Assignment 5 Report

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November 4, 2025

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I. Problem 5.1

Several derivative approximation methods were utilized to estimate $f'(1.5)$ for the single-variable function $f(x)$ defined in Eq. (1). These methods included: forward-, backward-, and central-difference approaches, a complex-step approach, forward-mode algorithmic differentiation via operator overloading, and analytic differentiation. The analytic result $f'(x)$ is expressed in Eq. (2); its derivation, which follows from the basic principles of calculus, is omitted here.

$$f(x) = \frac{e^x}{\sqrt{\sin^3 x + \cos^3 x}} \quad (1)$$

$$f'(x) = \frac{e^x(2s^3 + 2c^3 + 3sc^2 - 3s^2c)}{2(s^3 + c^3)^{1.5}} \quad (2)$$

$$(\text{for } s = \sin x \text{ and } c = \cos x)$$

Solving Eq. (2) for $x = 1.5$ yields the analytic result shown in Eq. (3).

$$f'(1.5) = 4.05342789389862 \quad (3)$$

This analytic derivative was used to calculate the relative error in all subsequently found estimates of $f'(1.5)$, per Eq. (3).

$$\varepsilon = \frac{|f'(1.5) - \tilde{f}'(1.5)|}{|f'(1.5)|} \quad (4)$$

Here, $f'(1.5)$ is the analytic derivative value as defined in Eq. (3) and $\tilde{f}'(1.5)$ is the numerically approximated derivative value. This nomenclature will continue throughout this investigation to differentiate between the true and approximate derivative values.

The first step of this investigation involved performing a step size study on the three finite-difference approaches as applied to this problem. Step sizes were defined from $h = 10^{-1}$ to $h = 10^{-25}$, inclusive, by assigning the elements of the vector of step sizes, $\underline{h}$, to be the terms in the finite sum defined by Eq. (5).

$$10^{-1} + \sum_{i=-25}^{-2} \sum_{j=1}^{9} j \cdot 10^i \rightarrow \text{elements}(\underline{h}) \quad (5)$$

By defining h-values in this manner, a relatively high-fidelity step size study was achieved over the range of interest. The relative error ε, as defined by Eq. (4), was calculated for each h-value, as defined by Eq. (5), for all three finite-difference methods. These ε-values were then plotted on a logarithm-scaled plot to allow for visualization of the efficacy of each method as well as the results of the step size study. This plot is shown in Fig. (1).

Examination of this plot reveals several interesting, though not necessarily unexpected, results. The most evident result is that the central-difference approach provides an error approximately three orders of magnitude smaller than the forward- or backward-difference approaches. Such a result
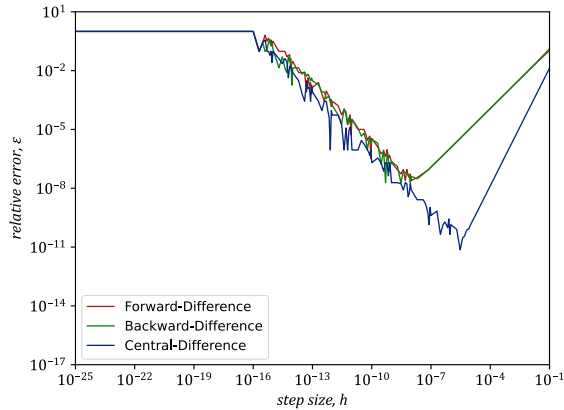
Figure 1. Plot of the relative error, as defined by Eq. (4), in estimations of $f'(1.5)$ *for* f(x) *defined by Eq. (1) via forward-, backward-, and central-difference methods. Notes the logarithmic scaling on both the ordinate and abscissa axes.*

is to be expected, as the central-difference method introduces an $\mathcal{O}(n^2)$ truncation error while the forward- and backward-difference methods introduce $\mathcal{O}(n)$ truncation errors.

One also notices from Fig. (1) that the ideal step size – that is, the step size for which ε is minimized – for the central-difference method is several orders of magnitude larger than the ideal step size for the forward- and backward-difference methods. Indeed, at relatively large step sizes the central-difference method provides significantly more accurate derivative estimations than the alternatives.

