepsilon)$. Thus there is no N such that $x_n \in [0, \varepsilon)$ for all $n \geq N$. So (x_n) does not converge in $\mathbb{R}_l$.

Example 1.8.3. In the discrete topology, the only convergent sequences are those that are eventually constant, e.g. $(0, 5, 1, 3, 1, 1, 1, 1, 1, \ldots)$

Question. *Is every limit point of A the limit of some sequence in A? And vice versa?*

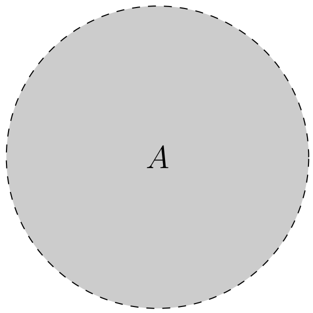
- In $\mathbb{R}^n$ with the standard topology, yes.
- In general, no. In fact, there exist limit points that are not limits of convergent sequences.
- Some sequences converge to more than one point.
- It is easier to converge in coarser spaces (there are less neighborhoods of each point to consider).

Theorem 1.8.4. *If X is Hausdorff then every convergent sequence has a unique limit.*

Proof. Let (x_n) converge to $x \in X$. Consider a point $y \in X$ that is distinct from x. Since X is Hausdorff, there are disjoint neighborhoods U_x and U_y containing x and y, respectively. Since $(x_n) \to x$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U_x$. But for all $n \geq N$, $x_n \notin U_y$ and thus (x_n) cannot converge to y. Since y was any point in X other than x, the limit $(x_n) \to x$ is unique. $\square$

1.9 Boundaries

Example 1.9.1. $A = \{x^2 + y^2 < 1 \mid x, y \in \mathbb{R}\} \subset \mathbb{R}^2$



then its boundary is $\{x^2 + y^2 = 1\}$

Definition. *Let $A \subset X$. Then the* **boundary** *of A is $\partial A = \bar{A} \setminus \mathrm{Int}(A)$.*

Example 1.9.2. $[0, 1) \subset \mathbb{R}$

$\partial A = \{0, 1\}$. In fact, for any interval in the standard topology, its boundary is the set of its endpoints.

Example 1.9.3. $[0, 1) \subset \mathbb{R}_l$

$\bar{A} = A$ and $\mathrm{Int}(A) = A$ since A is both open and closed. Thus $\partial A = \varnothing$. In general, if A is both open and closed then its boundary is empty.

Theorem 1.9.4. *$x \in \partial A$ if and only if every neighborhood of x intersects both A and $X - A$.*

Proof. By definition, $x \in \partial A$ implies that $x \in \bar{A}$ and $x \notin \mathrm{Int}(A)$. By the first, every neighborhood U of x intersects A. And by the second, $x \in X - \mathrm{Int}(A)$. By (2) of Theorem 1.6.7, $X - \mathrm{Int}(A) = \mathrm{Cl}(X - A)$. So since $x \in \mathrm{Cl}(X - A)$, every neighborhood U of x must intersect $X - A$. For the other direction, suppose every neighborhood U of x intersects X and $X - A$. Then $x \in \bar{A}$ since U intersects X. And since every such U intersects $X - A$, there does not exist a neighborhood $U \subset A$. So $x \notin \mathrm{Int}(A)$. Finally, we have that $x \in \bar{A} - \mathrm{Int}(A) = \partial A$. $\square$

Example 1.9.5. What is $\partial\mathbb{Q}$ in $\mathbb{R}$?

$\mathrm{Int}(\mathbb{Q}) = \varnothing$ and $\bar{\mathbb{Q}} = \mathbb{R}$ (the rationals are dense in the reals). Therefore $\partial\mathbb{Q} = \mathbb{R} - \varnothing = \mathbb{R}$. In other words, every neighborhood of $q \in \mathbb{Q}$ contains a rational and an irrational number, so $\mathbb{Q} \subset \partial\mathbb{Q}$. The same logic applies for all $x \notin \mathbb{Q}$.

Example 1.9.6. For $A \subset X_{discrete}$, $\partial A = \varnothing$ since A is both open and closed.

Example 1.9.7. For $A \subset X_{trivial}$ $(A \neq \varnothing, X)$, $\bar{A} = X$ and $\mathrm{Int}(A) = \varnothing$ so $\partial A = X$.

Heuristically, ∂A represents how far apart $\bar{A}$ and $\text{Int}(A)$ are.

- The coarser the topology, the "further" apart they are.
- In $\mathcal{T}_{triv}$, $\partial A = X$.
- In $\mathcal{T}_{disc}$, $\partial A = \varnothing$.

Proposition 1.9.8. *For $A \subset X$,*

1) ∂A is closed

2) $\partial A = \bar{A} \cap \overline{X - A}$

3) $\partial A \cap \text{Int}(A) = \varnothing$

4) $\partial \cup \text{Int}(A) = \bar{A}$

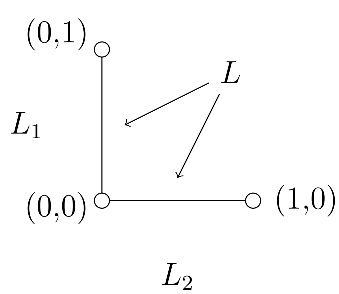
5) $\partial A \subset A \iff A$ is closed

6) $\partial A \cap A = \varnothing \iff A$ is open

7) $\partial A = \varnothing \iff A$ is both open and closed.

Proof omitted. □

Example 1.9.9. Consider the following set $L \subset \mathbb{R}^2$ with the vertical interval topology



Note that in $\mathbb{R}^2_{std}$, $\bar{L} = L \cup \{(0,0),(0,1),(1,0)\}$, and $\text{Int}(L) = \varnothing$, so $\partial L = \bar{L}$. However, in $\mathbb{R}^2_{VI}$, take $x \in L_1$ which is open in this topology. Then there exist neighborhoods of x that don't intersect $X - L$, so $L_1 \not\subset \partial L$. Note that $\bar{L_1} = \{0\} \times [0,1]$. Also, $\text{Int}(L_2) = \varnothing$ and since $\mathbb{R}^2 \setminus L_2$ is open, L_2 is closed. Thus $\bar{L_2} = L_2$. So in the vertical interval topology,

$$\begin{aligned} \partial L_1 &= \{0\} \times \{0,1\} \\ \partial L_2 &= L_2 \\ \Rightarrow \partial L &= (0,1) \times \{0\} \cup \{(0,0),(0,1)\}. \end{aligned}$$

1.10 Applications to GIS

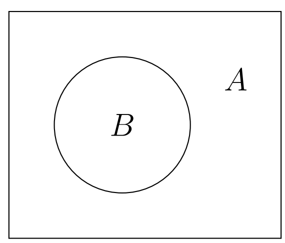
In real life examples, we may want to know if A intersects B.

Definition. *For $y \subset X$, we define* $C_Y = \begin{cases} 1 & \textit{if } Y \textit{ is nonempty} \\ 0 & \textit{if } Y \textit{ is empty.} \end{cases}$

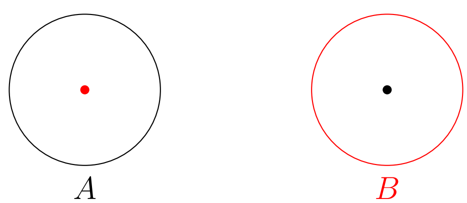
Definition. *The* **intersection quadruple** *for sets A and B is defined as*

$$I_{A,B} = \left(C_{\partial A \cap \partial B}, C_{\mathring{A} \cap \mathring{B}}, C_{\partial A \cap \mathring{B}}, C_{\mathring{A} \cap \partial B} \right).$$

Example 1.10.1. $I_{A,B} = (0,1,0,1)$

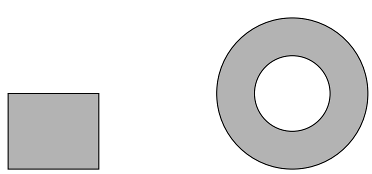


Example 1.10.2. $I_{A,B} = (0,0,1,1)$

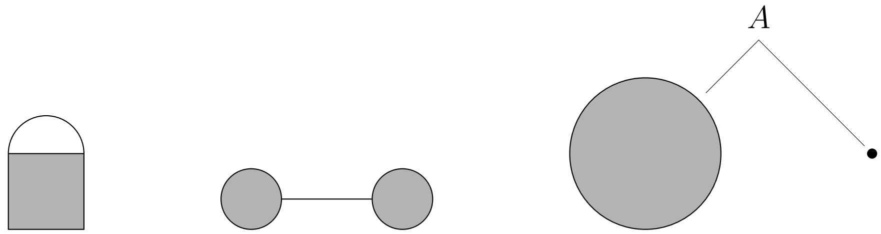


Definition. *A is* **regularly closed** *if* $A = \mathrm{Cl}(\mathrm{Int}(A))$.

Example 1.10.3. Some regularly closed sets.



Example 1.10.4. Some sets that are not regularly closed.



Remark. All boundary points are close to some interior point.

Theorem 1.10.5. *If A is regularly closed and $\partial A \cap \mathring{B} \neq \varnothing$ then $\mathring{A} \cap \mathring{B} \neq \varnothing$.*

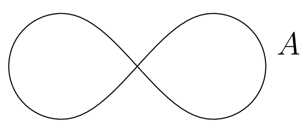
Proof omitted. □

However, it is still possible that $\partial A \cap \partial B = \varnothing$ but $\mathring{A} \cap \partial B \neq \varnothing$.

Corollary 1.10.6. *For A, B regularly closed, the following $I_{A,B}$ values are impossible:*

$$\begin{array}{ccc} (1,0,1,0) & (1,0,0,1) & (1,0,1,1) \\ (0,0,1,0) & (0,0,0,1) & (0,0,1,1). \end{array}$$

Furthermore, if we throw out sets like A:



and restrict the list to **planar spatial regions**, *then there are only 9 possible $I_{A,B}$ values.*

Explicitly,

Definition. *A* **planar spatial region** *is a proper subset of $\mathbb{R}^2$ whose interior is not a disjoint union of open sets, and is regularly closed.*

Note: "proper" excludes the case where $I_{A,B} = (0,1,0,0)$. Now, each of the 9 values for $I_{A,B}$ implies certain facts about A and B.

Example 1.10.7. $I_{A,B} = (1,1,1,0) \implies A \subset B$

Example 1.10.8. $I_{A,B} = (1,1,0,0) \implies A = B$

2 Creating New Topological Spaces

2.1 The Subspace Topology

Definition. *Let $Y \subset X$ where X is a topological space. Define the* **subspace topology** *on Y by $\mathcal{T}_Y = \{U \cap Y \mid U \text{ is open in } X\}$. Then $(Y, \mathcal{T}_Y)$ is itself a topological space.*

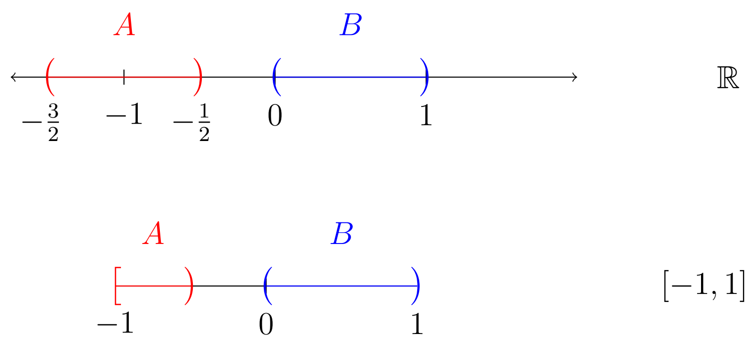
Claim. *$\mathcal{T}_Y$ is a topology on Y.*

Proof. (1) Since $\varnothing$ is open in X and $\varnothing \cap Y = \varnothing$, $\varnothing$ is open in Y. And since $X \cap Y = Y$, Y is open in Y.

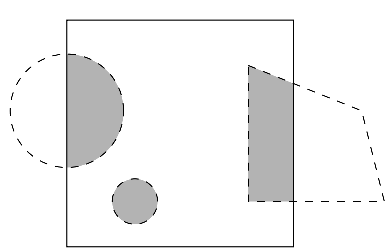
(2) Let $V = \bigcup\limits_{\alpha} V_\alpha$ where each V_α is open in Y. So each $V_\alpha = U_\alpha \cap Y$ where U_α is open in X. Thus $V = \bigcup\limits_{\alpha}(U_\alpha \cap Y) = \left(\bigcup\limits_{\alpha} U_\alpha\right) \cap Y$, and the arbitrary union of open sets in X is open, so $\bigcup\limits_{\alpha} U_\alpha = U$ is open, which implies $U \cap Y$ is open in Y.

(3) Let $W = \bigcap\limits_{i=1}^{n} W_i$ where each W_i is open in Y. Then each $W_i = U_i \cap Y$ for some U_i open in X. Then $W = \bigcap\limits_{i=1}^{n} W_i = \bigcap\limits_{i=1}^{n} (U_i \cap Y) = \left(\bigcap\limits_{i=1}^{n} U_i\right) \cap Y$, and since finite intersections of open sets in X are open, $\bigcap\limits_{i=1}^{n} U_i = U$ is open in X. Thus $U \cap Y$ is open in Y. We conclude that $\mathcal{T}_Y$ is indeed a topology on Y. $\square$

Example 2.1.1. $[-1, 1] \subset \mathbb{R}$



Example 2.1.2. $X = \mathbb{R}^2, Y = [0, 1] \times [0, 1]$



Example 2.1.3. If $Y \subset \mathbb{R}^n$, the standard topology on Y is the subspace topology inherited by Y from $\mathbb{R}^n_{std}$

Example 2.1.4. $\mathbb{Z}_{std}$ is a topological subspace of $\mathbb{R}$, and it is easy to show that $\mathbb{Z}_{std} = \mathbb{Z}_{discrete}$

Proposition 2.1.5. *Let $Y \subset X$ have the subspace topology and let $U \subset Y$. Then*

1) U is open in $X \implies U$ is open in Y

2) If Y is open in X, then U open in $Y implies U$ open in X.

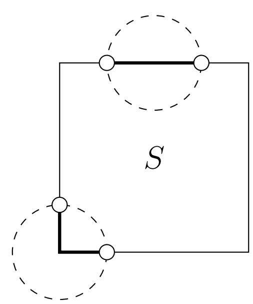
Proof omitted. Note that the converse of (1) is not true in general. □

Definition. *If C is* **closed in Y**, *then $C = D \cap Y$ for a closet set $D \subset X$.*

Theorem 2.1.6. *If $\mathcal{B}_X$ is a basis for X, then a basis for $(Y, \mathcal{T}_{sub})$ is $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}_X\}$.*

Proof. Let $\mathcal{B}_Y$ be a collection of open sets in Y given by $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}_X\}$. Consider any U open in Y. Then $U = V \cap Y$ for some V open in X. Let $Y \in U$. Then $y \in V$ and since $\mathcal{B}_X$ is a basis on X, there is some $B_1 \in \mathcal{B}_X$ such that $y \in B_1 \subset V$. Thus we have $y \in B_1$ and $y \in Y \implies x \in B_1 \cap Y$. By definition, $B_1 \cap Y \in \mathcal{B}_Y$, and $B_1 \cap Y \subset V \cap Y = U$. By Lemma 1.2.6, this is sufficient to show that $\mathcal{B}_X$ is a basis for the subspace topology on Y. □

Example 2.1.7. The square $S \subset \mathbb{R}^2$

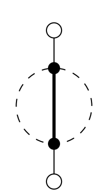


A basis for the subspace topology on S is

$$\{\text{open intervals on one edge of } S\} \cup \{\text{L-shaped open “intervals”}\}.$$

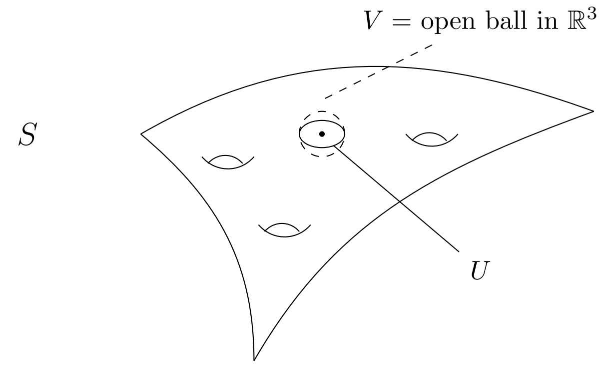
Example 2.1.8. $I = \{a\} \times (b, c)$

I

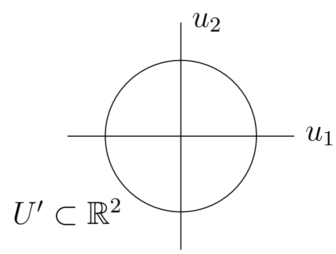


Consider I in the vertical interval topology on $\mathbb{R}^2$. Then the subspace topology on I inherited from $\mathbb{R}^2_{std}$ is equivalent to the subspace topology inherited from the vertical interval topology.

Example 2.1.9. Let $S \subset \mathbb{R}^3$ be a surface

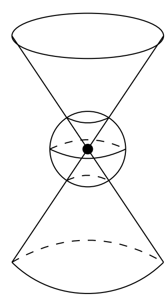


In differential geometry, we parametrize S by taking the subspace topology (where open sets are $U = V \cap S$ with V open in $\mathbb{R}^3$). Then the U pictured above is equivalent to:

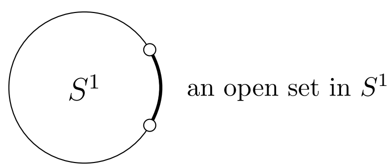


Then establishing coordinates u_1 and u_2 allows us to do calculus on S.

Example 2.1.10. The cone cannot be regularly parametrized



Example 2.1.11. $S^1 = \{x^2 + y^2 = 1 \mid x, y \in \mathbb{R}\} \subset \mathbb{R}^2$



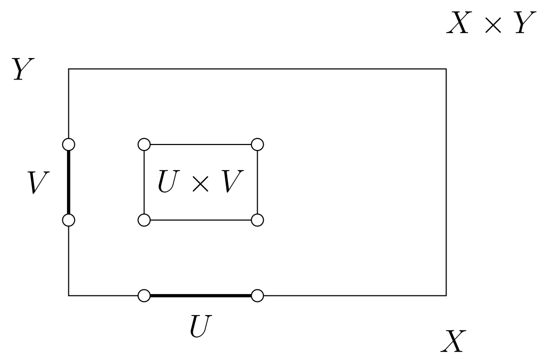
When we refer to "a circle", we mean anything that is topologically equivalent to S^1. A basis for the subspace topology on S^1 is $\mathcal{B} = \{\text{open arcs of } S^1\}$.

Proposition 2.1.12. *Let $Y \subset X$. If X is Hausdorff then Y is Hausdorff.*

Proof omitted. □

2.2 The Product Topology

Suppose we have two topological spaces X and Y and cross them:



If U is open in X and V is open in Y, we want $U \times V$ to be open in $X \times Y$. However, not all open sets in $X \times Y$ will be a product like $U \times V$.

Definition. *The basis for the* **product topology** *on $X \times Y$ is*

$$\mathcal{B} = \{U \times V \mid U \textit{ open in } X, V \textit{ open in } Y\}.$$

Proposition 2.2.1. *$\mathcal{B}$ is a basis for the product topology on $X \times Y$.*

Proof. (1) Since X is open in X and Y is open in Y, $X \times Y$ is a basis element and contains all points.

(2) Take $(x, y) \in B_1 \cap B_2$, where $B_1 = U_1 \cap V_1$ and $B_2 = U_2 \times V_2$. Then since $U_1 \cap U_2$ is open in X and $V_1 \cap V_2$ is open in Y,

$$B_1 \cap B_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times ((V_1 \cap V_2)$$

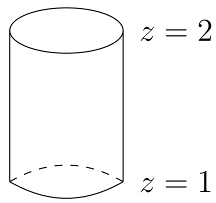
is a basis element containing (x, y), and is itself contained in $B_1 \cap B_2$. Thus $\mathcal{B}$ is a basis. □

Proposition 2.2.2. *$\mathcal{E} = \{C \in \mathcal{C}$, a basis for X, and $D \in \mathcal{D}$, a basis for $Y\}$ is a basis for the product topology on $X \times Y$.*

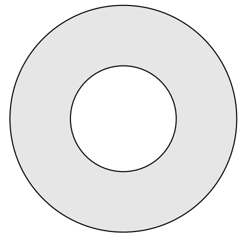
Proof. $\mathcal{E}$ is a collection of open sets on $X \times Y$. Suppose W is open in $X \times Y$; let $W = U \times V$. Take $p = (x, y) \in W$. Then $x \in U$ and $y \in V$. Since $\mathcal{C}$ is a basis for X, there is some $C \in \mathcal{C}$ such that $x \in C \subset U$. And likewise, there is some $D \in \mathcal{D}$ such that $y \in D \subset V$. Then $(x, y) \in C \times D \subset U \times V = W$. By Lemma 1.2.6, $\mathcal{E}$ is a basis that generates the product topology on $X \times Y$. □

To visualize $X \times Y$, imagine attaching a perpendicular copy of Y at each point $x \in X$ (or vice versa). In topology, $X \times Y$ and $Y \times X$ are equivalent (orientation in this sense is unimportant).

Example 2.2.3. $S^1 \times [1, 2]$



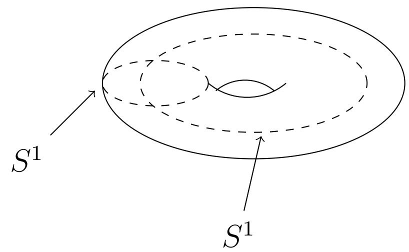
cylinder



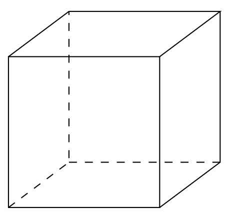
annulus

basis = {open sectors}

Example 2.2.4. $S^1 \times S^1 = T^2$, a torus

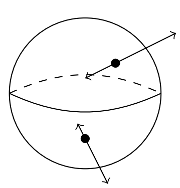


Likewise, $S^1 \times \bar{D}^2$, where $\bar{D}^2$ is a closed disk, produces a solid torus ("shell" plus the inside). In general, $(S^1)^n = S^1 \times S^1 \times \cdots \times S^1 = T^n$, an n-dimensional torus.



T^3 if opposite faces are glued together

Example 2.2.5. $S^2 \times \mathbb{R}$, where $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$



$S^2 \times \mathbb{R}$ is a 3-dimensional manifold, but it is *not* a subset of $\mathbb{R}^3$.

Proposition 2.2.6. *The following are properties of sets in the product topology.*

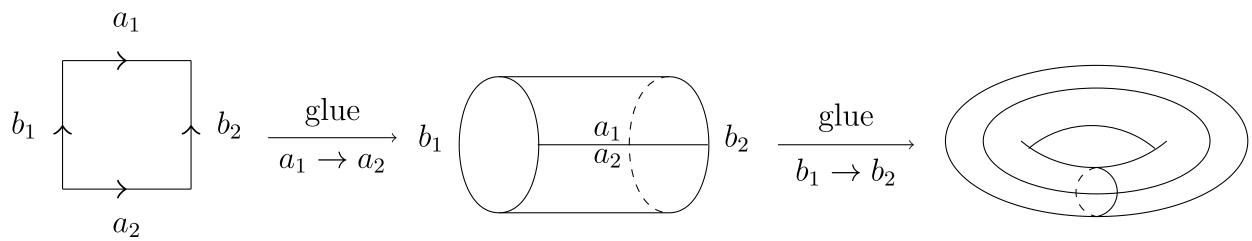
1) $\text{Int}(A \times B) = \text{Int}(A) \times \text{Int}(B)$

2) $\overline{A \times B} = \bar{A} \times \bar{B}$

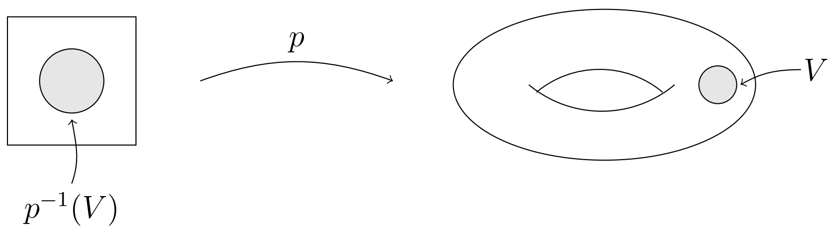
3) $\partial(A \times B) \neq \partial A \times \partial B$.

Proof omitted. $\square$

2.3 The Quotient Topology



Consider an open set on the torus:



Idea: in a quotient topology, open sets have open preimages.

Definition. *Let X be a topological space and let $p : X \to Q$ be a surjective map onto the set Q. The* **quotient topology** $\mathcal{T}_p$ *on Q is given by:*

- *$V \subset Q$ is open in $Q \iff p^{-1}(V)$ is open in X*
- *Q is then called a* **quotient space**
- *p is called a* **quotient map** *between topological spaces.*

Claim. *$\mathcal{T}_p$ is a topology.*

Proof. (1) Since $\varnothing \subset Q$ and $p^{-1}(\varnothing) = \varnothing$ is open in X, $\varnothing$ is open in Q. And since p is well-defined and onto, $p^{-1}(Q) = X$ so Q is open in Q.

(2) If V_α are open in Q then each $p^{-1}(V_\alpha)$ is open in X. Let $V = \bigcup\limits_\alpha V_\alpha$ and consider

$$p^{-1}(V) = p^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha p^{-1}(V_\alpha).$$

Note that we cannot do this in the other direction, i.e. $\bigcup\limits_\alpha p^{-1}(V_\alpha) \neq p^{-1}\left(\bigcup\limits_\alpha V_\alpha\right)$ in general. However in our case, this is the union of open sets in X, so $p^{-1}(V)$ is open in X. Hence arbitrary unions are open in Q.

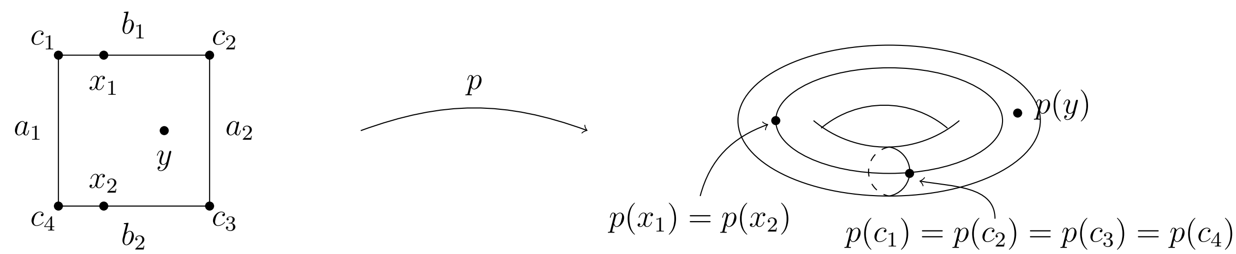
(3) If U_i are open in Q, then each $p^{-1}(U_i)$ is openin X. Let $U = \bigcap_{i=1}^{n} U_i$. Then

$$p^{-1}(U) = p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i)$$

since for each $x \in U_i$, $p^{-1}(x) \in p^{-1}(U_i)$. This is the finite intersection of open sets in X, so $p^{-1}(U)$ is open in X. Hence U is open in Q and we conclude that $\mathcal{T}_p$ is a topology. $\square$

Quotient spaces are closely related to equivalence classes. If we partition X by $\sim$, we can view $\sim: X \to \{\text{equivalence classes on } X\}$ as a surjective map between X and the quotient space formed by the equivalence classes of $\sim$ on X.

Example 2.3.1.



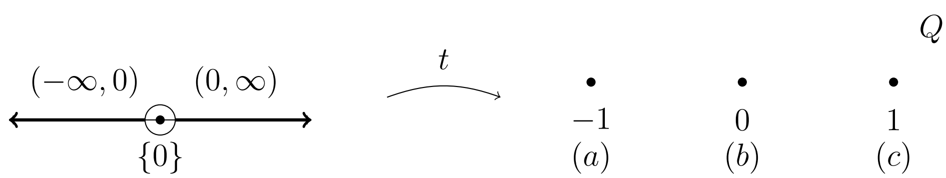
$[x_1] = \{x_1, x_2\}$
$[y] = \{y\}$
$[c_1] = \{c_1, c_2, c_3, c_4\}$

We can define $\sim$ by $x \sim y$ if $p(x) = p(y)$ and get the above equivalence classes.

Example 2.3.2. Consider the trichotomy property on $\mathbb{R}$: If a and b are real numbers, exactly one of the following is true:

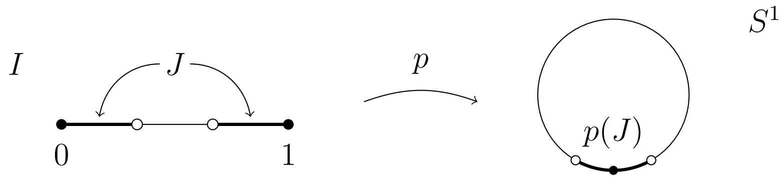
- $a < b$
- $a > b$
- $a = b$

Let $b = 0$. Then we can partition $\mathbb{R}$ into 3 pieces:



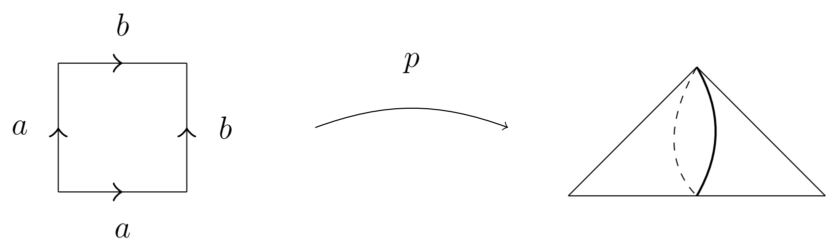
Define $x \sim y$ if either $x, y < 0$, $x, y > 0$ or $x = y = 0$. Then the classes $[-1]$, $[1]$ and $[0]$ partition $\mathbb{R}$. Thus we can describe the quotient topology $\mathcal{T}_t$ on Q. Notice that $t^{-1}(\{b, c\}) = [0, \infty)$, which is not open in $\mathbb{R}$. So $\{b, c\}$ is not open in Q. Likewise, $\{b\}$ and $\{a, b\}$ are not open. On the other hand, $t^{-1}(\{a, c\}) = \mathbb{R} \setminus \{0\}$ which is open in $\mathbb{R}$. Thus $\{a, c\}$ is open in Q, as are $\{a\}$ and $\{c\}$. As always, Q and $\varnothing$ are open, so $\mathcal{T}_t = \{\varnothing, Q, \{a\}, \{c\}, \{a, c\}\}$.

Example 2.3.3. Let $I = [0, 1] \subset \mathbb{R}$ and glue together 0 and 1. The result is a circle



J is an open set in I with the subspace topology. Thus the open arc $p(J)$ is open in the quotient space S^1.

Example 2.3.4.



Let p glue a sides together and b sides together. The result is a "hollow turnover" sort of shape, which is equivalent to a sphere.

Definition. *The* **real projective space** $\mathbb{R}P^n$ *is the quotient space formed by identifying points on the n-sphere* $S^n \subset \mathbb{R}^{n+1}$ *via the given equivalence relation* $\sim$: $\bar{x} \sim -\bar{x}$, *i.e. identifying antipodal points.*

Remark. We have developed a correspondence between three highly related concepts:

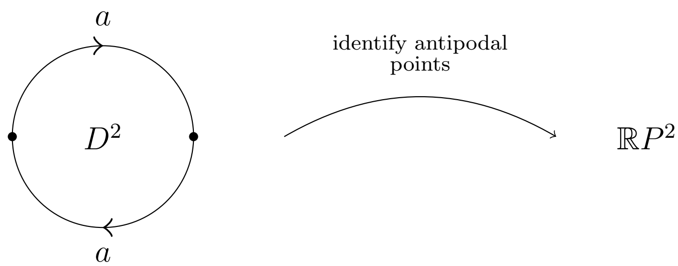
$$\begin{array}{c}\text{equivalence relations on } X \\ \updownarrow \\ \text{partitions of } X \\ \updownarrow \\ \text{quotient maps on } X\end{array}$$

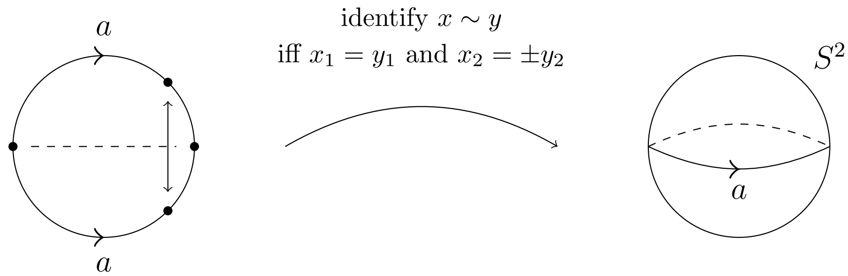
Definition. *Given an equivalence relation* $\sim$ *on* X, *define* $p : X \to X/\sim$, *where*

$$X/\sim \ = \{\textit{equivalence relations on } X \textit{ under } \sim\}$$

by $y \mapsto [y]$. *Then* p *is a quotient map and* $X/\sim$ *is a quotient space when equipped with the quotient topology. Similarly, given* $p : X \to Q$, *define an equivalence relation* $\sim$ *by* $x \sim y$ *iff* $p(x) = p(y)$.

Example 2.3.5. Quotient maps of the disk D^2

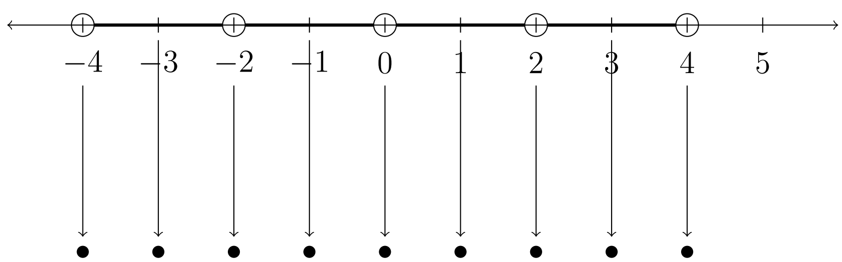




Example 2.3.6. The digital line revisited

Consider the map $p : \mathbb{R} \to \mathbb{Z}$ defined by

$$p(x) = \begin{cases} x & x \in \mathbb{Z} \\ 2n+1 & x \in (2n, 2n+2), n \in \mathbb{Z}. \end{cases}$$



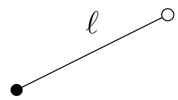
What sets are open in the quotient topology $\mathcal{T}_p$? For starters, $p^{-1}(\{1\}) = (0,2)$ which is open in $\mathbb{R}$, so $\{1\}$ is open in $\mathcal{T}_p$ (and this holds for any odd integer point-set). However, $p^{-1}(\{0\}) = \{0\}$ which is not open in $\mathbb{R}$, so $\{0\} \notin \mathcal{T}_p$. But we see that $p^{-1}(\{-1,0,1\}) = (-2,2)$, open in $\mathbb{R}$, so sets like $\{-1,0,1\}$ *are* open in $\mathcal{T}_p$. Therefore $\mathcal{T}_p$ is the quotient topology that corresponds to the digital line topology on $\mathbb{Z}$.

