- 7.01 Show that every set $A \subset R$ is a compact subset of \mathbb{R} in the finite complement topology on \mathbb{R} .
- 7.02 Prove Theorem 7.6: Let X be a topological space.
 - (a) If C_1, \ldots, C_n are each compact in X, then $U_{i=1}^n C_i$ is compact in X
 - (b) If X is Hausdorff, and $\{C_{\alpha}\}_{{\alpha}\in A}$ is a collection of sets that are compact in X, then $\cap_{{\alpha}\in A}C_{\alpha}$ is compact in X.
- 7.03 Provide an example demonstrating that an arbitrary union of compact sets in a topological space X is not necessarily compact.
- 7.07 Recall that the arithmetic progression topology on Z is generated by the basis $\mathcal{B} = \{A_{a,b}|a,b\in\mathbb{Z},b\neq 0\}$, where each

$$A_{a,b} = \{\dots, a-2b, a-b, a, a+b, a+2b, \dots\}$$

is an arithmetic progression. Determine whether or not \mathbb{Z} is compact in this topology.

- 7.12 Show that the Tube Lemma does not necessarily hold if we drop the assumption that Y is compact. That is, provide an example of a noncompact space Y and an open set U in $X \times Y$ such that U contains a slice $\{x\} \times Y \subset X \times Y$ but does not contain an open tube $W \times Y$ containing the slice.
- 7.17 Use compactness to prove that the plane is not homeomorphic to the sphere. (Recall, in Section 6.2 we distinguished between a number of pairs of spaces, including the line and the plane and the line and the sphere, but we indicated that we were not yet in a position to distinguish between the plane and the sphere. With compactness, we can now make that distinction.)
- 7.18 In this exercise we demonstrate that if we drop the condition that X is Hausdorff in Theorem 7.6, then the intersection of compact sets in X is not necessarily a compact set. Define the extra-point line as follows. Let $X = \mathbb{R} \cup (p_e)$, where p_e is an extra point, not contained in \mathbb{R} . Let \mathcal{B} be the collection of subsets of X consisting of all intervals $(a,b) \subset \mathbb{R}$ and all sets of the form $(c,0) \cup \{p_e\} \cup (0,d)$ for c < 0 and d > 0.
 - (a) Prove that \mathcal{B} is a basis for a topology on X.
 - (b) Show that the resulting topology on X is not Hausdorff.
 - (c) Find two compact subsets of X whose intersection is not compact. Prove that the sets are compact and that the intersection is not.
- 7.19 (a) Let (X, d) be a metric space. Prove that if A is compact in X, then A is closed in X and bounded under the metric d.
 - (b) Provide an example demonstrating that a subset of a metric space can be closed and bounded but not compact.

Summary