One interesting observation resulting from this step size study is that, for h-values smaller than the idea h-value for the forward- and backward-difference methods (which are approximately the same, as is expected), all three methods perform nearly identically. That is, for h-values for which subtractive cancellation dominates in all three cases, all three cases return almost identically poor results. Thus, generally speaking, we observe

that, for this problem formulation, there is little benefit to the additional computational cost of the central-difference approach if the h-value is chosen poorly (i.e. h is too small).

Before discussing the actual numerical results of the step size study, there is one other particularly interesting observation to be made from Fig. (1) which is not observed in the relative error plot included for this problem in *Engineering Design Optimization* (see Fig. (6.7) in text) [1]. By selecting a relatively fine vector of h-values, the error plot generating for this problem reveals a significant amount of noise which is present only in the region of h-values for which subtractive cancellation dominates. For h-values larger than the ideal step size, there is no observable noise whatsoever; for h-values smaller than the ideal step size, however, the relative error in the approximated derivative varies wildly and unpredictably between step sizes. Such phenomenon can most readily be attributed to the fact that when subtractive cancellation is the dominant effect, the subtractive error may in fact be beneficial for small perturbations of h. However, the error is just as likely to be disadvantageous for small perturbations of h, resulting in unpredictable noise in the error plot despite a general trend that decreasing h while subtractive cancellation dominates will increase the relative error in the derivative approximation.

With all interesting aspects of the error plot in Fig. (1) discussed, it is now necessary to turn away from visualization and attend to hard numbers. The ideal h-value for each finite-difference method is tabulated in Table (I)(a). It should be noted that small perturbations of h near this ideal step size may in fact decrease the relative error in the approximation; however, the approach of this study guarantees that the ideal step sizes

Table I(a)
Step Size Study Results

Method	Ideal h	Minimal ε
forward FD	$1 \cdot 10^{-8}$	$2.451 \cdot 10^{-8}$
backward FD	$9 \cdot 10^{-9}$	$1.931 \cdot 10^{-8}$
central FD	$3 \cdot 10^{-6}$	$7.213 \cdot 10^{-12}$

found are the best possible choices of the h-values defined by Eq. (5).

Examination of the results expressed in Table (I)(a) reveal nothing that wasn't seen from the plot: the forward- and backward-difference methods provide nearly the same error at nearly the same ideal step size, while the central-difference method provides an error several orders of magnitude less at an ideal step size several ordered of magnitude larger.

Comparing the results of this step size study to those found in *Engineering Design Optimization* for the same problem reveal no significant discrepancies nor cause for concern with these results [1]. With this preliminary investigation done, attention is now turned to the matters discussed in the problem formulation.

Firstly, a complex-step approach was taken to solve this problem, such that the results may be compared to the finite-difference methods in the same manner as in Fig. (6.9) in *Engineering Design Optimization* [1]. Applying the complex-step method to this problem was trivial since NumPy operators (for this problem, `np.exp()`, `np.sin()`, and `np.cos()`) are naturally overloaded to handle both real and complex arguments. As such, no modification to the function for evaluating $f(x)$ was required, and Eq. (6.14) from *Engineering Design Optimization* was implemented in a single line of code.

Unlike finite-difference methods, a complex-step approach can be effective across a much wider range of h-values. Here underflow, rather than subtractive cancellation, becomes the limiting factor for accuracy at very small step sizes; however, underflow begins to dominate in complex-step approaches at step sizes hundreds of orders of magnitude smaller than the h-values at which subtractive cancellation begins to dominate in finite-difference approaches.

