Basic Principle of Counting

How many diff mobile phone nums in SG? (1st digit either 8 or 9)	2 x 10 x 10 x 10 x 10 x 10 x 10 x 10
Num of 7-place license plates if 1st 2 letters are SF?	26 x 10 x 10 x 10 x 10
Num of committee of 1 Chinese, 1 M, 1 I from 6C, 4M, 2I	6 x 4 x 2

Permutations or Arrangement

Form 4-digit num using 1357	4!
Form 4-digit num using 1355	4!/2!
Form 8-digit num using 23445555	8! / (2!4!)
4M, 6W seated in a row if 4M must sit tgt	7!4!
2M, 4W seated in a row if 2M must not sit tgt	6!-5!2! OR 4!2!$\binom{5}{2}$
Arrangements of 123455 if 2 5's not tgt	6!/2!
Select r objs from set of n if order of selection is impt	n x (n-1) x... x (n-r+1) = $\frac{n!}{(n-r)!}$

Combinations

Committee of 2 from 4 men	Num of ways to select 2M if order impt = 4 x 3 Num of ways to select 2M if order not impt = (4x3)/2! — $\frac{4\times3}{2!} = \binom{4}{2}$
Select 6 num from 1,2,..., 45	$\binom{45}{6}$
5-card poker hands	$\binom{52}{5}$
Divide 4 men into 2 teams of 2 each	$\binom{4}{2}$ / 2!
Form committees of 3M, 2W from 6M, 5W	$\binom{6}{3}\binom{5}{2}$
Form committees of 3M, 2W from 6M, 5W if 1 of the M must be in committee	$\binom{5}{2}\binom{5}{2}$
Form committees of 3M, 2W from 6M, 5W if 2 of the W cannot serve tgt	$\binom{6}{3}\binom{2}{0}\binom{3}{2} + \binom{6}{3}\binom{2}{1}\binom{3}{1}$
Form committees of 3M, 2W from 6M, 5W if 2 of the W must be in committee if 1 of them is in	$\binom{6}{3}\binom{2}{0}\binom{3}{2} + \binom{6}{3}\binom{2}{2}\binom{3}{0}$
4M, 3W seated in a row if no 2 women tgt	$\binom{5}{3}$4!3!
4 black, 3 white marble in a row if no 2W are tgt	$\binom{5}{3}$
Expand $(x + y)^4$	$\binom{4}{0}x^0y^4 + \binom{4}{1}x^1y^3 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^3y + \binom{4}{4}x^4y^0$
Num of subsets of a set consisting of n elems	$\sum_{k=0}^{n}\binom{n}{k} = 2^n$
Binomial Theorem. $\boldsymbol{(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}}$ #Let i = k+1 and let i = k	Proof. By MI, n = 1: LHS = x + y. RHS = $\binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0$ = y + x. Suppose result true for n-1. Then $(x + y)^n = (x+y)(x+y)^{n-1} = (x + y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} = \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k} = \sum_{i=1}^{n}\binom{n-1}{i-1}x^i y^{n-i} + \sum_{i=0}^{n-1}\binom{n-1}{i}x^i y^{n-i} = x^n + \sum_{i=1}^{n-1}\binom{n-1}{i-1}x^i y^{n-i} + y^n + \sum_{i=1}^{n-1}\binom{n-1}{i}x^i y^{n-i} = x^n + y^n + \sum_{i=1}^{n-1}\left[\binom{n-1}{i-1} + \binom{n-1}{i}\right]x^i y^{n-i} = x^n + y^n + \sum_{i=1}^{n-1}\binom{n}{i}x^i y^{n-i} = \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}$

Multinomial Coefficients

Divide 9 ppl into 3 grps of size 2,3,4	$\frac{9!}{2!3!4!} = \binom{9}{2}\binom{7}{3}\binom{4}{4}$
Divide 9 ppl into 3 eql grps A,B and C	$\frac{9!}{3!3!3!}$
Divide 9 ppl into 3 eql grps	$(\frac{9!}{2!3!4!})$ / 3!
Expand $(x_1 + x_2 + x_3)^2$	$\binom{2}{2,0,0}x_1^2 + \binom{2}{0,2,0}x_2^2 + \binom{2}{0,0,2}x_3^2 + \binom{2}{1,1,0}x_1x_2 + \binom{2}{1,0,1}x_1x_3 + \binom{2}{0,1,1,}x_2x_3$

Num of Integer solns of Eqns

Num of distinct +ve integer valued vectors (x_1, x_2, x_3, x_4) satisfying $x_1 + x_2 + x_3 + x_4 = 9$	$\binom{9-1}{4-1}$
Divide 10 scouts into 5 teams A, B, C, D, E	5^{10}
Divide 10 scouts into 5 teams A, B, C, D, E if each team must receive 2 scouts	$\binom{10}{2,2,2,2,2} = \frac{10!}{2!2!2!2!2!}$
Divide 10 marbles into 5 boxes A, B, C, D, E	$\binom{10+5-1}{5-1}$
Divide 10 marbles into 5 boxes A, B, C, D, E if each box ≥ 1 marble	$\binom{10-1}{5-1}$

Extra

Committee of 6 ppl from 7M, 8W. Need ≥ 3W, ≥2M.	$\binom{7}{2}\binom{8}{4} + \binom{7}{3}\binom{8}{3}$
Diff linear arrangement of letters A,B,C for which A is before B	3!/2
8 ppl seated in a row if there are 4M, 4W and no 2M or 2W can sit tgt	4!4! x 2
8 ppl seated in a row if there are 4 married couple and each couple must sit tgt	4!2!2!2!2!
5 awards given to class of 30. How many outcomes if student can receive any num of awards?	30^5
5 awards given to class of 30. How many outcomes if student can receive ≤ 1 award?	30 x 29 x 28 x 27 x 26
Num of vectors $(x_1, ..., x_n)$ s.t. x_i is either 0 or 1 and $\boldsymbol{\sum_{i=1}^{n} x_i} \geq k$	For $\sum_{i=1}^{n} x_i = k$, vector must have k 1's and n-k 0's, num of such vector = $\binom{n}{k}$ For $\sum_{i=1}^{n} x_i \geq k$, $\binom{n}{k} + \binom{n}{k+1} + ... + \binom{n}{n}$
Num of vectors $(x_1, ..., x_n)$ s.t. $1 \leq x_i \leq n$ and $x_1 < x_2 < ... < x_k$	$\binom{n}{k}$ (choose k distinct integers for x_i and then only 1 way to arrange it)
Select committee of any size and chairperson for committee from n ppl	SUM (Select committee of size k and chairperson from this k ppl) = select chairperson from n ppl, remaining n-1 ppl either in or not in committee — $\sum_{k=1}^{n} k\binom{n}{k} = n \times 2^{n-1}$
Select committee of any size and chairperson and secretary (can same person as chairperson) for committee from n ppl	$\sum_{k=1}^{n}\binom{n}{k}k^2 = n \times 2^{n-1} + n(n-1)2^{n-2} = n(n+1)2^{n-2}$
Committee has 8 members. How many way to choose p, t, s if member can take one role only.	8 x 7 x 6
Num of way to choose p, t, s if A and B must serve tgt if either one selected. Member can take one role only	6 x 5 x 4 + 6 x 3! OR $\binom{6}{3}$3! + 6 x 3!
Arrangements of A, B, C, D, E, F, F if A must be btw 2 F's	5!
1st experiment can result in m diff outcomes. Suppose 1st experiment has outcome i, 2nd experiment can result in n_i diff outcomes. Total outcomes of 2 experiments?	$\sum_{i=1}^{m} n_i$
10 W, 12M. If 5M, 5W chosen and paired off, how many diff pairs	$\binom{12}{5}\binom{10}{5}$5!
20 ppl shake hands. How many handshakes	$\binom{20}{2}$
Start from point A to go to point B. Only can move up or right. How many paths from A to B	$\binom{7}{3}$ (total 7 moves of up and right, 3 right)
Go from A to B but must pass through certain point at 2U2R from A	$\binom{4}{2}\binom{3}{1}$
Prove $\boldsymbol{\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + ... + \binom{n}{r}\binom{m}{0}}$ Let m = r = n and $\boldsymbol{\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n}\binom{n}{k}^2}$	There are $\binom{n+m}{r}$ grps of size r. There are $\binom{n}{i}\binom{m}{r-i}$ grps of size r consisting of i men and r-i women
8 teachers divided among 4 schools, how many divisions possible?	4^8
What if each sch need 2 teachers?	$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$
3 weightlifters from US, 4 from Russia, 2 from China, 1 from Canada. If only considering countries of lifters, how many diff outcome possible.	$\frac{10!}{3!4!2!}$
How many outcomes if US has 1 in top 3 and 2 in bottom 3?	$\binom{3}{1}\binom{3}{2}\frac{7!}{4!2!}$

Sample Space

Diff types of sample spaces	S = {abcd: a,b,c,d = 0,1,...,9} (4D draw)	S = {x: 0≤ x < ∞} (lifespan)	S = {(H,H), (T,H), (T,H), (T,T)} (coin tossed 2 times)