Example 2.3.7. Graph theory

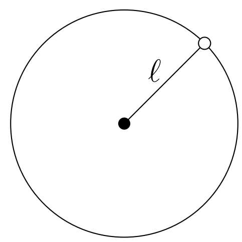
In topological graph theory, we can specify vertices a, b, c, d, e and edges, which are closed, bounded intervals with endpoints in $\{a, b, c, d, e\}$. Then a quotient space is obtained when endpoints are identified.

2.4 Configuration Spaces

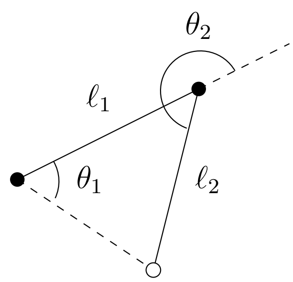
Say we have a robot consisting of just one arm:



What are all the possible states in which the arm can lie? Answer: a circle!



Now say we have two "arm" segments:



Assuming ℓ_1 and ℓ_2 can cross, θ_1 and θ_2 are each 2π periodic, so the configuration space would be $S^1 \times S^1$. However, if ℓ_1 and ℓ_2 are not allowed to cross, we obtain $S^1 \times (-\pi, \pi)$ as the configuration space.

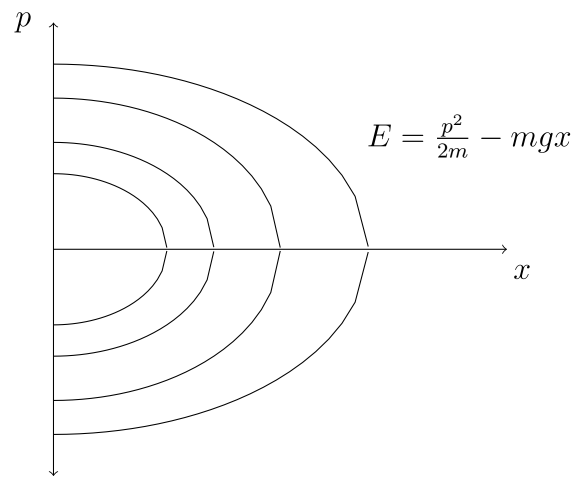
Definition. *The set of points where a given point can "travel", or end up, is called an* **operational space**. *This is a subset of the original space in which the point lies.*

Example 2.4.1. Phase spaces

In physics, a phase space is concerned with a particle's position and momentum, as described by such equations as

$$\begin{aligned} KE &= \frac{p^2}{2m} \\ PE &= -mgx \qquad \text{where } x \text{ is height} \\ \implies E &= \frac{p^2}{2m} - mgx. \end{aligned}$$

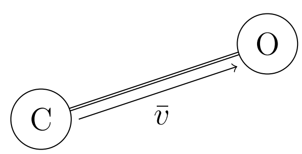
Look at the contours in the xp-plane:



This shows that $\frac{p^2}{2m} - mgx = c$, a constant. Furthermore, we could identify points of the same energy level and form a quotient space.

Example 2.4.2. Molecular configurations

Suppose we have a molecule CO:



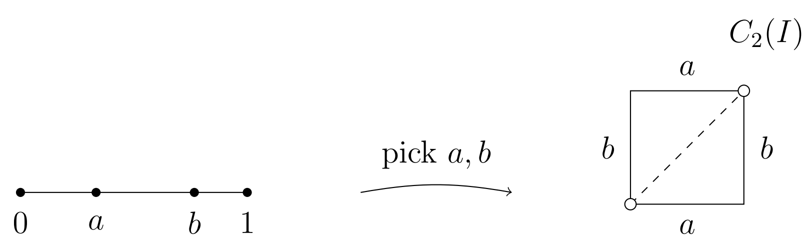
What are the possible configurations for this molecule in space? First we fix the location of C. This (operational) space is $\mathbb{R}^3$. Now let $\bar{v} = \overline{CO}$. Then the operational space for O is a sphere centered at the fixed position of C. Thus our configuration space for CO is $\mathbb{R}^3 \times S^2$.

Example 2.4.3. Rigid motions in n-dimensional space

Take an object in $\mathbb{R}^3$. The Euclidean rigid motions are described by the configuration space $\mathbb{R}^3 \times SO_3$, where SO_n is the set of special orthogonal $n \times n$ matrices. Note that for the $n = 3$ case, this configuration space is 6-dimensional. In general, the configuration space for the rigid motions of an object in $\mathbb{R}^n$ is $\mathbb{R}^n \times SO_n$, which is an $n \cdot \binom{n}{2}$-dimensional space.

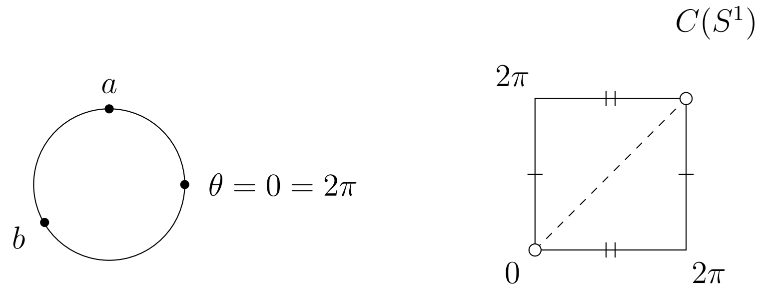
Definition. *For a space X, the* **configuration space** $C_n(X)$ *is the set of all n-tuples of distinct points in X.*

Example 2.4.4. $I = [0, 1]$

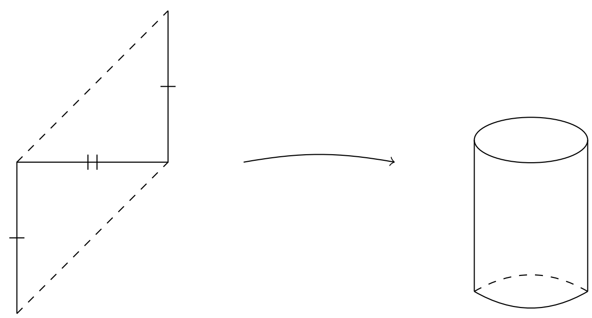


This configuration space is disconnected.

Example 2.4.5. $S^1 = \{x^2 + y^2 = 1\}$

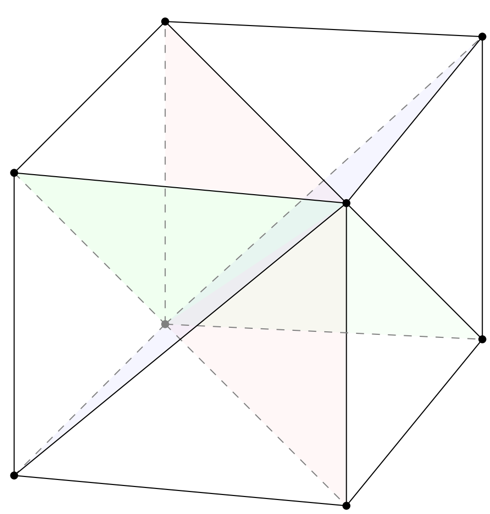


$C(S^1)$ is a connected space which is topologically equivalent to a cylinder with open top and bottom, i.e. $S^1 \times (0, 2\pi)$:



Example 2.4.6. $C_3(I)$

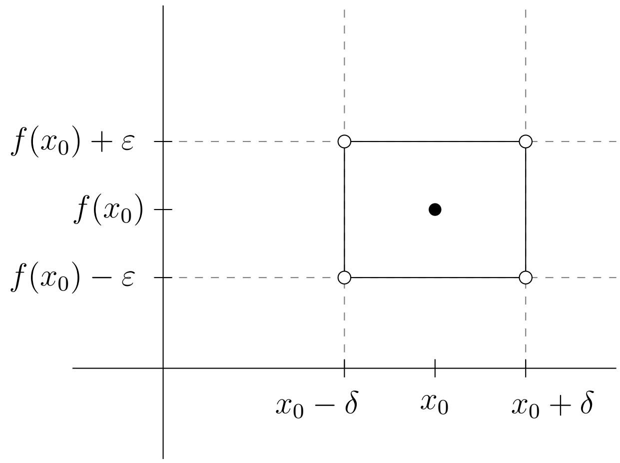
Let $I = [0,1]$ and consider $C_3(I)$. There are 3! different ways to order x, y, z. So there are 6 disconnected spaces in $C_3(I)$, which corresponds to the unit cube with planes slicing through the origin.



3 Topological Equivalence

3.1 Continuity

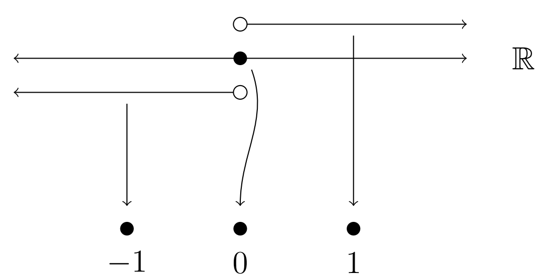
Recall in analysis, $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point x_0 if for every $\varepsilon > 0$ there is some $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.



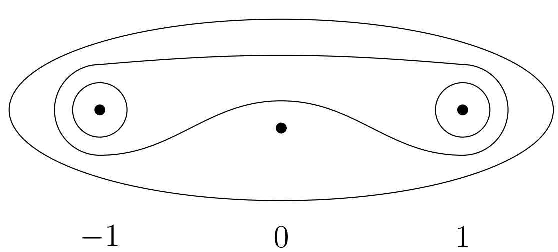
Definition. *For topological spaces X and Y, let $f : X \to Y$ be a well-defined function. Then f is* **continuous** *if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.*

Remark. For metric spaces, the ε–δ definition of continuity is equivalent to the topological definition.

Example 3.1.1. $sgn : \mathbb{R} \to \{-1, 0, 1\}$ defined by $sgn(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$



Define the topology on Y:



Then we see that

$$\begin{gathered} sgn^{-1}(\{-1\}) = (-\infty, 0),\ \text{open in } \mathbb{R} \\ sgn^{-1}(\{1\}) = (0, \infty),\ \text{open in } \mathbb{R} \\ sgn^{-1}(\{-1, 1\}) = (-\infty, 0) \cup (0, \infty),\ \text{open in } \mathbb{R}. \end{gathered}$$

Thus $sgn : \mathbb{R} \to Y$ is continuous. Notice that changing the topology on $\mathbb{R}$ or Y could make sgn discontinuous. In fact, $sgn : \mathbb{R} \to (Y, \mathcal{T}')$ is not continuous if $\mathcal{T}'$ is finer than the topology above.

Example 3.1.2. The identity function $id : X \to X$ defined by $id(x) = x$ is continuous for any topological space X.

Example 3.1.3. The constant map $C_k : X \to Y$ defined by $C_k(x) = k$ for some $k \in Y$ is continuous.

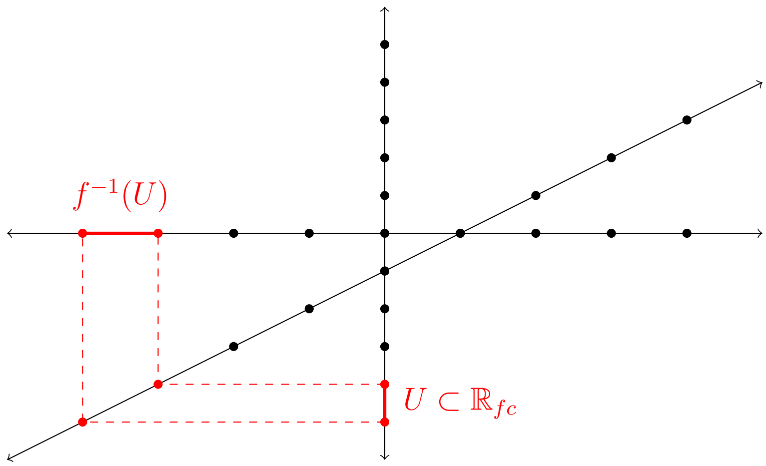
To see this, take any $U \subset Y$ which is open. Then $C_k^{-1}(U) = \begin{cases} X & k \in U \\ \varnothing & k \notin U. \end{cases}$

Example 3.1.4. Consider $f : (X, \mathcal{T}_{discrete}) \to Y$ between any spaces X and Y. Since $f^{-1}(U) \subset X$ is open for all $U \subset Y$, f is automatically continuous.

Example 3.1.5. Quotient maps are continuous

Let $p : X \to Q$. Then open sets in Q are precisely all U such that $p^{-1}(U) \subset X$ is open. In fact, the quotient topology is the finest topology on Q such that p is continuous.

Example 3.1.6. Define $f : \mathbb{R} \to \mathbb{R}_{fc}$ by $f(x) = \frac{x-1}{2}$



Let U be open in $\mathbb{R}_{fc}$. Then $U = \mathbb{R} \setminus \{\text{finite points}\}$. And $f^{-1}(U) = \mathbb{R} \setminus \{\text{finite points}\}$ too, since $\mathbb{R}_{fc} \subset \mathbb{R}$ and $f^{-1}(U)$ is open in $\mathbb{R}$. Hence f is continuous. However, if $f : \mathbb{R} \to \mathbb{R}_l$ then $f^{-1}([0,1)) = [1,3)$ which is not open in $\mathbb{R}$ ($\mathbb{R}$ is coarser than $\mathbb{R}_l$). Thus f is *not* continuous in this case.
Note that images of open sets under a continuous function may not be open. For example, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then $f((-1,1)) = [0,1)$, which is not open.

Definition. *An* **open map** *is a function that maps open sets to open sets.*

Proposition 3.1.7. *Let $\mathcal{T}_1 \subset \mathcal{T}_2$ and suppose $f : (X, \mathcal{T}_1) \to Y$ is continuous. Then $f : (X, \mathcal{T}_2) \to Y$ is continuous as well.*

Proof. Take an open set $U \subset Y$. Then $f^{-1}(U)$ is open in $(X, \mathcal{T}_1)$. Since $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, $f^{-1}(U)$ is open in $(X, \mathcal{T}_2)$ as well. Thus the preimage of open sets in Y are open in $\mathcal{T}_2$, so f is continuous from $(X, \mathcal{T}_2)$. □

Proposition 3.1.8. *Let $\mathcal{T}_1 \subset \mathcal{T}_2$ and suppose $g : X \to (Y, \mathcal{T}_2)$ is continuous. Then $g : X \to (Y, \mathcal{T}_1)$ is continuous as well.*

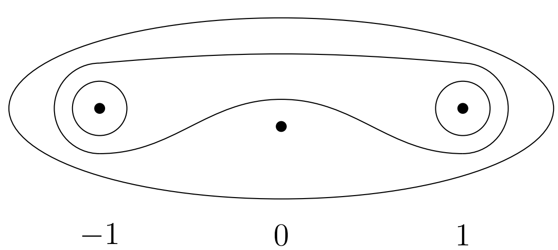
Proof. Similar to the previous proof, if U is open in $\mathcal{T}_2$ and $g^{-1}(U)$ is open in X, then we've covered all open sets in $\mathcal{T}_1$. Thus g is continuous from $X \to (Y, \mathcal{T}_1)$. □

Proposition 3.1.9. *Let X and Q be topological spaces and let $p : X \to Q$ be a continuous, surjective map. Then if p is an open map or a closed map, then it is a quotient map.*

Proof. We will show that $U \subset Q$ is open $\iff p^{-1}(U)$ is open in X. The forward direction is implied by continuity of p. For the other direction, let p be an open map (the case where p is closed is similar). Suppose $p^{-1}(U) \subset X$ is open. Since p is an open map, $p(p^{-1}(U)) = U$ is open in Y. Hence p is a quotient map. □

Example 3.1.10. $sgn : \mathbb{R} \to \{-1, 0, 1\} = Q$

As before, the *sgn* function induces the following topology on Q:



Note that sgn is open since

$$sgn((a,b)) = \begin{cases} Q & a < 0 < b \\ \{1\} & 0 \leq a < b \\ \{-1\} & a < b \leq 0 \end{cases}$$

which are all open. But sgn is not closed since for example $sgn(\{2\}) = \{1\}$ which is not closed in Q.

Example 3.1.11. $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$

The projection map $\pi_1((x,y)) = x$ is a quotient map that is both open and closed. However, if we restrict π_1 to $A = \{(x,y) \mid y = 0\} \cup \{(x,y) \mid x \geq 0\}$ then $\pi_1 : A \to \mathbb{R}$ is *neither open nor closed.*

Remark. Choosing a topology on a set X is fundamentally linked to deciding which maps are continuous from X (or to X).

Theorem 3.1.12. *Suppose we have a collection of maps $\{f_\alpha\}$ where each $f_\alpha : X \to Y$. Then*

(1) Given a topology on Y, there exists a topology on X, $\mathcal{T}_{min}$, such that any other topology on X for which $\{f_\alpha\}$ are continuous is finer than $\mathcal{T}_{min}$ (i.e. $\mathcal{T}_{min}$ is the coarsest topology for which $\{f_\alpha\}$ are all continuous).

(2) Given a topology on X, there exists a topology on Y, $\mathcal{T}_{max}$, that is the finest topology on Y for which $\{f_\alpha\}$ are continuous.

Proof. (1) Let $\mathcal{T}$ be a topology on Y and let $\{U_\beta\} = \mathcal{T}$ be the collection of open sets in this topology. Then $\{f^{-1}(U_\beta)\}$ form a subbasis for a topology which we will denote $\mathcal{T}_{min}$. By previous homework about subbases, $\mathcal{T}_{min}$ is the coarsest topology containing $\{f^{-1}(U_\beta)\}$. This is sufficient to show that $\mathcal{T}' \subset \mathcal{T}_{min}$ for all other $\mathcal{T}'$ on X.

(2) Suppose $\mathcal{T}'$ is a topology on X described by $\mathcal{T}' = \{V_\gamma\}$. Then if $U \subset Y$ is open, $f_\alpha^{-1}(U) = V_\gamma$ for each α and for some V_γ. The rest of the proof is similar to (1) and is left to the reader. $\square$

Theorem 3.1.13. *Given $f : X \to Y$, the following are equivalent.*

1) For all open sets $U \subset Y$, $f^{-1}(U)$ is open in X (definition of continuity)

2) Given a basis $\mathcal{B}$ on Y, for all $B \in \mathcal{B}$, $f^{-1}(B)$ is open in X

3) For all closed sets $C \subset Y$, $f^{-1}(C)$ is closed in X

4) For all subsets $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$

5) *For all $x \in X$ and for all neighborhoods $V \supset f(x)$, there is some neighbodhood U of x such that $f(U) \subset V$. This is a generalization of the ε–δ definition of continuity.*

Proof. We have proven ($1 \Leftrightarrow 3$). Moreover, $1 \Rightarrow 2$, and $2 \Rightarrow 1$ is implied by a previously proven identity: $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

($1 \Rightarrow 4$) Let $x \in \bar{A}$. Then $f(x) \in f(\bar{A})$. Take some neighborhood of $f(x)$, say V. Then $f^{-1}(V)$ is open by (1), and $x \in f^{-1}(V)$. Thus $f^{-1}(V)$ intersects A at some point x_0 (since $x \in \bar{A}$). This implies $f(x_0) \in V \cap f(A)$. Since V was arbitrary, every neighborhood of $f(x)$ intersects $f(A)$. Hence $f(x) \in \overline{f(A)} \implies f(\bar{A}) \subset \overline{f(A)}$.

($4 \Rightarrow 3$) Let $B \subset Y$ be closed. Let $A = f^{-1}(B)$. Then $f(A) = f(f^{-1}(B)) \subset B$ (note that this becomes an equality iff f is surjective). Now let $x \in \bar{A}$. Then $f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$ by (4) and since B is closed. So $x \in f^{-1}(B) = A$. Thus $\bar{A} \subset A \implies \bar{A} = A$. Hence A is closed.

($1 \Rightarrow 5$) Let $x \in X$ and V be open such that $f(x) \in V \subset Y$. This implies $f^{-1}(V)$ is an open set containing x. Then $f(f^{-1}(V)) \subset V$ so we have an open neighborhood of x whose image is contained in V.

($5 \Rightarrow 1$) Given an open set $V \subset Y$, take $x \in f^{-1}(V)$. We will show x must lie in some open $U_x \subset f^{-1}(V)$. Since V is a neighborhood of $f(x)$, by (5) there exists some neighborhood of x, say U_x, such that $f(U_x) \subset V$. This implies $f^{-1}(f(U_x)) \subset f^{-1}(V)$. But we also have that $f^{-1}(f(U_x)) \supset U_x$. So $U_x \subset f^{-1}(f(U_x)) \subset f^{-1}(V)$. Thus we have a neighborhood $U_x \ni x$ for any $x \in f^{-1}(V)$ such that $U_x \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is open, so preimages of open sets are open. □

Theorem 3.1.14. *If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $h = g \circ f : X \to Z$ is continuous.*

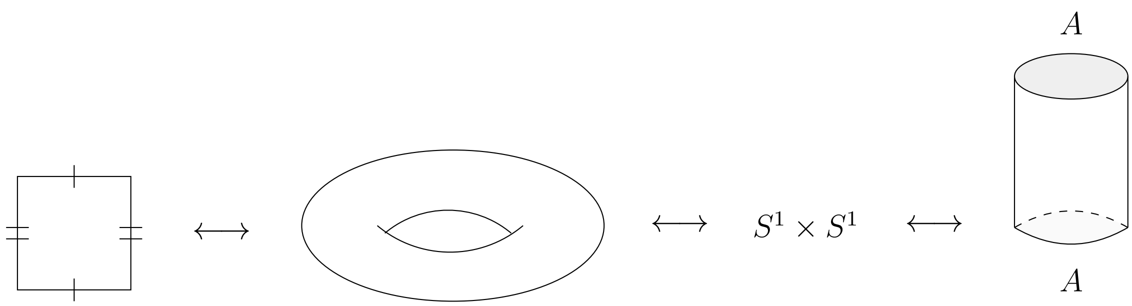
Proof omitted. □

Theorem 3.1.15. *Continuous functions map limits of sequences to limits of sequences. In other words, if $(a_n) \to x$ then $(f(a_n)) \to f(x)$.*

Proof omitted. □

3.2 Homeomorphisms

Motivation: when are two topological spaces equivalent?



All of the above are topologically equivalent: the square with sides identified, the torus T^2, $S^1 \times S^1$ and the cylinder with top and bottom identified.

Definition. *Let $f : X \to Y$ be a bijection between topological spaces. If both f and f^{-1} are continuous maps then f is a* **homeomorphism***, and X and Y are* **homeomorphic***.*

Remark. Homeomorphisms are pointwise and open set bijections.

Also note that $f(f^{-1}(U)) = U$ and $f^{-1}(f(V)) = V$ if and only f is a bijection. Some other useful properties of homeomorphisms are:

1. $id : X \to X$ is a homeomorphism.
2. If $f : X \to Y$ is a homeomorphism, then $f^{-1} : Y \to X$ is also a homeomorphism.
3. Compositions of homeomorphisms are homeomorphisms.
4. The property of being homeomorphic is an equivalence relation on topological spaces, so we can separate all topological spaces into their respective homeomorphism classes.

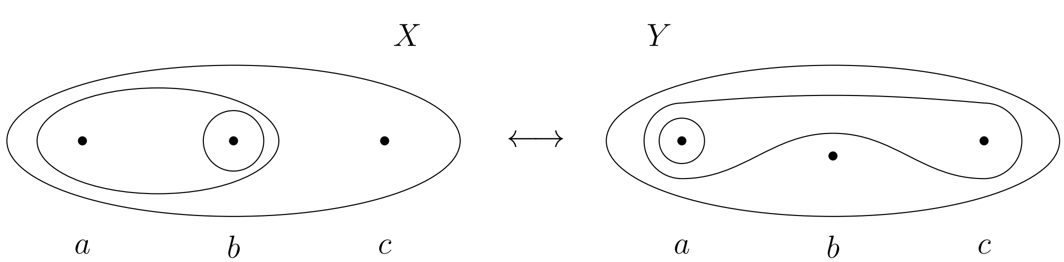
Definition. *A* **topological property** *is any property of a topological space that is invariant across a homeomorphism class.*

Example 3.2.1. The number of open sets (while not particularly useful) is a topological property. This extends to cardinality for infinite spaces.

Example 3.2.2. The Separation Axioms, including T1 and Hausdorff, are topological properties.

Example 3.2.3. Compactness, connectedness and metrizability are topological invariants which will be studied in detail in chapters to come.

Example 3.2.4.



Define $\pi : X \to Y$ by $\pi = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$. This is a homeomorphism, so $X \cong Y$.

Example 3.2.5. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$

Then $f^{-1}(x) = x^{1/3}$ which is bijective and continuous. Thus f is a homeomorphism.

Example 3.2.6. Define $g : (-1, 1) \to \mathbb{R}$ by $g(x) = \tan\left(\frac{\pi x}{2}\right)$

g is continuous, surjective and injective so g^{-1} exists. In fact $g^{-1}(x) = \frac{2}{\pi}\tan^{-1}(x)$ so $g^{-1} : \mathbb{R} \to (-1, 1)$ is continuous. Thus g is a homeomorphism, and $(-1, 1) \cong \mathbb{R}$. It follows that all open, bounded intervals are homeomorphic to the real line.

Example 3.2.7. There exist precisely 3 homeomorphism classes of intervals on $\mathbb{R}$ (not counting the degenerate interval $[a, a]$):

(i) Open intervals: $\mathbb{R}, (-\infty, a), (a, \infty), (a, b)$

(ii) Closed,bounded intervals: $[a, b]$

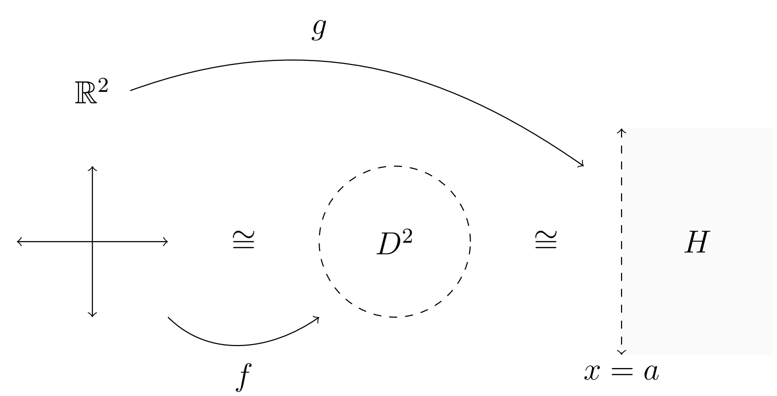
(iii) Half-open/closed, unbounded intervals: $[a, b), [a, \infty), (a, b], (-\infty, b]$

These classes are distinct, but showing that requires more topological properties.

Example 3.2.8. Every function $f(x)$ that is one-to-one and continuous is a homeomorphism between $\mathbb{R}$ and its image under f.

Example 3.2.9. We now have 4 equivalent views of a torus. (See figures at the beginning of the section.)

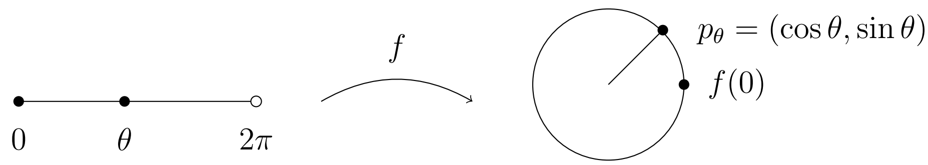
Example 3.2.10. The plane, open disk and half-plane are all homeomorphic.



Let $f : \mathbb{R}^2 \to D^2$ be given by $f(r, \theta) = \left(\frac{r}{1+r}, \theta\right)$. Then f is a homeomorphism. Next, define $g : \mathbb{R}^2 \to H$ by $g(x, y) = (e^x + a, y)$. It is easy to verify that this is also a homeomorphism. And since homeomorphism is an equivalence relation, all three spaces are topologically equivalent.

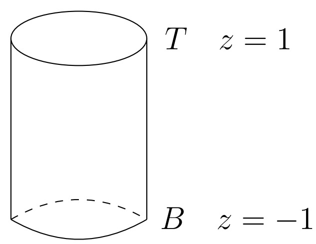
Question. *To show f is a homeomorphism, is it sufficient for f to be a continuous bijection?*

No. For a counterexample, define $f : [0, 2\pi) \to S^1$ by $f(\theta) = (\cos\theta, \sin\theta)$.

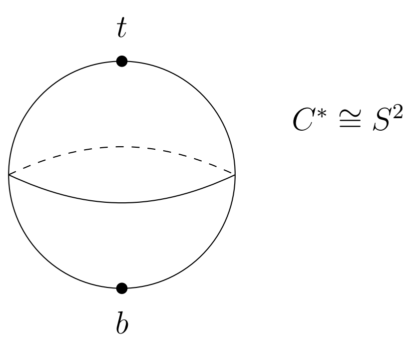


Then f is a continuous bijection, so f^{-1} exists. In fact, $f^{-1}(p_\theta) = \theta \mod 2\pi$. Let $V = \left[0, \frac{\pi}{2}\right)$, which is open in $[0, 2\pi)$. However, $(f^{-1})^{-1}(V) = f(V)$ is a half-open arc, which is not open in S^1. Thus f^{-1} is *not* continuous. These spaces are not homeomorphic.

Example 3.2.11. $C = \{(x, y, z) \mid x^2 + y^2 = 1, -1 \leq z \leq 1\}$



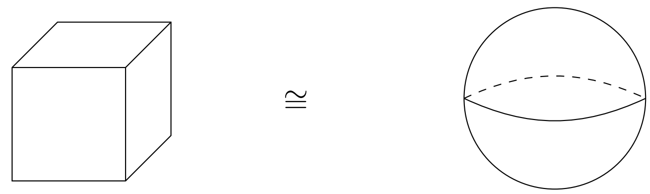
If we identify T to a point t and B to a point b, creating the quotient space C^*, then C^* is homeomorphic to the sphere S^2.



We can do this by constructing
$$\begin{tikzcd} C \arrow[r, "p"] \arrow[rr, bend right, "h"'] & C^* \arrow[r, "f"] & S^2 \end{tikzcd}$$
where we define $h(1, \theta, z) = (\sqrt{1 - z^2}, \theta, z)$.

Then h is *not* a homeomorphism, but it will help us show that f is. Specifically, take an open set $U \subset S^2$. Then $f^{-1}(U)$ is open in $C^* \iff p^{-1}(f^{-1}(U)) = h^{-1}(U)$ is open in C.

Example 3.2.12. The cube is homeomorphic to the sphere

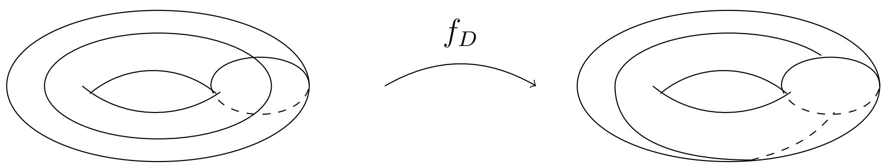


$C = [-1, 1]^3$ $\qquad$ $S^2 = (x, y, \sqrt{1 - x^2 - y^2})$

Define $f : C \to S^2$ by $f(p) = \frac{p}{||p||}$, so f takes p to the point on the sphere where the ray $\overrightarrow{Op}$ hits the sphere. Then f is a bijection, and more importantly f is a bijection on open sets, giving us a homeomorphism $C \cong S^2$.

Remark. Not every homeomorphism is a deformation.

Example 3.2.13. The Dehn twist



f_D is a homeomorphism, but clearly not a deformation. The actual transformation requires a cut along a meridian and a twist.

Definition. *An* **embedding** *is a function* $f : X \to Y$ *such that* $X \cong f(X)$.

Knots are a common example of embeddings of the circle S^1 into R^3. From this it is easy to see that all knots are homeomorphic, but not all pairs of knots are **isotopic** (deformable).

Theorem 3.2.14. *Hausdorff is a topological property.*

Proof. Let $f : X \to Y$ be homeomorphism. Note that $f^{-1} : Y \to X$ is also a homeomorphism, so it suffices to show the forward direction. Suppose X is Hausdorff. Take $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Then these pull back bijectively to $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. Since X is Hausdorff, there exist disjoint, open sets $U_1, U_2 \subset X$ such that $x_i \in U_i$ for $i = 1, 2$. Let $V_1 = f(U_1)$ and $V_2 = f(U_2)$. Then since f is a homeomorphism, $V_1, V_2 \subset Y$ are open, and $y_1 \in V_1$ and $y_2 \in V_2$. Take $z \in V_1 \cap V_2$. Then $f^{-1}(z) \in U_1 \cap U_2$, but this intersection is empty. Hence $V_1 \cap V_2$ is empty as well, so Y is Hausdorff. $\square$

Corollary 3.2.15. $\mathbb{R}$ *and* $\mathbb{R}_{fc}$ *are not homeomorphic.*

4 Metric Spaces

4.1 Metric Spaces

Definition. *A* **metric** d *on a topological space* X *is a function* $d : X \times X \to \mathbb{R}$ *satisfying the following for all* $x, y, z \in X$*:*

1) Positivity: $d(x, y) \geq 0$*, and* $d(x, y) = 0 \iff x = y$

2) Symmetry: $d(x, y) = d(y, x)$

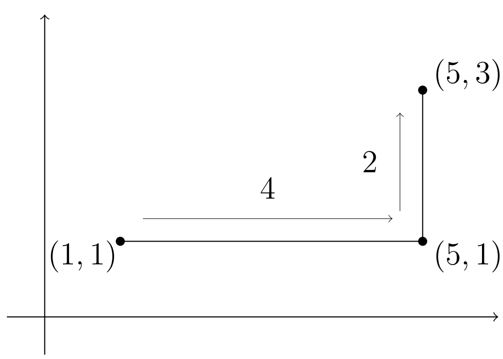
3) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Example 4.1.1. The Euclidean metric on R^2: for $\bar{x}, \bar{y} \in \mathbb{R}^2$, $d(\bar{x}, \bar{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

Example 4.1.2. The Euclidean metric on R^n: for $\bar{x}, \bar{y} \in \mathbb{R}^n$, $d(\bar{x}, \bar{y}) = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2}$

Example 4.1.3. Other metrics on $\mathbb{R}^n$

- The **taxicab metric**: $d_1(\bar{x}, \bar{y}) = |x_1 - y_1| + \ldots + |x_n - y_n|$



$d_1((1, 1), (5, 3)) = 4 + 2 = 6$

- The **sup metric**: $d_\infty(\bar{x}, \bar{y}) = \sup\limits_{1 \leq i \leq n} |x_i - y_i|$

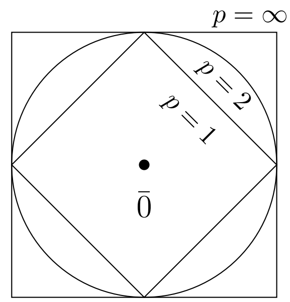
  In the above diagram, $d_\infty((1, 1), (5, 3)) = 4$.