For the purpose of this investigation, the onset of underflow was defined as the smallest h-value yielding machine precision for which all smaller h-values return derivative approximations with relative errors greater than machine precision. Here it is necessary to briefly discuss a limitation of the relative error definition in Eq. (4). If $f'(1.5)$ and $\tilde{f}'(1.5)$ are identical to machine precision, then Eq. (4) returns $\varepsilon = 0$. This is, however, not an entirely true result: by definition, a computer cannot guarantee a result beyond its machine precision: hence, zero error is fundamentally impossible with machine computing. In reality, the minimum error which a machine can guarantee is its machine precision; thus, in the event that Eq. (4) yields a result of $\varepsilon = 0$, the error result is instead defined as $\varepsilon = 10^{-16}$, the machine precision associated with the double-precision floating-point arithmetic executed in Numerical Python.

With this stipulation in the error calculation established, a discussion of the onset of underflow in the complex-step result is possible according to the above definition. Following results in *Engineering Design Optimization*, the complex-step method was applied for h-values ranging from 10^{-200} to 10^{-321}. The onset of underflow was identified from the relative errors values found for the

complex-step approximation in this range of step sizes; that underflow onset step size was determined to be $h = 10^{-307}$. Step sizes lower than this value yielded increasingly large relative errors while step sizes greater than this value yielded relative errors near machine precision. This behavior is shown graphically in Fig. (2), which shows the relation between ε and h for the complex-step approach in this range of incredibly small h-values.
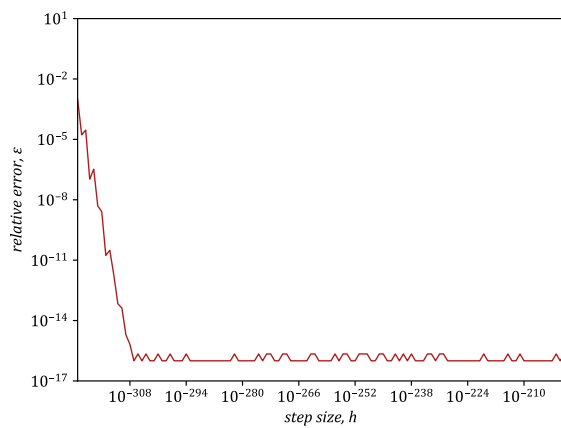

Figure 2. Plot of relative error, ε, of the complex-step derivative approximations for various step sizes, h*. Underflow behavior is observed for* h*-values less than* 10^{-307}*.*

Of course, more interesting than the h-value associated with the onset of underflow is the ideal h-value for the complex-step approach. As defined earlier, the ideal step size for a given method is considered to be the largest step size yielding the smallest relative error. In the case of the complex-step approach, many step sizes yield relative errors equaling machine precision, as seen in Fig. (2). Thus, the ideal step size for the complex-step approach is the largest step size yielding a relative error equal to machine precision. This result is shown in Table (I)(b).

Table I(b)
Step Size Study Results – Cont'd

Method	Ideal h	Minimal ε
complex-step	$1 \cdot 10^{-8}$	$1 \cdot 10^{-16}$

Examining the step size study result shown in Table (I)(b) in light of the results shown in Table (I)(a) reveals that the complex-step method return an approximation with a relative error eight orders of magnitude less than the forward-difference method despite both methods sharing the same ideal step size.

The relative errors of the forward-difference, central-difference, and complex-step results were plotted for h-values ranging from 10^{-25} to 10^{-1} to compare the efficacy of these methods at relatively larger h-values. This plot is shown in Fig. (3).
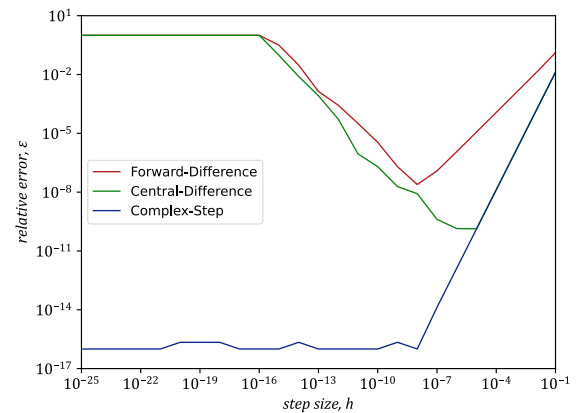

Figure 3. Plot of the relative errors of the forward-difference, central-difference, and complex-step results for relatively large step sizes. The plot is identical to the results shown in Fig. (6.9) of Engineering Design Optimization *for this range of step sizes* [1]*.*

Examination of Fig. (3) reveals some interesting phenomena. Firstly, the results

shown in Table (I) are confirmed graphically: the tabulated ideal step sizes are seen as the minima of the finite-difference curves and as the rightmost minima of the complex-step curve.