DeMorgan's Laws Proof. i. $(\bigcup_{i=1}^{n} \boldsymbol{E_i})^C = \bigcap_{i=1}^{n} \boldsymbol{E_i^C}$ Proof. Let x ∈ $(\bigcup_{i=1}^{n} \boldsymbol{E_i})^C$ ⇒ x ∉ $\bigcup_{i=1}^{n} \boldsymbol{E_i}$ ⇒ x ∉ E_1 and x ∉ E_2 and ... x ∉ E_n ⇒ x ∈ $\boldsymbol{E_1^C}$ and x ∈ $\boldsymbol{E_2^C}$ and ... x ∈ $\boldsymbol{E_n^C}$ ⇒ x ∈ $\bigcap_{i=1}^{n} \boldsymbol{E_i^C}$ ⇒ $(\bigcup_{i=1}^{n} \boldsymbol{E_i})^C \subset \bigcap_{i=1}^{n} \boldsymbol{E_i^C}$ Let x ∈ $\bigcap_{i=1}^{n} \boldsymbol{E_i^C}$ ⇒ x ∈ $\boldsymbol{E_1^C}$ and x ∈ $\boldsymbol{E_2^C}$ and ... x ∈ $\boldsymbol{E_n^C}$ ⇒ x ∉ E_1 and x ∉ E_2 and ... x ∉ E_n ⇒ x ∉ $\bigcup_{i=1}^{n} \boldsymbol{E_i}$ ⇒ x ∈ $(\bigcup_{i=1}^{n} \boldsymbol{E_i})^C$ ⇒ $(\bigcup_{i=1}^{n} \boldsymbol{E_i})^C \supset \bigcap_{i=1}^{n} \boldsymbol{E_i^C}$. Thus $(\bigcup_{i=1}^{n} \boldsymbol{E_i})^C = \bigcap_{i=1}^{n} \boldsymbol{E_i^C}$	ii. $(\bigcap_{i=1}^{n} E_i)^C = \bigcup_{i=1}^{n} E_i^C$ Proof. Using 1st law of DeMorgan, $\left(\bigcup_{i=1}^{n} E_i^C\right)^C = \bigcap_{i=1}^{n} \left(E_i^C\right)^C = \bigcap_{i=1}^{n} E_i$ (since $(E^C)^C$ = E) Thus, $(\bigcap_{i=1}^{n} E_i)^C = \bigcup_{i=1}^{n} E_i^C$

Some simple propositions

60% of students wear neither ring nor necklace. 20% wear ring and 30% wear necklace. Prob that a student wearing ring or necklace, P(R∪N) Prob that student wearing ring and necklace, P(RN)	Let R = {ring}, N = {necklace}. P((R∪N)^C) = 0.6, P(R) = 0.2, P(R) = 0.3 P(R∪N) = 1 - P((R∪N)^C) = 0.4 P(R∪N) = P(R) + P(N) - P(RN). P(RN) = 0.2 + 0.3 - 0.4
1. P(∅) = 0	Proof. Let E_1 = S, E_i = ∅ for i > 1, then S = $\bigcup_{i=1}^{\infty} E_i$ and E_i are mutually exclusive. Using axiom 3, $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$. P(S) = P(S) + P($E_2$) + P($E_3$) + ... = P(S) + P(∅) + P(∅) + ... ⇒ P(∅) = 0
2. $P(\bigcup_{i=1}^{n} \boldsymbol{E_i}) = \sum_{i=1}^{n} \boldsymbol{P(E_i)}$ (when sample space is finite)	Proof. Let E_i = ∅ fpr i > n and apply axiom 3
7. P(E^C) = 1 - P(E)	Proof. 1 = P(S) = P(E∪E^C) = P(E) + P(E^C)
8. If E ⊂ F, then P(E) ≤ P(F)	Proof. Let F = E∪E^CF. P(F) = P(E) + P(E^CF). Since P(E^CF) ≥ 0,...
9. P(E∪F) = P(E) + P(F) - P(E∩F)	Proof. Let E∪F = E∪E^CF. P(E∪F) = P(E) + P(E^CF) – (1). F = EF ∪ E^CF. P(F) = P(EF) + P(E^CF) – (2) Substitute 2 into 1, P(E∪F) = P(E) + P(F) - P(EF)
11. Inclusion-exclusion identity Proof P(E_1∪E_2∪...∪E_n) = $\sum_{i=1}^{n} \boldsymbol{P(E_i)}$ $-\sum_{i_1<i_2} \boldsymbol{P(E_{i_1}E_{i_2})}$ + ... + $(-1)^{r+1}\sum_{i_1<i_2<\ldots<i_r} \boldsymbol{P(E_{i_1}E_{i_2}\ldots E_{i_r})}$ + ... + $(-1)^{n+1}$P(E_1E_2...E_n)	If outcome is not in any E_i, then its prob don't contribute to either side of eqn Suppose outcome is in m of events E_i, m > 0, let prob of outcome = w, then i. outcome is in E_1∪E_2∪...∪E_n and w will be counted once on LHS ii. outcome is in m of E_i and w will be counted $\binom{m}{1}$ times in $\sum_{i=1}^{n} P(E_i)$ iii. outcome is in $\binom{m}{2}$ subsets of $E_{i_1}E_{i_2}$ and w counted $\binom{m}{2}$ times in $\sum_{i_1<i_2} P(E_{i_1}E_{i_2})$ and so on... w = $\binom{m}{1}$w – $\binom{m}{2}$w + $\binom{m}{3}$w – ... ± $\binom{m}{m}$w ⇒ $\binom{m}{0}$ = 1 = $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \ldots \pm \binom{m}{m}$ $\sum_{i=0}^{m} \binom{m}{i} (-1)^i = 0$ (always true, by using binomial theorem and let x = -1, y = 1)
12i. $P(\bigcup_{i=1}^{n} \boldsymbol{E_i}) \le \sum_{i=1}^{n} \boldsymbol{P(E_i)}$ Proof. Note P(E_i) = $P(\boldsymbol{E_1^C\ldots E_{i-1}^C E_i}) + P((\boldsymbol{E_1^C\ldots E_{i-1}^C})^C\boldsymbol{E_i})$ $P(\boldsymbol{E_1^C\ldots E_{i-1}^C E_i})$ = P(E_i) – $P(\boldsymbol{E_i}(\bigcup_{j<i} \boldsymbol{E_j}))$ – sub this result in	$\bigcup_{i=1}^{n} E_i = E_1 \cup E_1^C E_2 \cup E_1^C E_2^C E_3 \cup \ldots \cup E_1^C E_2^C \ldots E_{n-1}^C E_n$ $P(\bigcup_{i=1}^{n} E_i) = P(E_1) + P(E_1^C E_2) + P(E_1^C E_2^C E_3) + \ldots + P(E_1^C E_2^C \ldots E_{n-1}^C E_n)$ $= P(E_1) + \sum_{i=2}^{n} P(E_1^C \ldots E_{i-1}^C E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{i=2}^{n} P(E_i \cup_{j<i} E_j)$ Since $\sum_{i=2}^{n} P(E_i \cup_{j<i} E_j) \ge 0$, thus $P(\bigcup_{i=1}^{n} E_i) \le \sum_{i=1}^{n} P(E_i)$
12ii. $P(\bigcup_{i=1}^{n} \boldsymbol{E_i}) \ge \sum_{i=1}^{n} \boldsymbol{P(E_i)} - \sum_{j<i} \boldsymbol{P(E_iE_j)}$ Proof. Note $\sum_{i=2}^{n}\sum_{j=1}^{i-1} (\boldsymbol{E_iE_j})$ = [P(E_2E_1) + P(E_3E_1) + P(E_3E_2) + ... + P(E_nE_1) + P(E_nE_2) + ... + P(E_nE_{n-1})] = $\sum_{j<i} \boldsymbol{P(E_iE_j)}$	$P(E_i \cup_{j<i} E_j)$ = P(E_iE_1 ∪ E_iE_2 ∪ ... ∪ E_iE_{i-1}) = $P(\bigcup_{j<i} E_iE_j) \le \sum_{j<i} P(E_iE_j) = \sum_{j=1}^{i-1} (E_iE_j)$ (from 9i.) $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{i=2}^{n} P(E_i \cup_{j<i} E_j)$ (from proof of 9i.) $\ge \sum_{i=1}^{n} P(E_i) - \sum_{i=2}^{n}\sum_{j=1}^{i-1} (E_iE_j) = \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_iE_j)$

Sample Spaces having equally likely outcomes

Prob sum of 2 dice = 4	3/36	Select 8 chips from 10 defective and 90 non-defective. Prob ≤ 1 chip is defective	$\frac{\binom{80}{0}\binom{90}{8}}{\binom{100}{8}} + \frac{\binom{10}{1}\binom{90}{7}}{\binom{100}{8}}$

Form committee of 5 from 6M and 9W. P(A), Prob John, 2M and 2W are selected? P(B), Prob 3M (no John) and 2W selected?	P(A) = $\frac{\binom{5}{2}\binom{9}{2}}{\binom{15}{5}}$ P(B) = $\frac{\binom{5}{3}\binom{9}{2}}{\binom{15}{5}}$
Suppose 3 red and 2 blue balls arranged s.t. all 5! possible orderings are equally likely. If we now record result by listing colors of balls, show that all possible results remain equally likely.	total arrrangements of rrrbb = $\frac{5!}{3!2!}$, prob = 1/$\frac{5!}{3!2!}$ = $\frac{3!2!}{5!}$

A box contains n balls of which 1 is special. If k balls are withdrawn 1 at a time, prob that special ball is chosen?	Mtd 1: prob = $\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(n-k)!(k-1)!}}{\frac{n!}{(n-k)!k!}} = \frac{k}{n}$	Mtd 2: Let A_i = event that special ball = i[th] ball chosen, i = 1,2,...,k Since any of n balls is equally likely to be i[th] ball chosen, P(A_i) = 1/n OR Total num of outcomes = n(n-1)(n-2)...(n-k+1) = $\frac{n!}{(n-k)!}$ Num of outcomes of A_i = (n-1)(n-2)...(n-i+1)(1)(n-i)...(n-k+1) = $\frac{(n-1)!}{(n-k)!}$ P(A_i) = [$\frac{(n-1)!}{(n-k)!}$] / [$\frac{n!}{(n-k)!}$] = $\frac{(n-1)!}{n!}$ = 1/n P({special ball chosen}) = $P(\bigcup_{i=1}^{k} A_i) = \sum_{i=1}^{k} P(A_i)$ = k/n

P(full house) = $\frac{\boldsymbol{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}}{\boldsymbol{\binom{52}{5}}}$	P(straight) = $\frac{10(4^5-4)}{\binom{52}{5}}$ (A2345, 23456,...,10JQKA) x (4 suits^5 - 4 outcomes with all same suit(straight flush))	P(4 of a kind) = $\frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$	P(flush) = $\frac{\binom{4}{1}\left[\binom{13}{5}-10\right]}{\binom{52}{5}}$ 4 suits x (any 5 cards - 10 possible straight)