- For an integer $1 \leq p \leq \infty$, the **p-metric**: $d_p(\bar{x}, \bar{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$

  The Euclidean and sup metrics are special cases for $p = 2$ and $p = \infty$, respectively.

Example 4.1.4. What are "unit circles" with respect to different metrics on $\mathbb{R}^2$?

A unit circle with respect to a p-metric is the set of points $\{\bar{y} \mid d_p(\bar{0}, \bar{y}) = 1\}$. This gives us



Notice that for all p, $d_p(\bar{0}, (0,1)) = 1$. Furthermore, $d_p(\bar{0}, \bar{y}) \geq d_{p+\varepsilon}(\bar{0}, \bar{y})$ for all $\bar{y} \in \mathbb{R}^2$, $1 \leq p \leq \infty$ and $\varepsilon > 0$.

Each p-metric comes from the ℓ^p norm: $||\bar{y}||_p = d_p(\bar{0}, \bar{y})$. We can try to extend this definition to what we are tempted to call $\mathbb{R}^\infty$, i.e. the set of all real sequences. But for each finite p, some of the summations will diverge. For example, if $\bar{0} = (0, 0, 0, \ldots)$ (the zero sequence) and $\bar{y} = (1, 1, 1, \ldots)$, then $d_2(\bar{0}, \bar{y}) = \sqrt{1 + 1 + 1 + \ldots}$ which diverges. The sequences for which $d_p(\bar{0}, \bar{y})$ converges form **ℓ^p spaces**.

Remark. Any norm on a vector space induces a metric: let $|| \cdot ||$ denote a norm on a vector space X. Then define $d(x, y) = ||x - y||$. Note that a norm possesses:

- nonnegativity
- scalability
- the triangle inequality.

Note that every metric space is a topological space, but the converse is not true in general. The following definitions and proofs will help us establish this fact.

Definition. *For any metric space X, we define the* **open and closed metric balls** *B_d and $\bar{B}_d$ by*

$$B_d(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$$
$$\bar{B}_d(x, \varepsilon) = \{y \mid d(x, y) \leq \varepsilon\}.$$

Proposition 4.1.5. *$\bar{B}_d$ is closed and* $\mathrm{Cl}(B_d(x, \varepsilon)) \subset \bar{B}_d(x, \varepsilon)$ *(these are not equal in general).*

Proof omitted. □

Proposition 4.1.6. *The collection $\mathcal{B} = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$ is a basis for a topology on the metric space (X, d), called the* **metric topology**.

To verify this, we prove the following lemma:

Lemma 4.1.7. *For all $y \in B(x, \varepsilon)$ there is some $\delta > 0$ such that $B(y, \delta) \subset B(x, \varepsilon)$.*

Proof. Let $a = d(x, y)$ and $\delta = \varepsilon - a$. Then the triangle inequality implies that if $z \in B(y, \delta)$,

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< a + \delta \\ &= \varepsilon - \delta + \delta = \varepsilon \end{aligned}$$

from which it follows that $z \in B(x, \varepsilon)$. Hence $B(y, \delta) \subset B(x, \varepsilon)$. □

Corollary 4.1.8. *U is open in (X, d) if and only if for all $x \in U$ there is some $B_d(x, \varepsilon) \subset U$.*

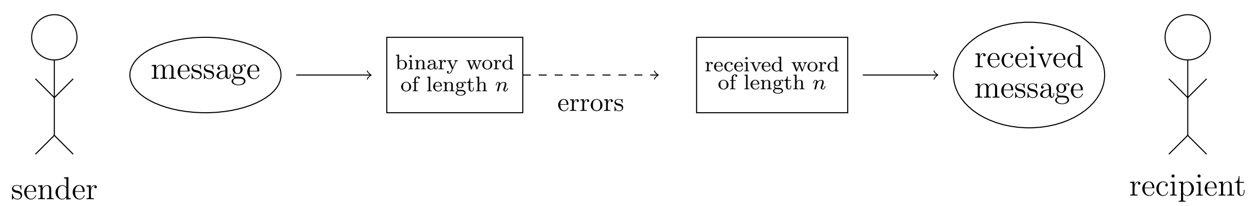
Example 4.1.9. Function spaces

Function spaces are an important type of topological space. For an interval $[a, b] \subset \mathbb{R}$, we define $C[a, b] = \{f \mid f \text{ is continuous on } [a, b]\}$. Consider two metrics on $C[a, b]$:

1. $d(f, g) = \max\limits_{x \in [a,b]} |f(x) - g(x)|$. Note that the Extreme Value Theorem says that continuous functions on closed, bounded intervals attain their maximum value(s), so we can write this metric using max instead of sup.

2. $p(f, g) = \displaystyle\int_a^b |f(x) - g(x)|\, dx$. This is in a sense the measure of the area between the two functions.

4.2 Error-Checking Codes



Metrics can help us detect errors and utilize self-correcting data.

Definition. *The* **Hamming distance** *D_H is a metric on the set of length n binary words $Vn = \{0, 1\}^n$, where $D_H(x, y) =$ the number of digits where x and y disagree.*

Example 4.2.1. If $x = 101010$ and $y = 110011$ then $D_H(x, y) = 3$.

We can define the metric topology induced by D_H on V^n by:

$$\begin{aligned} B_H(x,2) &= \{y \mid D_H(x,y) < 2\} \\ &= \{y \mid y \text{ disagrees with } x \text{ in precisely } n \text{ digits}\} \cup \{x\} \end{aligned}$$

where $|B_H(x,2)| = n+1$.

Note that $B_H(x,1) = \{x\}$, which implies that the metric topology is **discrete**. Also, $\mathrm{Cl}(B_H(x,1)) = \mathrm{Cl}(\{x\}) = \{x\} = B_H(x,1)$, and $\bar{B}_H(x,1) = B_H(x,2)$. (This is a counterexample to Proposition 4.1.5.)

Proposition 4.2.2. *The metric topology on a finite metric space (X,d) is discrete.*

Proof. $X \times X$ is finite, so $d(X \times X \setminus \Delta)$ is the set of positive distances on X (via d). Note that $d(\Delta) = \{0\}$ so we don't consider these points. Let $m = \min d(X \times X \setminus \Delta)$. Then $B(x,m) = \{x\}$ for any $x \in X$. And if all point-sets are open, this describes the discrete topology on X. □

Definition. *A* **code** *C is a subset of V^n, i.e. a set of "words", and a* **codeword** *is any element of C.*

Definition. *The* **minimum distance** *of C is the smallest value of $D_H(x,y)$ for distinct $x, y \in C$.*

Example 4.2.3. Consider the following code C in V^6:

001000
100111
111011
010010

The minimum distance for C is 3, which also happens to be the most common minimum distance selected for a code. Suppose we receive $y \in C$ and we assume there is at most one error. Then there will be a unique element $x \in C$ such that $D_H(x,y) = 1$ (by the triangle inequality). This allows us to partition incoming messages into one of the above codeword "boxes".

Theorem 4.2.4. *Given a code of length n with minimum distance m, any message y that has at most $\frac{m}{2}$ errors represents a unique codeword (i.e. we've corrected the error).*

Proof. Similar to Example 4.2.3. □

In practice, $n = 2^k - 1$ and $|C| = 2^{n-k}$ so each codeword has $n-k$ bits of data and the other k bits are used to check errors. A common error-checking scheme has $n = 7$, $k = 3$ and $|C| = 2^4 = 16$.

Example 4.2.5. Levenshtein distance and DNA

Let V be the set of all DNA, i.e. strings of A, C, T and G. We can have the following types of errors:

error	ACTG
insertion (i)	AC$\underline{\text{A}}$TG
deletion (d)	ATG
replacement (r)	A$\underline{\text{T}}$TG

Definition. *The* **Levenshtein distance** $D_L(x, y)$ *is the minimum of* $i_S + d_S + r_S$ *over all sequences* S *that change* x *into* y.

Proposition 4.2.6. *For any two binary words* $x, y \in V^n$, $D_L(x, y) \leq D_H(x, y)$. *(Equality does not hold in general.)*

Proof. The Levenshtein distance is $\min\limits_S(i_S + d_S + r_S)$, where i_S is the number of insertions, d_S is the number of deletions and r_S is the number of replacements, and this min is taken over all sequences S that change x into y. In particular, the replacement sequence R that flips every digit in x that disagrees with the corresponding digit in y actually describes the Hamming distance. Thus $D_H(x, y) = r_R \leq \min\limits_S(i_S + d_S + r_S) = D_L(x, y)$. □

4.3 Properties of Metric Spaces

Proposition 4.3.1. *Metric spaces are Hausdorff.*

Proof. Let (X, d) be a metric space. Take $x, y \in X$ where $x \neq y$ and let $l = d(x, y)$. Since $x \neq y$, $l > 0$. Consider the open balls $B_d\left(x, \frac{l}{2}\right)$ and $B_d\left(y, \frac{l}{2}\right)$. By the triangle inequality, these are disjoint. Hence X is Hausdorff. □

Corollary 4.3.2. *If a space is not Hausdorff, it is not a metric space (and cannot have a metric).*

This shows that $\mathbb{R}_{fc}$, the nested topology on $\mathbb{N}$, the digital line, the digital plane, the particular point topology PPX and the countable complement topology on $\mathbb{R}$ are all *not* metric spaces.

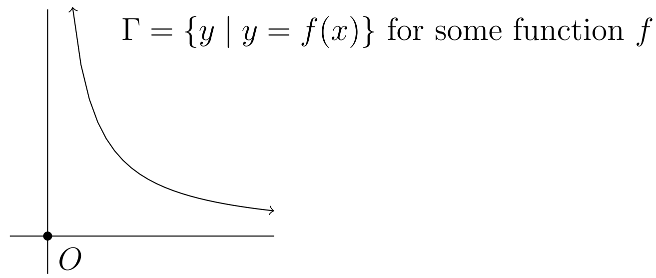
Proposition 4.3.3. *For metric spaces* (X, d_X) *and* (Y, d_Y), *a function* $f : X \to Y$ *is continuous if for all* $x \in X$, $\varepsilon > 0$, *there is some* $\delta > 0$ *such that* $d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$.

Proof omitted. This is a generalization of the standard ε–δ definition of continuity.

Definition. *In (X, d), the* **distance** *between subsets $A \subset X$ and $B \subset X$ is defined as*

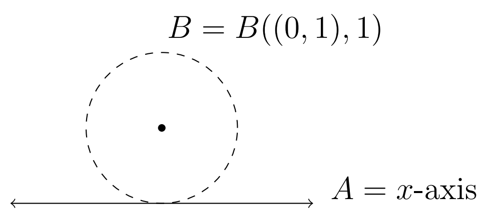
$$d(A, B) = \inf_{\substack{a \in A \\ b \in B}} \{d(a, b)\}.$$

Example 4.3.4. Consider a curve through the plane:



Then calculating $d(O, \Gamma)$ is a nice Calc II problem!

Example 4.3.5.



then $d(A, B) = 0$

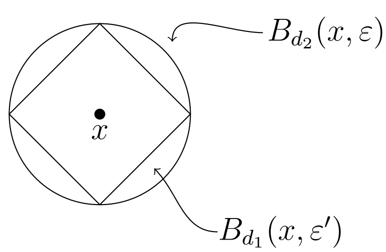
Theorem 4.3.6. *Let d and d' be metrics on X and let $\mathcal{T}$ and $\mathcal{T}'$ be their respective metric topologies. Then*

$$\mathcal{T} \subset \mathcal{T}' \iff \forall x \in X, \forall \varepsilon > 0, \exists \varepsilon' > 0 \text{ such that } B_{d'}(x, \varepsilon') \subset B_d(x, \varepsilon).$$

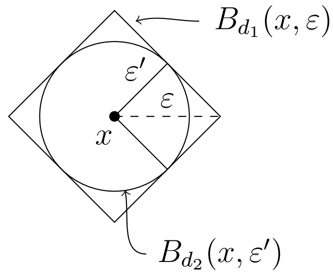
Proof. Follows directly from Theorem 1.2.7. □

Proposition 4.3.7. *In $\mathbb{R}^2$ (this holds for $\mathbb{R}^n$ in general), the taxi cab metric and the Euclidean metric generate the same topology.*

Proof. Let $\mathcal{T}_1$ be the topology generated by the taxi cab metric d_1 and let $\mathcal{T}_{std}$ be the standard topology, which we know is generated by the Euclidean metric d_2. For one direction, take a basis element $B_{d_2}(x, \varepsilon)$ and let $\varepsilon' = \varepsilon$. Then $B_{d_1}(x, \varepsilon') \subset B_{d_2}(x, \varepsilon)$ by picture:



So $\mathcal{T}_{std} \subset \mathcal{T}_1$. On the other hand, take $B_{d_1}(x, \varepsilon)$ and let $\varepsilon' = \frac{\varepsilon}{\sqrt{2}}$. Then by the next picture, $B_{d_2}(x, \varepsilon') = B_{d_2}\left(x, \frac{\varepsilon}{\sqrt{2}}\right) \subset B_{d_1}(x, \varepsilon)$:



Hence d_1 and d_2 generate the same topology on $\mathbb{R}^2$. $\square$

It is a simple geometric exercise to verify that the above containments actually hold.

Definition. *A metric d is* **bounded** *if there is some M such that for all $x, y \in X$, $d(x, y) \leq M$.*

Remark. In any space X with a metric bounded by M, $B(x, M + 1) = X$.

Definition. *The* **diameter** *of a subset $A \subset (X, d)$ is given by*

$$diam(A) = \sup_{x,y \in A} \{d(x, y)\}.$$

Proposition 4.3.8. *For any metric space X, there exists a bounded metric.*

Proof. Let (X, d) be a metric space and define $\bar{d} = \min(d, 1)$.

1) Positivity is inherited from d since $d(x, y) \geq 0 \implies \bar{d}(x, y) > 0$, and $d(x, y) = 0 \iff x = y$ implies the same for $\bar{d}$.

2) Symmetry is likewise inherited from d.

3) If either $\bar{d}(x, z)$ or $\bar{d}(y, z)$ is equal to 1, then $\bar{d}(x, y) \leq 1 + \bar{d}(x, z)$ for example. Otherwise

$$\begin{aligned} \bar{d}(x, z) + \bar{d}(y, z) &= d(x, z) + d(y, z) \\ &\geq d(x, y) \quad \text{by triangle inequality for } d \\ &\geq \bar{d}(x, y). \end{aligned}$$

Hence $\bar{d}$ is a metric. $\square$

Definition. *Let $f : X \to Y$ be a bijection between metric spaces such that for all $x_1, x_2 \in X$, $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$. Then f is an* **isometry** *and X and Y are said to be* **isometric**.

Remark. f^{-1} is an isometry as well, and isometries are continuous, therefore isometries are homeomorphisms. The converse is not true in general.

4.4 Metrizability

Definition. *A topological space X is* **metrizable** *is there exists a metric d on X inducing its topology.*

Example 4.4.1. As mentioned before, any space that is not Hausdorff is not metrizable.

Results:

i) Subspaces of metric spaces are metrizable (via the same metric).

ii) Countable products of metric spaces are metrizable.

iii) Some "order" topologies are metrizable but others are not. For example, the vertical interval topology on $\mathbb{R}^2$ is metrizable.

iv) Metrizability is a topological property.

Definition. *A topological space X is* **regular** *if*

1) One-point sets are closed (T1)

2) For all $a \in X$ and every closed set $C \subset X \setminus \{a\}$, there exist disjoint open sets U and V such that $a \in U$ and $C \subset V$.

Note: regularity implies Hausdorff, so regularity is a stronger condition (in fact it is strictly stronger, i.e. Hausdorff $\not\Longrightarrow$ regular).

Example 4.4.2. Let $\mathbb{R}$ have the topology given by the basis $\{(a, b)\} \cup \{(c, d) \cap \mathbb{Q}\}$

(Show this is a topology. Show the resulting space is Hausdorff but not regular.)

Theorem 4.4.3 (Urysohn Metrization Theorem)**.** *X is metrizable if X is regular and has some basis that is countable.*

Proof omitted. □

Example 4.4.4. $\mathbb{R}$ with the standard topology

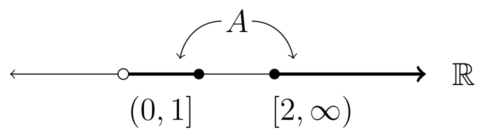
$\mathbb{R}$ is regular (easy to verify). Let $\mathcal{B}_Q = \{(a, b) \mid a, b \in \mathbb{Q}\}$. Then it is easy to see that $\mathcal{B}_Q$ is a countable basis for the standard topology on $\mathbb{R}$. Hence by the Urysohn Metrization Theorem, $\mathbb{R}$ is metrizable.

Theorem 4.4.5 (Nagata-Smirnov Metrizability Theorem)**.** *X is metrizable if and only if X is regular and has a basis that is countably locally finite.*

Proof *definitely* omitted. □

5 Connectedness

5.1 Connected Sets

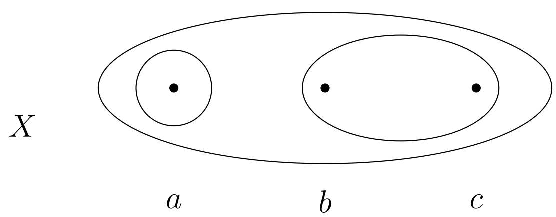


A is an example of a set that is disconnected

Definition. *A topological space X is* **disconnected** *if there exist nonempty open sets $U, V \subset X$ such that $U \cap V = \varnothing$ and $U \cup V = X$. Such open sets U and V are called a* **separation** *of X.*

Definition. *If no such open sets exist, then X is* **connected**.

Example 5.1.1.

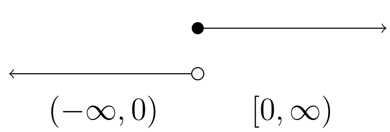


$U = \{a\}$
$V = \{b, c\}$

then U, V is a separation of X, so X is disconnected.

Example 5.1.2. $\mathbb{R}$ is connected. In fact, $\mathbb{R}^n$ is connected for all finite $n \geq 1$.

Example 5.1.3. $\mathbb{R}_l$



$\mathbb{R} = (-\infty, 0) \cup [0, \infty)$, which are open and disjoint in $\mathbb{R}_l$. Thus $\mathbb{R}_l$ is disconnected.

Example 5.1.4. The discrete topology

For any X with the discrete topology (and $|X| > 1$), X is disconnected. This is because for any nonempty subset $U \subset X$, U and $X - U$ are both open.

Example 5.1.5. The trivial topology

X with the trivial topology is connected because X is the only nonempty open set.

Theorem 5.1.6. *X is connected if and only if the only subsets of X that are both open and closed are $\varnothing$ and X.*

Proof. ($\Longleftarrow$) Suppose X is disconnected. Then there is some separation U, V of X, where U and V are nonempty. By definition, U is open and $X - U = V$ is open, which shows that each one is closed as well. And since they are both nonempty, $U \neq X, \varnothing$.

($\Longrightarrow$) Suppose there is some $U \neq X, \varnothing$ that is both open and closed. Then U and $X - U$ are both open, disjoint and nonempty. And $U \cup (X - U) = X$ so X is disconnected. $\square$

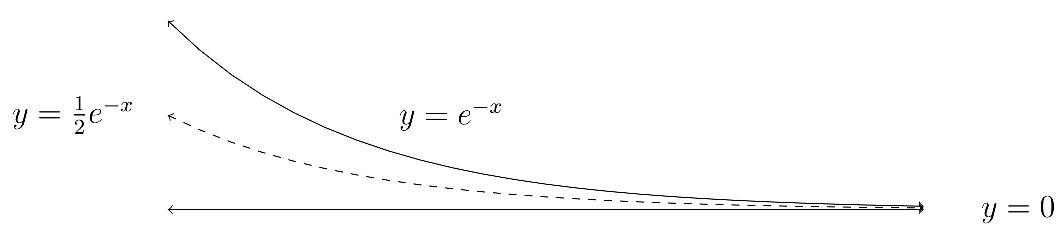
Corollary 5.1.7. *Let $\mathcal{T}_1 \subset \mathcal{T}_2$ be topologies on X.*

1) If $(X, \mathcal{T}_1)$ is disconnected then $(X, \mathcal{T}_2)$ is disconnected.

2) If $(X, \mathcal{T}_2)$ is connected then $(X, \mathcal{T}_1)$ is connected (by contrapositive).

Definition. *A subset $A \subset X$ is* **disconnected** *in X if there is some open sets $U, V \subset X$ such that each intersects A, $A \subset U \cup V$ and $U \cap V$ is disjoint from A.*

Example 5.1.8. Consider $A \subset \mathbb{R}^2$, the union of the graphs $y = 0$ and $y = e^{-x}$.



Let $U = \{(x, y) \mid y > \frac{1}{2}e^{-x}\}$ and $V = \{(x, y) \mid y < \frac{1}{2}e^{-x}\}$. Then U and V are both open and each intersects A such that $A \subset U \cup V$. And $U \cap V = \varnothing$, so A is disconnected.

Proposition 5.1.9. *A is disconnected in X $\iff$ A with the subspace topology inherited from X is disconnected.*

The idea here is that $A_1 = A \cap U$ and $A_2 = A \cap V$, where both are open in A as a subspace, form a separation iff U and V separate A in X.

Theorem 5.1.10. *Let X be a connected space and let $f : X \to Y$ be continuous. Then $f(X)$ is connected.*

Proof. Let $f : X \to Y$ be continuous and suppose $f(X)$ is disconnected. Then there exists a separation U, V of $f(X)$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Note that $X = f^{-1}(f(X)) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ since U, V are disjoint. Then $f(X)$ intersects U and V so there exist $u \in U$ and $v \in V$ such that $f^{-1}(\{u\}) \subset f^{-1}(U)$ and $f^{-1}(\{v\}) \subset f^{-1}(V)$. So $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Any $x \in f^{-1}(U) \cap f^{-1}(V)$ would imply $f(x) \in U \cap V$, but $f(x) \in f(X)$ and we know that $f(X) \cap (U \cap V)$ is empty, so no such x exists. Thus $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint, and form a separation of X, so X is disconnected. Hence we have proven the theorem by contrapositive. □

Lemma 5.1.11. *Let C, a connected space, be contained in $D \subset X$, which is disconnected. If U, V is a separation of D then either $C \subset U$ or $C \subset V$.*

Proof. Suppose C intersects both U and V. Since U, V separate D, we have

- U and V are open;
- $C \subset D \subset (U \cup V)$ so U and V cover C;
- $D \cap (U \cap V) = \varnothing$ so $C \cap (U \cap V)$ is empty as well.

Thus U, V separate C, so C is disconnected in X. The lemma follows by contrapositive. □

Theorem 5.1.12. *Let C be connected in X and let $A \subset X$, where $C \subset A \subset \bar{C}$. Then A is connected in X. In other words, adding limit points to a connectd set yields a connected set.*

Proof omitted. □

Theorem 5.1.13. *Let $\{C_\alpha\}$ be a collection of connected subsets of X, with $\bigcap\limits_\alpha C_\alpha$ nonempty. Then $\bigcap\limits_\alpha C_\alpha$ is connected.*

Proof. Suppose $\bigcup\limits_\alpha C_\alpha$ is disconnected; let U, V be a separation. Since $\bigcap\limits_\alpha C_\alpha$ is nonempty, $x \in \bigcap\limits_\alpha C_\alpha \implies x \in U$ or $x \in V$. Assume without loss of generality that $x \in U$ and $x \notin V$. Since $x \in C_\alpha$, Lemma 5.1.11 shows that each C_α is contained in U. Thus $\bigcap\limits_\alpha C_\alpha \subset U$ and $\bigcup\limits_\alpha C_\alpha$ is disjoint from V. This contradicts U, V separating $\bigcup\limits_\alpha C_\alpha$. Hence $\bigcup\limits_\alpha C_\alpha$ must be connected. □

For a space X, define the relation $\sim_C$ by $x \sim_C y$ if x and y both lie in some connected set $C' \subset X$.

Claim. *$\sim_C$ is an equivalence relation.*

Proof. Take any $x \in X$. Then $x \in \{x\}$ which is connected since there are no disjoint pairs of sets that both intersects $\{x\}$. Thus $x \sim_C x$ and reflexivity holds. Next, if $x \sim_C y$ then x and y lie in some connected set C', which is equivalent to saying $y, x \in C' \implies y \sim_C x$ so symmetry holds. Lastly, suppose $x \sim_C y$ and $y \sim_C z$. Then $x, y \in C_1$ and $y, z \in C_2$ so $y \in C_1 \cap C_2$. By Theorem 5.1.13, $C_1 \cup C_2$ is connected and $x, z \in C_1 \cup C_2$. So $x \sim_C z$, proving transitivity. Hence $\sim_C$ is an equivalence relation. □

Definition. *The equivalence classes under $\sim_C$ are called* **components** *(or connected components) of X.*

Note that if X is connected, there is only one component of X. Next we prove some basic properties of the components of a topological space.

Proposition 5.1.14. *For a topological space X,*

1) *Each component is connected.*
2) *If $A \subset X$ is connected then A lies in some component of X.*
3) *Each component is closed.*

Proof. (1) follows from the definition of $\sim_C$ and (3) results from the fact that each component C contains its limit points (by Theorem 5.1.12). We prove (2) by contrapositive. Suppose A intersects two components C_1 and C_2 such that $a_1 \in A \cap C_1$ and $a_2 \in A \cap C_2$. Then a_1 and a_2 are in different equivalence classes, so there is no connected set containing a_1 and a_2. In other words, every set containing both a_1 and a_2 is disconnected, so A is disconnected. □

Claim. *If X is infinite then X_{fc} is connected.*

Proof. Suppose X_{fc} is disconnected. Let U, V be a separation of X. Then $X = U \cup V$ and $U \cap V = \varnothing$. Since U is open, $X \setminus U = V$ is finite. Likewise, $X \setminus V = U$ is finite as well. The union of finite sets is finite, so $X = U \cup V$ is finite. Hence the claim is proven by contrapositive. □

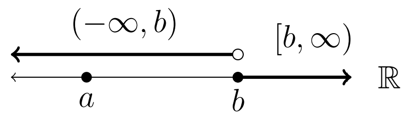
In particular, this tells us that X_{fc} has one component.

Definition. *X is* **totally disconnected** *if its components are the one-point sets.*

Example 5.1.15. For any X with the discrete topology, every one-point set is both open and closed, so the largest connected sets are the one-point sets. Hence $X_{discrete}$ is a totally disconnected space.

Example 5.1.16. What are the components of $\mathbb{R}_l$?

Take $a, b \in \mathbb{R}$ with $a < b$ and let A be a set containing them.



Let $U = (-\infty, b)$ and $V = [b, \infty)$. Then U, V are open, disjoint, and $a \in U$ and $b \in V$, so they are nonempty. Also, $A \subset \mathbb{R} = U \cup V$ so U, V separate A. Then a and b do not lie in any connected set together. Since a and b were arbitrary, this holds for any pair of points in $\mathbb{R}$. Thus we can conclude that the components of $\mathbb{R}_l$ are the one-point sets so $\mathbb{R}_l$ is totally disconnected.

Example 5.1.17. By homeomorphism, $\mathbb{R}_u$ is also totaly disconnected.

Example 5.1.18. $\mathbb{Q}$ and the Cantor set are other examples of totally disconnected sets.

By Proposition 5.1.14, components of X are closed, but they are not necessarily open. The following result characterizes spaces with a finite number of components.

Proposition 5.1.19. *If X has n components for finite n, then they are open and there are $2^{n-1} - 1$ different separations of X.*

Proof. Suppose X has finitely many components $C_1, C_2, \ldots, C_n$. Then since each is closed and $X = \bigcup_{k=1}^{n} C_k$, for each k we have that

$$C_k = X \setminus (C_1 \cup C_2 \cup \cdots \cup C_{k-1} \cup C_{k+1} \cup \cdots \cup C_n).$$

Also, the finite union of closed sets is closed, implying each component C_k is open. Now suppose U, V is a separation of X. Then each C_k lies in precisely one of U, V. The total number of possibilities for where components lie is 2^n. However, U and V must be nonempty so we throw out $U = \varnothing$ and $V = \varnothing$, reducing the number to $2^n - 2$. Moreover, we count $U \cup V$ and $V \cup U$ as one separation, so to fix this we divide the expression by 2. Therefore the total number of possible separations of X is $\dfrac{2^n - 2}{2} = 2^{n-1} - 1$. □

An important result is that $\mathbb{R}$ is connected. This generalizes nicely to $\mathbb{R}^n$:

Lemma 5.1.20. *Finite products of connected spaces are connected.*

Proof omitted. □

To prove $\mathbb{R}$ is connected, we first state some relevant properties of $\mathbb{R}$:

1. **The Completeness Axiom**: Every bounded set $A \subset \mathbb{R}$ possesses a least upper bound, $\sup A$.

2. For all $x, y \in \mathbb{R}$ where $x < y$, there is some $z \in \mathbb{R}$ such that $x < z < y$.

Theorem 5.1.21. *$\mathbb{R}$ is connected.*

Proof. Suppose U, V separate $\mathbb{R}$. Let $u \in U$ and $v \in V$, and without loss of generality assume $u < v$. Let $U' = [u, v] \cap U$ and $V' = [u, v] \cap V$. Then U', V' are a separation of $[u, v]$. Completeness tells us that $\sup U'$ exists in $\mathbb{R}$. Then $u \leq \sup U' \leq v$. Suppose $\sup U' \in U'$. Since U' is open in $[u, v]$, $\sup U'$ is contained in an open basis element $(\sup U' - \varepsilon, \sup U' + \varepsilon) \subset U'$. This contradicts $\sup U'$ being the least upper bound of U', so $\sup U' \notin U'$. On the other hand, suppose $\sup U' \in V'$. Since V' is open, there is some open interval $(\sup U' - \varepsilon, \sup U' + \varepsilon) \subset V'$. And $U' \cap V' = \varnothing$ so $\sup U' - \varepsilon$ is an upper bound for U'. But again this contradicts $\sup U'$ being the least upper bound of U', so $\sup U' \notin V'$ either. Hence U', V' do not cover $[u, v]$, so no separation of $\mathbb{R}$ exists. Hence $\mathbb{R}$ is connected. □

5.2 Applications of Connectedness

Now we have the tools to begin proving some of the most important results in general topology. First we prove that connectedness is preserved under homeomorphism.

Theorem 5.2.1. *Connectedness is a topological property, and if $X \cong Y$ then X and Y have the same cardinality of components.*

Proof. Let X be a connected space and let $f : X \to Y$ be a homeomorphism. We have shown that if f is continuous, $f(X)$ is connected, so this implies Y is connected. Since f is a bijection, this establishes X connected $\iff$ Y connected. Next, let $\{C_\alpha\}$ be the collection of components of X and let $\Gamma_\alpha = f(C_\alpha)$. Since C_α is connected and f is continuous, Γ_α is connected. Thus Γ_α lies in some component of Y. Take $y \in \Gamma_\alpha$. Then $y \in A_y$, a component of Y, and $\Gamma_\alpha \subset A_y$. Since components are connected, A_y is connected which implies $f^{-1}(A_y)$ is also connected. Thus $f^{-1}(A_y)$ lies in some component of X, namely the one containing $f^{-1}(y)$. Also note that $f^{-1}(\Gamma_\alpha) = C_\alpha$ which contains $f^{-1}(y)$ so $f^{-1}(A_y) \subset C_\alpha$. And since f is a bijection, $f(f^{-1}(A_y)) \subset f(C_\alpha) \implies A_y \subset \Gamma_\alpha$. Therefore $A_y = \Gamma_\alpha$ which proves that f is bijective on components of X and components of Y. □

Theorem 5.2.2. *The n-sphere S^n for $n \geq 1$ is connected.*

Proof. Stereographic projection extends in the natural way to arbitrary finite dimensions, so the punctured n-sphere $S^n \smallsetminus \{p\}$ is homeomorphic to $\mathbb{R}^n$ for $n \geq 1$. It follows from Lemma 5.1.20 that $\mathbb{R}^n$ is connected, so by Theorem 5.2.1, $S^n \smallsetminus \{p\}$ is connected. Note that $\mathrm{Cl}(S^n \smallsetminus \{p\}) = S^n$, and closures of connected spaces are connected. Hence S^n is connected. □

Note that the 0-sphere S^0 is actually a two-point set, which is disconnected. The last few theorems now allow us to characterize many connected spaces.

Example 5.2.3. If X is connected and $p : X \to Q$ is a quotient map then Q is connected.

As a result, the following are all connected spaces:

- cylinder
- annulus
- Möbius strip
- sphere S^n
- Klein bottle
- torus
- projective plane $\mathbb{R}P^2$

Example 5.2.4. The square $[0,1]^2$ is connected since $[0,1]$ is connected and products of connected spaces are connected.

Example 5.2.5. The punctured plane $\mathbb{R}^2 \smallsetminus \{O\}$ is connected.

Proof. Note that the half-planes H_u, H_ℓ, H_R and H_L are all homeomorphic to $\mathbb{R}^2$. Unions of connected spaces are connected, thus $H_u \cup H_\ell \cup H_R \cup H_L = \mathbb{R}^2 \smallsetminus \{O\}$ is connected. □

Definition. *Let X be connected. A* **cutset** *is a subset $S \subset X$ such that $X \smallsetminus S$ is disconnected.*

Definition. *A* **cutpoint** *for a connected space X is a point $p \in X$ such that $X \smallsetminus \{p\}$ is disconnected.*

Example 5.2.6. Any point $x \in (0,2)$ is a cutpoint of $(0,2)$.

Example 5.2.7. $\mathbb{R}^2 \smallsetminus S^1$ is disconnected so S^1 is a cutset of $\mathbb{R}^2$.