Fig. (3) also provides insight into the effect of truncation error on these relative error values. The truncation error in the forward-difference approach is $\mathcal{O}(n)$ while the truncation error in the central-difference and complex-step approach is $\mathcal{O}(n^2)$. Thus for step sizes for which truncation error dominates, it is expected that the central-difference and complex-step methods have identical errors while the error in the forward-difference approach is much larger. Indeed, in examining Fig. (3) it is seen that for h-values greater than $\sim 10^{-6}$, the central-difference and complex-step curves overlap while the forward-difference curves sits above them; this aligns with the expected behavior based on the order of the truncation errors in each approach.

Fig. (3) also reveals the major benefit of the complex-step approach: the lack of error due to subtractive cancellation. In the region where the relative errors in the finite-difference methods begin to rise due to subtractive cancellation dominance, the relative error in the complex-step method continues to fall; this is because, expectedly, the complex-step method does not suffer from subtractive cancellation.

Equations exist to estimate the optimal step size for the forward- and central-difference approaches based on ε_f, the precision of the objective function calculation (in this case, $\varepsilon_f = 10^{-16}$ is machine precision since f is found analytically with double-precision arithmetic). The ideal step size for the forward-difference approach, and its associated error bound, are given identically by Eq. (6). The ideal step size for the central-difference approach is given by Eq. (7) and the associated error bound is given by Eq. (8) [1].

$$\sqrt{\varepsilon_f} = \sqrt{10^{-16}} = 1 \cdot 10^{-8} \qquad (6)$$

$$\sqrt[3]{\varepsilon_f} = \sqrt[3]{10^{-16}} \cong 5 \cdot 10^{-6} \qquad (7)$$

$$\sqrt[3]{\varepsilon_f^2} = \sqrt[3]{(10^{-16})^2} \cong 2 \cdot 10^{-11} \qquad (8)$$

These theoretical ideal step sizes and error bounds are compared to those found numerically for this problem in Table (II).

Table II
Theoretical v. Numerical Comparison for Finding Ideal Step Size in FD Methods

Method	Value	Theoretical	Numerical
forward-	h	$1 \cdot 10^{-8}$	$1 \cdot 10^{-8}$
	ε	$1 \cdot 10^{-8}$	$2 \cdot 10^{-8}$
central-	h	$5 \cdot 10^{-6}$	$3 \cdot 10^{-6}$
	ε	$2 \cdot 10^{-11}$	$7 \cdot 10^{-12}$

Examination of the results shown in Table (II) reveals that the theoretical estimations for the ideal step size and associated minimum relative error in the forward- and central-difference methods are relatively accurate. The estimation of the forward-difference ideal step size matched the numerical value exactly, and the associated estimation of the associated minimal relative error was on the same order of magnitude as the numerical value. The central-difference results were slightly further apart, but the theoretical values still matched the numerical values to within one order of magnitude. When examining the plot in Fig. (3) in light of these theoretical values, no clear discrepancies or anomalies are apparent.

The final part of this investigation involved implementing a forward-mode operator-overloading AD approach to calculating this derivative. Python functions were written to handle the basic arithmetic and trigonometric operations involved in finding $f(x)$ in Eq. (1). These functions took data in the form of a 2-long tuple, list, or NumPy array where the first element was the argument value, as a float data type, and the second element was the argument derivative, also as a float data type. In the case of operators requiring two arguments, such as addition or division, the functions would take two of these 2-long data values. These functions returned a 2-long tuple with the first element being the operation result, as a float data type, and the second being the operation derivative, also as a float data type.