P(2 pairs) = $\frac{\boldsymbol{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}}{\boldsymbol{\binom{52}{5}}} = \frac{\boldsymbol{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{44}{1}}}{\boldsymbol{\binom{52}{5}}(2!)}$ (since 2 grps)	P(1 pair) = $\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$	P(3 of a kind) = $\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$

birthday problem. If n ppl are in a room, prob that no 2 of them have the same b.d.	$\frac{(365)(364)(363)\ldots(365-n+1)}{365^n}$. If n ≥ 23, prob is less than 1/2
Cards are turned up 1 at a time until 1st ace appears. Is the next card more likely to be ace of spades or 2 clubs?	P(ace) = $\frac{51!}{52!} = \frac{1}{52}$ (arrange the 51 cards 1st: 51!. Then put ace spades after 1st ace: only 1 way) P(2 clubs) = 1/52 also

Football team has 20 offensive and 20 defensive players. Players are paired in grps of 2.	P(no O-D pair) = $\frac{\frac{\binom{20}{2,2,\ldots,2}}{10!}\cdot\frac{\binom{20}{2,2,\ldots,2}}{10!}}{\frac{\binom{40}{2,2,\ldots,2}}{20!}}$	P(2 O-D pair) = $\frac{\binom{20}{2}\binom{20}{2}2!\left[\frac{\binom{18}{2,2,\ldots,2}}{9!}\right]^2}{\frac{\binom{40}{2,2,\ldots,2}}{20!}}$, choose 2 O, 2 D, then 2! to arrange these grps

Suppose N men randomly select a hat.	Prob that none select their own hat? Let E_i = event i[th] man select own hat, i = 1,2,...,N P(≥ 1 man select own hat) = $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2})$ + ... + $(-1)^{r+1}\sum_{i_1<i_2<\ldots<i_r} P(E_{i_1}E_{i_2}\ldots E_{i_r})$ + ... + $(-1)^{N+1}$P(E_1E_2...E_N) (point 11) For event $E_{i_1}E_{i_2}\ldots E_{i_r}$, let the r men ($i_1$, i_2,..., i_r) select own hats first, other (N-r) men have (N-r)! ways of selecting remaining (N-r) hats	Prob exactly k men select their own hat? num of ways only k men select own hats = N choose k men * [P(N - k don't select own hat) * total num of ways N - k men select hat] = $\binom{N}{k}$[1 - 1 + $\frac{1}{2!}$ - $\frac{1}{3!}$ + ... + $(-1)^{N-k}$ $\frac{1}{(N-k)!}$](N - k)!

$P(E_{i_1}E_{i_2}\dots E_{i_r}) = \frac{(N-r)!}{N!}$

$\sum_{i_1<i_2<\dots<i_r} P(E_{i_1}E_{i_2}\dots E_{i_r}) = \binom{N}{r}\frac{(N-r)!}{N!} = \frac{1}{n!}$, where $\binom{N}{r}$ = num of terms

$P\left(\bigcup_{i=1}^{N} E_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{N+1}\frac{1}{N!}$

$P\left(\left(\bigcup_{i=1}^{N} E_i\right)^C\right) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^{N}\frac{1}{N!} \to e^{-1} \approx 0.37$ as $N \to \infty$

P(only k men select own hat) =

$\frac{\binom{N}{k}(N-k)![1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^{N-k}}{(N-k)!}]}{N!} = \frac{\frac{N!}{k!(N-k)!}(N-k)![1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^{N-k}}{(N-k)!}]}{N!} = \frac{1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^{N-k}}{(N-k)!}}{k!} \to e^{-1}/k!$ as $N \to \infty$

Prob as a cts set fn

Incr seq: $E_1 \subset E_2 \subset E_3 \subset \dots \subset E_n \subset E_{n+1}\dots$ Hence $\lim_{n\to\infty} \boldsymbol{E_n} = \bigcup_{i=1}^{\infty} \boldsymbol{E_i}$

Decr seq: $E_1 \supset E_2 \supset\dots\supset E_n \supset E_{n+1} \supset\dots$ And $\lim_{n\to\infty} \boldsymbol{E_n} = \bigcap_{i=1}^{\infty} \boldsymbol{E_i}$

If $\{E_n, n \geq 1\}$ is either incr or decr seq, then $\lim_{n\to\infty} \boldsymbol{P(E_n)} = P(\lim_{n\to\infty} \boldsymbol{E_n})$

Proof. Suppose $\{E_n, n \geq 1\}$ is incr seq. Let $F_1 = E_1$, $F_2 = E_2E_1^C\dots$

$F_n = E_n\left(\bigcup_{i=1}^{n-1} E_i\right)^C = E_nE_{n-1}^C$ since $E_{n-1} = \bigcup_{i=1}^{n-1} E_i$

Note F_n are mutually exclusive events and $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i \ \forall\, n \geq 1$

$P(\bigcup_{i=1}^{\infty} F_i) = P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(F_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} F_i) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} E_i) = \lim_{n\to\infty} P(E_n)$

Extra

Deck of 20 cards numbered 1,2,3,..., 20 shuffled. Card selected 1 at a time without replacement until all 20 selected.

P(3rd card is 9) = 1/20 OR 19!/20!

P(1st 3 cards all odd num) = $\frac{10}{20} \times \frac{9}{19} \times \frac{8}{18}$ OR $\frac{\binom{10}{3}3!17!}{20!}$

P(last 3 cards all odd num) = $\frac{10}{20} \times \frac{9}{19} \times \frac{8}{18}$

Roll 5 dice.

P(no two same)	P(1 pair)	P(2 pair)	P(3 same)	P(full house)	P(4 same)	P(5 same)
$= \frac{\binom{6}{5}5!}{6^5}$	$= \frac{6\binom{5}{3}5!/2!}{6^5}$	$= \frac{\binom{6}{2}\binom{4}{1}\frac{5!}{2!2!}}{6^5}$	$= \frac{6\binom{5}{2}5!/3!}{6^5}$	$= \frac{6\binom{5}{1}\frac{5!}{2!3!}}{6^5}$	$= \frac{6\binom{5}{1}5!/4!}{6^5}$	$= \frac{\binom{6}{1}}{6^5}$

P(neither you nor dealer has blackjack). Let A = {you have blackjack}, B = {dealer has blackjack}. Ans = 1- P(A∪B)

P(A∪B) = P(A) + P(B) - P(A∩B) = 2 x $\frac{\binom{4}{1}\binom{16}{1}}{\binom{52}{2}} - \frac{\binom{4}{1}\binom{16}{1}\binom{3}{1}\binom{15}{1}}{\binom{52}{2}\binom{50}{2}}$

P(2nd die higher value than 1st)

1 = P(1st is higher) + P(2nd is higher) + P(same)
1 = 2 x P(2nd is higher) + 6/36
P(2nd is higher) = 5/12

Woman has n keys, of which 1 will open door. Tries opening door, discarding those that don't work, P(open on kth try)

P(open on kth try) = $\frac{(n-1)(n-2)\dots(n-k+1)}{n(n-1)\dots(n-k+1)} = \frac{1}{n}$

P(open on kth try without discarding key) = $\frac{(n-1)^{k-1}}{n^k}$

Num of ppl in room for P(at least 2 of them have same birthday month) to be ≥ 0.5

P(all months diff) = $\frac{12*11*\dots*(13-n)}{12*12*\dots*12}$. When n = 5, this prob < 0.5

Closet contains 10 pairs of shoes. If 8 shoes randomly selected,

P(no complete pair) = $\frac{\binom{10}{8}\binom{2}{1}^8}{\binom{20}{8}}$

P(exactly 1 complete pair) = $\frac{\binom{10}{1}\binom{2}{2}\binom{9}{6}\binom{2}{1}^6}{\binom{20}{8}}$

$(\bigcap_1^{\infty} \boldsymbol{E_i}) \cup \boldsymbol{F} = \bigcap_1^{\infty}(\boldsymbol{E_i} \cup \boldsymbol{F})$

Let x ∈ $(\bigcap_1^{\infty} \boldsymbol{E_i}) \cup \boldsymbol{F}$ ⇒ x ∈ $(\bigcap_1^{\infty} \boldsymbol{E_i})$ or x ∈ $\boldsymbol{F}$ ⇒ x ∈ E_1 and x ∈ E_2 and .... or x ∈ F ⇒ x ∈ E_1 or x ∈ F and x ∈ E_2 or x ∈ F and .... ⇒ x ∈ $\bigcap_1^{\infty}(\boldsymbol{E_i} \cup \boldsymbol{F})$

Let x ∈ $(\bigcap_1^{\infty} \boldsymbol{E_i} \cup \boldsymbol{F})$ ⇒ x ∈ $E_1 \cup \boldsymbol{F}$ and x ∈ $E_2 \cup \boldsymbol{F}$ and .... ⇒ x ∈ E_1 and x ∈ E_2 and .... or x ∈ F ⇒ x ∈ $\bigcap_1^{\infty}(\boldsymbol{E_i})$ or x ∈ F ⇒ x ∈ $(\bigcap_1^{\infty} \boldsymbol{E_i}) \cup \boldsymbol{F}$

$(\bigcup_1^{\infty} E_i)F = \bigcup_1^{\infty} E_i F$ can be proven similarly

$P(EF^C \cup E^CF) = P(EF^C) + P(E^CF)$ (disjoint events). $P(E) = P(EF^C \cup EF) = P(EF^C) + P(EF)$ and $P(F) = P(E^CF) + P(EF)$
$P(EF^C \cup E^CF) = P(E) - P(EF) + P(F) - P(EF)$

Consider matching problem, Let A_N = num of ways N men can select hats so no man select own
Let B_N = num of ways N men can select among N hats that does not contain the hat of 1 of these men

$A_N = (N-1)B_{N-1} \Rightarrow B_{N-2} = A_{N-1}/(N-2)$
$A_N = (N-1)[A_{N-2} + (N-2)B_{N-2}]$ (N-1: num of ways 1st man select a hat not his own, A_{N-2}: extra man choose hat of 1st man, N-2: num of ways extra man choose hat that is not the 1st man hat
$A_N = (N-1)[A_{N-2} + A_{N-1}]$

Parallel sys with n components. P(component work) = 0.5. P(component 1 works|sys working)

P(component 1 works|sys working) = $\frac{P(\text{component 1 working})}{1-P(\text{all component not working})} = \frac{1/2}{1-(1/2)^n}$

4 freshman boys, 6 freshman girls, 6 sophomore boys. How many sophomore girls for sex and class to be indep when student selected at random?