A famous result in general topology is the Jordan Curve Theorem:

Theorem 5.2.8 (Jordan Curve Theorem)**.** *Any embedding of S^1 into $\mathbb{R}^2$ is a cutset of $\mathbb{R}^2$.*

Although the statement of the theorem is elementary, most proofs require the machinery of algebraic topology. A proof can be found in Section 10.2.

Theorem 5.2.9. *If S is a cutset of X and $f : X \to Y$ is a homeomorphism then $f(S)$ is a cutset of Y.*

Proof omitted. □

Example 5.2.10. Show $\mathbb{R}^2 \ncong \mathbb{R}$.

Proof. Suppose $\mathbb{R}^2 \cong \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^2$ is a homeomorphism. Any $x \in \mathbb{R}$ is a cutpoint, and since f is a bijection, $f(x)$ is a point in $\mathbb{R}^2$. But we proved the punctured plane is still connected, so $f(x)$ is not a cutpoint of $\mathbb{R}^2$. By Theorem 5.2.9, $\mathbb{R}^2 \ncong \mathbb{R}$. □

This is another result that can be generalized to $\mathbb{R}^m$ and $\mathbb{R}^n$ for $m \neq n$; however this proof requires more advanced techniques.

Theorem 5.2.11 (Intermediate Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then for all y between $f(a)$ and $f(b)$ there exists $c \in (a, b)$ such that $f(c) = y$.*

The IVT is one of the most important theorems in calculus. It can be generalized to continuous maps from any connected spaces, which we prove next.

Theorem 5.2.12 (Generalized Intermediate Value Theorem)**.** *Let X be connected and let $f : X \to \mathbb{R}$ be continuous. Then for all $y_1, y_2 \in f(X)$, if $z \in [y_1, y_2]$ there is some $c \in X$ such that $f(c) = z$.*

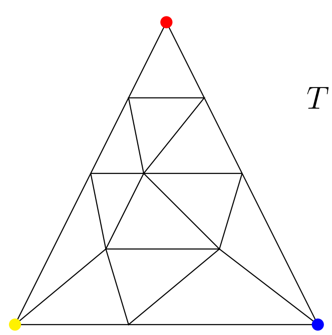
Proof. Suppose not. Then $(-\infty, z), (z, \infty)$ is a separation of $f(X)$. But f is continuous so $f(X)$ must be connected. Hence there is some $c \in X$ such that $f(c) = z$ for any z. □

Fixed points are vital in several areas of mathematics, including differential equations, functional analysis and even economics. Many of the original results for fixed point problems came out of topology. One of the most famous, Brouwer's Fixed Point Theorem, paved the way for the development of market equilibria in the 1950s. Its proof isn't complicated; however we will delay discussion until Section 8.3.

Theorem 5.2.13 (Brouwer's Fixed Point Theorem)**.** *Any smooth (continuous) map $f : \overline{B^n} \to \overline{B^n}$, where B^n is the unit n-ball, has a fixed point.*

Triangulate a triangle T and color each vertex one of three different colors, then fill in the vertices of the triangulation on the interior of T with the three colors. Sperner's Lemma is an interesting result in graph theory:

Lemma 5.2.14 (Sperner)**.** *Any triangulated triangle T with vertices colored one of three colors will contain an interior triangle whose vertices are each a different color.*



One of the standard "fundamental theorems" in general topology is the Borsuk-Ulam Theorem, which establishes a surprising property for continuous functions on spheres.

Theorem 5.2.15 (Borsuk-Ulam)**.** *Let $f : S^n \to \mathbb{R}^m$, where $m \le n$, be a continuous function from the n-sphere into Euclidean space. Then there exists a pair of antipodes $c, -c \in S^n$ such that $f(c) = f(-c)$.*

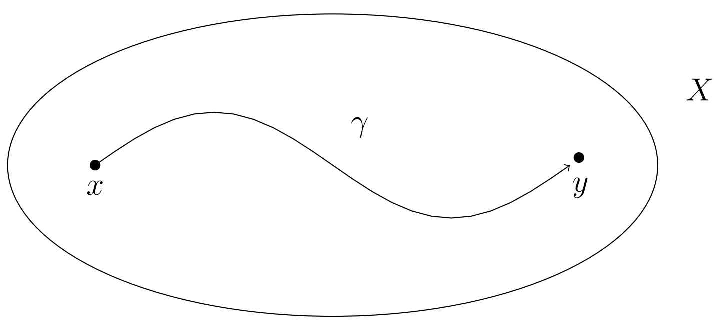
Proof. We prove the case when $m = 1$. Define $g(x) = f(x) - f(-x)$. Then $g : S^n \to \mathbb{R}$ is a continuous function. Consider a pair of antipodal points $x_0, -x_0 \in S^n$. If $g(x_0) = 0$ then $f(x) = f(x_0)$ and we're done. On the other hand, if $g(x_0) > 0$ then $g(-x_0) < 0$; likewise if $g(x_0) < 0$ then $g(-x_0) > 0$. Let γ be a smooth arc in S^n that contains x_0 and $-x_0$. Since γ is connected, by the Generalized IVT (5.2.12) there exists a point $c \in \gamma$ such that $g(c) = 0$. Hence $f(c) = f(-c)$ and the result holds in all cases. $\square$

Example 5.2.16. At any given time on Earth's surface there exists a pair of antipodal points that have the same temperature and humidity.

5.3 Path Connectedness

While connectedness characterizes a great number of topological spaces, there are stronger forms of connectedness.

Definition. *A* **path** *from x to y, where $x, y \in X$, is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.*



If X is disconnected, paths between points may not exist.

Definition. *X is* **path connected** *if for every $x, y \in X$ there exists a path from x to y.*

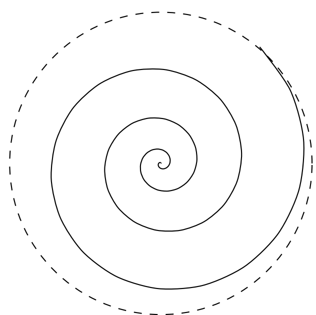
Example 5.3.1. $\mathbb{R}^n$, contractible subsets of $\mathbb{R}^n$, S^n and any space with the trivial topology are all path connected.

Theorem 5.3.2. *If X is path connected, then X is connected.*

Proof. Suppose X is path connected. Then for any $x, y \in X$ there exists a path f from x to y such that $\mathrm{Im}(f) = \gamma \subset X$. Also, $[0,1]$ is connected and f is continuous so γ is connected. Each connected subset of X lies in a single component of X, so since x and y were chosen arbitrarily, every pair of points in X (the endpoints of a path) lie in the same component of X. Hence X is connected. □

The converse is *false*! There exist connected sets that are not path connected, as the following examples show.

Example 5.3.3. The Topologist's Whirlpool



Let A be the graph of $f(\theta) = \left(\dfrac{\theta}{\theta+1}, \theta\right)$ for $\theta \geq 0$. Note that A is asymptotic to S^1 since $r = \dfrac{\theta}{\theta+1} < 1$ but $r \to 1$ as θ increases. Define **the Topologist's Whirpool** $W = A \cup S^1$.

Claim 1: $\bar{A} = W$.

Proof. Use the Intermediate Value Theorem (5.2.11) and/or Archimedes Principle to show that every neighborhood of a point $p \in W$ must intersect A. □

Claim 2: A is connected.

Proof. $A = f([0,\infty)]$, and $[0,\infty)$ is connected. Since f is continuous, A is connected. □

Claim 3: W is connected.

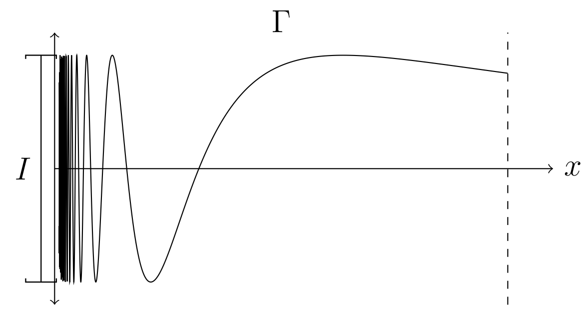
Proof. $W = \bar{A}$ and the closure of a connected space is connected. □

Claim 4: W is not path connected.

Proof. Consider a path $p : [0,1] \to W$ such that $p(0) = z \in S^1$, say $z = (0,1)$. We will show that $p^{-1}(S^1)$ is both open and closed in $[0,1]$. First, $\mathbb{R}^2 \setminus S^1$ is open, so S^1 is closed in $\mathbb{R}^2$ and in any subspace of $\mathbb{R}^2$ that contains it. In particular, S^1 is closed in W. And p is continuous, so $p^{-1}(S^1)$ is closed in $[0,1]$. Now take $t \in p^{-1}(S^1)$ and let N be a neighborhood of $p(t)$. We may assume $N = \{(r,\theta) \mid r \in (1-\varepsilon, 1+\varepsilon), \theta \in (\theta_0 - \delta, \theta_0 + \delta)\}$ where θ_0 is the angle of $p(t) \in S^1$. Then $N' = W \cap N$ is open in W. This implies $p^{-1}(N')$ is open in $[0,1]$ since p is continuous. Since $t \in p^{-1}(N')$ there exists some open basis element $U = (a,b) \subset [0,1]$ — note here that if $t = 0$, this set will be $[0,b)$ which is fine. Now (a,b) is an interval, so it's connected and $p(U)$ is connected by continuity. Suppose $p(U)$ intersects A, the spiral.

Let $q \in p(U) \cap A$. Then we can find some $p \in (1-\varepsilon, 1)$ such that the arc of $A \cap N$ containing q is separated from S^1 by the circle $\{(r,\theta) \mid r = p\}$. This gives a separation $\widetilde{U} = \{(r,\theta) \mid r < p\} \cap N$ and $\widetilde{V} = \{(r,\theta) \mid r > p\} \cap N$ of N', so long as N' does not intersect the circle. In particular q and $p(t)$ are separated by the circle. Therefore any set containing q and $p(t)$ will be separated by $\widetilde{U}$ and $\widetilde{V}$, so these points must lie in different components. Note that $p(U)$ is connected, so it must lie in the component of N' containing $p(t)$. Thus $q \notin p(U)$ which implies $p(U)$ and A dont intersect. Since $p(U) \subset W = A \cup S^1$ this further implies that $p(U)$ is contained completely in S^1. Therefore $U \subset p^{-1}(S^1)$ so each $t \in p^{-1}(S^1)$ lies in an open set contained in $p^{-1}(S^1)$. Hence $p^{-1}(S^1)$ is both open and closed in $[0,1]$. Now, since $[0,1]$ is connected and $p^{-1}(S^1)$ is nonempty it must be the whole interval. In other words, the path does not leave S^1. Therefore we can conclude that no point on the spiral A has a path connecting it to any point in S^1, which shows W is not path connected. $\square$

Example 5.3.4. The Topologist's Sine Curve



Define Γ to be the graph of $f(x) = \sin\left(\frac{1}{x}\right)$ for $0 < x < 1$ and let I be the unit interval on the y-axis. The Topologist's Sine Curve is defined to be $T = \Gamma \cup I$. As in Example 5.3.3, T is connected but not path connected.

Example 5.3.5. $\mathbb{R}^n$ is path connected. $\mathbb{R}^n$ with a point deleted is still path connected. In fact, $\mathbb{R}^n \smallsetminus C$, where C is a countable set of points, is path connected.

Theorem 5.3.6. *If $f : X \to Y$ is continuous and surjective and X is path connected, then Y is also path connected.*

Proof. Take $y_1, y_2 \in Y$. Since f is surjective, there exist $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. By path connectedness of X there exists a path $p : [0,1] \to X$ such that $p(0) = x_1$ and $p(1) = x_2$. Define $\widetilde{p} : [0,1] \to Y$ by $\widetilde{p} = f \circ p$. Then $\widetilde{p}(0) = f(p(0)) = f(x_1) = y_1$ and $\widetilde{p}(1) = f(p(1)) = f(x_2) = y_2$. Thus $\widetilde{p}$ is a path connecting y_1 and y_2 so Y is path connected. $\square$

Note: the hypothesis that f is continuous is required since the composition $f \circ p$ needs to be continuous.

As with connectedness, we may define a relation $\sim_p$ on X by $x \sim_p y$ iff there exists a path from x to y.

Proposition 5.3.7. $\sim_p$ *is an equivalence relation.*

Proof omitted. $\square$

Definition. *The equivalence relation* $\sim_p$ *partitions* X *into equivalence classes called* **path components**.

Proposition 5.3.8. *Let* X *be a topological space.*

1) *Every path component is path connected, and therefore connected.*
2) *Every path component lies inside of some (connected) component of* X*.*
3) *A path connected subset of* X *lies inside of some path component.*

Proof omitted. $\square$

6 Compactness

6.1 Compact Sets

Definition. *A collection $\mathcal{O}$ of subsets of X is a* **cover** *of X if the union of all sets in $\mathcal{O}$ equals X. $\mathcal{O}$ is an* **open cover** *if all sets in $\mathcal{O}$ are open in X. For a subset $A \subset X$, $\mathcal{O}$* **covers** *A if A is contained in the union of all the sets in $\mathcal{O}$.*

Definition. *$\mathcal{C}$ is a* **subcover** *of $\mathcal{O}$ if $\mathcal{C}$ covers X and $\mathcal{C} \subset \mathcal{O}$.*

Example 6.1.1. Existence of covers and open covers

If $(X, \mathcal{T})$ is a topological space then $\mathcal{T}$ is an open cover of X. Thus (open) covers exist for all topological spaces. If $\mathcal{B}$ is a basis for $\mathcal{T}$ then $\mathcal{B}$ is also an open cover of X.

Example 6.1.2. Open covers of $\mathbb{R}$

Let $\mathcal{O}_1 = \{(n-1, n+1)\}_{n\in\mathbb{Z}}$ and let $\mathcal{O}_2 = \{(-\infty, 1), (0, \infty)\}$. Then $\mathcal{O}_1$ and $\mathcal{O}_2$ are both open covers of $\mathbb{R}$.

Definition. *A topological space X is* **compact** *if every open cover of X has a finite subcover.*

Example 6.1.3. $\mathbb{R}$ is not compact.

In Example 6.1.2, $\mathcal{O}_1$ covers $\mathbb{R}$ but there is no finite subcover of $\mathcal{O}_1$ since each integer n lies in only one open set in $\mathcal{O}_1$. Therefore $\mathbb{R}$ is not compact.

Example 6.1.4. If X is finite then X is compact.

Suppose $|X| = n < \infty$. Then $|\mathbb{P}(X)| = 2^n$ so any open cover $\mathcal{O}$ is a subcollection of $\mathbb{P}(X)$, hence finite.

Definition. *A subset $A \subset X$ is* **compact in X** *if it is compact in the subspace topology on A inherited from X.*

Equivalently, $A \subset X$ is compact $\iff$ every open cover of A by sets in X has a finite subcover.

Example 6.1.5. $A = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ is compact in $\mathbb{R}$.

Proof. Take an open cover $\mathcal{O}$ of A. Then there is some $U_0 \in \mathcal{O}$ which contains the origin. U_0 is open in $\mathbb{R}$ so it contains some interval $(-\varepsilon, \varepsilon)$. Then there is a smallest $\frac{1}{n}$ such that $\frac{1}{n+1}, \frac{1}{n+2}, \ldots$ are all inside U_0. In fact for all $n < \frac{1}{\varepsilon}$, $\frac{1}{n}$ lies outside U_0. Thus there are only finitely many elements of $A \setminus U_0$ and each one is covered by some $U_i \in \mathcal{O}$. Therefore $\{U_0, U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{O}$, which shows that A is compact. $\square$

Example 6.1.6. $B = (0, 1] \subset \mathbb{R}$ is not compact.

Proof. Consider $\mathcal{O} = \left\{\left(\frac{1}{n}, 2\right)\right\}_{n\in\mathbb{N}}$. Note that $\mathcal{O}$ covers B. Any finite subcollection $\mathcal{C} \subset \mathcal{O}$ will have some $N = \max\{n\}$ such that $\frac{1}{N}$ is an endpoint. Then $\mathcal{C}$ does not intersect $\left(0, \frac{1}{N}\right)$ so it doesn't cover B. Therefore B is not compact. □

As with connectedness, we can use compactness to classify topological spaces up to homeomorphism.

Theorem 6.1.7. *Compactness is a topological property.*

Proof. Suppose $f : X \to Y$ is a homeomorphism and X is compact. Take any open cover $\mathcal{O}$ of Y. Then $\mathcal{P} = \{f^{-1}(U) \mid U \in \mathcal{O}\}$ covers X and each $f^{-1}(U)$ is open by continuity, so in fact $\mathcal{P}$ is an open cover of X. Since X is compact there exists a finite subcover $\mathcal{P}' \subset \mathcal{P}$. Define $\mathcal{O}' = f(\mathcal{P})$. Since f is a bijection on open sets, $|\mathcal{O}'|$ is finite and a subcover of $\mathcal{O}$. Hence Y is compact. □

Theorem 6.1.8. *Let $A \subset X$ be compact and suppose $f : X \to Y$ is continuous. Then $f(A) \subset Y$ is compact.*

The previous proof is easily modified to show this. □

Corollary 6.1.9. *Let $p : X \to Q$ be a quotient map and suppose X is compact. Then Q is compact.*

Theorem 6.1.10. *The following are properties of compact sets:*

1) Finite unions of compact sets are compact.

2) If X is Hausdorff, then arbitrary intersections of compact sets are compact.

Proof omitted. □

Example 6.1.11. In $\mathbb{R}^n$, compact $\implies$ closed, but closed $\not\Longrightarrow$ compact.

In general, a compact set does not have to be closed. As a counterexample, consider $\mathbb{R}_{fc}$ (or any space under the finite complement topology). We proved that any $A \subset \mathbb{R}_{fc}$ is compact, so in particular $\mathbb{R}_{fc} \setminus \{0\}$ is compact but not closed. It turns out that Hausdorff is required for this implication to hold.

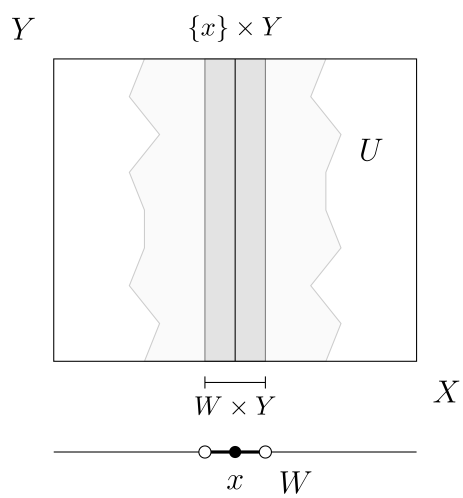
Theorem 6.1.12. *If X is Hausdorff and $A \subset X$, then A compact $\implies$ A closed.*

Proof. Hausdorff implies for each $a \in A$ and $x \in X \smallsetminus A$ there exist open, disjoint sets $U_a, V_a \subset X$ such that $x \in U_a$ and $a \in V_a$. Then $\{V_a\}_{a \in A}$ is an open cover of A. Since A is compact, there exists a finite subcover $\{V_1, V_2, \ldots, V_n\}$ of $\{V_a\}_{a \in A}$. Let $U = \bigcap_{i=1}^{n} U_i$. Then U is open, $x \in U$ and U is disjoint from $\bigcup_{i=1}^{n} V_i \supset A$. So $U \subset X \smallsetminus A$. We have thus found an open set containing any point $x \in X \smallsetminus A$ which is itself contained in $X \smallsetminus A$. Therefore $X \smallsetminus A$ is open, proving the theorem. □

Theorem 6.1.13. *Let C be closed and $C \subset D$ which is compact in X. Then C is compact. In other words, closed subsets of compact sets are compact.*

Proof. Take an open cover $\mathcal{O}$ of C. Then $\mathcal{O}' = \mathcal{O} \cup X \smallsetminus C$ is an open cover of D. Since D is compact, there exists a finite subcover $\mathcal{A}'$ of $\mathcal{O}'$. Then $\mathcal{A} = \mathcal{A}' \smallsetminus (X \smallsetminus C)$ is a finite subcover of C (i.e. throw out $X \smallsetminus C$ if it's in $\mathcal{A}'$). Hence C is compact. □

Lemma 6.1.14 (Tube Lemma)**.** *Let Y be compact. If U is open in $X \times Y$ and for some $x \in X$, $\{x\} \times Y$ is contained in U, then there exists a neighborhood W containing x such that $W \times Y \subset U$.*



The Tube Lemma is used to prove the following result.

Theorem 6.1.15. *If X and Y are compact spaces then $X \times Y$ is compact.*

In the next section we will prove that $[0, 1]$ is compact, so the last few results show that the following are compact spaces:

- The square $[0, 1]^2$
- Sphere

- Torus
- Klein bottle
- Projective space $\mathbb{R}P^2$
- Möbius band
- Cylinder/annulus.

Theorem 6.1.16 (Tychonoff)**.** *Products of uncountably many compact sets are compact.*

Proof omitted. □

6.2 Results in Analysis

One of the most important results in the study of compact spaces is the Heine-Borel Theorem, which establishes necessary and sufficient conditions for compact subsets of any finite Euclidean space $\mathbb{R}^n$. In order to prove this theorem, we need the next two results.

Lemma 6.2.1 (Nested Intervals)**.** *Let $\mathcal{C} = \{[a_n, b_n]\}_{n\in\mathbb{N}}$ be a collection of nested intervals, i.e. $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n. Then $\bigcap_{n=1}^{\infty}[a_n, b_n]$ is nonempty.*

Proof. Nested implies $a_1 \leq a_2 \leq \ldots \leq a_n \leq b_n \leq \ldots \leq b_2 \leq b_1$. Let $A = \sup\{a_n\}$ and $B = \inf\{b_n\}$. Since $a_n \leq b_n$ for all n, $A \leq B$ so $[A, B]$ is nonempty. By definition, $\bigcap_{n=1}^{\infty}[a_n, b_n] = [A, B]$ and therefore this intersection is nonempty. □

Theorem 6.2.2. *$[a, b] \subset \mathbb{R}$ is compact.*

Proof. Let $\mathcal{O}$ be an open cover of $[a, b]$ and suppose no finite subcover exists. Divide $[a, b]$ in half: $[a, b_1] \cup [b_1, b]$. Then at least one half won't have a finite cover by sets in $\mathcal{O}$. Suppose without loss of generality that $[a, b_1]$ doesn't have a finite cover by $\mathcal{O}$. Repeat division, obtaining nested intervals $\{[a_n, b_n]\}_{n\in\mathbb{N}}$. By the Nested Intervals Lemma (6.2.1), their intersection is nonempty so take $x \in \bigcap_{n=1}^{\infty}[a_n, b_n]$. Then there exists some open $U \in \mathcal{O}$ containing x. This implies $(x - \varepsilon, x + \varepsilon) \subset U$ for some $\varepsilon > 0$. Let N be sufficiently large so that $\frac{b-a}{2^N} < \varepsilon$. Then $x \in [a_N, b_N]$. This implies $[a_N, b_N] \subset (x - \varepsilon, x + \varepsilon) \subset U$ so U covers $[a_N, b_N]$, a contradiction. Hence $\mathcal{O}$ must have a finite subcover and $[a, b]$ is compact. □

Theorem 6.2.3 (Heine-Borel)**.** *Let A be a subset of $\mathbb{R}^n$ for finite, positive n. Then A is compact if and only if A is closed and bounded.*

Proof. ($\Longrightarrow$) Since $\mathbb{R}^n$ is Hausdorff, we proved that compact implies closed. Consider the open cover $\{B(0,n)\}$ of A (this covers A since it covers all of $\mathbb{R}^n$). Since A is assumed to be compact, there exists a finite subcover $\{B(0,n_1), B(0,n_2), \ldots, B(0,n_k)\}$ where we may assume $B(0,n_1) \subset B(0,n_2) \subset \cdots \subset B(0,n_k)$. Since this is an open cover of A, $A \subset B(0,n_k)$ Thus A is bounded.

($\Longleftarrow$) Take $a \in A$, where $a = (a_1, \ldots, a_n)$. Let M be the bound of A, i.e. $d(x,y) \leq M$ for all $x, y \in A$. Then $A \subset [a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M] \subset \mathbb{R}^n$. This is the product of compact spaces, so it is compact. And A is closed and contained in a compact set, so by Theorem 6.1.13 A must be compact. □

Example 6.2.4. The n-sphere $S^n \subset \mathbb{R}^{n+1}$ is compact. S^n is sometimes referred to as the 1-point compactification of $\mathbb{R}^n$.

Example 6.2.5. Every knot is an embedding $S^1 \hookrightarrow \mathbb{R}^3$ and S^1 is compact, so knots are compact.

Example 6.2.6. The torus T^2 is compact

We can now prove this in three ways:

a) T^2 is a quotient space of $[0,1]^2$ which is compact.

b) $T^2 = S^1 \times S^1$, each of which is compact.

c) T^2 is closed and bounded in $\mathbb{R}^3$ so it is compact by the Heine-Borel Theorem (6.2.3).

Example 6.2.7. The n-torus T^n is compact.

A natural question related to the Heine-Borel Theorem is: Does the correspondence between closed-and-boundedness and compactness extend to arbitrary metric spaces? The answer is no in general, although one implication holds:

Theorem 6.2.8. *For a metric space (X,d), A compact in $X \implies A$ closed and bounded.*

Proof omitted. □

Compactness has important implications in other areas of analysis, beginning with the Extreme Value Theorem taught in beginning calculus courses. Before proving this, we prove the following lemma.

Lemma 6.2.9. *A compact set $A \subset \mathbb{R}$ contains a maximum and a minimum.*

Proof. A compact $\implies$ A closed and bounded by the Heine-Borel Theorem (6.2.3). In particular, A is bounded above so $\sup A$ exists. Let $M = \sup A$ and suppose $M \notin \bar{A}$. Then there is some $U \ni M$ which is disjoint from A. And since U is open, $(M - \varepsilon, M + \varepsilon) \subset U$ for some $\varepsilon > 0$. But M is an upper bound for A and $U \cap A = \varnothing$, so $M - \varepsilon$ is also an upper bound for A. This contradicts $M = \sup A$. Hence $M \in \bar{A} = A$. The proof is similar for a minimum. □

Theorem 6.2.10 (Extreme Value Theorem)**.** *Let X be compact. If $f : X \to \mathbb{R}$ is continuous then f achieves both its maximum and its minimum values at some points $a_1, a_2 \in X$.*

Proof. Since X is compact and f is continuous, $f(X) \subset \mathbb{R}$ is compact. Then by Lemma 6.2.9, $f(X)$ contains its maximum and minimum values, say M and m, respectively. Thus there exist $a_1 \in f^{-1}(M)$ and $a_2 \in f^{-1}(m)$ where the max. and min. are achieved. □

Some further applications of compactness in analysis are stated below.

Theorem 6.2.11. *For a continuous function $f : X \to Y$, X compact $\implies$ f uniformly continuous.*

Definition. *A metric space X is* **complete** *if every Cauchy sequence in X converges.*

Theorem 6.2.12. *Every compact metric space is complete.*

Besides the topological definition of compactness given in this chapter, there are several other definitions which are more useful in different settings.

Definition. *A subset $A \subset X$ is* **sequentially compact** *if every sequence (a_n) in A has a convergent subsequence (a_{n_k}).*

Definition. *A subset $A \subset X$ is* **limit point compact** *if every infinite subset of A has a limit point.*

Theorem 6.2.13. *Compactness implies sequential compactness and limit point compactness. Moreover, if X is metrizable then all three are equivalent. In general however, they are not equivalent.*

7 Manifolds

7.1 Topological Manifolds

Definition. *A* **topological n-manifold** *is a Hausdorff topological space X^n which has a countable basis, and each $x \in X$ lies in some neighborhood U which is homeomorphic to B^n, an open Euclidean ball.*

Definition. *A* **PL n-manifold** *(piecewise linear) is a topological manifold where all homeomorphisms are piecewise linear.*

Example 7.1.1. Polyhedra are PL 2-manifolds.

Definition. *A* **smooth n-manifold** *is a topological n-manifold where the homeomorphisms are* **diffeomorphisms***: f and f^{-1} are smooth, i.e. they have continuous derivatives of every order.*

In general, we have the following relation among the three types of manifolds:

$$\left\{ \begin{matrix} smooth \\ n{-}manifolds \end{matrix} \right\} \subset \left\{ \begin{matrix} PL \\ n{-}manifolds \end{matrix} \right\} \subset \left\{ \begin{matrix} topological \\ n{-}manifolds \end{matrix} \right\}$$

Considered up to homeomorphism, they're all equivalent for $n \leq 3$, but for higher dimensions the containment is strict.

Definition. *An* **exotic sphere** *is a topological space X^n which is homeomorphic, but not diffeomorphic, to $S^n \subset \mathbb{R}^{n+1}$.*

Exotic spheres exist for most, but not all, dimensions $n \geq 7$. An important result relating to these manifolds is

Theorem 7.1.2 (Milnor, 1956)**.** *Exotic spheres exist for $n = 7$.*

Example 7.1.3. $\mathbb{R}$ with the standard topology is a 1-manifold, and it is easy to see that no other topology on $\mathbb{R}$ forms a 1-manifold. Moreover, any open interval (a, b) is homeomorphic to $\mathbb{R}$ so they are all 1-manifolds as well.

Example 7.1.4. S^1 is a 1-manifold.

Theorem 7.1.5 (Classification of 1-Manifolds)**.** *Let X be a 1-manifold. Then up to homeomorphism, X is one of the following:*

1) If X is connected and compact, then $X \cong S^1$

2) If X is connected but not compact, then $X \cong \mathbb{R}$

3) If X is disconnected, then X has a countable number of components, each of which is homeomorphic to either S^1 or $\mathbb{R}$.

7.2 Classification of Surfaces

We classified 1-manifolds in the previous section. What's the current state of affairs for classification of higher dimensional manifolds?

n = 2. Surface Classification: Theorem 7.2.3 (we will prove this in Section 7.3)

n = 3. Poincaré Conjecture (proven by G. Perelman), Geometrization Conjecture

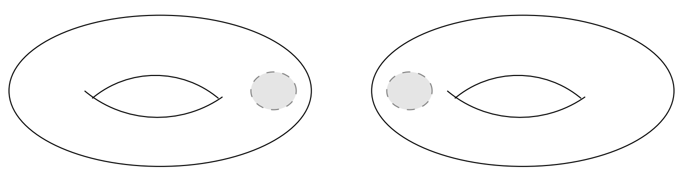
n = 4. Classification unknown. In fact, this has been proven *impossible.*

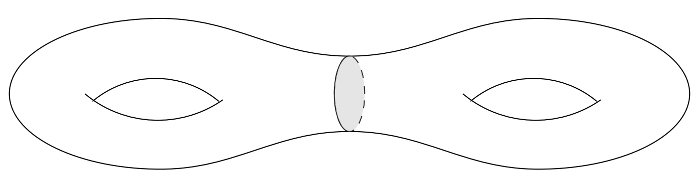
Definition. *A* **surface** *is defined as a topological 2-manifold. Loosely, a surface is* **orientable** *if it has 2 sides, and* **non-orientable** *if it has only 1 side.*

Example 7.2.1. The Möbius strips, Klein bottle and the projective plane $\mathbb{R}P^2$ are all examples of non-orientable surfaces.

Example 7.2.2. S^2, $\mathbb{R}^2$, T^2, the double (triple, etc.) torus and an annulus are all orientable.

An important operation in the classification of surfaces (and in topology in general) is the connect sum. Given two surfaces, remove a disk from each. Glue together the remaining parts of the surfaces along the boundary circle of the disk. This is a well-defined quotient space:





Definition. *The quotient space obtained by identifying the described points on two surfaces* Σ_1 *and* Σ_2 *is called the* **connect sum** *of the two surfaces, denoted* $\Sigma_1 \# \Sigma_2$.

Some useful facts about connect sums:

1) Connect sums do not depend on the disks chosen

2) $S_1 \# S_2 \cong S_2 \# S_1$ (connect sum commutes)

3) $\#$ is associative

4) For any surface Σ, $S^2 \# \Sigma \cong \Sigma$ (identity)

5) Inverses under # do not exist, so # on the set of surfaces forms a **commutative semigroup**

6) S_1, S_2 orientable $\iff$ $S_1 \# S_2$ orientable.

The main result, which we will prove in the Section 7.3, is the following theorem.

Theorem 7.2.3 (Surface Classification Theorem)**.** *A connected, compact surface without boundary is homeomorphic to one of the following:*

1) A sphere S^2

2) A g-holed torus $\underbrace{T \# \cdots \# T}_{g} = \Sigma_g$

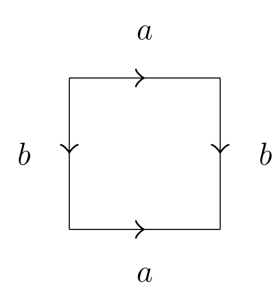
3) A connect sum of projective planes $P \# \cdots \# P$*.*

Moreover, (1) and (2) are orientable and (3) is non-orientable.

The remainder of this chapter is devoted to outlining the proof of surface classification which can be found in Massey (1967). The general strategy involves assigning "words" to polygons:

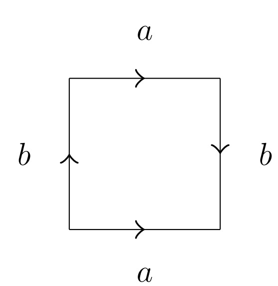
- Pick a vertex and a direction.
- Write down each letter as "x" if the arrow points in the direction chosen, and "x^{-1}" if the arrow is opposite.

Example 7.2.4. The torus



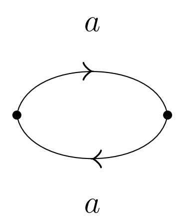
The canonical word for the torus is $aba^{-1}b^{-1}$.

Example 7.2.5. The Klein bottle



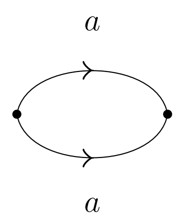
The canonical word for the Klein bottle is $aba^{-1}b$.

Example 7.2.6. The projective plane



P can be formed as a quotient space of a bigon, shown above with word aa (or $a^{-1}a^{-1}$).