A Python function was then written to express Eq. (1) in terms of these overloaded operators. When this function was passed the tuple `1.5,1` – corresponding to $x = 1.5$ and $\frac{dx}{dx} \triangleq 1$ – the function returns a tuple containing both the function evaluation, $f(1.5)$, and the forward-mode AD evaluation, $f'(1.5)$. Note that f' here is not denoted with a tilde as $\tilde{f}'$ since AD yields an exact solution so long as the overloaded operators all employ analytic derivatives. The resultant AD derivative, expectedly, matched the analytic result defined in Eq. (3). However, due to the previously-discussed limitations of machine computing, its relative error was defined to be $\varepsilon = 10^{-16}$.

Thus in this investigation, three methods yielded exact solutions (to machine precision): analytic differentiation, as defined by Eq. (2), the complex-step method, and a forward-mode operator-overloading AD approach. The results of these three exact methods are summarized in Table (III).

Table III
Summary of Exact Solutions of $f'(x)$

Method	Result, $f'(1.5)$
analytic solution	4.05342789389862
complex-step	4.05342789389862
forward-mode AD	4.05342789389862

II. Problem 5.2

We consider the following function, f, given by Eq. (9), which depends on the eccentric anomaly, E, and the mean anomaly, M, of a Keplerian system. We additionally consider Kepler's equation, expressed in Eq. (10), which relates these two parameters.

$$f(E, M) = E - M \qquad (9)$$
$$E - e \sin E = M \qquad (10)$$

Note that e in Eq. (10) is the eccentricity scalar, which is constant for a given Keplerian system. We seek an analytic form for $\frac{df}{dM}$ derived from the unified derivatives equation.

This approach first requires definition of residual equations representing the system. The first residual, r_1, is found by enforcing $M = \tilde{M}$, where M is the variable in the UDE system and $\tilde{M}$ is the M-value which the system is evaluated at. The last two residuals, r_2 and r_3, are found by simply rearranging Eq. (9) and (10) to get zero on the right-hand side. These residual equations are expressed in Eq. (11) through (13).

$$r_1(M, \tilde{M}) = M - \tilde{M} = 0 \qquad (11)$$
$$r_2(E, M) = E - e \sin E - M = 0 \qquad (12)$$
$$r_3(E, M, f) = f - (E - M) = 0 \qquad (13)$$

We define M to be the input, E to be the state, and f to be the output, such that $\hat{u}$, as defined in Eq. (6.69) of *Engineering Design Optimization*, is given by Eq. (14). Likewise, $\hat{r}$, as defined in Eq. (6.70) of the text, is given by Eq. (15).

$$\hat{u} = [M \quad E \quad f]^{\mathrm{T}} \tag{14}$$

$$\hat{r} = \begin{bmatrix} M - \tilde{M} \\ E - e\sin E - M \\ f - (E - M) \end{bmatrix} = 0 \tag{15}$$

The Jacobian matrix of the UDE system, whose elements are given as $\frac{\partial \hat{r}_i}{\partial \hat{u}_j}$ for row i and column j, is then given by Eq. (16).

$$\frac{\partial r}{\partial u} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 - e\cos E & 0 \\ 1 & -1 & 1 \end{bmatrix} \tag{16}$$

Application of the forward form of the UDE, given by Eq. (6.65) in the text, requires definition of the 3×3 matrix $\frac{du}{dr}$, which for this problem is defined by Eq. (17).

$$\frac{du}{dr} = \begin{bmatrix} \frac{dM}{dM} & \frac{dM}{dr_2} & \frac{dM}{df} \\ \frac{dE}{dM} & \frac{dE}{dr_2} & \frac{dE}{df} \\ \frac{df}{dM} & \frac{df}{dr_2} & \frac{df}{df} \end{bmatrix} \tag{16}$$

Solving for $\frac{df}{dM}$ via the forward form of the UDE requires only the bottom row of this matrix and the bottom row of the 3×3 identity matrix, resulting in the linear system given by Eq. (17) and equivalently in Eq. (18).