P(Boy, F) = $\frac{4}{16+x}$, P(Boy) = $\frac{10}{16+x}$, P(F) = $\frac{10}{16+x}$. For independence, P(Boy, F) = P(Boy)P(F)
$\frac{4}{16+x} = \frac{10}{16+x}\frac{10}{16+x}$. 4x = 36 and x = 9.

Stock price moves up by 1 unit with prob p, down 1 unit with prob 1-p. P(original price after 2 days)
P(up by 1 after 3 days)
P(up on 1st day|up by 1 after 3 days)

P(original price after 2 days) = p(1-p) + (1-p)p = 2p(1-p).
P(up by 1 after 3 days) = $\binom{3}{2}p^2(1-p) = 3p^2(1-p)$
P(up on 1st day|up by 1 after 3 days) = $\frac{P(\text{up on 1st day,original after 2nd and 3rd day})}{P(\text{up by 1 after 3 days})} = \frac{p[2p(1-p)]}{3p^2(1-p)} = \frac{2}{3}$

P(heads) = p. P(HHHH), P(THHH), P(THHH before HHHH)

P(HHHH) = p^4. P(THHH) = $(1-p)p^3$. P(THHH before HHHH) = 1 - P(HHHH) (since HHHH can only appear in 1st 4 flips) = $1 - p^4$

Suppose E and F are mutually exclusive events. If indep trials are performed, show E will occur before F with prob P(E)/[P(E)+P(F)]

P(E before F) = P(E before F|E 1st)P(E 1st) + P(E before F|F 1st)P(F 1st) + P(E before F|E and F not 1st)(1-P(E 1st)-P(F 1st) = P(E 1st) + P(E before F)(1-P(E 1st)-P(F 1st)) = P(E) + P(E b f|1-P(E)-P(F). So P(E b F) = P(E)/[P(E)+P(F)]

Die A has 4r, 2w faces. Die B has 2r, 4w faces. Fair coin is flipped. If head, die A is used, else die B. Show P(red) = 0.5. P(r on 3rd|1st 2 r). P(A|1st 2 r)

P(red) = (1/2)(4/6) + (1/2)(2/6) = 1/2. P(r on 3rd|1st 2 r) = $\frac{P(rrr)}{P(rr)} = \frac{(1/2)(4/6)^3 + (1/2)(2/6)^3}{(1/2)(4/6)^2 + (1/2)(2/6)^2} = \frac{3}{5}$

P(A|rr) = $\frac{P(rr|A)P(A)}{P(rr)} = \frac{(4/6)^3(1/2)}{(1/2)(4/6)^2 + (1/2)(2/6)^2} = \frac{4}{5}$

Let S = {1,2,..., n}. Suppose A and B are independently, equally likely to be any of the 2^n subsets (including null set and S itself) of S.
Show P(A⊂B) = $(3/4)^n$. Show P(AB = ∅) = $(3/4)^n$

P(A⊂B) = $\sum_{i=0}^{n} P(A \subset B \mid |B| = i)P(|B| = i) = \sum_{i=0}^{n} \frac{2^i}{2^n}\frac{\binom{n}{i}}{2^n} = \frac{1}{4^n}\sum_{i=0}^{n}\binom{n}{i}2^i1^{n-i} = \frac{1}{4^n}(2+1)^n = \left(\frac{3}{4}\right)^n$

P(AB = ∅) = P(A⊂B^C) = $\left(\frac{3}{4}\right)^n$ since B^C also equally likely to be any of the subsets.

Person is declared guilty if at least 2 out of 3 judges vote guilty. Suppose person is in fact guilty, each judge will vote guilty w prob 0.7. If person actually innocent, prob vote guilty = 0.2. If 70% of people are guilty, P(judge 3 vote guilty|j1 and 2 vote guilty)
P(j3 vote guilty|j1 and 2: 1 guilty and not guilty vote)
P(j3 vote guilty|j1 and 2 both vote not guilty)

P(judge 3 vote guilty|j1 and 2 vote guilty) = $\frac{0.7(0.7)^3+0.3(0.2)^3}{0.7(0.7)^2+0.3(0.2)^2} = \frac{97}{142}$

P(j3 vote guilty|j1 and 2: 1 guilty and not guilty vote) = $\frac{0.7(0.7)^2 2(0.3)+0.3(0.2)^2 2(0.8)}{0.7(2)(0.7)(0.3)+0.3(2)(0.2)(0.8)} = \frac{15}{26}$

P(j3 vote guilty|j1 and 2 both vote not guilty) = $\frac{0.7(0.7)(0.3)^2+0.3(0.2)(0.8)^2}{0.7(0.3)^2+0.3(0.8)^2} = \frac{33}{102}$

$P(E_1) = P(E_2) = 0.7(0.7) + 0.3(0.2) = 0.55$
$P(E_1E_2) = 0.7(0.7)^2 + 0.3(0.2)^2 = 0.355 \neq P(E_1)P(E_2)$. Similarly, all 3 events are not indep.
$P(E_1E_2E_3|\text{guilty}) = P(E_1|\text{guilty})P(E_2|\text{guilty})P(E_3|\text{guilty})$

Let E_i be event judge i cast guilty. Are events indep? Are they conditionally indep?	$P(E_1E_2 \mid guilty) = P(E_1 \mid guilty)P(E_2 \mid guilty)$... show for E_1E_3 and E_2E_3, so conditionally indep.		
Prove if $E_1, E_2, ..., E_n$ are indep, then $P(E_1 \cup E_2 \cup_{...} \cup E_n) = 1 - \prod_{i=1}^{n}[\mathbf{1} - \boldsymbol{P(E_i)}]$	$P(E_1 \cup E_2 \cup_{...} \cup E_n) = 1 - P(\bigcap_1^n E_i^C) = 1 - \prod_{i=1}^{n}[1 - P(E_i)]$		
Fair coin tossed 2 times. A = {1st toss heads}. B = {2nd toss head}. C = {both toss on same side}. Show A, B, C are pairwise indep, but not indep.	P(A) = P(B) = P(C) = 1/2. P(AB) = P(AC) = P(BC) = 1/4. So P(AB) = P(A)P(B)... However, P(ABC) = 1/4 ≠ P(A)P(B)P(C). So pairwise indep, but not indep		
If $0 \le a_i \le 1$, i = 1,2,..., show $\sum_{i=1}^{\infty}[\boldsymbol{a_i} \prod_{j=1}^{i-1}(\mathbf{1} - \boldsymbol{a_j})] + \prod_{i=1}^{\infty}(\mathbf{1} - \boldsymbol{a_i}) = 1$	Suppose want to calculate prob of flipping coin until head appears. $a_i \prod_{j=1}^{i-1}(1 - a_j)$ = P(1st head on i^{th} flip). $\prod_{i=1}^{\infty}(1 - a_i)$ = P(all tails).		
Weather tmr will be same as weather tdy with prob p. If weather is dry on Jan 1, show P_n = prob dry n days later satisfies $P_n = (2p-1)P_{n-1} + (1-p)$, $P_0 = 1$. Prove $P_n = 1/2 + 1/2(2p-1)^n$	$P_n = pP_{n-1} + (1-p)(1-P_{n-1}) = (2p-1)P_{n-1} + (1-p)$. Using MI, suppose P_{n-1} is true. $P_n = (2p-1)P_{n-1} + (1-p) = (2p-1)[1/2 + (1/2)(2p-1)^{n-1}] + 1-p = (2p-1)/2 + (2p-1)^n/2 + 1-p = 1/2 + (1/2)(2p-1)^n$		
Chessboard has 64 squares. Prob placing 8 rooks wont all be in same row or col.	$\frac{64*49*36*25*16*9*4*1}{64*63*62*61*60*59*58*57}$	P(blackjack)	$\frac{4*16*2}{52*51}$ (order matters) OR $\frac{\binom{4}{1}\binom{16}{1}}{\binom{52}{2}}$ (order dont matter)
2 symmetric dice both have 2 sides R, 2B, 1Y, 1W. P(both same color)	$P(RR) + P(BB) + P(YY) + P(WW) = \frac{2}{6}\frac{2}{6} + \frac{2}{6}\frac{2}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6}$		
20 families. 4 have 1 child. 8 have 2 child. 5 have 3 child. 2 have 4 child. 1 has 5 child. Total 48 children	P(child chosen come from family with i children)?, i = 1,2,3,4,5. Let B = no of children in child family. P(B = 1) = 4/48, P(B = 2) = 16/48, P(B = 3) = 15/48, P(B = 4) = 8/48, P(B = 5) = 5/48		
2 sch chess club contain 8, 9 players. 4 members from each club randomly chosen to join contest. Random player from 1 team paired with those from other team. Suppose R and E are in diff sch.	P(R and E paired) = $\frac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} = \frac{1}{18}$. P(R and E chosen but not paired) = $\frac{\binom{7}{3}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} - \frac{1}{18} = \frac{1}{6}$ P(exactly one of R and E chosen) = $\frac{\binom{7}{3}\binom{8}{4} + \binom{7}{4}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}}$		
Urn contains 3R, 7B balls. A and B select balls consecutively w/o replacement until R ball chosen	P(A select R) = $\frac{3*9! + 7*6*3*7! + 7*6*5*4*3*5! + 7*6*5*4*3*2*3*3!}{10!}$		
Forest has 20 elk, of which 5 are captured, tagged and then released. 4 elk then captured.	P(2 of these 4 are tagged) = $\frac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}}$	Bridge: 13 cards, Yarborough: bridge with no cards higher than 9 (exclude ace)	$\frac{\binom{20}{0}\binom{32}{13}}{\binom{52}{13}}$
Teacher gives class 10 qns. Test contain 5 of those qns. If student knows how to ans 7 qns.	P(all 5 qns correct) = $\frac{\binom{7}{5}\binom{3}{0}}{\binom{10}{5}}$. P(at least 4 correct) = $\frac{\binom{7}{5}\binom{3}{0}}{\binom{10}{5}} + \frac{\binom{7}{4}\binom{3}{1}}{\binom{10}{5}}$	Die roll 4 times. P(6 appears at least once)?	$1 - \frac{5^4}{6^4}$
n socks, 3R in drawer. Value of n if when 2 socks chosen, P(RR) = .5	$\frac{\binom{3}{2}}{\binom{n}{2}}$ = .5. n = 4	5 hotels. If 3 ppl check into hotels in a day, P(all diff hotel)?	$\frac{5*4*3}{5*5*5}$
A B C D E arranged in linear order.	P(1 person btw A and B) = $\frac{\binom{3}{1}3!2!}{5!}$. P(2 ppl btw) = $\frac{\binom{3}{2}2!2!2!}{5!}$. P(3 ppl btw) = 3!2!	6M, 6W divided into 2 groups of 6 each. P(both grp same num of men)?	$\frac{\binom{6}{3}\binom{6}{3}}{\binom{12}{6}}$
P(bridge is void of at least 1 suit). S: void of spade, H, D, C...	$P(S \cup H \cup D \cup C)$ = [P(S) + P(H) + P(D) + P(C)] - [P(SH) + P(SD) + P(SC) + P(HD) + P(HC) + P(DC)] + [P(SHD) + P(SHC) + P(SDC) + P(HDC)] - P(SHDC) = $\frac{\binom{4}{1}\binom{39}{13}}{\binom{52}{13}} - \frac{\binom{4}{2}\binom{26}{13}}{\binom{52}{13}} + \frac{\binom{4}{3}\binom{13}{13}}{\binom{52}{13}} - 0$		