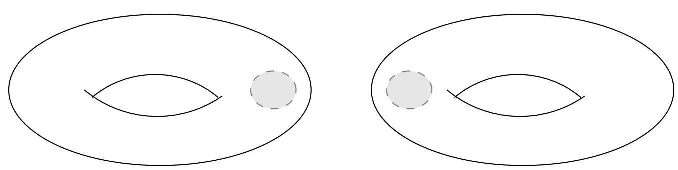
Example 7.2.7. The sphere



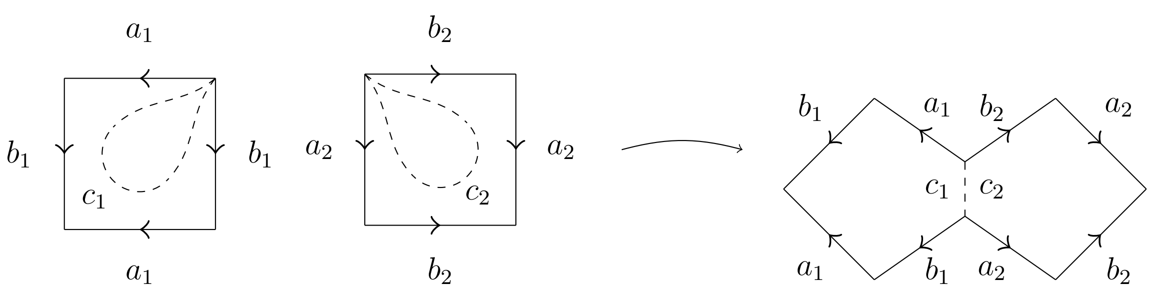
Similarly, S^2 is the quotient space of a bigon with word aa^{-1}. Recall that the sphere acts as the identity in the semigroup of all surfaces, so this makes sense.

Example 7.2.8. The connect sum of two tori

Consider $T\#T$. Note what happens to the associated word, $(a_1b_1a_1^{-1}b_1^{-1})\#(a_2b_2a_2^{-1}b_2^{-1})$.



Glue together c_1 and c_2.



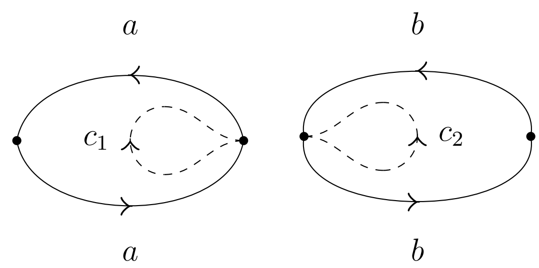
The result is an octagon with sides identified. The word is $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$.

This suggests the following theorem:

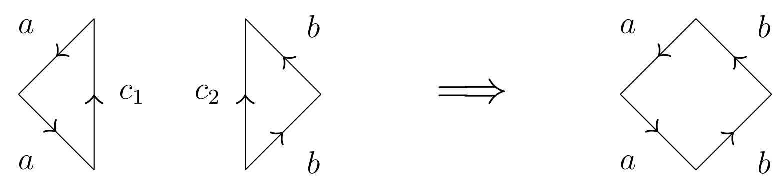
Theorem 7.2.9. *Let S and T be surfaces, and let W_S and W_T be words representing S and T, respectively. Then $S\#T$ is represented by $W_S W_T$.*

Proof omitted. □

Example 7.2.10. The connect sum of two projective planes



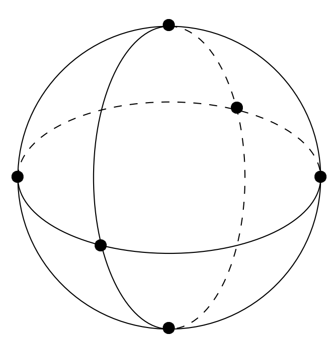
Start with P and P, represented by the words aa and bb. Split open along c_1 and c_2 and glue the figures together.



From this, we see that $P\#P$ is represented by $aabb$.

Definition. *A* **triangulation** *of a compact surface S is a collection of closed subsets $T_1, \ldots, T_n$ that cover S such that each T_i is homeomorphic to a triangle $T_i' \subset \mathbb{R}^2$. Moreover, any distinct triangles T_i, T_j are either disjoint, share exactly one edge or share exactly one vertex.*

Example 7.2.11. A legal triangulation of the sphere using 8 triangles



Theorem 7.2.12 (Rado, 1925)**.** *Every compact surface is triangulable.*

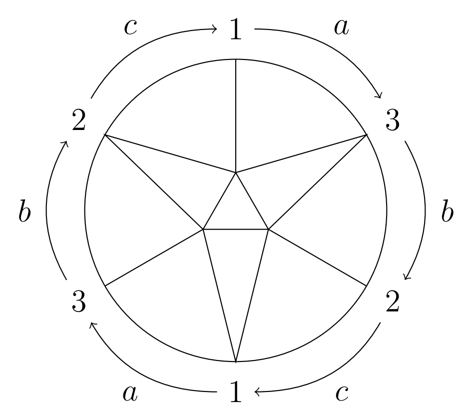
This can be proven using the Jordan Curve Theorem (5.2.8).

7.3 Euler Characteristic and Proof of the Classification Theorem

Definition. *The* **Euler characteristic** *of a surface S is $\chi(S) = V - E + T$, where V is the number of vertices, E is the number of edges and T is the number of triangles in a legal triangulation of S.*

Remark. The Euler characteristic is a topological property. As a result, $\chi(S)$ is independent of any particular triangulation of S that is chosen.

Example 7.3.1.

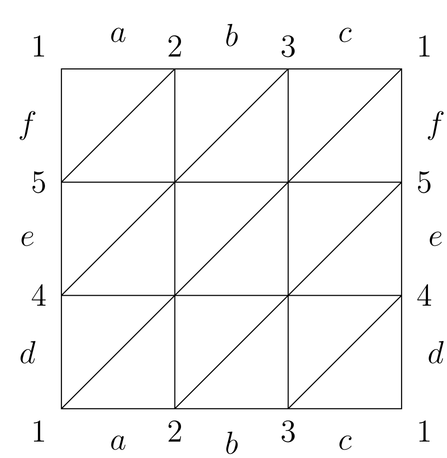


The Euler characteristic is calculated as

$$\begin{aligned} V &= 6 \\ E &= 15 \\ T &= 10 \\ \chi(S) &= 1. \end{aligned}$$

Notice that the word for this "polygon" is *abcabc* which has the form xx, so this is a triangulation of the projective plane. By the remark, $\chi(P) = 1$.

Example 7.3.2.

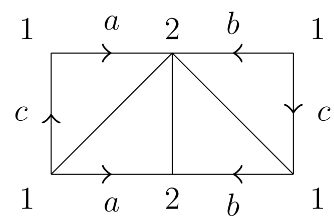


The Euler characteristic is

$$\begin{aligned}
V &= 9 \\
E &= 27 \\
T &= 18 \\
\chi &= 0.
\end{aligned}$$

Notice that the word for this figure is $abcdefc^{-1}b^{-1}a^{-1}f^{-1}e^{-1}d^{-1}$ which is of the form $xyx^{-1}y^{-1}$, the canonical word for a torus. Then by the remark, $\chi(T) = 0$.

Example 7.3.3.



As in Example 7.3.2, this surface has Euler characteristic 0:

$$\begin{aligned}
V &= 2 \\
E &= 6 \\
T &= 4 \\
\chi &= 0.
\end{aligned}$$

The word is $ab^{-1}cba^{-1}c = xyx^{-1}y$, the word for a Klein bottle. But we know that the Klein bottle and the torus are not homeomorphic (one is orientable, the other is not), so this shows that Euler characteristic is not enough to completely classify all surfaces.

Example 7.3.4. Sometimes reducing words directly reveals a surface's homeomorphism type

Reduce the following word:

$$\begin{aligned}
bacc^{-1}dea^{-1}e^{-1}d^{-1}fg^{-1}f^{-1}gb &= badea^{-1}e^{-1}d^{-1}fg^{-1}f^{-1}gb \\
&= adea^{-1}(de)^{-1}fg^{-1}f^{-1}gbb \\
&= (xyx^{-1}y^{-1})(fg^{-1}f^{-1}g)bb
\end{aligned}$$

by the substitution $x = a, y = de$. Then we recognize this surface as $T\#T\#P$.

Remark. In any triangulation, each edge belongs to exactly 2 triangles. Since each triangle has 3 edges, this implies that $3T = 2E$. Note that this only holds for compact surfaces *without boundary.*

Example 7.3.5. For all even $n \geq 4$, the sphere S^2 is triangulable with n triangles.

This is a result of the Euler characteristic equation. In particular, for the sphere we must have $T = 2k$, $E = 3k$ and $V = k + 2$ since $\chi(S^2) = 2$. Visually, each time k increases by 1, it's like dropping another vertex onto the equator of the sphere and creating two new triangles.

Proposition 7.3.6. *For any compact surface,*

$$\begin{aligned} E &= 3(V - \chi) \\ V &\geq \frac{7 + \sqrt{49 - 24\chi}}{2}. \end{aligned}$$

Proof. By the remark, $3T = 2E$, so

$$\chi = V - E + \frac{2}{3}E = V - \frac{1}{3}E.$$

Solving for E gives $E = 3V - 3\chi$ as desired. Next, observe that the number of distinct edges is bounded above by the total number of pairs of vertices, which is given by

$$\binom{V}{2} = \frac{V!}{(V-2)!\,2!} = \frac{(V-1)V}{2}.$$

Then by the above work, we have

$$\begin{aligned} E &\leq \frac{(V-1)V}{2} \\ 3(V - \chi) &\leq \frac{V^2 - V}{2} \\ 6V - 6\chi &\leq V^2 - V \\ 0 &\leq V^2 - 7V + 6\chi. \end{aligned}$$

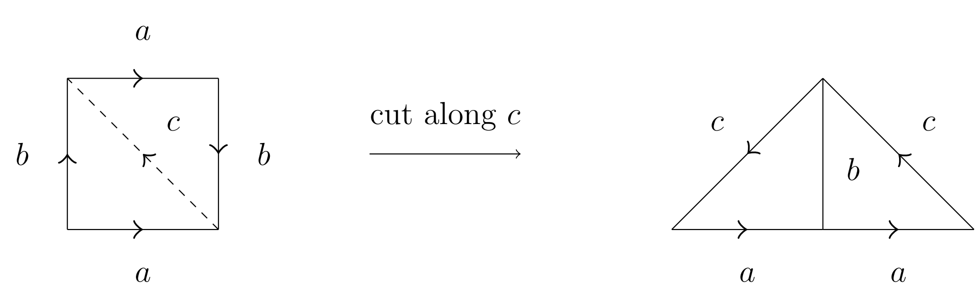
This can be solved using the quadratic formula:

$$V \geq \frac{7 + \sqrt{49 - 24\chi}}{2}.$$

$\square$

Example 7.3.7. Cutting and pasting polygons

In order to prove the classification theorem (7.2.3), we may have to cut and reattach some polygons. For example, consider the Klein bottle ($aba^{-1}b$):



Then the word becomes $aacc$, so $K \cong P\#P$.

We are now ready to prove the main theorem:

Theorem (7.2.3)**.** *A connected, compact surface without boundary is homeomorphic to one of the following:*

1) A sphere S^2

2) A g-holed torus $\underbrace{T\#\cdots\#T}_{g} = \Sigma_g$

3) A connect sum of projective planes $P\#\cdots\#P$.

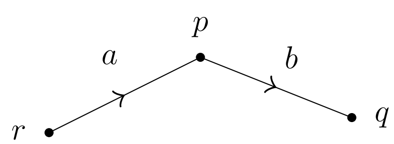
Moreover, (1) and (2) are orientable and (3) is non-orientable.

Proof of Surface Classification (7.2.3): By Rado's Theorem, every compact, connected surface can be triangulated, so take any collection of triangles with sides identified. Any surface may be classified by following the six step algorithm detailed here.

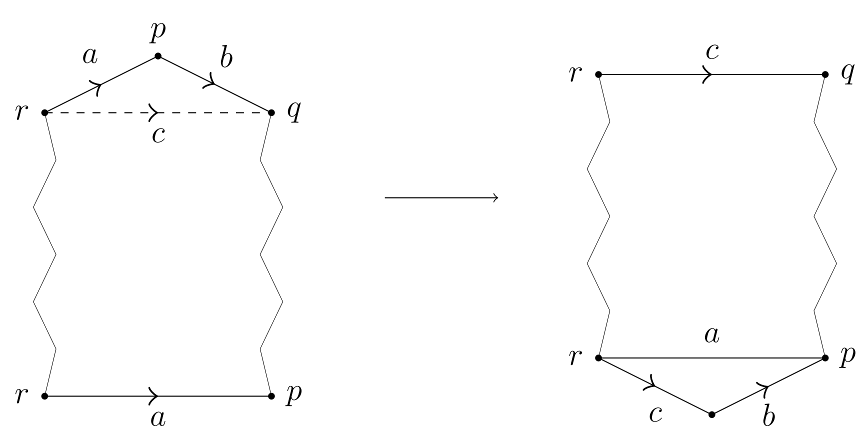
1) Cut along some edges until the figure can be flattened into the plane. Form an algebraic word representing this planar triangulation. Note that each exterior edge appears twice, so the planar figure is a $2n$-gon.

2) Cancel any pairs xx^{-1} and repeat this step as often as possible.

3) Transform the polygon so all vertices glue to the same point. If we have more than one equivalence class of vertices, then there is some adjacent pair of vertices p and q that are not identified.



Then $a = b^{-1}$ by Step 2, and $p \neq q \implies a \neq b$ either. This implies a is found in one other place in the figure. Suppose we have the following (the symmetric case is shown similarly):

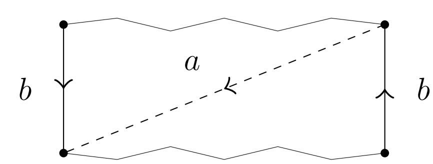


Cut along c between r and q and glue $a \leftrightarrow a$. This moves a to the interior, reduces the number of vertices labelled "p" by one, and add a vertex "q". Repeat until there are no p's left. Repeat for all distinct vertices until all exterior vertices are labelled the same. Again, use Step 2 whenever possible.

4) We are left with two types of edges:

   - Edges of the first kind: $a \dots a$
   - Edges of the second kind: $a \dots a^{-1}$.

   For example, in $abca^{-1}bc^{-1}$, a and c are edges of the second kind and b is of the first kind. For each b of the first kind, cut from an endpoint of b to the corresponding endpoint and glue so that the pair is adjacent: $\dots bb \dots$



   Repeat until all edges of the first kind are adjacent to their partners: $\dots aa \dots bb \dots$

5) Assume the remaining edges of the second kind occur in interleafed pairs: $a \dots b \dots a^{-1} \dots b^{-1}$ Make cuts to push these sides together and cancel by Step 2. See Massey for further details.

6) **Lemma:** $T\#P \cong P\#K \cong P\#P\#P$.

This algorithm will reduce any polygon so that it represents a word in standard form:

i) The sphere S^2: $aa^{-1} = 1$

ii) The g-holed torus Σ_g: $a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}$

iii) n copies of P: $a_1a_1a_2 \cdots a_na_n$.

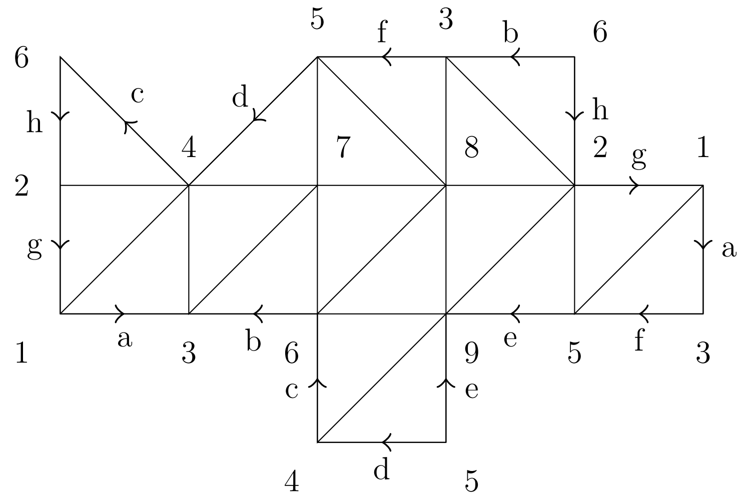
This completely classifies all connected, compact surfaces without boundary. □

Example 7.3.8. Given a triangulation

124	236	134	246	367	238
347	469	459	698	678	135
457	259	289	578	358	125

we can use the algorithm in the Surface Classification proof to determine the surface's homeomorphism type.

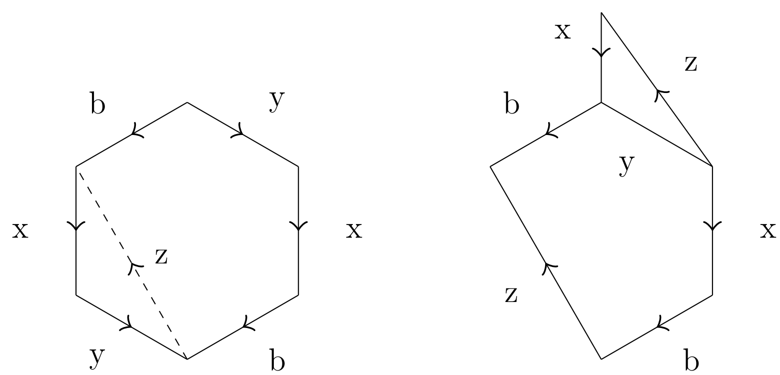
First we form a polygon from the above triangles:



The word we obtain for this planar 16-gon is $bfdchgab^{-1}c^{-1}d^{-1}ee^{-1}f^{-1}a^{-1}g^{-1}h^{-1}$. If we make the substitutions $x = fdc$ and $y = hga$, we can rewrite this as

$$\begin{aligned} bfdchgab^{-1}c^{-1}d^{-1}ee^{-1}f^{-1}a^{-1}g^{-1}h^{-1} &= bfdchgab^{-1}c^{-1}d^{-1}f^{-1}a^{-1}g^{-1}h^{-1} \\ &= bxyb^{-1}x^{-1}y^{-1}. \end{aligned}$$

Putting this word onto a hexagon and making a cut along z, then gluing the triangle back on, we obtain



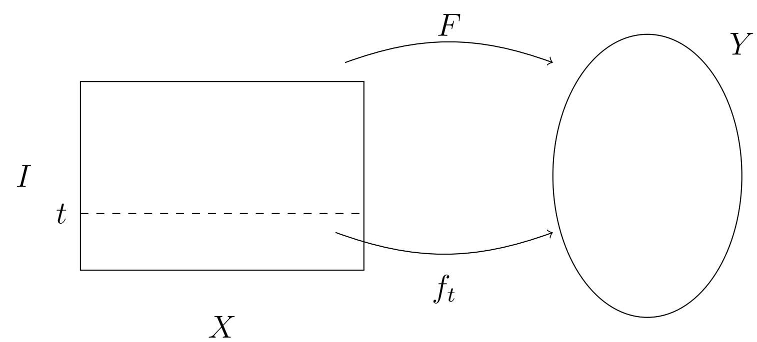
This new polygon has word $zxbz^{-1}b^{-1}x^{-1}$, which using the substitution $w = xb$ becomes $zwz^{-1}w^{-1}$. Hence this surface is equivalent to a torus.

8 Homotopy Theory

8.1 Homotopy

Our fundamental notion of topological equivalence is homeomorphism. However, in algebraic topology we often want to consider two spaces to be equivalent, or have the 'same shape', in a much broader sense than homeomorphism.

Definition. *A* **homotopy** *is a family of maps $f_t : X \to Y$ for $t \in I = [0, 1]$ such that the function $F : (X \times I) \to Y$ defined by $F(x, t) = f_t(x)$ is continuous.*



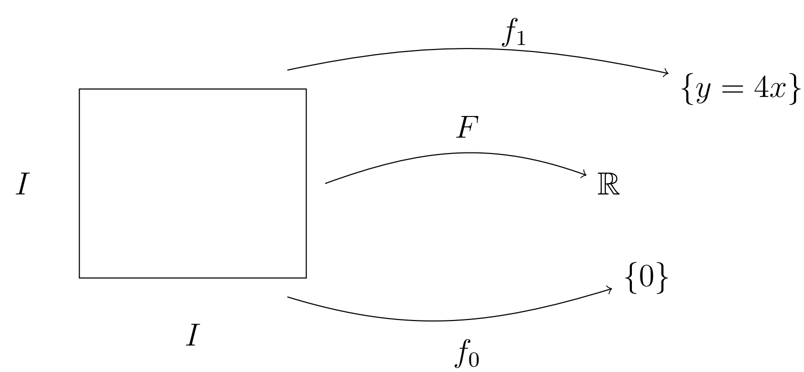
The basic question can be stated as: Can we find a continuous rectangle that maps continuously to Y?

Definition. *Two maps $g, h : X \to Y$ are said to be* **homotopic***, denoted $g \simeq h$, if there is a homotopy $F : X \times I \to Y$ such that $f_0 = g$ and $f_1 = h$.*

Example 8.1.1. Let $X = I$, $Y = [0, 1]$, $f(x) = x$ and $g(x) = x$.

Define $f_t(x) = x$ for all $t \in I$. Then $F(x, t) = x$ is a projection map, so F is continuous and thus a homotopy. This shows that $f(x) \simeq g(x)$.

Example 8.1.2. Let $X = I$, $Y = \mathbb{R}$, $f(x) = 0$ and $g(x) = 4x$.



Define $F(x, t) = 4tx$. Then $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Since F is continuous on $X \times I$, this proves that f and g are homotopic.

Example 8.1.3. Let $X = S^1$, $Y = S^1$, $f(\theta) = 0$ and $g(\theta) = \theta$. It turns out that f and g are *not* homotopic, but this is much harder to show (see Section 8.3).

Example 8.1.4. Let $X, Y = \mathbb{R}$, $f(x) = x$ and $g(x) = 2x$.

To show these maps are homotopic, we want a homotopy $F : \mathbb{R} \times I \to \mathbb{R}$ such that $F(x, 0) = x$ and $F(x, 1) = 2x$. Let $F(x, 1) = (1 + t)x$. Then F satisfies the criteria of being a homotopy, so $f \simeq g$.

Theorem 8.1.5. *Homotopy is an equivalence relation.*

Proof. (1) Let $F(x, t) = f(x)$. Then F is clearly continuous, so $f(x) \simeq f(x)$. This shows that homotopy is reflexive.

(2) Suppose $f(x) \simeq g(x)$. Then there is a family of functions $F(x, t)$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Let $G(x, t) = F(x, 1 - t)$. Then $G(x, 0) = g(x)$ and $G(x, 1) = f(x)$ by construction. Moreover, G inherits continuity from F, so G is a homotopy. This shows symmetry.

(3) Finally, suppose $f(x) \simeq g(x)$ and $g(x) \simeq h(x)$. Then there is a map $F(x, t)$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, and there is another function $G(x, t)$ such that $G(x, 0) = g(x)$ and $G(x, 1) = h(x)$. Define

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \le t \le \frac{1}{2} \\ G(x, 2t - 1) & \frac{1}{2} < t \le 1. \end{cases}$$

Then $H : X \times I \to Y$ is continuous since $\lim\limits_{t \to 1/2} H(x, t) = g(x)$ and H is continuous on each piece. Moreover, $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$, so $f(x) \simeq h(x)$. This establishes transitivity, so homotopy is an equivalence relation. □

8.2 The Fundamental Group

In this section we explore the earliest, and perhaps most important, connection between topological spaces and algebraic structures: the fundamental group. In order to do so, we first develop the concepts of homotopy type and path composition.

Definition. *A map $f : X \to Y$ is called a* **homotopy equivalence** *if there exists a map $g : Y \to X$ such that fg is homotopic to the identity on Y and gf is homotopic to the identity on X. If there exists such a homotopy equivalence between X and Y, then we say X and Y have the same* **homotopy type**.

Proposition 8.2.1. *Topological equivalence (i.e. homeomorphism) implies homotopy equivalence.*

Proof. Let $f : X \to Y$ be a homeomorphism. Then f^{-1} exists and is continuous, and for any $x \in X$, $y \in Y$, we have $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(y) = y$. Hence f is a homotopy equivalence. □

Example 8.2.2. $\mathbb{R}$ is homotopic to itself.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$ and $g(x) = 2x$. Then $fg = gf = 4x$, which is *not* the identity itself. However, we proved that $0 \simeq 4x$ and $0 \simeq x$, so by transitivity $4x \simeq x$, the identity on $\mathbb{R}$. Thus $\mathbb{R}$ is homotopic to itself (in a really roundabout way!)

Example 8.2.3. $\mathbb{R}$ is homotopic to a point.

Let $X = \{0\}$ and $Y = \mathbb{R}$. Define $f : X \to Y$ by $f(x) = x$ and $g : Y \to X$ by $g(x) = 0$. Then $gf(0) = 0$ so gf is the identity on $\{0\}$. However, $fg(x) = f(0) = 0$, which is not the identity on $\mathbb{R}$. But we did show that $0 \simeq x$, so again f is a homotopy equivalence, and $\{0\}$ and $\mathbb{R}$ have the same homotopy type.

Example 8.2.4. $\mathbb{R}$ and S^1 do not have the same homotopy type.

Let $X = \mathbb{R}$ and $Y = S^1$. Define $f : X \to Y$ by $f(x) = e^{2\pi i x}$ and $g : Y \to X$ by $g(\theta) = 0$. Then $fg = (1, 0)$ and $gf = 0 \simeq x$. But as we mentioned, $(1, 0)$ is not homotopic to the identity on S^1. In fact, there are *no* homotopy equivalences between $\mathbb{R}$ and S^1, so these spaces do not have the same homotopy type.

Theorem 8.2.5. *Let $B \subset \mathbb{R}^n$ and X be any topological space and suppose $f, g : X \to B$ are maps such that for all $x \in X$, the line segment between $f(x)$ and $g(x)$ lies in B. Define H to be the function that traces out the line segment at a constant rate from $t = 0$ to 1, i.e.*

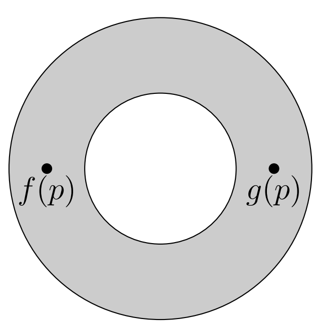
$$H(x, t) = f(x) + t(g(x) - f(x)).$$

Then H is a homotopy, called the **straight line homotopy**.

Proof. It suffices to show that H is continuous on $X \times I$. This is routine, and left to the reader. □

Example 8.2.6. If B is convex, such a straight line homotopy always exists.

Example 8.2.7. Let $X = \{p\}$ and Y be an annulus.



Suppose $f(p)$ and $g(p)$ are as shown. Then there is no notion of a "straight line" between these points (at least not in the space Y), but we *can* find a homotopy by wrapping around Y in some fashion.

Theorem 8.2.8. *Two functions $f : \{p\} \to Y$ and $g : \{p\} \to Y$ are homotopic if Y is path connected.*

Proof omitted. $\square$

In a path connected space, homotopies rather easy to manipulate since we can simply shift the base points of the homotopy along the interval $[0, 1]$. This leaves out important information like the "hole" in an annulus, which distinguishes such a space from convex spaces. For this reason we introduce the concept of a homotopy relative to a base point, or more generally a set of base points.

Definition. *Let $F : X \times I \to Y$ be a homotopy and A be a subspace of X. If each of the f_t fix A, then F is called a* **homotopy relative to A**, *written homotopy rel A.*

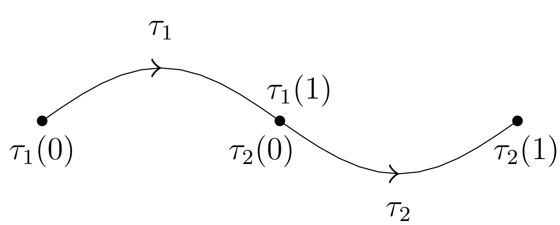
Suppose we have homotopic maps $f, g : X \to Y$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$

Then a point $x \in X$ could only belong to such a subset A if $f(x) = g(x)$. Some further notation:

- $rel(\partial)$ means A is taken to be the boundary of X.
- $rel(0, 1)$ means $X = [0, 1]$ and $A = \partial I$ (the endpoints) is fixed.

We are commonly interested in **paths** in X, by which we mean continuous maps $\tau : I \to X$. A natural question is how to combine two paths.



To "add" these paths (and maintain continuity), trace the first path from 0 to $\frac{1}{2}$ twice as fast, then trace the second path from $\frac{1}{2}$ to 1 also twice as fast. This is made explicit in the following definition.

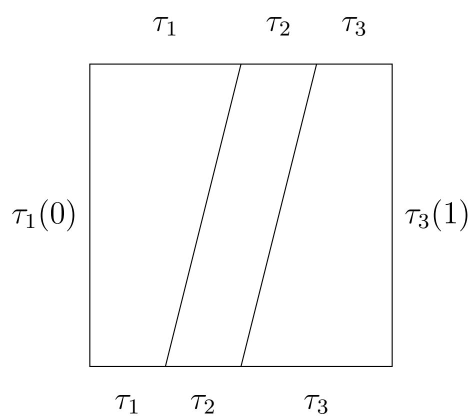
Definition. *Given two paths $\tau_1, \tau_2 : I \to X$ such that $\tau_1(1) = \tau_2(0)$, we define their product, or* **path composition** *$\tau_1\tau_2 : I \to X$ by*

$$\tau_1\tau_2(t) = \begin{cases} \tau_1(2t) & 0 \le t \le \frac{1}{2} \\ \tau_2(2t-1) & \frac{1}{2} < t \le 1. \end{cases}$$

Proposition 8.2.9. *Path composition is associative, i.e.* $\tau_1(\tau_2\tau_3) = (\tau_1\tau_2)\tau_3$.

Proof. Let τ_1, τ_2 and τ_3 be paths in X such that $\tau_1(1) = \tau_2(0)$ and $\tau_2(1) = \tau_3(0)$. Define

$$F(t,y) = \begin{cases} \tau_1\left(\frac{4t}{y+1}\right) & 0 \le y \le 4t-1 \\ \tau_2(4t-y-1) & 4t-1 < y \le 4t-2 \\ \tau_3\left(1 - \frac{4(1-t)}{2-y}\right) & 4t-2 < y \le 1. \end{cases}$$



Then we have the following:

$$F(0,t) = \begin{cases} \tau_1(4t) \\ \tau_2(4t-1) \\ \tau_3(2t-1) \end{cases} = (\tau_1\tau_2)\tau_3$$

$$F(1,t) = \begin{cases} \tau_1(2t) \\ \tau_2(4t-2) \\ \tau_3(4t-3) \end{cases} = \tau_1(\tau_2\tau_3)$$

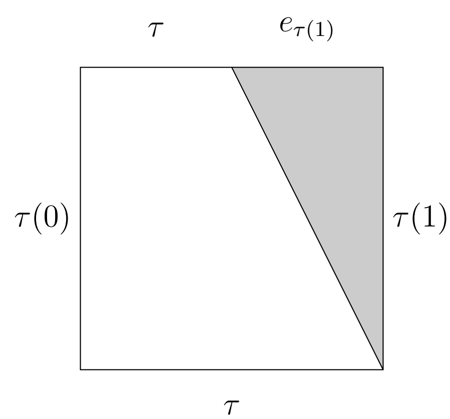
$$F(y,0) = \tau_1(0)$$
$$F(y,1) = \tau_3(1).$$

Moreover, at $y = 4t-1$, $\tau_1\left(\frac{4t}{y+1}\right) = \tau_1(1) = \tau_2(0) = \tau_2(4t-y-1)$. And at $y = 4t-2$, $\tau_2(4t-y-1) = \tau_2(1) = \tau_3(0) = \tau_3\left(1 - \frac{4(1-t)}{2-y}\right)$. So $F(t,y)$ is continuous. Thus $(\tau_1\tau_2)\tau_3 \simeq \tau_1(\tau_2\tau_3)$, so association holds for path composition. $\square$

Next, we attempt to define an identity map. If τ is a path in X, define the right identity $e_{\tau(1)} = \tau(1)$. Similary, we can define a left identity $e_{\tau(0)} = \tau(0)$. We claim that $\tau e_{\tau(1)} \simeq \tau$ and $e_{\tau(0)}\tau \simeq \tau$.

For the first homotopy, define

$$F(t,y) = \begin{cases} \tau((y+1)t) & 0 \le y \le 2-2t \\ \tau(1) & 2-2t < y \le 1. \end{cases}$$



Then we have

$$\begin{aligned} F(0,t) &= \tau(t) \\ F(1,t) &= \tau(2t) = \tau e_{\tau(1)}(t) \\ F(y,0) &= \tau(0) \\ F(y,1) &= \tau(1). \end{aligned}$$

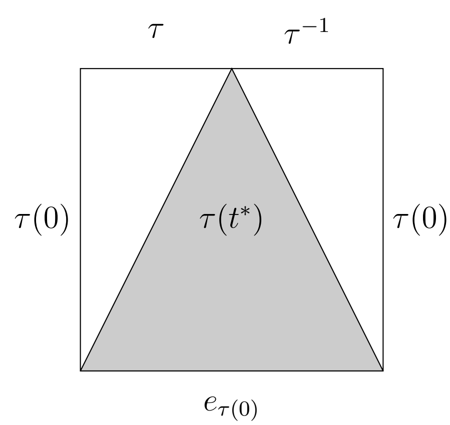
But at the seam $y = 2-2t$, note that $\tau(-2t^2+3t) \neq \tau(1)$ so F fails to be continuous. This is a good example of how it's important to be careful with continuity around "seams" in the square $I \times I$.

If we instead define

$$F(t,y) = \begin{cases} \tau\left(\frac{2t}{2-y}\right) & 0 \leq y \leq 2-2t \\ \tau(1) & 2-2t < y \leq 1 \end{cases}$$

the same identities hold as above. This time, at $y = 2-2t$ we have $\tau\left(\frac{2t}{2-y}\right) = \tau(1)$ so $F(t,y)$ is continuous on the whole square. This shows that $\tau e_{\tau(1)} \simeq \tau$, and the proof is similar for $e_{\tau(0)}\tau \simeq \tau$.

The next step in describing a group structure on paths in X is to define an inverse map. Let τ be a path in X and define $\tau^{-1}(t) = \tau(1-t)$. We will show that $\tau\tau^{-1}$ is homotopic to the left identity $e_{\tau(0)}$ and remark that the proof is similar for showing that $\tau^{-1}\tau \simeq e_{\tau(1)}$. Consider the diagram



(where t^* denotes wherever you stop moving through the τ phase). The lines shown are $y = 2t$ and $y = 2 - 2t$. Define

$$F(t, y) = \begin{cases} \tau(2t) & 0 \le t \le \frac{y}{2} \\ \tau(y) & \frac{y}{2} < t \le \frac{2-y}{2} \\ \tau^{-1}(2t-1) & \frac{2-y}{2} < t \le 1. \end{cases}$$

Then we have

$$\begin{aligned} F(0, y) &= \tau(0) \\ F(1, y) &= \tau^{-1}(1) = \tau(0) \\ F(t, 0) &= \tau(0) \\ F(t, 1) &= \tau\tau^{-1}. \end{aligned}$$

At $y = 2t$, $F(t, y) \to \tau(2t)$ on both pieces. And at $y = 2 - 2t$, $F(t, y) \to \tau(2 - 2t) = \tau^{-1}(2t - 1)$, so F is continuous. This proves that $\tau\tau^{-1} \simeq e_{\tau(0)}$.

