4 Continuous Functions and Homeomorphisms

4.1 Definition of Continuous

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if for every $x_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists $a\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$

4.2 Open Set Definition of Continuity

Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous** if $f^{-1}(V)$ is open in X for every open set V in Y.

We call this the **open set definition of continuity**. Paraphrased, it states that f is continuous if the preimage of every open set is open.

4.3 Theorem that a function is continuous if and only if the preimage of the basis elements is open

Let X and Y be topological spaces and $\mathcal{B}$ be a basis for the topology on Y . Then $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

Proof on page 132

4.4 Theorem that every polynomial is continuous

Let $\mathbb{R}$ have the standard topology. Then every polynomial function $p : \mathbb{R} \to \mathbb{R}$, with $p(x) = a_n x^n + \ldots + a_1 x + a_0$, is continuous.

4.5 Theorem that says the closure of a subset maps to part of the closure of the superset

Let $f : X \to Y$ be continuous and assume that $A \subset X$. If $x \in Cl(A)$, then $f(x) \in Cl(f(A))$.

Proof on page 134

4.6 Translation of $\varepsilon - \delta$

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if, for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$.

$\forall x \in X$ and every open set U containing $f(x), \exists$ neighborhood V of x, such that $f(V) \subset U$

4.7 Theorem that a function is continuous if and only if every element has a neighborhood containing $f(x)$, there exists a neighbor$V$ of x such that $f(V) \subset U$

A function $f : X \to Y$ is continuous in the open set definition of continuity if and only if for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$

Proof on page 135

4.8 Theorem that converges points will converge given a function

Assume that $f : X \to Y$ is continuous. If a sequence $(x_1, x_2, \ldots)$ in X converges to a point x, then the sequence $(f(x_1), f(x_2), \ldots)$ in Y converges to $f(x)$.

Proof on page 135

4.9 Theorem that we can map closed sets between each other

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$.

4.10 Theorem that function composition works for continuity

Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then the composition function, $g \circ f : X \to Z$, is continuous.

Proof on page 136

4.11 The Pasting Lemma

Let X be a topological space and let A and B be closed subsets of X such that $A \cup B = X$. Assume that $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for all x in $A \cap B$. Then $h : X \to Y$, defined by

$$h(x) = \begin{cases} f(x) \text{ if } x \in A \\ g(x) \text{ if } x \in B \end{cases}$$

is a continuous function.

Proof on page 137

4.12 Definition of a Homeomorphism

Let X and Y be topological spaces, and let $f : X \to Y$ be a bijection with inverse $f^{-1} : Y \to X$. If both f and f^{-1} are continuous functions, then f is said to be a **homeomorphism**. If there exists a homeomorphism between X and Y, we say that X and Y are **homeomorphic** or **topologically equivalent**, and we denote this by $X \cong Y$.

4.13 Facts about Homeomorphisms

(i) The function $id : X \to X,$ defined by $id(x) = x,$ is a homeomorphism.
(ii) If $f : X \to Y$ is a homeomorphism, then so is $f^{-1} : Y \to X$.
(iii) If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then so is $g \circ f : X \to Z$