Conditional Prob and Reduced Sample Space

Draw cards one at a time without replacement from deck. Let A = {1^{st} card = ace}, B = {2^{nd} card = ace}	P(A) = 4/52. P(B | A) = 3/51. P(B) = 4/52	
Toss fair die twice. Given 1st die is 5, what is conditional prob that sum of 2 dice is even?	P(E | F) = 3/6 (since 1^{st} already 5, 6 outcomes for 2^{nd} die) P(E) = 18/36 (consider whole sample space for unconditional prob)	
Toss fair die twice. Given 1^{st} die is num < 3, prob sum of 2 dice is > 7	P(E | F) = 1/12 (only 1 possible outcome; (2,6))/ (1st die 1: 6 outcomes + 1st die 2: 6 outcomes)	
Take 8 balls from 4 black and 6 white balls sequentially. Given 5 of the 8 chosen are white, prob that 1st ball chosen is white?	P(1^{st} ball W | 5 of 8 balls are W) = 5/8 (since 1st ball equally likely to be any of 8 chosen balls)	OR Let B = {1st ball chosen W}. B_5 = {5 W balls chosen}. P(B) = $\frac{6}{10}$. $P(B_5) = \frac{\binom{6}{5}\binom{4}{3}}{\binom{10}{8}}$. $P(B_5 \mid B) = \frac{\binom{6-1}{4}\binom{4}{3}}{\binom{10-1}{7}}$. $P(B \mid B_5) = \frac{P(BB_5)}{P(B_5)} = \frac{P(B)P(B_5 \mid B)}{P(B_5)} = \frac{5}{8}$
P(first 2 cards are aces). Let A = {1st card ace}, B = {2nd card ace}	P(BA) = P(A)P(B | A) = $\frac{4}{52} \times \frac{3}{51}$	
Divide 52 cards into 4 piles of 13 each. Prob each pile has exactly one ace? Note $P(E_4) = P(E_1E_2E_3E_4)$	Let E_1 = {ace spades in any pile}, E_2 = {ace spades, ace hearts in diff pile}, E_3 = {ace spades, hearts diamonds diff pile}, E_4 = {all aces diff pile} $P(E_1) = 1$. $P(E_2 \mid E_1) = \frac{39}{12+39}$. $P(E_3 \mid E_1E_2) = \frac{26}{26+12+12}$. $P(E_4 \mid E_1E_2E_3) = \frac{13}{12+12+12+13}$ $P(E_1E_2E_3E_4) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1E_2)P(E_4 \mid E_1E_2E_3) = 0.105$	OR $P(E_4) = \frac{4!\binom{48}{12,12,12,12}/4!}{\binom{52}{13,13,13,13}/4!} = 0.105$
Multiplication rule: $P(E_1E_2...E_n) = P(E_1)\,P(E_2 \mid E_1)P(E_3 \mid E_1E_2)...P(E_n \mid E_1E_2...E_{n-1})$	Proof. RHS = $P(E_1) \times \frac{P(E_1E_2)}{P(E_1)} \times \frac{P(E_1E_2E_3)}{P(E_1E_2)} \times ... \times \frac{P(E_1E_2...E_n)}{P(E_1E_2...E_{n-1})} = P(E_1E_2...E_n)$	

Thrm of Total Prob and Bayes' Thrm

Blood Test is 95% TP. 1% FP. 6% of pop has disease	P(test +ve) = 0.06 x 0.95 + (1-0.06) x 0.01 = 0.0664	P(disease | test +ve) = (0.06 x 0.95) / 0.0664 = 0.06 P(no disease | test -ve) = (1-0.06) x (1-0.01) / (1-0.664) = 0.94
Urn 1 initially has n red balls and urn 2 has n blue balls. Remove 1 ball from urn 1, take 1 ball from urn 2 and put in urn 1. Repeat until all molecules removed from both urn. Let R = {last ball removed from urn 1 is red}	Let 1 of the n red balls be a special one. F = {s ball is final one selected}. N_i = {s ball not i^{th} ball removed} $P(N_1) = \frac{n-1}{n}$. $P(N_2 \mid N_1) = \frac{n-1}{n}$... $P(F \mid N_1N_2...N_n) = \frac{1}{n}$. Note $F \subset N_1N_2...N_n$. $P(F) = P(N_1N_2...N_nF)$ = $P(N_1)P(N_2 \mid N_1)...\ P(N_n \mid N_1N_2...N_{n-1})P(F \mid N_1N_2...N_n)$ (multiplication rule) = $\frac{n-1}{n} \times \frac{n-1}{n} ... \times \frac{1}{n} = \left(1 - \frac{1}{n}\right)^n \frac{1}{n}$ Since special ball can be any of n red balls, P(R) = P(F) x n = $\left(1 - \frac{1}{n}\right)^n \to e^{-1}$ as $n \to \infty$	
If urn 1 now has r_1 red balls and b_1 blue balls, urn 2 has r_2 red balls and b_2 blue balls. Find P(R)	Let 1 of the balls in urn 1 be special one. Now $P(F) = P(N_1N_2...N_{r_2+b_2}F) = P(N_1)P(N_2 \mid N_1)...\ P(N_{r_2+b_2} \mid N_1N_2...N_{r_2+b_2-1})P(F \mid N_1N_2...\ N_{r_2+b_2}) = \frac{r_1+b_1-1}{r_1+b_1} \times ... \times \frac{r_1+b_1-1}{r_1+b_1} \times \frac{1}{r_1+b_1} = \left(1 - \frac{1}{r_1+b_1}\right)^{r_2+b_2} \frac{1}{r_1+b_1}$ Let O = {last ball removed is one of the ball originally in urn 1} $P(O) = \left(1 - \frac{1}{r_1+b_1}\right)^{r_2+b_2} \frac{1}{r_1+b_1} \times (r_1 + b_1) = \left(1 - \frac{1}{r_1+b_1}\right)^{r_2+b_2}$ $P(R) = P(R \mid O)P(O) + P(R \mid O^C)P(O^C) = \frac{r_1}{r_1+b_1} \times \left(1 - \frac{1}{r_1+b_1}\right)^{r_2+b_2} + \frac{r_2}{r_2+b_2} \times \left[1 - \left(1 - \frac{1}{r_1+b_1}\right)^{r_2+b_2}\right]$	
Plant A, B, C produce 20%, 30%, 50% of RAM chips. % of defective chips by A, B and C are 1%, 2%, 4% respectively.	P(defective) = 0.2(0.01) + 0.3(0.02) + 0.5(0.04) = 0.028 P(plant A | defective) = 0.2(0.01)/0.028 = 0.071	

Couple with 2 children. Suppose we encounter mother walking with one of her child. If child is a girl, prob that both are girls?
So actually prob depends on how mom choose child to walk with her
If mom only chose elder child, then $P(G|G_1B_2) = 1$, $P(G|B_1G_2) = 0$ and $P(G_1G_2|G) = \frac{1}{2}$
If mom only choose girl, then $P(G|G_1B_2) = 1$, $P(G|B_1G_2) = 1$ and $P(G_1G_2|G) = \frac{1}{3}$

Let G_1 = {1st child girl}, G_2 = {2nd child girl}, G = {child seen with mom is girl}...for boys also...
$P(G_1G_2|G) = \frac{P(G_1G_2G)}{P(G)} = \frac{P(G_1G_2)}{P(G)}$ since $G_1G_2 \subset G$
$P(G) = 0.25 * 1 + 0.25 * P(G|G_1B_2) + 0.25 * P(G|B_1G_2) + 0.25 * 0$
$P(G_1G_2|G) = \frac{0.25}{0.25+0.25P(G|G_1B_2)+0.25P(G|B_1G_2)} = \frac{1}{1+P(G|G_1B_2)+P(G|B_1G_2)}$

Conditioning formula: $P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$

Proof. $E = EF^C \cup EF$. $P(E) = P(EF^C) + P(EF) = P(E|F^C)P(F^C) + P(E|F)P(F)$

Independent Events

Draw cards from 52 cards without replacement. C = {2nd card is ace}. A = {1st card ace}
C = {2nd card is ace}. A = {1st card red color}
C = {2nd card is ace}. B = {1st card diamond}

$P(C) = 4/52$. $P(C|A) = 3/51$. So A and C not indep
$P(C) = 4/52$. $P(C|A) = \frac{2\times3+24\times4}{26\times51} = P(C)$. A and C indep
$P(C) = 4/52$. $P(C|B) = \frac{1\times3+12\times4}{13\times51} = P(C)$. B and C indep

Toss 2 fair dice. A = {1st die 3}, B = {sum is 5}, C = {sum is 8}, D = {sum is 7}
$P(D) = 6/36$. $P(D|A) = 1/6$. A and D indep

$P(B) = 4/36 = 1/9$. $P(B|A) = 1/6$. A and B not indep
$P(C) = 5/36$. $P(C|A) = 1/6$. A and C not indep

E, F and G are indep ⇒ E and $F \cup G$ are indep

E.g. $P(E(F \cup G)) = P(EF \cup EG) = P(EF) + P(EG) - P(EFG) = P(E)P(F) + P(E)P(G) - P(E)P(F)P(G)$ (∵ indep) $= P(E)[P(F) + P(G) - P(F)P(G)] = P(E)P(F \cup G)$

Seq of Bernoulli trials(Binomial). Infinite seq of indep. trials are performed. Success has prob p, failure: 1-p.