So far we have consider paths on X, which have a starting point $\tau(0)$ and an ending point $\tau(1)$. This induces two identities, which cannot happen if we are to describe a group structure on paths. To resolve this issue, we will pick a **base point** x_0 in X and restrict our focus to **loops** based at x_0, which are just paths that start and end at x_0.

Definition. *The* **fundamental group** *of a topological space X with base point x_0 is the group of homotopy classes of loops based at x_0, denoted $\pi_1(X, x_0)$.*

Proposition 8.2.10. *The fundamental group is a group under path composition.*

Proof. By picking a base point x_0, path composition is well defined, and closure is guaranteed. Let $e = e_{\tau(0)} = e_{\tau(1)}$, which is the identity by our work above. Also define τ^{-1} as above, which was shown to be the inverse of τ for any path in general. Hence $\pi_1(X, x_0)$ is a group. $\square$

A natural question to ask is whether the fundamental group of a space changes if we pick a different base point. It is clear that if X is path connected, or at the very least if x_0 and x_1 lie in the same path component, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$, but in general this is not true if the points lie in different components. In the case that X is path connected, we usually drop the base point from our notation for the fundamental group and simply write $\pi_1(X)$.

Example 8.2.11. The fundamental group of the plane

We have shown that in $\mathbb{R}^2$, all loops are homotopic to the identity, so there is only one homotopy class. Thus $\pi_1(\mathbb{R}^2, x_0)$ is trivial.

Definition. *A map f that is homotopic to a constant map is said to be* **nullhomotopic**.

In $\mathbb{R}^n$, all loops are nullhomotopic. In fact, a stronger result is true: for any convex subset $X \subset \mathbb{R}^n$, every loop in X is nullhomotopic. Therefore $\pi_1(X, x_0) = \{0\}$. However, we saw in Example 8.2.6 that not all loops in the annulus are nullhomotopic.

Definition. *A space with the homotopy type of a point is called* **contractible**. *This is equivalent to saying the identity map on this space is nullhomotopic.*

Theorem 8.2.12. *If X is a contractible space, then its fundamental group is trivial.*

Proof omitted. □

So far we have not been able to prove any examples of spaces with nontrivial fundamental group. An important example is the circle S^1, which we prove in detail in the next section.

Definition. *A space X is called* **simply connected** *if it is path connected and has trivial fundamental group.*

Definition. *For a map $r : X \to X$ with $r(X) = A$ such that $r(a) = a$ for all $a \in A$, we say A is a* **retract** *of X, and r is a* **retraction map**.

Example 8.2.13. For an annulus A, the inner circle is a retract of A via the retraction map r that takes every point on the annulus to the closest point on the circle.

Unfortunately, this definition implies that every space contains a retract that is a single point. Thus a stricter notion of retract is needed.

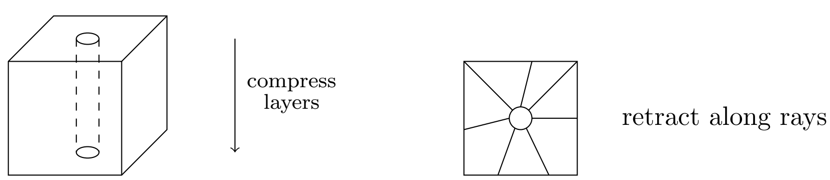
Definition. *A* **deformation retract** *from X onto A is a family of maps $f_t : X \to X$ indexed by $t \in [0,1]$ such that f_0 is the identity, $f_1(X) = A$ and $f_t(a) = a$ for all t and for all $a \in A$. In other words, a deformation retract is a homotopy rel A.*

Proposition 8.2.14. *Retract and deformation retract are transitive.*

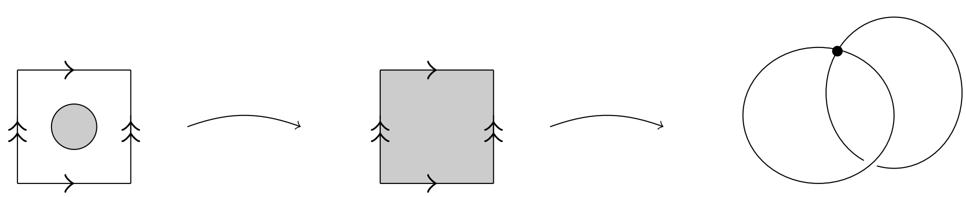
Proof omitted. □

Example 8.2.15. There is a circle that is a deformation retract of the Möbius band.

Example 8.2.16. A cube with a hole punched out deformation retracts onto a circle.



Example 8.2.17. Deleting a disk from a torus yields the **wedge** of two circles



This gives a space that is homeomorphic to a figure eight.

8.3 The Fundamental Group of the Circle

In the previous section we defined the fundamental group $\pi_1(X)$ for any path connected topological space X (and extended to any path disconnected space with base point x_0 in some path component), but so far we have not seen any examples of spaces with nontrivial fundamental group. In this section we prove that the fundamental group of the circle is isomorphic to the integers $\mathbb{Z}$.

Before embarking on a proof of this fact, we introduce a concept from the study of compact metric spaces.

Definition. *Let X be a metric space and $\mathcal{O}$ be an open cover of X. Then ε is a* **Lebesgue number** *for $\mathcal{O}$ if any subset of X whose diameter is less than ε is contained in at least one element of $\mathcal{O}$.*

Example 8.3.1. The collection $\mathcal{O} = \{(n, n+2) \mid n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$. The Lebesgue number for $\mathcal{O}$ is 1, and in fact this is the largest possible Lebesgue number for this particular open cover.

Example 8.3.2. In $\mathbb{R}^2$, an open cover is $\mathcal{O} = \{\text{circles of radius 2 with center } (x, y) \in \mathbb{Z}^2\}$. As in the previous example, a Lebesgue number for $\mathcal{O}$ is 1.

The Lebesgue number originates from the following result for compact metric spaces:

Theorem 8.3.3 (Lebesgue's Number Lemma)**.** *Every open cover of a compact metric space X has a Lebesgue number.*

We will use this fact in our study of $\pi_1(S^1)$. Let $f(t) = e^{2\pi i t} = \cos(2\pi t) + i\sin(2\pi t)$. This maps I onto the circle S^1 surjectively and *almost* bijectively ($f(0) = f(1)$). Think of S^1 as a subspace of $\mathbb{R}$ and let

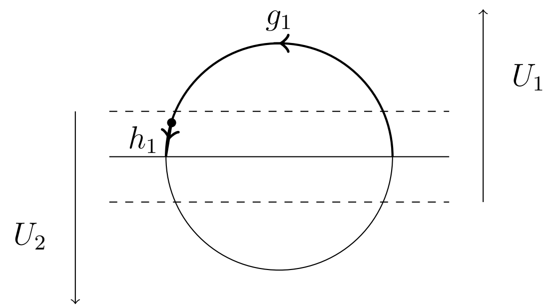
$$U_1 = \left\{(x, y) \in \mathbb{R}^2 \mid y > -\tfrac{1}{10}\right\} \cap S^1$$
$$U_2 = \left\{(x, y) \in \mathbb{R}^2 \mid y < \tfrac{1}{10}\right\} \cap S^1.$$

Then U_1, U_2 is an open cover of S^1. Note that $U_1 \cap U_2$ is an open (and disconnected) neighborhood of $f(0)$ and $f\left(\frac{1}{2}\right)$. Moreover, each U_i is connected and contractible even though S^1 is neither.

Let $\pi_1(S^1)$ be the fundamental group of the circle — remember S^1 is path connected so we don't have to specify a base point — and take an element g in this group. As mentioned above, $g : I \to S^1$ with $g(0) = g(1)$. By construction $g^{-1}(U_1)$ and $g^{-1}(U_2)$ form an open cover of I. Since I is compact, Lebesgue's Lemma says that $\mathcal{O} = \{g^{-1}(U_1), g^{-1}(U_2)\}$ has a Lebesgue number ε. This allows us to divide the interval I into $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ such that the following hold:

- $g([t_i, t_{i+1}]) \subset U_1$ or $g([t_i, t_{i+1}]) \subset U_2$ for all $0 \leq i \leq n-1$.
- $g([t_i, t_{i+1}])$ and $g([t_{i+1}, t_{i+2}])$ do not lie in the same U_i.

Now for each $i = 0, \ldots, n-1$ define $\beta_i = [g_i]$, the homotopy class of $g_i = g([t_i, t_{i+1}])$. Also set $\beta = [g(I)]$. We claim that $\beta = \beta_0\beta_1\beta_2 \cdots \beta_n$. To see this, let $\gamma_i = [h_i]$, where h_i is any path with $h_i(0) = g(t_{i+1})$ and $h_i(1) = (\pm 1, 0)$ such that the entire path h_i lies in $U_1 \cap U_2$. (This is pictured below.)



Let $\delta_0 = \beta_0\gamma_0$ and for each subsequent i, let $\delta_i = \gamma_{i-1}^{-1}\beta_i\gamma_i$, so that $\delta_n = \gamma_{n-1}^{-1}\beta_n$. Then

$$\delta_0\delta_1\delta_2 \cdots \delta_n = (\beta_0\gamma_0)(\gamma_0^{-1}\beta_1\gamma_1)(\gamma_1^{-1}\beta_2\gamma_2) \cdots (\gamma_{n-1}^{-1}\beta_n) = \beta_0\beta_1\beta_2 \cdots \beta_n.$$

But $\delta_0\delta_1\delta_2 \cdots \delta_n = \beta$ by definition, so we have proven the claim.

Notice that U_1 and U_2 are simply connected, and so contractible. Thus the fundamental group for each U_i is trivial: all paths between any pair of points in U_i are homotopic. For $i = 1, 2$ let η_i be the unique path up to homotopy between $(1, 0)$ and $(-1, 0)$, i.e. η_1 is from $(1, 0)$ to $(-1, 0)$ and η_2 is the other direction. Call a full rotation $\alpha = \eta_1\eta_2$. Then our options for the δ_i are the trivial path, η_i or η_i^{-1} $(i = 1, 2)$. In fact this fully describes the possibilities for β:

- β is trivial
- $\beta = \eta_1\eta_2\eta_1\eta_2 \cdots = \alpha^s$ (after cancelling pairs of inverses)
- $\beta = \eta_2^{-1}\eta_1^{-2}\eta_2^{-1}\eta_1^{-1} \cdots = \alpha^{-s}$ (again after cancelling pairs).

We have thus proven that $\pi_1(S^1)$ is generated by α, but we need to verify that there are no relations on $\langle\alpha\rangle$ in order to prove that $\pi_1(S^1) \cong \mathbb{Z}$. To do so, we define the degree of a path in S^1.

Definition. *For a path $g : I \to S^1$ that is surjective and bijective except for $g(0) = g(1)$, we define the* **degree** *of g to be $\frac{\theta}{2\pi}$ where θ is the signed angle covered by g, i.e. its arc length. Intuitively, degree measures how many times a path goes around the circle, including direction (+ or −).*

Proposition 8.3.4. *Degree is invariant under homotopy.*

Proof. First, define $D(f_i, f_j) = \sup\limits_{t \in I}\{d(f_i(t), f_j(t))\}$ for any $f_i, f_j : I \to S^1$. Now suppose $f \simeq g$. Then there exists a homotopy $F : S^1 \times I \to S^1$ which is uniformly continuous. Thus for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d((x,t),(x',t')) < \delta$ implies $d(F(x,t), F(x',t')) < \varepsilon$. For example, let $\varepsilon = \frac{\pi}{4}$. Partition I into $0 = t_1 < t_2 < \cdots < t_{n+1} = 1$ such that $d(f(t), f(t')) < \frac{\varepsilon}{2}$ whenever t and t' are in the same partition. For some $h(t) = F(x, t)$, we have

$$\begin{aligned} d(h(t), h(t')) &\leq d(h(t), f(t)) + d(f(t), f(t')) + d(f(t'), h(t')) \\ &< \tfrac{\pi}{4} + \tfrac{\pi}{2} + \tfrac{\pi}{4} = \pi. \end{aligned}$$

So for all h such that $d(f(t), h(t)) < \delta$, we have $d(h(t), h(t')) < \pi$ for any pair t, t' in the same interval of the partition. Let a_i denote the signed arc length along S^1 from $f(t_i)$ to $f(t_{i+1})$ and let b_i denote the same for h. Then we compute the degrees of f and h to be

$$\begin{aligned} \deg(f) &= \tfrac{1}{2\pi}(a_1 + a_2 + a_3 + \ldots + a_n) \\ \deg(h) &= \tfrac{1}{2\pi}(b_1 + b_2 + b_3 + \ldots + b_n). \end{aligned}$$

Notice that $D(f, h) < \varepsilon = \frac{\pi}{4}$ so $|a_i - b_i| < \frac{\pi}{2}$ for all i — this is because a_i is bounded by $\frac{\pi}{2}$ and b_i is bounded by π. We claim that

$$|(a_1 + a_2 + \ldots + a_k) - (b_1 + b_2 + \ldots + b_k)| < \tfrac{\pi}{4}$$

for all k. The base case is easy: $|a_1 - b_1| < \frac{\pi}{4}$ since $f(t_1) = h(t_1)$ and there can only be a leap of $\frac{\pi}{4}$ for $|f(t_2) - h(t_2)|$. Induction on k proves the claim.

Now letting $k = n$ shows that $\deg(f) - \deg(h) < \dfrac{\frac{\pi}{4}}{2\pi} = \dfrac{1}{8}$. But degree is an integer, so this implies $\deg(f) = \deg(h)$. Thus moving up δ on the homotopy doesn't change $\deg(f)$. Repeating this argument prove that $\deg(f) = \deg(g)$. □

This brings us to the main result.

Theorem 8.3.5. *$\pi_1(S^1) = \langle \alpha \rangle \cong \mathbb{Z}$, where α is one full rotation on S^1.*

Proof. We proved that the fundamental group of S^1 is generated by α. But by definition of degree, $\deg(\alpha^n) = n$ for every $n \in \mathbb{Z}$. Then Proposition 8.3.4 shows that there are no relations on the homotopy classes for distinct powers of α. Hence $\pi_1(S^1) \cong \mathbb{Z}$. □

We now present some interesting results that may be proven by knowing the fundamental group of the circle.

Corollary 8.3.6 (The Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial in $\mathbb{C}[x]$ has a root in $\mathbb{C}$.*

Proof. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ with $a_i \in \mathbb{C}$. Since $\mathbb{C}$ is a field, we may assume $a_n = 1$. Suppose p has no roots in $\mathbb{C}$. Then for all $r \in \mathbb{R}$, $r \geq 0$, we can define a loop

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

in the unit circle $S^1 \subset \mathbb{C}$ based at $(1, 0)$. As we vary r, f_r represents a homotopy class of loops based at $(1, 0)$, with f_0 as the trivial loop and $[f_r] = [e] \in \pi_1(S^1)$ for all r. For a large enough value of r, say $r > |a_{n-1}| + \ldots + |a_0|$ and at least bigger than 1, the set of $z \in \mathbb{C}$ such that $|z| = r$ satisfies

$$|z^n| = r^n = r \cdot r^{n-1} > (|a_{n-1}| + \ldots + |a_0|) \cdot |z^{n-1}| \geq |a_{n-1}z^{n-1} + \ldots + a_0|.$$

In particular, this shows that each polynomial $p_t(z) = z^n + t(a_{n-1}z^{n-1} + \ldots + a_0)$ for $0 \leq t \leq 1$ has no roots on the circle $|z| = r$. Now if we replace p by p_t in the formula for $f_r(s)$ above and vary t between 0 and 1, we obtain a homotopy $f_r \simeq e^{2\pi ins}$. Since $\pi_1(S^1) = \langle \alpha \rangle$, where α may be taken to be the path $e^{2\pi is}$, $0 \leq s \leq 1$, we see that $[f_r] = [\alpha^n]$. But as we showed above, $[f_r]$ is trivial for all r. Hence n must be 0, so that $e^{2\pi ns} = 1$. Therefore the only polynomials that have no roots in $\mathbb{C}$ have degree 0, i.e. the constant polynomials. □

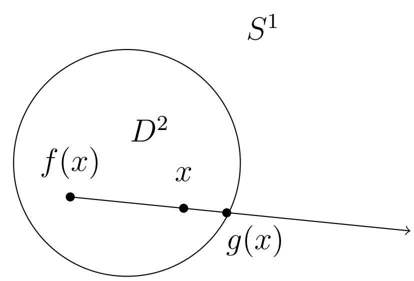
Next, we prove Brouwer's Fixed Point Theorem (5.2.13). Before we do this, we need a fact about the fundamental group which is an interesting result in its own right:

Theorem 8.3.7. *Every continuous map $f : X \to Y$ between topological spaces induces a homomorphism $f^* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.*

Proof omitted. □

Theorem (5.2.13)**.** *Let $f : D^2 \to D^2$ be any continuous map on the closed unit disk in $\mathbb{R}^2$. Then f has a fixed point, that is, there is some $p \in D^2$ such that $f(p) = p$.*

Proof. Suppose $f(p) \neq p$ for any $p \in D^2$. Define $g : D^2 \to S^1$ by letting $g(x)$ be the point on the circle where the ray from $f(x)$ through x intersects S^1.



Since we assumed f has no fixed points, g is well-defined and inherits continuity from f. However, notice that for any $z \in S^1$, $g(z) = z$ as well. Thus g is a retract of D^2 onto S^1. Let $\iota : S^1 \hookrightarrow D^2$ be the natural inclusion, i.e. $\iota(z) = z$ for all $z \in S^1$. Then clearly $g \circ \iota : S^1 \to S^1$ is the identity. By Theorem 8.3.7, the induced map $(g \circ \iota)^*$ is an isomorphism from $\pi_1(S^1)$ to itself. However, $g^* : \pi_1(D^2) \to \pi_1(S^1)$ is a trivial map since $\pi_1(D^2)$ is itself trivial. Thus $g^* \circ \iota^* = 0 \neq (g \circ \iota)^*$, a contradiction. Hence every continuous map from the closed unit disk to itself must have a fixed point. □

Remarkably, Brouwer proved his fixed point theorem in 1910, well before the development of homotopy theory. He in fact proved the result for D^n, using a generalization of degree for loops in S^n.

For the remainder of the section, we state several important results about the fundamental group which were not proven in class. See Hatcher (pp. 34-37) for details.

Proposition 8.3.8. *If X and Y are path connected, then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.*

Proof omitted. □

Proposition 8.3.9. *For $n \geq 2$, $\pi_1(S^n)$ is trivial.*

Proof omitted. □

Recall that for all $n \geq 1$, $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$. The last two results tell us that $\pi_1(\mathbb{R}^n \setminus \{0\}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})$, so that $\pi_1(\mathbb{R}^n \setminus \{0\})$ is $\mathbb{Z}$ for $n = 2$ and trivial for all $n > 2$. A further application of these facts is:

Proposition 8.3.10. *For all $n \neq 2$, $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$.*

Proof. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism. We showed the case $n = 1$ in Example 5.2.10, so let us assume $n > 2$. By assumption we must have $\mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}^n \setminus \{f(0)\}$, but by the comments above the fundamental groups of these two spaces are $\mathbb{Z}$ and 0. Hence they cannot be homeomorphic. □

To further explore the group theory behind the fundamental group, we have the following characterization of retraction maps.

Proposition 8.3.11. *If there exists a subspace $A \subset X$ such that A is a retract of X, then the inclusion $i : A \hookrightarrow X$ induces a homomorphism $i^* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ which is injective. Furthermore, if A is a deformation retract then i^* is an isomorphism.*

Proof omitted. □

In fact, every retraction induces a homomorphism ρ of $G = \pi_1(X)$ onto a subgroup $H \leq G$ such that ρ restricts to the identity on H.

8.4 The Seifert-van Kampen Theorem

We made progress in the last section, especially with the last few results, but there still remain a large number of spaces for which we cannot yet compute the fundamental group. This brings us to the Seifert-van Kampen Theorem, which allows us to calculate $\pi_1(X)$ when X decomposes into simpler spaces whose fundamental groups are known.

Theorem 8.4.1 (Seifert-van Kampen)**.** *Suppose $X = A \cup B$, where $A \cap B$ is path connected, $\pi_1(A \cap B)$ is finitely generated and*

$$\pi_1(A) = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$$
$$\pi_1(B) = \langle b_1, \ldots, b_p \mid s_1, \ldots, s_q \rangle.$$

Then the fundamental group of X is

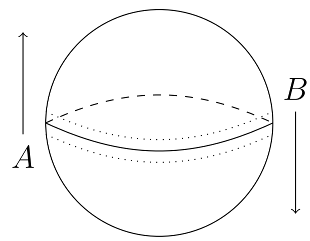
$$\pi_1(X) = \langle a_1, \ldots, a_m, b_1, \ldots, b_p \mid r_1, \ldots, r_n, s_1, \ldots, s_q, u_1 = v_1, \ldots, u_t = v_t \rangle$$

where u_i and v_i are the expressions for the generators of $\pi_1(A \cap B)$ in terms of $\pi_1(A)$ and $\pi_1(B)$, respectively.

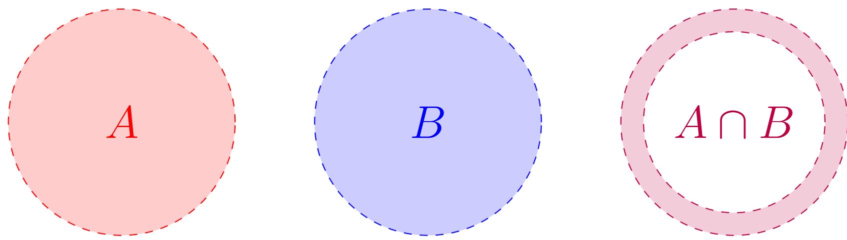
The original version of this theorem was known as van Kampen's Theorem, and it dealt with the special case when $\pi_1(A \cap B)$ is trivial. The proof of Seifert-van Kampen may be found in Section 1.2 of Hatcher. What we will focus on is applications and examples.

Example 8.4.2. The sphere

We write $S^2 = A \cup B$, where A is the northern hemisphere plus a little extra past the equator, and B is the southern hemisphere plus a little past the the equator.



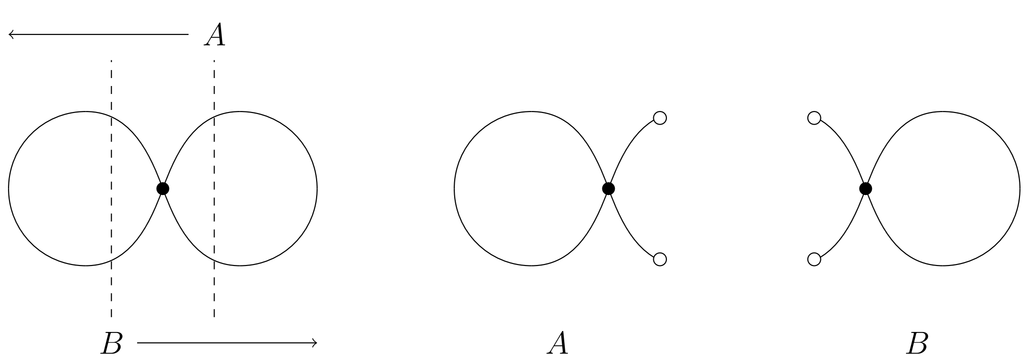
Up to homeomorphism, A, B and $A \cap B$ look like:



First note that S^2 and each of these sets defined above are all path connected, so we may write fundamental groups without reference to base points. Clearly A and B are contractible so their fundamental groups are trivial. Their intersection $A \cap B$ deformation retracts onto a circle, so by Proposition 8.3.11, $\pi_1(A \cap B) \cong \pi_1(S^1) = \mathbb{Z}$. Now by Seifert-van Kampen (Theorem 8.4.1), $\pi_1(A \cup B)$ is formed by taking the generators of $\pi_1(A)$ and $\pi_1(B)$ (there are none) and expressing the generator of $\pi_1(A \cap B) = \langle \alpha \rangle$ in terms of the fundamental groups of A and B. In both cases $\alpha = e$, so overall $\pi_1(A \cup B) = 0$. Hence the sphere has trivial fundamental group.

Example 8.4.3. The figure eight, i.e. the wedge of two circles

Recall the informal definition of the **wedge sum** of two spaces as the quotient space obtained by identifying one point from each space. Then the figure eight may be thought of as the wedge of two circles $S^1 \vee S^1$. Divide the space into the following subsets:



Since A and B both deformation retract onto S^1, $\pi_1(A) \cong \pi_1(B) \cong \mathbb{Z}$, so we can write $\pi_1(A) = \langle \alpha \rangle$ and $\pi_1(B) = \langle \beta \rangle$. Notice that $A \cap B$ is contractible, so $\pi_1(A \cap B) = 0$. Thus the fundamental group of the wedge of two circles is just the free product of the fundamental groups for A and B: $\pi_1(S^1 \vee S^1) = \langle a, b \rangle \cong \mathbb{Z}^2$.

Example 8.4.4. Similarly, $S^1 \vee S^2$ has fundamental group isomorphic to $\mathbb{Z}$.

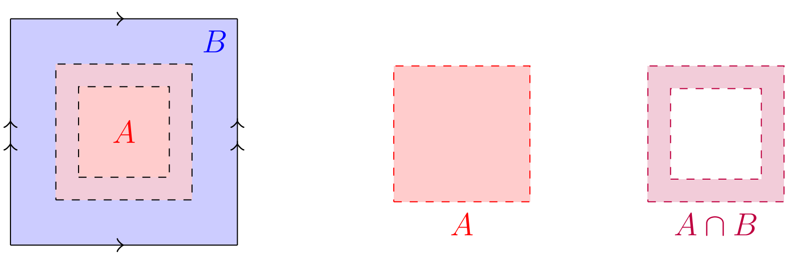
These examples suggest the following, which is sometimes known simply as the van Kampen Theorem (it was proven by van Kampen for covers A, B whose intersection is contractible; Seifert later generalized the result which became known as the Seifert-van Kampen Theorem):

Corollary 8.4.5. *If $A \cap B$ is contractible (or in general if it has trivial fundamental group), then $\pi_1(A \cup B)$ is just the free product of $\pi_1(A)$ and $\pi_1(B)$.*

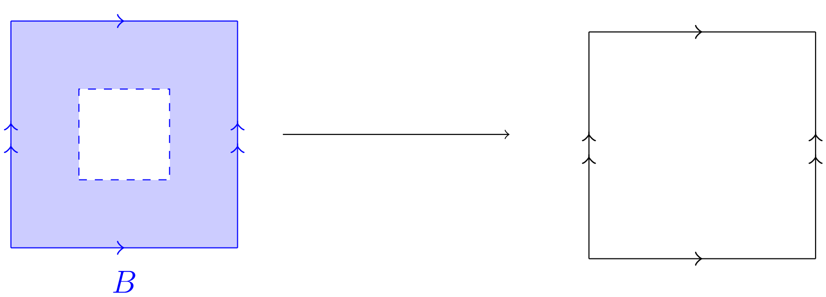
Here's an example where Corollary 8.4.5 doesn't apply.

Example 8.4.6. The fundamental group of the torus

We shall consider T as the quotient space obtained by identifying sides of a square. Let A be a region inside the square, and let B be a region containing the edges of I^2 plus some of the interior of I^2 (think of B like a picture frame).



First notice that A has trivial fundamental group, and $A \cap B$ deformation retracts onto a circle so its fundamental group is $\langle \gamma \rangle \cong \mathbb{Z}$. On the other hand, consider B:



B deformation retracts onto the frame with sides identified, which we know is homeomorphic to $S^1 \vee S^1$ and has fundamental group $\langle \alpha, \beta \rangle \cong \mathbb{Z}^2$. By Seifert-van Kampen (8.4.1), $\pi_1(T)$ is found by taking the generators α, β and forming relations from the generator γ. In $\pi_1(A)$, $[\gamma] = [e]$ since everything is trivial. But in $\pi_1(B)$, $[\gamma] = [\alpha\beta\alpha^{-1}\beta^{-1}]$. Hence we have

$$\pi_1(T) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} \rangle$$

the infinite abelian group on two generators.

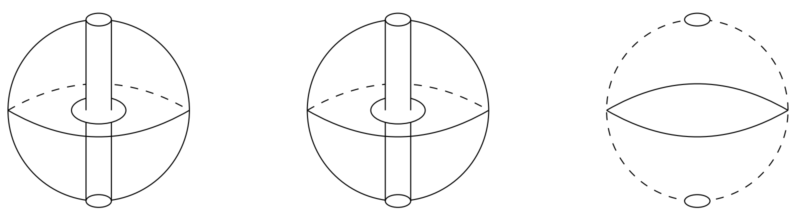
8.5 The Fundamental Group and Knots

This section assumes the reader is familiar with basic definitions and concepts in knot theory. For most of these questions, one can turn to Colin Adams' *The Knot Book*. Since every knot is an embedding of S^1 into $\mathbb{R}^3$, they all have the same fundamental group $\mathbb{Z}$. A more interesting notion is that of the fundamental group of the *complement* of a knot in $\mathbb{R}^3$, or equivalently in S^3, the one-point compactification of $\mathbb{R}^3$. Thus when we talk about the **fundamental group of a knot** K, we will be referring to $\pi_1(\mathbb{R}^3 - K)$.

In fact, topologists usually consider the **exterior** of a knot, which is formed by taking a small neighborhood of K, denoted $N(K)$, and considering the complement of its interior, $S^3 - \mathring{N}(K)$. Keep in mind that the following all have the same fundamental group:

- The knot complement $\mathbb{R}^3 - K$.
- The knot complement $S^3 - K$.
- The knot exterior in $\mathbb{R}^3$: $\mathbb{R}^3 - \mathring{N}(K)$.
- The knot exterior in S^3: $S^3 - \mathring{N}(K)$.

Consider the simplest example: the unknot in S^3. In order to apply the Seifert-van Kampen Theorem (8.4.1), it is useful to treat S^3 as the quotient space of two 2-dimensional spheres, A and B. This is known as a **Heegaard splitting**.



A B $A \cap B$

Remove a neighborhood of the unknot, which can be visualized above as two tubes running through the spheres. Then A and B each deformation retract to the circle around the tube, as shown. Thus $\pi_1(A) = \langle a \rangle$ and $\pi_1(B) = \langle b \rangle$. Additionally, $A \cap B$ deformation retracts to the equator of a sphere, so $\pi_1(A \cap B) = \langle c \rangle$. In the fundamental groups for A and B, c just equals each of their generators. Hence a reduced presentation for the fundamental group of the unknot is

$$\pi_1(S^3 - K) = \langle a \rangle \cong \mathbb{Z}.$$

This generalizes to any knot through an algorithm called the **Wirtinger presentation** of a knot, which we describe here.

1) Orient the knot. Check each crossing for right- or left-handedness:

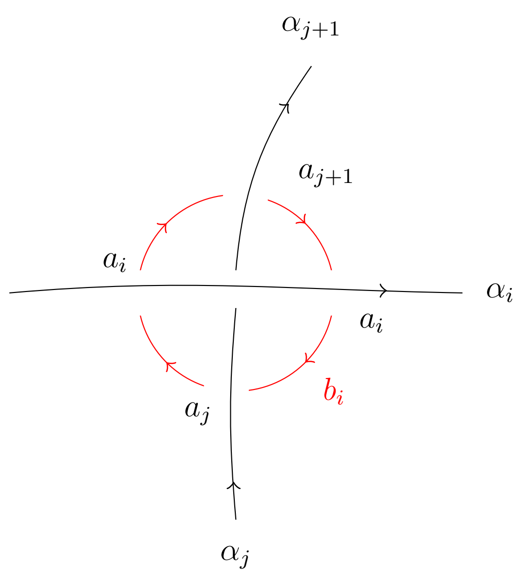
right-handed left-handed

2) Consider each overstrand as a tube deleted from one ball of our Heegaard splitting of S^3, which we will refer to as the 'top ball'. Then each understrand may be considered as a "divot" in the bottom ball.

3) The divots in the bottom ball don't change the fundamental group of the ball — to see this, imagine pumping air into the ball until the divots pop back out (why is this a deformation retract?) On the other hand, each tube in the top ball increases the genus of the ball by 1, so there will be a generator of the fundamental group corresponding to each tube (i.e. each overstrand). Meanwhile, the divots remaining in the top ball don't change its fundamental group either.

4) Label the strands $\alpha_1, \alpha_2, \ldots, \alpha_n$ in order going around the knot. For each generator a_i in the fundamental group of the top ball, choose its orientation according to the right hand rule for its corresponding strand α_i.

5) The intersection of the two balls is a sphere with a hole deleted for each divot (i.e. each crossing). Let $b_1, b_2, \ldots, b_n$ be the generators of its fundamental group. These are trivial in the bottom ball, since the fundamental group there is trivial, but in the other ball, orientation plays a role. For each b_i, fix an orientation. Then calculate the relation by the right hand rule:



The relation given by b_i above is $a_i a_{j+1}^{-1} a_i^{-1} a_j$. (If this was a left-handed crossing, it would produce the relation $a_i a_{j+1} a_i^{-1} a_j^{-1}$.)

6) This gives a full presentation of $\pi_1(S^3 - K)$ by the Seifert-van Kampen Theorem (8.4.1).

Note that $\pi_1(S^3 - K)$ is invariant across:

- Ambient isotopy
- Reidemeister moves, i.e. different projections of K (see Adams' *The Knot Book*)

- Reflection and inversion.

Together these imply that the fundamental group of a knot is unique and we may consider any configuration of the knot's crossings we wish — in practice it is best to choose a reduced projection of the knot since then there are less relations to compute!

Another consequence of these facts is that if two knots have non-isomorphic fundamental groups, then they are not equivalent. This is quite powerful; however, in general it is a hard problem to determine whether two groups are isomorphic or not.