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 - e\cos E & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{df}{dM} \\ \frac{df}{dr_2} \\ \frac{df}{df} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{17}$$

$$\therefore \begin{cases} \dfrac{df}{dM} - \dfrac{df}{dr_2} + \dfrac{df}{df} = 0 \\[2mm] (1 - e\cos E)\dfrac{df}{dr_2} - \dfrac{df}{df} = 0 \\[2mm] \dfrac{df}{df} = 1 \end{cases} \tag{18}$$

This is a system of three equations and three unknowns from which $\frac{df}{dM}$ may be solved for trivially. The result is given in Eq. (19).

$$\frac{df}{dM} = \frac{1}{1 - e\cos E} - 1 \tag{19}$$

To verify this result, a central-difference approximation of $\frac{df}{dM}$ was employed, as defined by Eq. (20).

$$\frac{df}{dM} \cong \frac{f(M + h\,; E) - f(M - h\,; E)}{2h} \tag{20}$$

For this comparison, Newton's method was used to solve Kepler's equation for E for a fixed e-value, allowing for computation of $\frac{df}{dM}$, per Eq. (19), for a given M-value and estimation of $\frac{df}{dM}$, per Eq. (20), about that same M-value via perturbation of M. In lieu of a full step size study for the central-difference method, the theoretical ideal step size for a central-difference approach with double-precision arithmetic, expressed in Table (II) to be $h = 5 \cdot 10^{-6}$, was employed. Since this

comparison only seeks to prove that Eq. (19) is the correct analytic form for $\frac{df}{dM}$, no great deal of precision is required in the finite-difference approximation. Additionally, the eccentricity scalar is held constant at an arbitrarily-selected value of $e = 0.8$. The results of this comparison are tabulated in Table (III).

Table III

Analytic v. Numerical Results for $\frac{df}{dM}$

M	Analytic $\frac{df}{dM}$	Numerical $\frac{df}{dM}$
$\frac{\pi}{3}$	-0.16591136609	-0.16591136609
$\frac{\pi}{2}$	-0.3236314525	-0.3236314525
$\frac{3\pi}{4}$	-0.4195765037	-0.4195765037

Examination of Table (III) reveals that the analytic and approximate values of $\frac{df}{dM}$ are identical to eleven digits for three arbitrarily chosen M-values and a constant eccentricity scalar, $e = 0.8$. This is robust evidence that the analytic result for $\frac{df}{dM}$ found via the forward form of the UDE is correct.

III. Problem 5.3

The objective of this problem was to calculate the 10×10 stress Jacobian $\frac{d\sigma}{dA}$ for the ten-bar truss finite-element problem defined in Appendix D.2.2 of *Engineering Design Optimization* [1]. The problem formulation and solution method are not relevant to the solution methodology and as such are not discussed here.

The Jacobian $\frac{d\sigma}{dA}$, whose rows correspond to the gradients of each stress constraint, σ_i, in the finite-element problem definition. This Jacobian was computed in four different manners: a central-difference approach, a complex-step approach, an direct method approach, and an adjoint method approach.

The implementation of each approach follows directly from the text and need not be discussed in detail here. For the central-difference approach, the theoretical ideal step size $h = 5 \cdot 10^{-6}$ (see *Section I*) was employed. For the complex-step approach, the step size $h = 10^{-200}$ was utilized, as recommended by the text [1]. All values were calculated in SI units – no scaling was applied to the problem. For each approach, the stress Jacobian was calculated with the cross-sectional area of each bar being equal to unity. This was an arbitrary selection; since the purpose of this investigation is to compare the results of the different differentiation methods, the selection of design variables is not significant.

The full 10×10 stress Jacobians found from each approach were identical to zero decimal places (since all solutions were on the order of 10^5 or higher, and thus similarity to zero decimal places translates to a considerable degree of identical significant figures). The stress Jacobian found by all four approaches is shown in Fig. (4) with units of $kPa \cdot m^{-1}$.