P(at least 1 success occur in 1st n trials)= $1 - (1-p)^n$
P(exactly k success occur in 1st n trials) = $\binom{n}{k}p^k(1-p)^{n-k}$
P(all trials are success) = pppp... = $\lim_{n\to\infty} p^n = \begin{cases} 0, 0 \le p \le 1 \\ 1, p = 1 \end{cases}$

Sys contains 3 components arranged in parallel. If component i functions with prob p_i, i = 1,2,3

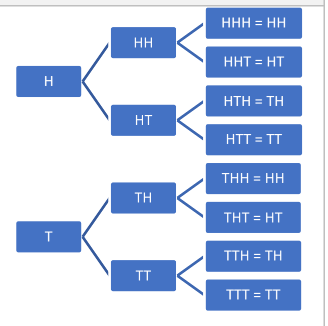
P(sys functions) = 1 - P(sys don't fn) (since need ≥ 1 component functioning) = $1 - (1-p_1)(1-p_2)(1-p_3)$
P(sys functions in series) = $p_1p_2p_3$

Box contains 5 balls, 2 are W. A draw first, then B with replacement. Winner is 1st to draw W

P(A wins) = $\frac{2}{5} + \left(\frac{3}{5}\right)^2\frac{2}{5} + \left(\frac{3}{5}\right)^4\frac{2}{5} + \ldots = \frac{2}{5}[1 + \left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^4 + \ldots] = \frac{2}{5}[1/(1-\left(\frac{3}{5}\right)^2)] = \frac{5}{8}$

P(A wins) = $\frac{2}{5} + \left(\frac{3}{5}\right)^2$P(A wins) (recursive solution)
P(A wins)$[1-\left(\frac{3}{5}\right)^2] = \frac{2}{5}$

Player A picks one of the following: HHH, HHT, HTH, HTT, THH, TTH, THT, TTT
Player B then picks one of the remaining 7 patters. A fair coin is tossed until either A or B pattern appears
If A picks HHH, B THH. P(A wins) = 1/8 (only 1 route). A picks HHT, B picks THH. P(A wins) = 1/4 (HHH or HHT)
A picks HTH, B picks HHT, P(B wins) = 2/3. P(B wins|HH) = .5P(B wins|HH) + .5(1). P(B wins|HH) = 1 – (1).
P(B wins|HT) = .5(0) + .5P(B wins|TT). P(B wins|TT) = 2P(B wins|HT) – (2)
P(B wins|TH) = .5P(B wins|HH) + .5P(B wins|HT) = .5 + .5P(B wins|HT) (from (1)). 2P(B wins|TH) = 1 + P(B wins|HT) – (3)
P(B wins|TT) = .5P(B wins|TH) + .5P(B wins|TT). P(B wins|TT) = P(B wins|TH) – (4)
Sub (4) into (3): 2P(B wins|TT) = 1 + P(B wins|HT) - (5)
Sub (2) into (5): 4P(B wins|HT) = 1 + P(B wins|HT). P(B wins) = 1/3 – (6)
Sub (6) into (5): P(B wins|TT) = 2/3 – (7). Sub (7) into (4): P(B wins|TH) = 2/3
P(B wins) = .25P(B wins|HH) + .25P(B wins|HT) + .25P(B wins|TH) + .25P(B wins|TT) = 2/3



Problem of the points - Fermat and Pascal. Independent trials, with Success prob p, failure: 1-p. Probability n success before m failures?

Pascal: Let $P_{n,m}$ = P(n success before m failures)
$P_{n,m} = pP_{n-1,m} + (1-p)P_{n,m-1}$, n ≥ 1, m ≥ 1
$P_{n,0} = 0$. $P_{0,m} = 1$. $P_{1,1} = pP_{0,1} + (1-p)P_{1,0} = p$

Femat: winner must emerge after m+n-1 trials
For n success before m failures, need at least n success in n+m-1 trials
$P_{n,m} = \sum_{k=n}^{m+n-1}\binom{m+n-1}{k}p^k(1-p)^{m+n-1-k}$

Gambler's ruin problem. A with \$M, B with \$N. P(success) = p, P(failure) = 1-p.
Success: B gives A \$1. Failure: A gives B \$1. Game stops when either A or B wins

Let P_i = P(someone wins all M+N starting with \$i). $P_0 = 0$, $P_{M+N} = 1$
$P_i = 0.5P_{i-1} + 0.5P_{i+1} \Rightarrow P_{i+1} = 2P_i - P_{i-1}$
$P_2 = 2P_1 - P_0 = 2P_1$. $P_3 = 2P_2 - P_1 = 3P_1$. $P_i = 2P_{i-1} - P_{i-1} = iP_1$
So $P_{M+N} = (M+N)P_1 = 1 \Rightarrow P_1 = 1/(M+N)$
P(A wins) = $P_M = MP_1 = M/(M+N)$. P(B wins) = $P_N = NP_1 = N/(M+N)$

E and F are independent $P(E|F) = P(E)$

$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(E)P(F)}{P(F)}$ if E and F are indep = P(E)

E and F indep ⇒ E and F^C indep

$E = EF \cup EF^C$. $P(E) = P(EF) + P(EF^C) = P(E)P(F) + P(EF^C)$ (since E, F indep) $P(EF^C) = P(E) - P(E)P(F) = P(E)(1-P(F)) = P(E)P(F^C)$

P(E|F) is a prob

Seq of Bernoulli trials. P(success) = p, P(failure) = 1-p. P(n consecutive success before m consecutive failures)?
Let E = {n consecutive successes before m consecutive failures}
H = {first trial is success}. F = {trials 2 to n all success}
G = {trials 2 to m all failures}

$P(E) = P(H)P(E|H) + P(H^C)P(E|H^C) = pP(E|H) + (1-p)P(E|H^C)$
$P(E|H) = P(E|FH)P(F|H) + P(E|F^CH)P(F^C|H)$
$= 1 * p^{n-1} + P(E|H^C)(1-p^{n-1})$ (since once fail, restart from beginning)
$P(E|H^C) = P(E|GH^C)P(G|H^C) + P(E|G^CH^C)P(G^C|H^C)$
$= 0(1-p)^{m-1} + P(E|H)[1-(1-p)^{m-1}]$ –solve simultaneous eqn and sub into P(E)

k+1 coins in box. P(i^{th} coin = heads) = i/k, i = 0,1,...,k. Coin randomly selected from box and then repeatedly flipper. P(n+1 flip = head|first n flips all head)
Let c_i = {i^{th} coin selected}. F_n = {1st n flips all heads}. H = {n+1 flip = head}

If k = 3, $\int_0^1 x^{n+1}dx \approx \frac{1}{3}\left(\frac{1}{3}\right)^{n+1} + \frac{1}{3}\left(\frac{2}{3}\right)^{n+1} + \frac{1}{3}\left(\frac{3}{3}\right)^{n+1}$ (use rectangle to estimate area) = $\frac{1}{3}\sum_{i=0}^{3}\left(\frac{i}{3}\right)^{n+1}$

$P(H|F_n) = \sum_{i=0}^{k} P(H|F_nc_i)P(c_i|F_n)$
$P(H|F_nc_i) = P(H|c_i) = i/k$
$P(c_i|F_n) = \frac{P(c_iF_n)}{P(F_n)} = \frac{P(F_n|c_i)P(c_i)}{\sum_{j=0}^{k}P(F_n|c_j)P(c_j)} = \frac{\left(\frac{i}{k}\right)^n\frac{1}{k+1}}{\sum_{j=0}^{k}\left(\frac{j}{k}\right)^n\frac{1}{k+1}} = \frac{\left(\frac{i}{k}\right)^n}{\sum_{j=0}^{k}\left(\frac{j}{k}\right)^n}$
$P(H|F_n) = \sum_{i=0}^{k}\frac{\frac{i}{k}\left(\frac{i}{k}\right)^n}{\sum_{j=0}^{k}\left(\frac{j}{k}\right)^n} = \frac{\sum_{i=0}^{k}\left(\frac{i}{k}\right)^{n+1}}{\sum_{j=0}^{k}\left(\frac{j}{k}\right)^n} \approx \frac{\int_0^1 x^{n+1}dx}{\int_0^1 x^n dx} = \frac{1/(n+2)}{1/(n+1)} = \frac{(n+1)}{(n+2)}$

Extra

Deck of cards numbered 1,2,..., 20 selected 1 at a time w/o replacement until all 20 selected.

P(3rd, 4th card are odd|1st, 2nd card are odd) = $\frac{8}{18}\frac{7}{17} = \frac{28}{153}$
P(1st, 2nd card are odd|3rd, 4th card are odd) = $\frac{28}{153}$
P(card numbered 10 is among last 5 cards|1st, 2nd card are odd) = $\frac{5*17!}{18!} = \frac{5}{18}$

Urn A contains 2W, 4R balls; urn B contains 8W, 4R balls; urn C contains 1W, 3R balls. 1 ball is selected from each urn.

$P(A = W|2W) = \frac{P(A=W,2W)}{P(2W)} = \frac{P(A=W,B=W,C=R) + P(A=W,B=R,C=W)}{P(2W)} = \frac{\frac{2\,8\,3}{6124}+\frac{2\,4\,1}{6124}}{\frac{2\,8\,3}{6124}+\frac{2\,4\,1}{6124}+\frac{4\,8\,1}{6124}} = \frac{7}{11}$

Divide 52 cards into 4 piles of 13 each. Prob each pile has exactly one ace?Let E_i = event i^{th} hand has exactly 1 ace. Find $p = P(E_1E_2E_3E_4)$

$P(E_1) = \binom{4}{1}\binom{48}{12}/\binom{52}{13}$, $P(E_2|E_1) = \binom{3}{1}\binom{36}{12}/\binom{39}{13}$, $P(E_3|E_1E_2) = \binom{2}{1}\binom{24}{12}/\binom{26}{13}$,
$P(E_4|E_1E_2E_3) = 1$. $p = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3)$ (by multiplication rule)

48% of the women and 37% of the men of a class remained nonsmokers for at least 1 year. These ppl attended a success party. If 62% of original class were male,

P(women|attending party) = $\frac{P(women\ and\ attending\ party)}{P(attending\ party)} = \frac{0.48*0.38}{0.37*0.62+0.48*0.38} \approx 0.443$
P(attending party) = 0.48*0.38 + 0.37*0.62 = 0.4118

Urn I contains 2W, 4R balls, urn II contains 1W, 1R ball. Ball randomly chosen from urn I and put into urn II and ball then randomly selected from urn II.