Example 8.5.1. The trefoil knot

Let K be the trefoil. We may assume a minimal projection of K, which has 3 crossings. If A and B are the top and bottom spheres, respectively, in the Wirtinger presentation of K then $\pi_1(A) = \langle a_1, a_2, a_3\rangle$, $\pi_1(B) = 0$ and $\pi_1(A \cap B) = \langle c_1, c_2, c_3\rangle$. These generators of $\pi_1(A \cap B)$ produce the following relations:

$$e = c_1 = a_1 a_3^{-1} a_2^{-1} a_3 \qquad e = c_2 = a_3 a_2^{-1} a_1^{-1} a_2 \qquad e = c_3 = a_2 a_1^{-1} a_3^{-1} a_1$$

and so the fundamental group of the trefoil has the following presentation:

$$\pi_1(S^3 - K) = \langle a_1, a_2, a_3 \mid a_1 a_3^{-1} a_2^{-1} a_3 = a_3 a_2^{-1} a_1^{-1} a_2 = a_2 a_1^{-1} a_3^{-1} a_1 = e\rangle.$$

To get a better idea what this group looks like, we perform some reductions. Note that the first relation can be written as $a_1 = a_3^{-1} a_2 a_3$, so we can remove a_1 as a generator and rewrite the other relations:

$$\begin{aligned} e &= a_3 a_2^{-1} (a_3^{-1} a_2 a_3)^{-1} a_2 = a_3 a_2^{-1} a_3^{-1} a_2^{-1} a_3 a_2 \\ \text{and} \quad e &= a_2 (a_3^{-1} a_2^{-1} a_3) a_3^{-1} (a_3^{-1} a_2 a_3) = a_2 a_3^{-1} a_2^{-1} a_3^{-1} a_2 a_3. \end{aligned}$$

Notice that both become $a_2 a_3 a_2 = a_3 a_2 a_3$, so they are the same relation! Furthermore, if we let $y = a_3 a_2$ and $x = a_2 a_3 a_2$, then $a_2 a_3 a_2 = x$ and $a_3 a_2 a_3 = y^3 x^{-1}$, so our final presentation of the fundamental group is

$$\pi_1(S^3 - K) = \langle x, y \mid x^2 = y^3\rangle.$$

8.6 Covering Spaces

The final topic in this chapter is covering spaces. For details that are left out, one may consult Hatcher's online algebraic topology notes.

When we calculated the fundamental group of the circle, we saw that the group is generated by a clockwise rotation of 2π radians, i.e. one complete rotation. To distinguish each rotation, one may visualize the process of travelling around the circle as instead travelling down (or up) a helix in $\mathbb{R}^3$ which lies over the circle, or "covers" it. This is our first example of a covering space.

In general, covering spaces are a useful tool for computing fundamental groups of spaces. And as before, the algebraic structure of the fundamental group can be interpreted via the topology of covering spaces. In fact, there is a fascinating correspondence between the covering spaces of X and the subgroups of $\pi_1(X)$ which has a strong similarity to the correspondence between field extensions and subgroups of the Galois group in Galois theory.

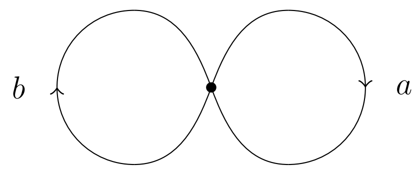
Definition. *A* **covering space** *of X is a space $\widetilde{X}$ equipped with a map $p : \widetilde{X} \to X$ such that there exists an open cover $\mathcal{O} = \{U_\alpha\}$ of X where each $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, and the restriction of p to $p^{-1}(U_\alpha)$ is a homeomorphism onto U_α.*

Example 8.6.1. Covering spaces of S^1

The helix arises from the map $p : \mathbb{R} \to S^1$ given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$. In this case, $\mathcal{O}$ can be the collection of pairs of open arcs that cover S^1, and $\widetilde{X}$ is the helix-shaped copy of $\mathbb{R}$ sitting inside $\mathbb{R}^3$. Another example is the map $p : S^1 \to S^1$, $p(z) = z^n$, where z is a complex number with $|z| = 1$.

Example 8.6.2. The figure eight, or the wedge of two circles

Let $X = S^1 \vee S^1$. We may view this space as a graph with a single vertex and two edges; call them a and b, and pick an orientation for each edge.



Let $\widetilde{X}$ be any graph with four edges meeting at each vertex with orientations chosen so that each vertex looks locally like X, i.e. every edge in $\widetilde{X}$ is labelled with either an a or a b, and at each vertex there is one of each coming in and one of each going out. Given such a graph, let $p : \widetilde{X} \to X$ be the map that sends all vertices to the vertex of X, and each edge in $\widetilde{X}$ to the edge of X with the same orientation and label. Clearly this makes $\widetilde{X}$ into a covering space for X. More importantly, *every* covering space of X is a graph constructed in this way. It is a theorem in graph theory that every graph (including infinite graphs) with four edges meeting at each vertex can be labelled and oriented in the way we described.

Definition. *Let $\widetilde{X}$ be a covering space of X via $p : \widetilde{X} \to X$. A* **lift** *of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \widetilde{X}$ such that $p\tilde{f} = f$.*

Theorem 8.6.3 (Homotopy Lifting Property)**.** *Given a covering space $p : \widetilde{X} \to X$, a homotopy $f_t : Y \to X$ and a map $\tilde{f}_0 : Y \to \widetilde{X}$ lifting f_0, there exists a unique homotopy $\tilde{f}_t : Y \to \widetilde{X}$ of $\tilde{f}_0$ that lifts f_t.*

Proof omitted. □

In particular, if $f : I \to X$ is a path in X and $\tilde{x}_0$ is a lift of the starting point $f(0) = x_0$, there is a unique path $\tilde{f} : I \to \widetilde{X}$ lifting f and starting at $\tilde{x}_0$. The uniqueness of lifts implies for example that the lift of any constant path is constant. This suggests the following.

Proposition 8.6.4. *The map $p^* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ induced by a covering space $p : \widetilde{X} \to X$ taking $\tilde{x}_0 \mapsto x_0$ is injective. The image subgroup $p^*(\pi_1(\widetilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops based at x_0 whose lifts to $\widetilde{X}$ are loops with basepoint $\tilde{x}_0$.*

Proof. If $\tilde{f}_0 : I \to \widetilde{X}$ is in the kernel of p^* then there exists a homotopy $f_t : I \to X$ of $f_0 = p\tilde{f}_0$ such that f_1 is trivial. By the uniqueness of lifts, this implies that f_t lifts to a homotopy $\tilde{f}_t$ starting at $\tilde{f}_0$ and ending with the trivial loop in $\widetilde{X}$. Hence $[\tilde{f}_0] = 0$ in $\pi_1(\widetilde{X}, \tilde{x}_0)$ which shows p^* is injective. The rest of the proposition follows from the definitions and the homotopy lifting property. □

Note that for any covering space $\widetilde{X}$ of X, the cardinality of the set $p^{-1}(x)$ for any $x \in X$ is locally constant over X. Hence if X is connected, $|p^{-1}(x)|$ is constant over the entire space.

Definition. *For a connected space X with covering space $p : \widetilde{X} \to X$, the cardinality of $p^{-1}(x)$ is called the number of* **sheets** *of the covering.*

Proposition 8.6.5. *Let X be a path connected space and suppose $p : \widetilde{X} \to X$ is a path connected covering space such that $p(\tilde{x}_0) = x_0$. Then the number of sheets of $\widetilde{X}$ is equal to $[\pi_1(X) : p^*(\pi_1(\widetilde{X}))]$.*

Proof omitted. □

Proposition 8.6.6. *Given a covering space $p : \widetilde{X} \to X$ and a map $f : Y \to X$, where Y is path connected, a lift $\tilde{f} : Y \to \widetilde{X}$ exists if and only if $f^*(\pi_1(Y)) \subset p^*(\pi_1(\widetilde{X}))$.*

Proof omitted. □

Proposition 8.6.7. *Given a covering space $p : \widetilde{X} \to X$ and a map $f : Y \to X$, if two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \widetilde{X}$ of f agree at one point of Y and Y is connected, then they agree on all of Y.*

Proof omitted. □

The main result for covering spaces is a classification of all covering spaces of a fixed space X. Since any such space may be divided into its path components, we will focus on classifying covering spaces of path connected spaces. This classification comes in a form similar to Galois correspondence in field theory; here we will describe a correspondence between connected covering spaces of X and subgroups of $\pi_1(X)$.

Proposition 8.6.8. *If X is path connected and locally path connected, then two path connected covering spaces $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are isomorphic if and only if $p_1^*(\pi_1(\widetilde{X}_1, \tilde{x}_1)) = p_2^*(\pi_1(\widetilde{X}_2, \tilde{x}_2))$.*

Proof omitted. □

Definition. *We say X is* **semilocally simply-connected** *if every $x \in X$ has a neighborhood U such that the map $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion $U \hookrightarrow X$ is trivial.*

Since this section only serves as a brief overview of covering spaces, we will skip straight to the classification theorem. For more details, one may consult Section 1.3 of Hatcher.

Theorem 8.6.9. *Let X be path connected, locally path connected and semilocally simply-connected. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\text{path connected covering spaces}\\ p:\widetilde{X}\to X, \tilde{x}_0\mapsto x_0\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{subgroups of}\\ \pi_1(X,x_0)\end{matrix}\right\}$$

which associates a covering space $p:\widetilde{X}\to X$ with the subgroup $p^*(\pi_1(\widetilde{X},\tilde{x}_0))$, where $p(\tilde{x}_0)=x_0$. Moreover, the conjugacy classes of subgroups of $\pi_1(X,x_0)$ corresponds to isomorphism classes of path connected covering spaces $p:\widetilde{X}\to X$.

A consequence of the uniqueness of lifts (Proposition 8.6.7) is that a simply-connected covering space of a path connected, locally path connected space X is a covering space of *every other* path connected covering space of X. These are unique up to isomorphism, so we are able to define a "largest" such cover.

Definition. *A simply-connected covering space of X is called a* **universal cover***.*

More generally, there is a partial ordering on the set of path connected covering spaces of X, which describes which spaces cover which other ones. This translates algebraically to the partial ordering of subgroups of $\pi_1(X,x_0)$.

Definition. *For a covering space $p:\widetilde{X}\to X$, the automorphisms $\widetilde{X}\to\widetilde{X}$ are called* **deck transformations***.*

The deck transformations of a covering space form a group $G(\widetilde{X})$ under composition.

Example 8.6.10. The helix covering S^1

Recall that the map $p:\mathbb{R}\to S^1$ projecting a vertical helix onto the circle is a covering space. Then every deck transformation may be realized as a vertical translation of the helix onto itself, and so $G(\widetilde{X})\cong\mathbb{Z}$ in this case.

Proposition 8.6.7 implies that when $\widetilde{X}$ is path connected, a deck transformation is completely determined by where it sends a single point.

Definition. *A covering space is* **normal** *if for every $x\in X$ and lifts $\tilde{x}$ and $\tilde{x}'$ of x, there is a deck transformation taking $\tilde{x}\mapsto\tilde{x}'$.*

Theorem 8.6.11. *Let $p:(\widetilde{X},\tilde{x}_0)\to(X,x_0)$ be a path connected covering space of a path connected, locally path connected space X, and let $H=p^*(\pi_1(\widetilde{X},\tilde{x}_0))\le\pi_1(X,x_0)$. Then*

1) $\widetilde{X}$ is a normal covering space of X if and only if H is a normal subgroup of $\pi_1(X,x_0)$.

2) In this case, $G(\widetilde{X})\cong\pi_1(X,x_0)/H$.

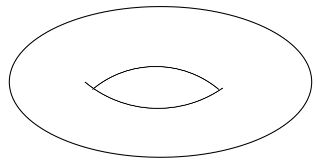
In particular, $G(\widetilde{X})\cong\pi_1(X)$ when $\widetilde{X}$ is a universal cover.

Example 8.6.12. In Example 8.6.10, with the helix $\widetilde{X}$ covering S^1, we saw that $G(\widetilde{X})\cong\mathbb{Z}=\pi_1(S^1)$. By Theorem 8.6.11, the helix must be a universal cover of S^1.

9 Surfaces Revisited

9.1 Surfaces With Boundary

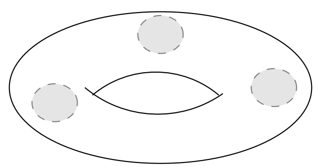
In Chapter 7 we classified all surfaces, which we previously defined to be 2-manifolds without boundary. For example, the torus is a surface without boundary:



Here we will define manifolds more generally, so as to include those with boundary.

Definition. *An* **n-manifold** *(possibly with boundary) is a topological space such that for every point x there is a neighborhood of x that is homeomorphic to a neighborhood of some point in* $\mathbb{R}^{n+} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

A particular example of interest is a surface with boundary, which is simply a 2-manifold with boundary. The torus with several disks deleted is one such surface:



a compact surface with 3 boundary components

In this section we will classify all surfaces *with* boundary, which is a remarkably similar statement to the earlier classification theorem (7.2.3) for surfaces without boundary. First recall that we restricted ourselves to compact surfaces in Chapter 7. We will do the same here, since classification is much more complicated if a surface is not compact. Next, we define what it means to "puncture" a surface. This is the deletion of a regular neighborhood of some point on the surface, which we may take to be an open disk. Note that each puncture creates a boundary component; in the picture above, the 3-times-punctured torus has 3 boundary components.

Theorem 9.1.1 (Classification of Surfaces with Boundary)**.** *Every compact surface with boundary is homeomorphic to one of the following:*

1) A sphere with m punctures

2) A connect sum of n tori with m punctures

3) A connect sum of n projective planes with m punctures.

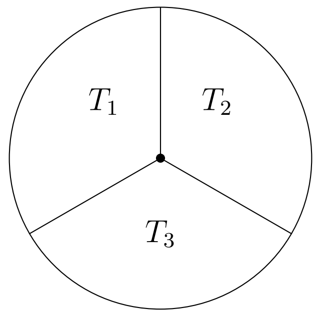
Our proof will follow Massey (1967, 1977) which continues the proof for surfaces without boundary. We will redefine some terms along the way, including:

Definition. *A* **triangulation** *of a surface with boundary components $C = \{c_1, \ldots, c_n\}$ consists of a set of triangles $T = \{T_1, \ldots, T_m\}$ satisfying:*

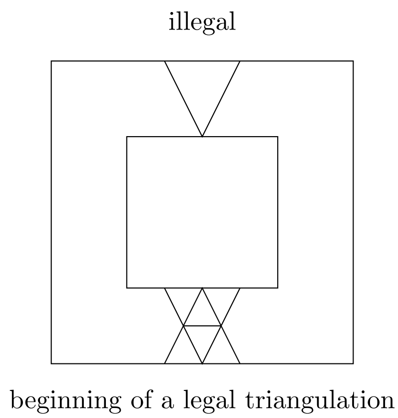
a) For all $i \neq j$, $T_i \cap T_j$ is either empty or equals exactly one vertex.

b) For all i, $T_i \cap \left(\bigcup_{k=1}^{n} c_k \right)$ is either empty, or equals exactly one point or exactly one edge.

Example 9.1.2. Triangulating a circle with boundary



Example 9.1.3. Legal and illegal triangulations of an annulus, viewed as a square frame



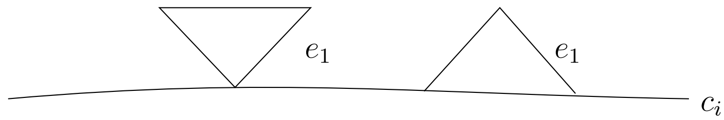
Theorem 9.1.4. *F is a compact surface with boundary if and only if it has a boundary and has a triangulation with a finite number of triangles.*

Proof omitted. This is Radó's Theorem for surfaces with boundary. □

To prove the classification theorem, we may thus start with a collection of triangles and instructions for how they "fit together". We now describe the algorithm which classifies every compact surface with boundary.

Proof of Classification Theorem: Start with triangles $T = \{T_1, \ldots, T_m\}$ and instructions on how to glue them together. Let $C = \{c_1, \ldots, c_n\}$ be the collection of boundary components of the surface F.

1) For each boundary component c_i, take a triangle T_1 that intersects c_i and choose an edge e_1 that intersects c_i but is not contained in c_i. The two possibilities (up to symmetry) for Step 1 are:



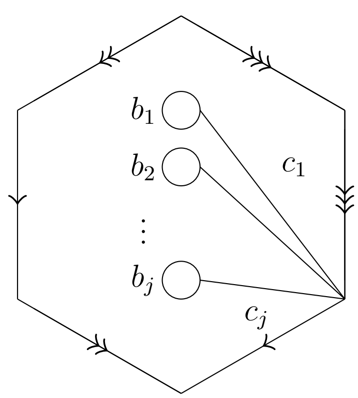
2) Let T_2 be the other triangle containing e_1. Note that T_2 also intersects c_i. Pick a new edge e_2 in $T_1 \cup T_2$ that intersects but is not contained in c_i, and let T_3 be the next triangle containing e_2. Repeat until all edges intersecting c_i have been used. This produces a polygon P_i for each boundary component c_i such that every P_i is composed of triangles and has exactly one puncture – in other words, P_i is homeomorphic to an annulus.

3) We have some number of triangles remaining; relabel these $T_1, \ldots, T_r$ – we are finished with the triangles which are now part of polygons so this should not cause confusion. Now glue together all of the polygons $P_1, \ldots, P_n$ and the remaining triangles $T_1, \ldots, T_r$ which together form a larger polygon with m holds and an even number of sides identified in pairs, just like in the proof for surfaces without boundary.

4) Recall the normal forms for surfaces without boundary:

$$\begin{array}{cl} x_1y_1x_1^{-1}y_1^{-1}\cdots x_ny_nx_n^{-1}y_n^{-1} & \text{when } F = T\#\cdots\#T \\ x_1x_1\cdots x_nx_n & \text{when } F = P\#\cdots\#P \\ xx^{-1} & \text{when } F = S^2. \end{array}$$

To find a normal form for surfaces with boundary, we perform the following move.



Cut along a segment from a fixed vertex to each boundary component b_i and label this segment c_i. Separate along these "seams" to form a figure with only one boundary component. In the figure above, the word formed is

$$xy^{-1}x^{-1}y^{-1}zzc_1b_1c_1^{-1}c_2b_2c_2^{-1}\cdots c_jb_jc_j^{-1}.$$

In general, the normal form (algebraic word) for a surface with boundary is just the normal form for some surface without boundary (connect sum of tori, projective planes or spheres) with a suffix of the form

$$c_1b_1c_1^{-1}\cdots c_jb_jc_j^{-1}$$

where j is the number of boundary components.

5) Now we can apply the algorithm from Section 7.3 to *any* surface to put the outside of the polygon into normal form.

This proves the Classification Theorem. Namely, any surface with boundary is homeomorphic to one of

- A sphere with m punctures
- A connect sum of tori with m punctures
- A connect sum of projective planes with m punctures.

$\square$

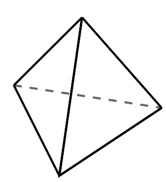
9.2 Euler Characteristic Revisited

In this section we describe the Euler characteristic from Section 7.3 for surfaces with boundary. In a triangulation $\mathcal{T}$ of a surface F, let V, E and T be the number of vertices, edges and (triangle) faces, respectively. Define $\chi(F, \mathcal{T})$ to be $V - E + T$. Then one can prove χ is the same for all triangulations of F, and we can define

Definition. *The* **Euler characteristic** *of a surface F is $\chi(F) = V - E + T$.*

This definition corresponds with the one given in Section 7.3 for surfaces without boundary.

Example 9.2.1. A legal triangulation of the sphere (viewed as a tetrahedron) is



Here we have $V = 4$, $E = 6$ and $T = 4$, so $\chi(S) = 4 - 6 + 4 = 2$.

In fact, we can "polygulate" a surface and the Euler characteristic still holds! (We just need to make sure that every polygonal face has connected boundary.)

It is also useful to be able to compute the Euler characteristic of the connect sum of two surfaces. To do so, let the disk cut out from each surface be the face of a triangle in a triangulation of the surfaces. If F_1 and F_2 are the two surfaces, then

$$\chi(F_1 \# F_2) = (V_1 + V_2 - 3) - (E_1 + E_2 - 3) + (T_1 + T_2 - 2) = \chi(F_1) + \chi(F_2) - 2.$$

Theorem 9.2.2. *A surface F with n punctures has Euler characteristic*

$$\chi(F) = \chi(F^*) - n$$

where F^* is the underlying surface without boundary.

All of this combines to show that the homeomorphism class of any surface is completely determined by

- Orientability
- Euler characteristic
- Number of boundary components.

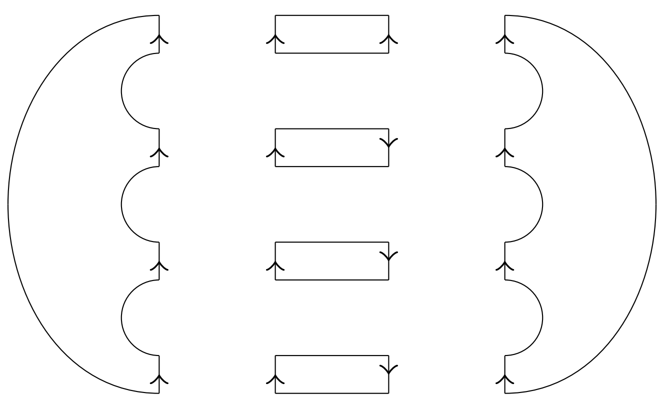
The first few surfaces, with and without boundary, are shown below. To consolidate notation, we will denote F with n punctures by F_n.

χ	orientable surface	non-orientable surface
2	S	
1	S_1	P
0	T, S_2	$P\#P, P_1$
-1	T_1, S_3	$P\#P\#P, (P\#P)_1, P_2$
-2	$T\#T, T_2, S_4$	$P\#P\#P\#P, (P\#P\#P)_1, (P\#P)_2, P_3$

To determine orientability, note the following useful characteristics:

- Every non-orientable surface contains a Möbius band.
- Equivalently, if some embedding of a circle in the surface passes through an odd number of twists, then the surface is non-orientable. Otherwise the surface is orientable.
- In particular, if *every* loop passes through an even number of twists then the surface is orientable.

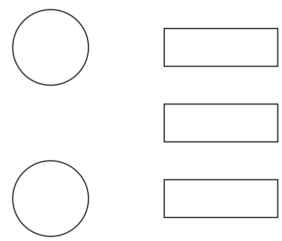
Example 9.2.3. Say we start with the following polygulation of a surface F:



There are 16 distinct vertices, 24 edges and 6 polygonal faces. Thus $\chi(F) = 16-24+6 = -2$. Tracing the boundary, we see that there is only one boundary component, so if F^* is the unpunctured version of F, $\chi(F^*) = -1$. Hence $F^* \cong P\#P\#P$ and we don't even have to examine orientability. However, it is easy to see that passing from the 'top side' of one of the large disks to the other and returning through another band gets you to the bottom side, showing non-orientability. In any case, F is non-orientable with one puncture and $\chi(F) = -2$ so the Surface Classification Theorem says that F is homeomorphic to the connect sum of three projective planes with one puncture.

9.3 Constructing Surfaces and Manifolds of Higher Dimension

In this section we describe a method to construct any surface using basic buildling blocks called **handles**. In the simplest two-dimensional case, we start with disks (0-handles) and bands (1-handles):



Glue the ends of the bands to the union of the boundaries of the disks to form a new surface. The only rule is you must attach 1-handles to existing 0-handles.

Using this construction we will prove

Theorem 9.3.1 (Seifert)**.** *Every knot K bounds an orientable embedded surface in S^3.*

Such a surface is called a **Seifert surface**. It is important to note the following fact.

Theorem 9.3.2. *The unknot is the only knot that bounds an embedded disk.*

Proof omitted. □

Definition. *The* **genus of a knot** *K is the least genus of any Seifert surface bounded by K.*

For example, the genus of the unknot is 0 since it bounds an embedded disk (genus 0). Any nontrivial knot has genus ≥ 1 since every orientable, punctured surface that is not a sphere must be the connect sum of some tori.

Seifert's theorem is proven by the following algorithm.

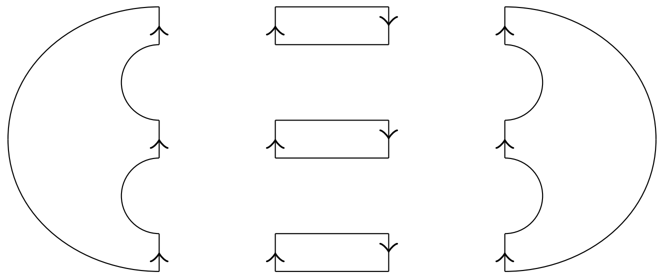
1) Start with a projection of a knot K and choose an orientation for K.

2) Smooth the crossing by detaching the strands and reattaching them with orientation preserved, but without adding the crossing back in. This gives us a collection of **Seifert circles** which bound 0-handles.

3) Attach 1-handles at each of the smoothed crossings according to the direction of the crossing. This gives a Seifert surface bounded by the original knot.

Note that not every surface produced by this algorithm will have the least possible genus. However, the algorithm always produces an orientable surface bounded by the knot. To see this, consider the case of two Seifert circles connected by 1-handles. This is always two-sided, no matter how the bands are twisted. This can be generalized to any number of disks and bands.

Example 9.3.3. To calculate the genus of the trefoil knot, employ Seifert's algorithm. Smooth the crossings with orientation preserved to form two 0-handles, and reattach the 1-handles twisting in the direction of the original crossings. This produces a Seifert surface which is a quotient space of the following figure:



A quick count shows that $V = 12$, $E = 18$ and $T = 5$, so $\chi = -1$. There is one boundary component (the knot) and the surface is orientable by Seifert's theorem. Therefore the surface is a once-punctured torus which has genus 1. The trefoil is known to be nontrivial, so as we said above its genus is ≥ 1. Hence we proven that the genus of the trefoil is 1.

Definition. *If K_1 and K_2 are knots, their* **connect sum** $K_1 \# K_2$ *is formed by connecting the knots with two direct strands in the simplest way possible, i.e. without adding crossings.*

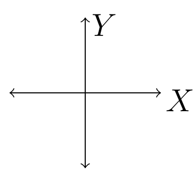
We will prove

Theorem 9.3.4. *If K_1 has genus g_1 and K_2 has genus g_2 then the genus of $K_1 \# K_2$ is $g_1 + g_2$.*

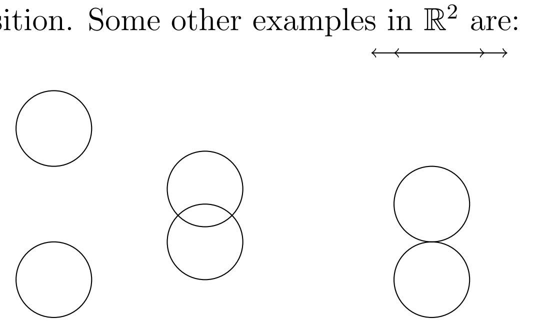
But first we need to describe a property of manifolds that is useful across all of topology.

Definition. *Two manifolds X and Y are said to be in* **general position** *(GP) in a larger manifold M if there is an $\varepsilon > 0$ such that $X \cup Y$ and $X \cap Y$ cannot be topologically changed by moving one of the manifolds by a factor less than ε.*

Example 9.3.5. Let X and Y be lines in $\mathbb{R}^2$.

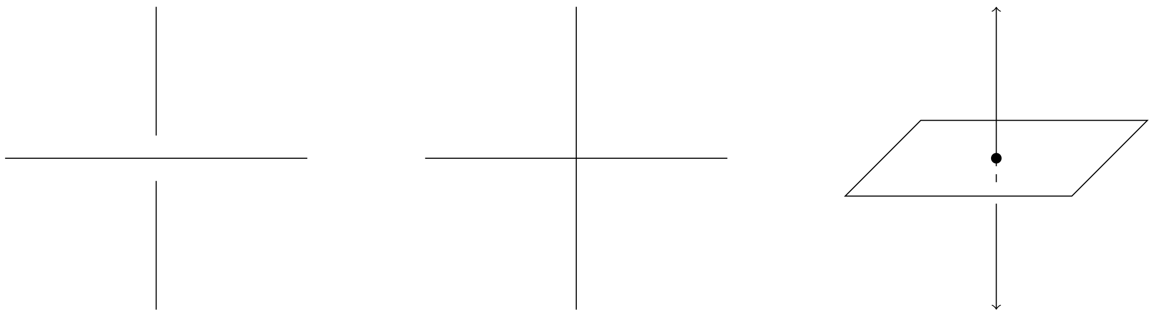


These are in general position. Some other examples in $\mathbb{R}^2$ are:



general position not general position

Example 9.3.6. In $\mathbb{R}^3$, any two curves (1-manifolds) in general position must have empty intersection.



general position not general position general position

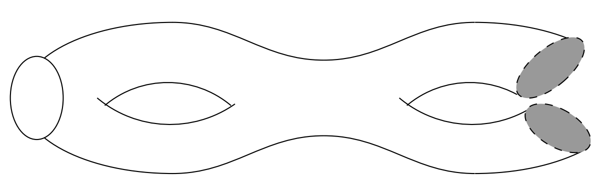
If two objects are chosen at random in a manifold M, there is a 0% chance that they will *not* be in general position. In other words we may always assume that any two manifolds are in general position.

To apply this to Seifert surfaces, note that general position implies that two surfaces living in a 3-manifold can only intersect in a 1-manifold or be disjoint. Moreover, two compact surfaces can only intersect in compact 1-manifolds, i.e. circles and closed intervals. We are now ready to prove the main result.

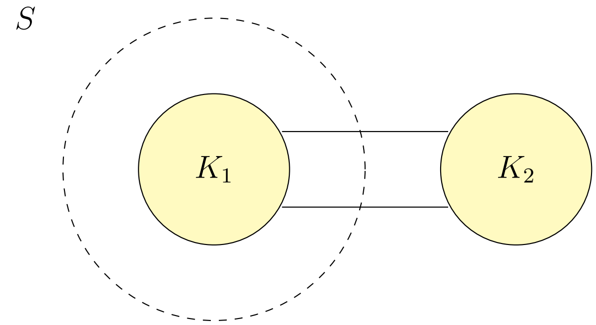
Theorem 9.3.7. *If K_1 and K_2 are knots with genus g_1 and g_2, respectively, then the genus of $K_1 \# K_2$ is $g_1 + g_2$.*

Proof. Let F be a least genus Seifert surface bounded by $K_1 \# K_2$ that intersects $S = S^2$ sitting inside $\mathbb{R}^3$ (or S^3). We may assume F and S are in general position *at the location of the connect sum*, i.e. S separates the K_1 and K_2 components of $K = K_1 \# K_2$. Their intersection must be a collection of circles and closed intervals by the preceding remarks: $F \cap S = \{a_1, \ldots, a_n, s_1, \ldots, s_m\}$ where a_i is an arc and s_j is a circle. Assume $F \cap S$ has the smallest number of intersections over all possible choices of GP. Note that the boundary of each a_i lies in $K \cap S$, but this consists of just two points. Thus we only have one arc ($n = 1$) and the rest are circles.

Take an innermost circle on S, i.e. one that bounds a disk that is disjoint from the rest of the s_j. Call this circle c and the disk it bounds D. Cut F open along c and glue in two copies of D:



This may cause the genus to decrease (as pictured) but it does not increase. This new surface is bounded by $K_1 \# K_2$, but that contradicts our minimality assumption above. Therefore there are no circles of intersection, and only one arc:



This means S separates F into F_1 and F_2, where F_1 is a Seifert surface for K_1 and F_2 is a Seifert surface for K_2.

Clearly $\text{genus}(F_1) + \text{genus}(F_2) = \text{genus}(F)$ so $g_1 + g_2 \leq \text{genus}(F)$. If the inequality was strict, then we could split up $K_1 \# K_2$ into two Seifert surfaces of minimal genus. But then $g(F_1') + g(F_2') = g(F)$ again, contradicting minimality of $g(F)$. Hence $g_1 + g_2 = g(K_1 \# K_2)$. □

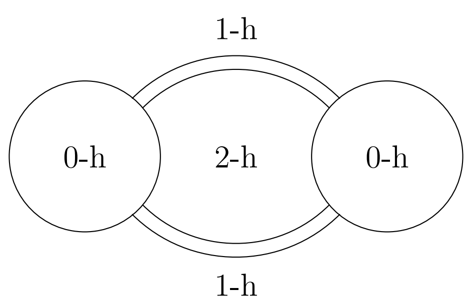
The band/disk construction for surfaces generalizes to the construction of n-manifolds.

- 0-handles are $B^n \times B^0$, where B^0 is interpreted as a point.
- 1-handles are $B^{n-1} \times B^1$.
- 2-handles are $B^{n-2} \times B^2$.
- In general, a j-handle is $B^{n-j} \times B^j$.

Note that each handle is homeomorphic to B^n, but written in a different way. This will allow us to define the rules for attaching handles.

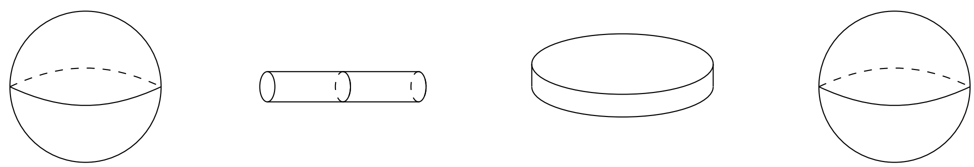
The attachment rule is this: a j-handle $B^{n-j} \times B^j$ must attach along $B^{n-j} \times \partial B^j = B^{n-j} \times S^{j-1}$ to the boundary of the existing construction. Attachments are made one at a time. In fact, we can always start with all the 0-handles required, and attach 1-handles, then 2-handles, etc.

Example 9.3.8. For surfaces (2-manifolds), 0-handles and 2-handles are B^2 (disks) and 1-handles are $B^1 \times B^1$ (bands).



Notice that attaching a 2-handle to a 1-handle ‘cancels out’, creating a surface with one less boundary component. So if we connect everything up with 0- and 1-handles and have only circular boundary components left, then attaching enough 2-handles will create a surface without boundary.

Example 9.3.9. For 3-manifolds, 0-handles are closed balls, 1-handles are solid cylindrical rods, 2-handles are plates and 3-handles are again closed balls.



0-handle $B^3 \times B^0$ 1-handle $B^2 \times B^1$ 2-handle $B^1 \times B^2$ 3-handle $B^0 \times B^3$

10 Additional Topics

Finally we include several additional topics of interest. These were taken from student presentations at the end of Dr. Howards' course and cover many topics left out in the preceding material.

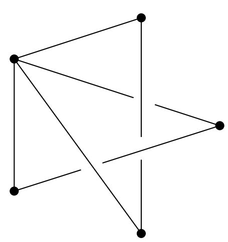
10.1 Topological Graph Theory

The content of this section comes from a presentation by Joel Barnett.