-862	-2.43	-120	3.63	18.6	-2.43	-173	158	10.3	-6.87
-118	-180	12.4	-31.4	16.4	21.0	17.9	-16.3	-88.7	59.4
114	-2.43	904	3.63	18.6	-2.43	-173	158	10.3	-6.87
-118	21.0	12.4	268	16.4	21.0	17.9	-16.3	-88.7	59.5
102	18.6	-107	-27.8	-142	18.6	-155	141	-78.5	52.6
-118	21.0	12.4	-31.4	16.4	-180	17.9	-16.3	-88.7	59.5
-162	3.44	169	-5.13	-26.3	3.44	-495	-223	-14.5	9.72
-162	3.44	169	-5.13	-26.3	3.44	245	451	-14.5	9.72
16.7	-29.7	-17.5	44.4	-23.3	-29.7	-25.3	23.1	-298	-84.1
16.7	-29.7	-17.5	44.4	-23.3	-29.7	-25.3	23.1	126	200

Figure 4. 10×10 stress Jacobian for the ten-bar truss finite-element as computed by central-difference, complex-step, direct, and adjoint methods with $A = \{1, \ldots, 1\}^{\mathrm{T}}$.

The maximum absolute error of the central-difference and complex-step solutions, in $Pa \cdot m^{-1}$, were found by subtracting the direct method solution (which is assumed to be exact) from the central-difference and complex-step solutions, then taking the maximum element from the resultant matrix. The adjoint method solution is also assumed to be exact, so its error is not considered. The maximum absolute errors for the central-difference and complex-step solutions are shown in Table (IV).

Table IV
Maximum Absolute Errors in Central-Difference and Complex-Step Solutions

Method	Max. Error [$Pa \cdot m^{-1}$]
central-difference	$2.361 \cdot 10^{-4}$
complex-step	$3.201 \cdot 10^{-10}$

Examination of the results shown in Table (IV) reveal that, while there is error in the results found by the central-difference and complex-step methods, this error is miniscule. Considering that the solutions are on the order of 10^5 or greater, errors on the order of 10^{-4} and 10^{-10} are negligible. While these solutions are not exact, they are more than sufficient for the purposes of this problem.

This leads to the question: is there any benefit to pursuing the exact solutions offered by the two implicit analytic methods? Both of these approaches are significantly lengthier and more complicated than the central-difference and complex-step approaches. Table (V) outlines the number of lines of code required to implement each method in Python. As this table demonstrates, the implicit analytic methods require much more coding – and thus much more user time – to implement.

Table V
Code Weight of Each Method

Method	Lines of Code
central-difference	10
complex step	7
implicit, direct	34
implicit, adjoint	36

These methods are also much more complex mathematically, and require a much deeper understanding of the fundamental material to implement correctly. For many users, this is prohibitive simply from a time perspective.

The implicit method also proved to be much more computationally expensive than their simpler counterparts. While all method required additional FEM computations, the implicit methods required the most additional computations; the complex -step approach required the least, with only one additional FEM solution required.

In light of the results shown in Table (IV) and (V), and the computational cost consideration, it is clear that the complex-step method is generally the ideal approach to derivative calculations for the ten-bar truss problem. The maximum error in the complex-step method is very near machine precision, the method requires only seven additional lines of code and no complex mathematical knowledge to implement, and requires only a single additional solution of the FEM system. The central-difference method, while similarly easy to implement, is slightly less accurate and more computationally expensive when compared to the complex-step approach; as such, it has little merit. The implicit direct and adjoint methods are considerably more complicated and computationally expensive while only being marginally more accurate

when compared to the complex-step approach; as such, they ought only to be considered when accuracy is the foremost priority.

IV. Disclosures

As required by the syllabus, a disclosure is included here regarding AI use on this assignment. U-M GPT was used as an aide when writing the Python scripts employed in this assignment. The tool was used primarily in place of a search engine to look up syntax for the various modules used in the scripts. The tool was also used in the debugging process when other measures failed. U-M GPT was also used to aide in the solution of Problem (5.2), though the work is still principally based from the textbook and my own derivation.

V. References

[1] J. R. Martins and A. Ning, Engineering Design Optimization, Cambridge: Cambridge University Press, 2020.