P(ball selected from urn II is white) = P(W|W transferred) + P(W|W not transferred) = $\frac{2}{6}\frac{2}{3} + \frac{4}{6}\frac{1}{3} = \frac{4}{9}$
P(W transferred|W selected) = $\frac{P(W\ selected\ and\ W\ transferred)}{P(W)} = \frac{\frac{2}{6}*\frac{2}{3}}{4/9}$ = 1/2

2 balls are either painted black or gold with prob = 0.5 and placed in a box.

P(both gold|at least 1 gold) = $\frac{1/4}{1-1/4} = \frac{1}{3}$
If box tips over and 1 gold ball falls out. P(both gold) = 1/2 since no other info about ball in box.

5% of M and 0.25% of W are color blind. Assume there are equal num of M and F. What is there were twice as many M as F

P(M|color blind) = $\frac{P(C|M)P(M)}{P(C|M)P(M)+P(C|F)P(F)} = \frac{0.05p}{0.05p+0.0025(1-p)}$
When p = 1/2, P(M|C) = 0.9524. If p = 2/3, P(M|C) = 0.9756

Deck of cards turned over 1 at a time until 1st ace appears. Let E = 1st ace is 20th card, A = {21st card is ace spades}, D = {21st card is ace}, C = {2 clubs in 1st 20 card}, B = {21st card is 2 clubs}

P(A|E) = P(D|E)P(A|DE) + P(D^C|E)P(A|D^CE) = $\frac{3}{32}\frac{1}{4} + \frac{29}{32}(0) = \frac{3}{128}$
P(B|E) = P(C|E)P(B|CE) + P(C^C|E)P(B|C^CE) = $\frac{19}{48}(0) + \frac{29}{48}\frac{1}{32} = \frac{29}{1536}$

3 coins in a box. One is 2-headed coin; another is fair coin; third is biased coin with P(head) = 75%.

P(2-headed coin|head) = $\frac{\frac{1}{3}(1)}{\frac{1}{3}(1)+\frac{1}{3}\frac{1}{2}+\frac{1}{3}\frac{3}{4}} = \frac{4}{9}$

10 coins, coin i P(head) = i/10, i = 1,2,..., 10.

P(5th coin|head) = $\frac{\frac{1}{10}\frac{5}{10}}{\sum_{i=1}^{10}\frac{1}{10}\frac{i}{10}} = \frac{5}{11}$

2 identical cabinets has 2 drawers. A contains a silver coin in each drawer, B contains silver coin in 1 drawer and gold coin in the other.

P(other drawer has silver coin|drawer has silver coin) = $\frac{1/2}{1(1/2)+1/2(1/2)}$ = 2/3

Prob that good, ave, bad risk person would be in accident is 0.05, 0.15, 0.30. If 20% of pop is good, 50% is ave, 30% is bad, what proportion of ppl have accident?

P(accident) = 0.2*0.05 + 0.5*0.15 + 0.3*0.3 = 0.175
P(good|no accident) = $\frac{0.95*0.2}{1-0.175}$. P(ave|no accident) = $\frac{0.85*0.5}{0.825}$

Day	P(mail|accepted)	P(mail|rejected)
M	.15	.05
T	.20	.1
W	.25	.1
Th	.15	.15
F	.1	.20

P(accepted) = .6

P(M) = P(M|A)P(A) + P(M|R)P(R) = .15(.6) + .05(.4) = .11
P(T|M^C) = $\frac{.2(.6)+.1(.4)}{1-.11} = \frac{16}{89}$
P(A|$M^CT^CW^C$) = $\frac{P(M^CT^CW^C|A)P(A)}{P(M^CT^CW^C)} = \frac{(1-.15-.2-.25)(.6)}{(1-.15-.2-.25)(0.6)+(1-.1-.1-.05)(0.4)} = \frac{12}{27}$
P(A|Th) = $\frac{P(Th|A)P(A)}{P(Th)} = \frac{.15(.6)}{.15(.6)+(.15)(.4)} = \frac{3}{5}$
P(A|no mail) = $\frac{P(no\ mail|A)P(A)}{P(no\ mail)} = \frac{(1-0.15-0.2-0.25-0.15-0.1)(0.6)}{(1-0.15-0.2-0.25-0.15-0.1)(0.6)+(1-0.5-0.1-0.1-0.15-0.2)(0.4)} = \frac{9}{25}$

Box 1 contains 2W, 3B balls. Box 2 contains 4W, 3B balls. Fair die is tossed, if num is 1 or 2, ball is randomly selected from box 1. Else, ball selected from box 2.

P(W) = $\frac{2}{6}\frac{2}{5} + \frac{4}{6}\frac{4}{7} = \frac{18}{35}$. P(box 2|W) = $\frac{P(W|box\ 2)P(box\ 2)}{P(W)} = \frac{(4/7)(4/6)}{18/35} = \frac{20}{27}$
Same experiment carried out twice. P(both same color) = P(WW) + P(BB) = $\frac{18}{35}\frac{18}{35} + \frac{17}{35}\frac{17}{35} = \frac{613}{1225}$

2 fair dice are rolled. P(one 6|dice lands on diff num)?

P(one 6|dice lands on diff num) = 10/(36-6)

P(1st die 6|sum = 7) = (1/36)/(6/36). P(1st die 6|sum = 8) = (1/36)/(5/36). P(1st die 6|sum = 9) = (1/36)/(4/36). P(1st die 6|sum = 10) = (1/36)/(3/36). P(1st die 6|sum = 11) = (1/36)/(2/36). P(1st die 6|sum = 12) = (1/36)/(1/36)

2 cards chosen without replacement. B = {both cards aces}. A_S = {ace spades chosen}. A = {at least 1 ace chosen}

P(B|A_S) = $\frac{\binom{1}{1}\binom{3}{1}/\binom{52}{2}}{\binom{1}{1}\binom{51}{1}/\binom{52}{2}} = \frac{1}{17}$. P(B|A) = $\frac{P(B)}{P(A)}$ (since B subset of A) = $\frac{\binom{4}{2}/\binom{52}{2}}{[\binom{4}{1}\binom{48}{1}+\binom{4}{2}]/\binom{52}{2}} = \frac{1}{33}$

Urn contains 5W, 7B balls. Each time ball is selected, its color is noted and replaced in the urn along with 2 other balls of the same color.

P(1st 2 balls are B and next 2 W) = $\frac{7}{12} * \frac{9}{14} * \frac{5}{16} * \frac{7}{18}$. P(of the 1st 4 balls selected, only 2 are black) = P(WWBB, WBWB, WBBW, BWWB, BWBW, BBWW)

36% own dog. 22% of families that own dog also cat. 30% own cat. P(D) = 0.36. P(C|D) = 0.22. P(C) = 0.3. P(C) = P(CD) + P(CD^C) = P(C|D)P(D) + P(C|D^C)P(D^C). P(C|D^C) = (0.3-0.22*0.36)/0.64 = 0.345

P(random selection own dog and cat) = P(CD) = P(C|D)P(D) = 0.22*0.36 = 0.0792
P(D|C) = $\frac{P(DC)}{P(C)} = \frac{0.0792}{P(CD)+P(C|D^C)P(D^C)} = \frac{0.0792}{0.0792+0.345*0.64}$ = 0.264

46% independents. 30% liberals. 24% conservative. In an election, 35% of I, 62% of L, and 58% of C voted.

P(V) = 0.35*0.46 + 0.62*0.3 + 0.58*0.24 = 0.4862. P(I|V) = 0.35*0.46/0.4862 = 0.331. P(L|V) = 0.3*0.62/0.4862 = 0.383. P(C|V) =0.24*0.58/0.4862 = 0.286.

Let E = {1st ace is 20th card}. A = {21st card is ace of spade}. D = {20th card is ace of spades}. C = {2 of club among 1st 20 cards}. B = {21st card is 2 club}

P(A|E) = P(A|DE)P(D|E) + P(A|D^CE)P(D^C|E) = 0 + $\frac{1}{32}\frac{3}{4} = \frac{3}{128}$
P(B|E) = P(B|CE)P(C|E) + P(B|C^CE)P(C^C|E) = 0*$\frac{19}{48} + \frac{1}{32}\frac{29}{48}$ (29 position to place 2 clubs)

B hits target with prob p_1. D hits target with prob p_2. Suppose they shoot simultaneously at same target. Assume both shots are indep

P(both hit|at least 1 hit) = $\frac{P(both\ hit)}{P(at\ least\ 1\ hit)} = \frac{p1p2}{1-(1-p1)(1-p2)}$
P(B hit|at least 1 hit) = $\frac{p1}{1-(1-p1)(1-p2)}$

Current score: B (87wins, 72lost), G (86, 73), D (86, 73)
G has 3 more games to play against D.
B has 3 more games to play againts P. (P cannot win division). Given prob of winning a game is .5. If 2 teams tie for 1st place, have additional game with same prob.

P(B wins 3) = 1/8. P(B wins 2) = 3(1/8) = 3/8. P(B wins 1) = 3/8. P(B wins 0) = 1/8
P(G wins 3) = 1/8 = P(G wins 0). P(G wins 2) = 3/8 = P(D wins 1)....
P(B wins division) = P(B wins 3) + P(B wins 2)[P(G wins 3)*.5 + P(G wins 2) + P(G wins 1) + P(G wins 0)*.5] + P(B wins 1)[P(G wins 2)*.5 + P(G wins 1)*.5] = 1/8 + 3/8*[1/16+3/8+3/8+1/16] + 3/8[3/16+3/16] = 38/64

Council contains 7 members, of which 3 members are in a steering committee. A legislation is first revied by committee members, and then by whole council if at least 2 of 3 committee approve. At council, at least 4 of 7 approve for legislation to pass. Prob of approval is p. P(given steering committee member vote is decisive)? Decisive if person vote is reversed, legislature reversed.