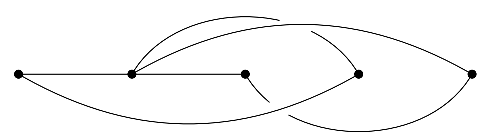
Definition. *A* **graph** *G is a set of vertices $V(G)$ and edges $E(G)$, where each edge is an unordered pair of vertices.*

Definition. *Two graphs G and H are said to be* **isomorphic** *if there is an isomorphism between them, i.e. a bijective function preserving the structure of the graph.*

Example 10.1.1. The following graphs are isomorphic

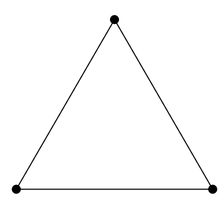


and



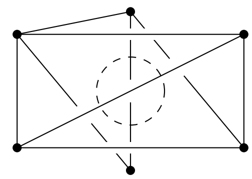
Definition. *K_n is the* **complete graph** *on n vertices, meaning there exists an edge for every possible pair of a set of n vertices.*

Example 10.1.2. K_3 is the triangular graph:



Theorem 10.1.3. *Every embedding of K_6 contains at least one pair of linked triangles.*

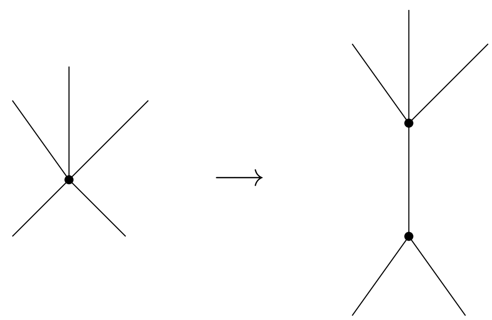
Proof. Combinatorially, there are only 10 pairs of disjoint triangles among the 6 vertices of K_6. Put an orientation on the 10 triangles so that the linking number of each pair may be computed. Let $u = \sum |L|$ where L is the linking number of a triangle pair and the sum is over all $\binom{10}{2}$ pairs. Note that isotopy doesn't change L, so u is invariant under isotopy. Define $v = u \mod 2$; we will show that v is invariant across all embeddings of K_6. Consider what happens when we change a crossing: the crossing must be between two non-adjacent edges



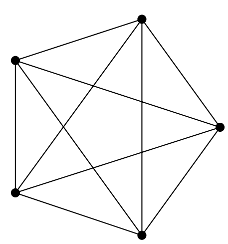
Switching the crossing (as pictured) can only change u by ± 2 (or 0, i.e. leave it unchanged), so v is invariant under crossing changes. Note that in the above picture, which just shows a subset of K_6, $u = v = 1$ so in particular u can never be 0. □

A related result is that for every n-partite graph $K_{n_1,n_2,\ldots,n_r}$ (e.g. $K_{3,3,1}$) there is always a pair of linked polygons in every embedding. Such a graph is said to be **intrinsically linked**.

Definition. *An* **expansion** *of a graph G is obtained by the following move*



There are certain 'simple' graphs, called **Petersen graphs**, that show up as subgraphs of many more complicated graphs. An example of a Petersen graph is K_5:



Theorem 10.1.4. *A graph G is intrinsically linked if and only if G contains a Petersan graph or an expansion of one.*

Proof omitted. □

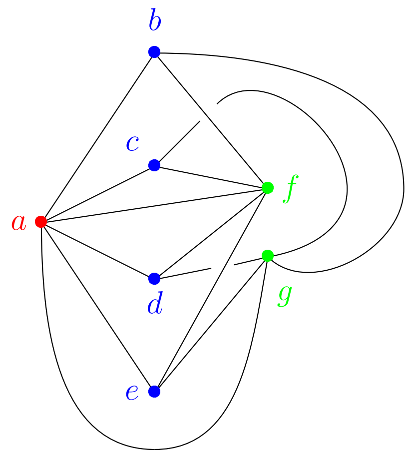
Definition. *A* **Hamiltonian cycle** *is a cycle that uses each vertex in $V(G)$ exactly once.*

Theorem 10.1.5. *Every embedding of K_7 and $K_{5,5}$ has a knotted Hamiltonian cycle, i.e. they are intrinsically linked.*

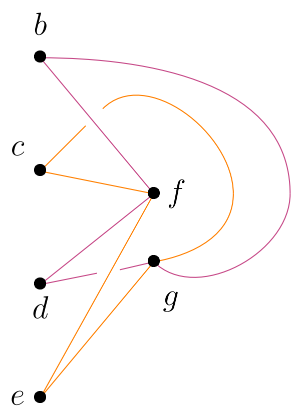
Proof omitted. □

Interestingly, K_7 and $K_{5,5}$ are the *only* known graphs that are intrinsically knotted.

Example 10.1.6. $K_{4,2,1}$ is intrinsically linked.



Label the vertices as shown. Consider the polygons $cgef$ and $bfdg$, colored orange and magenta, respectively:



We can see that these are linked, so $K_{4,2,1}$ in this embedding is intrinsically linked.

10.2 The Jordan Curve Theorem

This section covers a proof of the Jordan Curve Theorem that I presented in Dr. Howards' class.

The classic Jordan Curve Theorem (5.2.8) states that every simple closed curve in $\mathbb{R}^2$ divides the plane into two pieces, an inside and an outside. Equivalently, any embedding of S^1 into $\mathbb{R}^2$ separates the plane. We will prove a generalization of the theorem for embeddings of S^{n-1} into $\mathbb{R}^n$.

Theorem 10.2.1. *Let M be an embedding of S^{n-1} into $\mathbb{R}^n$. The complement of M in $\mathbb{R}^n$ consists of two connected open sets, the "outside" D_0 and the "inside" D_1. Moreover, the closure of D_1 is a compact manifold with M as its boundary.*

To prove this, we will develop the concept of winding number, which tracks the number of times the embedding of the circle, or more generally S^{n-1}, winds around a particular point $z \in \mathbb{R}^n$. The first definition is from intersection theory:

Definition. *For a smooth map $f : X \to Y$ where X is a compact manifold and Y is connected, we define the* **mod 2 intersection number** *for any closed manifold $Z \subset Y$ to be the number of points in $f^{-1}(Z)$ mod 2, denoted $I_2(f, Z)$.*

When X and Y are both n-manifolds, $I_2(f, \{y\})$ is the same for all $y \in Y$. In this case the number is called the **mod 2 degree** of f, denoted $\deg_2(f)$. Some useful properties of the mod 2 degree are:

- If $f : X \to Y$ and $g : X \to Y$ are homotopic, then $\deg_2(f) = \deg_2(g)$.
- If $X = \partial W$ and $f : X \to Y$ may be extended to all of W then $\deg_2(f) = 0$.

Definition. *Let X be a compact $(n-1)$-manifold and let z be a point in $\mathbb{R}^n$. For any map $f : X \to \mathbb{R}^n$ such that $z \notin f(X)$, define the* **winding number** *of f around z to be $W(f, z) = \deg_2(f)$.*

If the main theorem is true, then for any $z \in \mathbb{R}^n$ not on the boundary of M, the winding number should be 1 or 0 according to:

$$W(f, z) = \begin{cases} 1 & z \in M \\ 0 & z \notin M \end{cases}$$

where $f : S^{n-1} \to \mathbb{R}^n$ is an embedding with $f(S^{n-1}) = M$.

Lemma 10.2.2. *Let $z \in \mathbb{R}^n - M$. Prove that if x is any point of M and U any neighborhood of x in $\mathbb{R}^n$ then there exists a point of U that may be joined to z by a curve not intersecting M.*

Proof. Let $\mathcal{C}$ be the set of all points in M for which the lemma holds. We will show that $\mathcal{C}$ is open, closed and nonempty in M, which will imply $\mathcal{C} = M$ by connectedness. First, every neighborhood of each point in $\mathcal{C}$ intersects M by construction, so $\mathcal{C}$ is closed. On the other hand, M is a manifold so each point $x \in \mathcal{C}$ has a neighborhood $U \subset M$ that is homeomorphic to $\mathbb{R}^{n-1}$. This implies that $\mathcal{C}$ is open. To see that $\mathcal{C}$ is nonempty, consider the straight line from z to the nearest point $x \in M$; then every point on this line that is contained in any neighborhood U of x satisfies the lemma. □

Lemma 10.2.3. *$\mathbb{R}^n$ has at most two connected components.*

Proof. Let B be a small ball such that $B - M$ has two components, and fix two points z_0 and z_1 such that one is in each component. Then by Lemma 10.2.2, each point $z \in \mathbb{R}^n - M$ may be joined to either z_0 or z_1 by a curve that doesn't intersect M. □

This doesn't rule out the possibility that z_0 and z_1 may themselves be connected by such a curve. The next two lemmas will show exactly when this happen.

Lemma 10.2.4. *If z_0 and z_1 lie in the same component of $\mathbb{R}^n - M$ then $W(M, z_0) = W(M, z_1)$.*

NOTE: Here we adopt the notation $W(M, z_i)$ to denote the winding number of the embedding $S^{n-1} \hookrightarrow M$ around each point.

Proof. Use the property that homotopic maps have the same mod 2 degree. $\square$

Lemma 10.2.5. *For any two points $z_0, z_1 \in \mathbb{R}^n - M$, let ℓ be the number of times the ray $\overrightarrow{z_0 z_1}$ intersects M between z_0 and z_1. Then $W(M, z_0) = W(M, z_1) + \ell \mod 2$.*

Proof omitted. $\square$

Now we can prove the main theorem:

Theorem (5.2.8)**.** *Let M be an embedding of S^{n-1} into $\mathbb{R}^n$. The complement of M in $\mathbb{R}^n$ consists of two connected open sets, the "outside" D_0 and the "inside" D_1. Moreover, the closure of D_1 is a compact manifold with M as its boundary.*

Proof. By Lemmas 10.2.4 and 10.2.5, $\mathbb{R}^n - M$ has precisely two components:

$$D_0 = \{z \in \mathbb{R}^n \mid W(M, z) = 0\} \qquad \text{and} \qquad D_1 = \{z \in \mathbb{R}^n \mid W(M, z) = 1\}.$$

It is routine to verify that D_0 and D_1 are in fact open and connected. Now consider $\bar{D}_1$. By construction $\bar{D}_1 = D_1 \cup M$ so $M = \partial D_1$ as desired. Observe that for $z \in \mathbb{R}^n$ such that $|z|$ is large enough, $W(M, z)$ will always be zero. So $\bar{D}_1$ is closed and bounded, and we can apply the Heine-Borel Theorem (6.2.3) to show that $\bar{D}_1$ is compact. To prove the parameterization property is more difficult so we will end the discussion here. $\square$

10.3 Knot Polynomials

This section surveys knot polynomials, as presented by Elena Palesis.

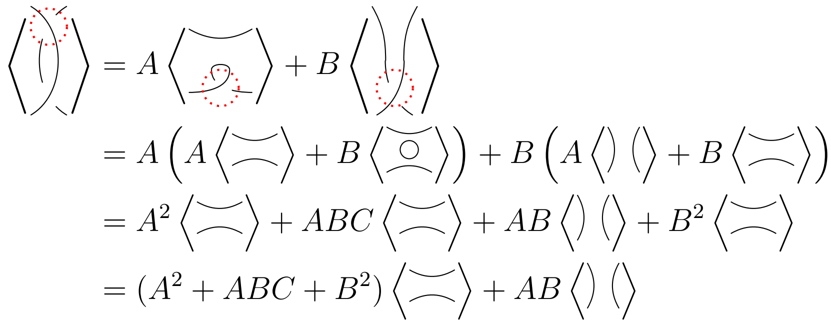
Definition. *For a link L, the* **bracket polynomial** $\langle L \rangle$ *is defined by the following rules:*

1) *For the unknot,* $\left\langle \bigcirc \right\rangle = 1$.
2) $\left\langle \times \right\rangle = A \left\langle)\,(\right\rangle + B \left\langle \asymp \right\rangle$ *and* $\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + B \left\langle)\,(\right\rangle$ *for some A, B.*
3) $\left\langle L \cup \bigcirc \right\rangle = C \langle L \rangle$ *for some C.*

Remark. The above rules imply that the unlink with m components has bracket polynomial

$$\left\langle \bigcirc \cup \bigcirc \cup \cdots \cup \bigcirc \right\rangle = C^{m-1}.$$

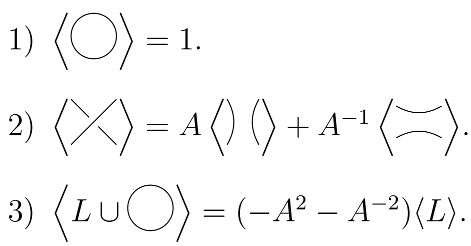
We want the bracket polynomial to preserve the 2nd Reidemeister move, so consider



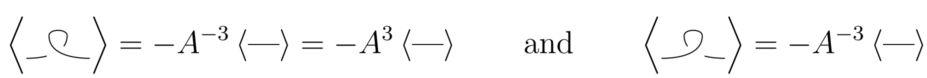
For this to preserve the Reidemeister move, we want the above to be equal to  so $AB = 1$, or $A^{-1} = B$, and

$$\begin{aligned} A^2 + ABC + B^2 &= 0 \\ A^2 + C + A^{-2} &= 0 \\ \implies C = -A^2 - A^{-2}&. \end{aligned}$$

Therefore the rules can be restated as



The bracket polynomial is invariant under R2 and R3, *but* it changes when crossings are added, i.e. R1 changes the bracket:



To account for this, we define

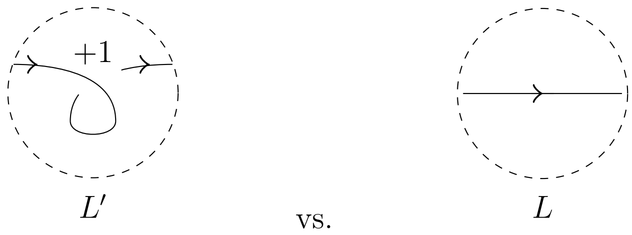
Definition. *For an oriented knot K, assign $+1$ to each right-handed crossing and -1 to each left-handed crossing. The sum of the values at each crossing is called the* **writhe** *of K, denoted $\omega(K)$. The writhe $\omega(L)$ of a link L is defined in the same way.*

It's easy to verify that $\omega(L)$ is invariant under R2 and R3, and on a particular component of L the writhe does not depend on orientation. However, R1 changes the writhe by $+1$ or -1. The next polynomial accounts for this.

Definition. *The* **X polynomial** *of a link L is defined as $X(L) = (-A^3)^{-\omega(L)}\langle L\rangle$.*

Proposition 10.3.1. *$X(L)$ is a knot (and link) invariant.*

Proof. $X(L)$ is invariant under R2 and R3 since it is defined in terms of the bracket. Thus it suffices to consider R1:



Then $\omega(L') = \omega(L) + 1$ and

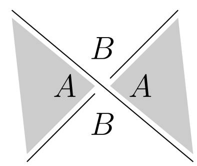
$$X(L') = (-A^3)^{-\omega(L')}\langle L'\rangle = (-A^3)^{-\omega(L)-1}(-A^3\langle L\rangle) = (-A^3)^{-\omega(L)}\langle L\rangle = X(L).$$

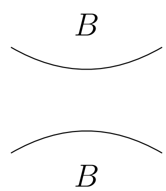
Hence $X(L)$ is invariant under R1–R3 so it is a knot and link invariant. $\square$

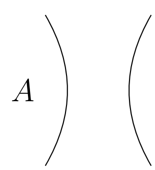
Definition. *The* **Jones polynomial** $V(L)$*, named for its discoverer Vaughan Jones, is obtained from the* X *polynomial by the change of variable* $A \mapsto t^{-1/4}$*.*

The calculation of $X(L)$ can be greatly simplified using **Skein relations**:

- For each (unoriented) crossing, twist the overstrand counterclockwise until it rests on top of the understrand and label the region crossed as A; the remainder of the knot plane is labelled B.

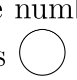


- Define an A-split 

and a B-split 

.

- Let $\{S_i\}$ be the set of all possible states, i.e. diagrams of L with all crossings smoothed. There are 2^n states for a knot/link with n crossings.

- Let $a(S_i)$ be the number of A-splits in a state S_i; likewise let $b(S_i)$ be the number of B-splits in S_i. Also let $|S_i|$ denote the number of unknotted components  in S_i. Compute $\langle L\rangle$ by

$$\langle L\rangle = \sum_{i=1}^{2^n} A^{a(S_i)-b(S_i)}(-A^2 - A^{-2})^{|S_i|-1}.$$

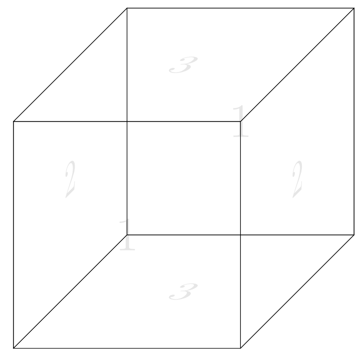
Definition. *The* **span** *of a knot* K *is defined as the difference between the highest and lowest exponent in* $\langle K\rangle$*.*

Span is a knot invariant, since $\langle K\rangle$ is invariant under all 3 Reidemeister moves. For any reduced, alternating projections of a knot, one can show that $\mathrm{Span}(K) = 4n$ where n is the number of crossings in the projection. This may be used to prove

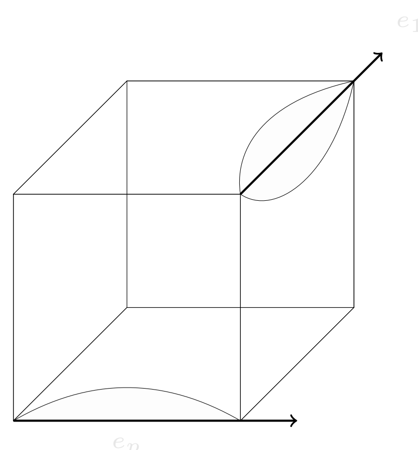
Theorem 10.3.2. *Two reduced, alternating projections of the same knot have the same number of crossings.*

10.4 Manifolds and Cosmology

This section describes a way to study the shape of the universe using 3-manifolds. This topic was presented by Elliott Hollifield. Given any gluing of a fundamental domain



and a set of equivalent edges $\{e_1, \ldots, e_p\}$, let E^* be the set resulting from gluing the interior of each edge.



We do this by producing a **face-edge sequence**:

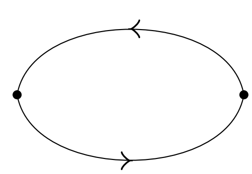
$$F[1,1], e_1, F[1,2], F[2,1], e_2, \ldots, F[p,1], e_p, F[1,p]$$

subject to the properties

i) Edge e_i glues to faces $F[i,1]$ and $F[i,2]$.

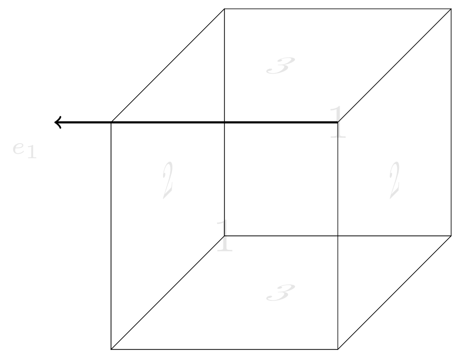
ii) e_1 glues to e_p.

If we give e_1 an orientation, the other edges inherit an orientation from the sequence. For example, the face-edge sequence in the figure above produces $\mathbb{R}P^2$, the projective plane:



In this case, where the directions of e_1 and e_p disagree, the shaded neighborhood is not homeomorphic to an open 3-ball, so this gluing would not produce a manifold.

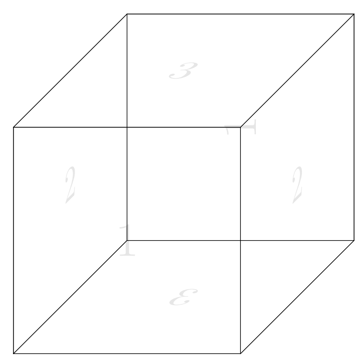
Example 10.4.1. A face-edge sequence that produces the torus



Denote the faces by T (top), B (bottom), L (left), R (right), F (front) and B' (back). Then the face edge sequence is

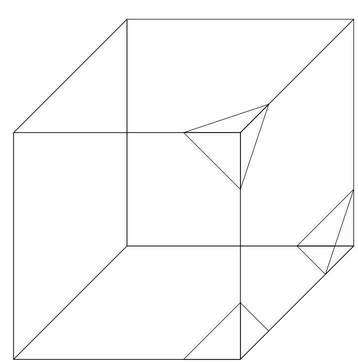
$$Te_1FB'e_2TB'e_3FB'e_4B'.$$

Example 10.4.2. A fundamental domain which doesn't have a face-edge sequence



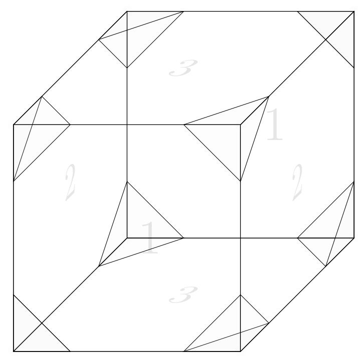
Consider $Te_1FB'e_2LRe_3B'Fe_4B$. In order to glue T and B, the shape must pass through itself, similar to a Klein bottle. So this fails the face-edge sequence criteria.

Next we describe a similar test for the vertices.



Let B be the quotient space obtained from gluing the tetrahedra in each equivalence class of vertices. We know that ∂B is a compact surface, so if ∂B is a sphere, we're good. However if $\partial B = T\#T\#\cdots$ or $P\#P\#\cdots$ then B will fail to be a 3-manifold.

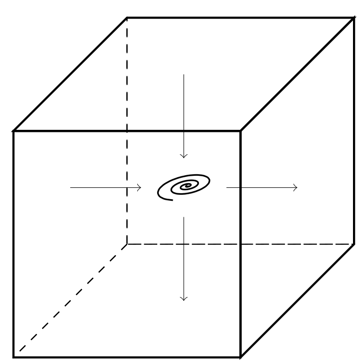
Example 10.4.3. We compute the Euler characteristic of B in the following gluing to determine if it yields a manifold.



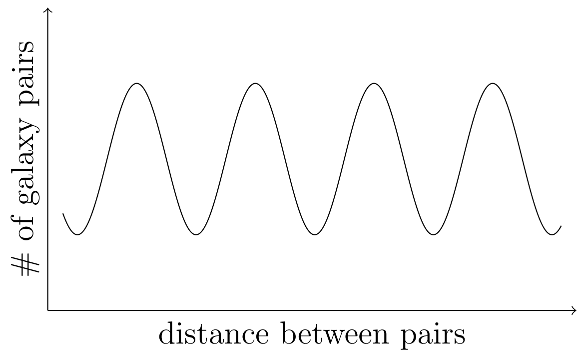
Note that $F = 8$ (the number of pyramids) and $E = \frac{3-8}{2} = 12$. To compute V, note that before gluing there are 8 vertices, but after gluing, there are 2 vertices for each equivalence class of edges, so $V = 6$. Then $\chi = 2$ so this is a manifold.

The Shape of the Universe

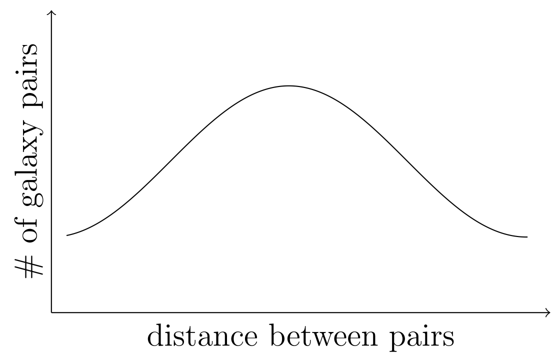
Imagine the Milky Way galaxy is sitting inside the 3-torus.



If we look out in any direction, we would see ourselves (assuming the curvature of the universe is 0). In fact, we'd see ourselves many times over. So when we look up into the night sky and see billions of stars, are we just seeing many copies of a few galaxies? This would have the following galaxy distribution:



However if the actual distribution were something like



this would be evidence in support of a Euclidean universe. Cosmologists are currently using this method of **intergalactic superdistances** to calculate the distribution for our universe.

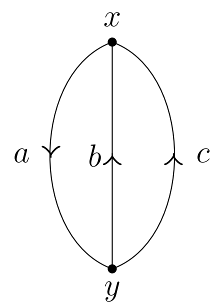
In 1965, astronomers Arno Penzias and Robert Wilson at Bell Labs were researching radio waves in the Milky Way and discovered the **cosmic microwave background** which emanates from every direction in the universe. Cosmologists are currently studying the pattern of CMB radiation in comparison with spheres in different fundamental domains to discern what the pattern should look like.

10.5 Introduction to Homology

We conclude these notes with an introductory look at the homology, as presented by Stewart Curry.

The fundamental group π_1 describes equivalence classes of loops. This can be generalized to higher dimensions, using π_2 (loops of loops), π_3, π_4, etc. These groups become complicated quickly. Instead, a more practical approach utilizes homology.

For a topological space X, the nth homology group $H_n(X)$ will in some sense 'count' the number of n-dimensional holes in the space. Consider the space X:



$a+b$, $a+c$ and $b-c$ are called **cycles**; they capture 'holes' in the graph. Define C_0 to be the free abelian group on vertices x, y and C_1 to be the free abelian group on edges a, b, c. Then cycles are simply linear combinations of the free generators.

Next define the **boundary operator** $\partial : C_1 \to C_0$ taking a cycle in C_1 to its boundary, ordered according to orientation. For example, in the above figure $\partial(a) = y-x$, $\partial(b) = x-y$ and $\partial(c) = x-y$. Then the cycles are precisely those edges (or edge sequences) that lie in $\ker \partial$.

Proposition 10.5.1. *∂ is a group homomorphism.*

Proof omitted. □

Definition. *The n***th homology group** *of a topological space X is defined as*

$$H_n(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1},$$

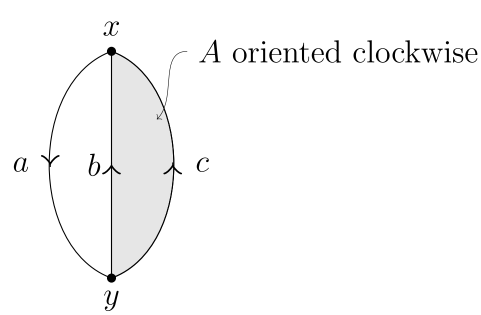
where $\partial_n : C_n \to C_{n-1}$ is the nth boundary operator. In other words, $\ker \partial_n$ is the set of n-cycles and $\operatorname{Im} \partial_{n+1}$ the set of $(n+1)$-boundaries.

Notice that $\alpha a + \beta b$ is a cycle $\iff$ $\partial(\alpha a + \beta b) = 0$, i.e. $\{\text{cycles}\} = \ker \partial$. In the space above, what are the cycles in C_0? Take any linear combination $\alpha a + \beta b + \gamma c$ and compute

$$\begin{aligned}
\partial(\alpha a + \beta b + \gamma c) &= \alpha\partial(a) + \beta\partial(b) + \gamma\partial(c) \\
&= \alpha(y-x) + \beta(x-y) + \gamma(x-y) \\
&= x(-\alpha+\beta+\gamma) + y(\alpha-\beta-\gamma).
\end{aligned}$$

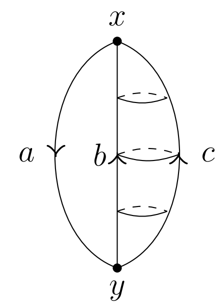
The solution to this is $\alpha = \beta + \gamma$ which tells us that $\{a+b, a+c\}$ generates $\ker \partial$. It turns out that the first homology group for X is $H_1(X) = \langle a+b, a+c \rangle \cong \mathbb{Z} \times \mathbb{Z}$.

To compute homology groups in higher dimensions, let's fill in a disk:



C_2 will be the free group on 2-dimensional objects. Here, A gives us $b - c = 0 \implies b = c$. So $H_2(X) = \langle a + b, a + c \rangle = \langle a + b \rangle \cong \mathbb{Z}$.

Next we add another dimension to X.

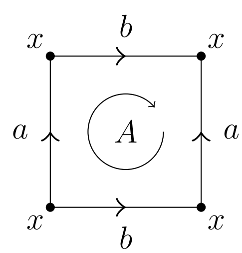


Then for a linear combination $\alpha A + \beta B$,

$$\begin{aligned} \partial_2(\alpha A + \beta B) &= \alpha \partial_2(A) + \beta \partial_2(B) \\ &= \alpha(b - c) + \beta(b - c) \\ &= b(\alpha + \beta) + c(-\alpha - \beta) = 0 \end{aligned}$$

so $\alpha A + \beta B = \alpha(A + B)$ and the second homology group is $H_2(X) = \langle A + B \rangle \cong \mathbb{Z}$.

Example 10.5.2. The homology of the torus



The vertex, edge and face sets are

$$C_0 = \langle x \rangle \qquad C_1 = \langle a, b \rangle \qquad C_2 = \langle A \rangle.$$

We have the sequence $0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$. Let's compute the homology groups:

$$\begin{aligned} H_0(X) &= \ker \partial_0 / \operatorname{Im} \partial_1 = \langle x \rangle / \{0\} = \langle x \rangle \cong \mathbb{Z} \\ H_1(X) &= \ker \partial_1 / \operatorname{Im} \partial_2 = \langle a, b \rangle / \langle a + b - a - b \rangle = \langle a, b \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \\ H_2(X) &= \ker \partial_2 / \operatorname{Im} \partial_3 = \langle A \rangle / \{0\} = \langle A \rangle \cong \mathbb{Z}. \end{aligned}$$

Notice that in the above example, $H_1(X) = \pi_1(X)$, the fundamental group. In general, the first homology group $H_1(X)$ is the *abelianization* of $\pi_1(X)$ but in this case the fundamental group was already abelian.

Example 10.5.3. The projective plane has the following homology groups

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

The theory of (finitely generated) abelian groups tells us that each homology group $H_i(X)$ of a space X may be decomposed into

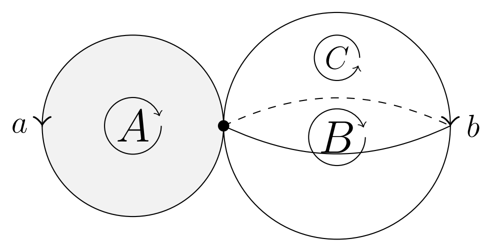
$$H_i(X) = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \times (\mathbb{Z}/a_1\mathbb{Z}) \times (\mathbb{Z}/a_2\mathbb{Z}) \times \cdots$$

for integers $a_k \geq 2$. The number of free summands (copies of $\mathbb{Z}$) is called the **rank** of $H_i(X)$. The ranks of homology groups play an important role in computing the Euler characteristic of a topological space.

Theorem 10.5.4. *For a space X, its Euler characteristic χ is computed by*

$$\chi = \sum_{i=1}^{\infty} (-1)^i \mathit{rank}(H_i(X)).$$

Example 10.5.5. The wedge of a sphere and a disk



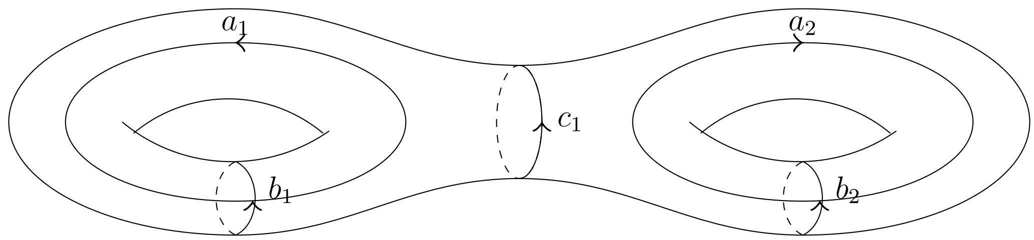
Here we have $C_0 = \langle x \rangle$, $C_1 = \langle a, b \rangle$ and $C_2 = \langle A, B, C \rangle$. Then the sequence of boundary operators is

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Since a and b do not have a boundary, $\partial_1(\alpha a + \beta b) = \alpha\partial_1(a) + \beta\partial_1(b) = 0$. For the faces, we have $\partial_2(\alpha A + \beta B + \gamma C) = \alpha\partial_2(A) + \beta\partial_2(B) + \gamma\partial_2(C) = \alpha a - (\beta + \gamma)b$ which tells us that $\alpha A + \beta B + \gamma C \in \ker \partial_2 \iff \alpha a = (\beta + \gamma)b$. In other words, $B - C$ generates the kernel of ∂_2. This allows us to compute the homology groups:

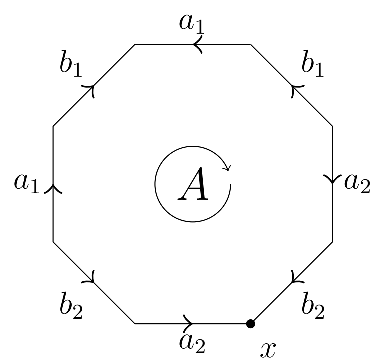
$$\begin{aligned} H_0(X) &= \ker \partial_0 / \operatorname{Im} \partial_1 = \langle x \rangle / \{0\} \cong \mathbb{Z} \\ H_1(X) &= \ker \partial_1 / \operatorname{Im} \partial_2 = \langle a, b \rangle / \langle a, b \rangle = \{0\} \\ H_2(X) &= \ker \partial_2 / \operatorname{Im} \partial_3 = \langle B - C \rangle / \{0\} \cong \mathbb{Z}. \end{aligned}$$

Example 10.5.6. The two-holed torus



We have the sequence $0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$. Note that $a_1, a_2, b_1, b_2, c_1 \in \ker \partial_1$, but a_1 is not the boundary of any subsurface of $T\#T$, so $a_1 \notin \operatorname{Im} \partial_2$. Likewise we see that $a_2, b_1, b_2 \notin \operatorname{Im} \partial_2$. However, c_2 bounds each punctured torus in the connect sum, so $c_1 \in \operatorname{Im} \partial_2$.

Consider $X = T\#T$ as the quotient space of an octagon:



We have $C_0 = \langle x \rangle$, $C_1 = \langle a_1, b_1, a_2, b_2 \}$ and $C_2 = \langle A \rangle$. Also,

$$\begin{aligned}
&\ker \partial_0 = \langle x \rangle \\
&\operatorname{Im} \partial_1 = \{0\} \qquad \ker \partial_1 = \langle a_1, b_1, a_2, b_2 \rangle \\
&\partial A = a_1 + b_1 - a_2 - b_2 + a_2 + b_2 - a_1 - b_1 = 0.
\end{aligned}$$

The homology groups are computed to be

$$\begin{aligned}
H_0(X) &= \ker \partial_0 / \operatorname{Im} \partial_1 = \langle x \rangle / \{0\} \cong \mathbb{Z} \\
H_1(X) &= \ker \partial_1 / \operatorname{Im} \partial_2 = \langle a_1, b_1, a_2, b_2 \rangle / \{0\} \cong \mathbb{Z}^4.
\end{aligned}$$