Given member	Other 2 member of steering committee	Other 4 council member
for	both for	3 against
for	1 for, 1 against	at least 2 for
against	1 for, 1 against	at least 2 for
against	both for	3 against

P(decisive) = (p^3)4p(1-p)^3 + [p*2p(1-p)][6p^2(1-p)^2 + 4p^3(1-p) + p^4] + [(1-p)*2p(1-p)][6p^2(1-p)^2 + 4p^3(1-p) + p^4] + [(1-p)p^2][4p(1-p)^3]

Urn contains 12 balls, of which 4 are W. A, B, C draw from urn successively. Winner is 1st 1 to draw W ball.

If ball is put back after drawing. P(A wins) = $\frac{4}{12} + \frac{8}{12}\frac{8}{12}\frac{8}{12}$P(A wins). P(A wins) = $\frac{4/12}{[1-(8/12)^3]}$.
If balls not replaced after drawing. P(A wins) = $\frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6}$

Urn contains n W and m B balls. Balls withdrawn 1 at a time until only those of the same color are left. Show P(W left) is n/(n+m)

Balls left all W if last ball drawn is W. Any of the n+m balls could be the last ball, so P = n/(n+m)

Pond contains R,B,G fish. r R, b B, g G fishes. Fish randomly removed. P(R fish 1st to be completely removed).

P(R) = P(RBG) + P(RGB)
P(RBG) + P(G last)P(RBG|G last) = $\frac{g}{r+b+g} * \frac{r}{r+b}$. P(RGB) = P(B last)P(RGB|B last) = $\frac{b}{r+b+g} * \frac{g}{r+g}$

If E_1 and E_2 are indep. then they are conditionally independent given F. Prove of give counterexample.

E_1 and E_2 indep ⇒? $P(E_1E_2|F) = P(E_1|F)P(E_2|F)$
Suppose fair die is tossed twice, Let E_1 = {1st toss 1}, E_2 = {2nd toss 2}, F = {sum = 4}
$P(E_1E_2|F) = 0$. But $P(E_1|F)$ and $P(E_2|F) > 0$. So statement is false

Applications of r.v.

John claims to have extrasensory perception. Card is randomly drawn from 4 cards. Test is done 10 times. If Joh gets 7 out of 10 correct, does he have ESP?
Let X = num of times out of 10 guess correctly

X is binomial dist with n = 10, p = 0.25
$P(X \geq 7) = \sum_{k=7}^{10}\binom{10}{k}0.25^k0.75^{10-k} = 0.0035$
Very unlikely, so John most likely has ESP

Channel transmits digits 0 and 1. P(digit incorrectly received) = 0.2. To reduce error, 00000 is transmitted instead of 0 and 11111 instead of 1. Receivers uses 'majority' decoding, e.g. 1 = 11111 -> 10111 = 1 (true), 1 = 11111 -> 10100 = 0 (false). Let X = num of digits out of 5 that are received incorrectly.

X is binomial dist with n = 5 and p = 0.2
P(digit wrong when decoded) = $P(X \geq 3) = \sum_{k=3}^{5}\binom{5}{k}0.2^k0.8^{5-k} = 0.058$
If only 1 digit was sent P(wrong) = 0.2, so improved accuracy

Coupon problem. There are N distinct types of coupons and selection is random. Let T = num of coupons that need to be collect until 1 obtain complete set of each type. Let A_j = event no type j coupon is obtained in first n coupons collected. P(T = n)?
Note P(T>n) = 1 if n < N
For 1 ≤ n ≤ N, $\sum_{i=1}^{N-1}(-1)^{i+1}\binom{N}{i}\left(\frac{N-i}{N}\right)^n = 1$
Now, let D_n = num of distinct types of coupons obtained in 1st n selections. $P(D_n = k)$?. Let A = event coupon is one of these k types; B = event all k types appear at least once

Consider $P(T>n-1) = P(T\geq n) = P(T=n) + P(T>n)$. $P(T=n) = P(T>n-1) - P(T>n)$
By inclusion-exclusion identity, $P(T>n) = P(\cup_{j=1}^{N} A_j) = \sum_j P(A_j) - \sum_{j_1<j_2} P(A_{j_1}A_{j_2}) + \ldots + (-1)^{k+1}\sum_{j_1<j_2<\ldots<j_k} P(A_{j_1}A_{j j_2}\ldots A_{j_k}) + \ldots + (-1)^{N+1}P(A_1A_2\ldots A_N)$
$P(A_j) = \left(\frac{N-1}{N}\right)^n$ (i.e. none of the n coupons are j)
$P(A_{j_1}A_{j_2}) = \left(\frac{N-2}{N}\right)^n$, $P(A_{j_1}A_{j j_2}\ldots A_{j_k}) = \left(\frac{N-k}{N}\right)^n$, $^1P(A_1A_2\ldots A_N) = 0$
$P(T>n) = N\left(\frac{N-1}{N}\right)^n - \binom{N}{2}\left(\frac{N-2}{N}\right)^n + \ldots + (-1)^N\binom{N}{N-1}\left(\frac{1}{N}\right)^n + 0 = \sum_{i=1}^{N-1}(-1)^{i+1}\binom{N}{i}\left(\frac{N-i}{N}\right)^n$
$P(A) = \left(\frac{k}{N}\right)^n$. $P(B|A) = P(T \leq n$; from k types instead of N) $= 1 - \sum_{i=1}^{k-1}(-1)^{i+1}\binom{k}{i}\left(\frac{k-i}{N}\right)^n$
$P(D_n = k) = \binom{N}{k}P(AB) = \binom{N}{k}\left(\frac{k}{N}\right)^n\left[1 - \sum_{i=1}^{k-1}(-1)^{i+1}\binom{k}{i}\left(\frac{k-i}{N}\right)^n\right]$

Pmf & Cdf

Find cdf of discrete r.v.

a	1	2	4
P(X=a)	0.5	0.25	0.25

$$F(a) = \begin{cases} 0, & a < 1 \\ 0.5, & 1 \leq a < 2 \\ 0.75, & 2 \leq a < 4 \\ 1, & a \geq 4 \end{cases}$$

$E(X) = \sum_a aP(X=a) = 1(0.5) + 2(0.25) + 4(0.25) = 2$

OR $E(X) = \sum_{i=1}^{\infty} P(X \geq i) = P(X\geq1) + P(X\geq2) + P(X\geq3) + P(X\geq4) + P(X\geq5) + \ldots = 1 + 0.5 + 0.25 + 0.25 + 0 + \ldots = 2$

E(X)

Find E(X) of indicator variable I. $I = \begin{cases} \mathbf{1, if\ event\ A\ occurs} \\ \mathbf{0, if\ event\ A^C\ occurs} \end{cases}$

$P(I = 1) = P(A)$. $P(I = 0) = P(A^C)$. $E(I) = 1P(I = 1) + 0P(I = 0) = P(A) + 0 = P(A)$

E[g(X)]

Let X be discrete r.v. with pmf: Find $E(X^2)$

x	-2	-1	0	1	2
P(X=x)	0.05	0.1	0.3	0.2	0.35

Let $Y = X^2$. $P(Y=0) = P(X=0) = 0.3$
$P(Y=1) = P(X=-1 \text{ or } X=1) = 0.1+0.2 = 0.3$
$P(Y=4) = P(X=-2 \text{ or } X=2) = 0.05+0.35 = 0.4$
$E(X^2) = E(Y) = 0(0.3) + 1(0.3) + 4(0.4) = 1.9$

OR $E(X^2) = (-2)^2(0.05) + (-1)^2(0.1) + 0^2(0.3) + 1^2(0.2) + 2^2(0.35) = 1.9$

Discrete r.v. $E[g(X)] = \sum_i \boldsymbol{g(x_i)p(x_i)}$

Proof. $\sum_i g(x_i)p(x_i) = \sum_j\sum_{i:g(x_i)=y_j} g(x_i)p(x_i) = \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i) = \sum_j y_j * P(g(X) = y_j) = E(g(X))$

Variance

Find Var(X) and Var(Y). P(X = 50) = 1. P(Y = 0) = P(Y = 100) = 0.5

Hence E(X) = E(Y) = 50. $Var(X) = E(X^2) - [E(X)]^2 = 50^2(1) - 50^2 = 0$
$Var(Y) = E(Y^2) - [E(Y)]^2 = 0^2(0.5) + 100^2(0.5) - 50^2 = 2500$

$Var(X) = E(X - \boldsymbol{\mu})^2 = E(X^2) - [E(X)]^2$

Proof. $Var(X) = E(X - \mu)^2 = \sum_x (x-\mu)^2p(x) = \sum_x (x^2 - 2\mu x + \mu^2)p(x) = \sum_x x^2p(x) - 2\mu\sum_x xp(x) + \mu^2\sum_x p(x) = E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$

$Var(aX + b) = a^2Var(X)$

Proof. $Var(aX+b) = E\{[(aX+b) - E(aX+b)]^2\} = E\{[(aX+b) - (a\mu+b)]^2\} = E\{a^2(X-\mu)^2\} = a^2E(X-\mu)^2 = a^2Var(X)$

Applications

Pepys problem. More likely to get at least an ace in 6 rolls of a die or at least 2 aces in 12 rolls of a die

$P(\geq 1 \text{ ace in 6 roll}) = 1 - (5/6)^6 = 0.67$
$P(\geq 2 \text{ ace in 12 rolls}) = 1 - (5/6)^{12} - 12(1/6)(5/6)^{11} = 0.62$

Multiple choice test contains 20 qns with 5 choices for each qn. If guess randomly...
Let X = num of correct ans, then X ~ Binomial(20, 0.2)

$P(X > 10) = \sum_{i=11}^{20}\binom{20}{i}\left(\frac{1}{5}\right)^i\left(\frac{4}{5}\right)^{20-i} = 0.0006$
E(X) = 20(0.2) = 4

Person buys a particular 4D number 3 times a week. How long will the person need to strike 1st prize? Let X = num of trials until person strikes 1st prize. X ~ Geometric(1/10000)

E(X) = 1/(1/10000) = 10000
10000/3 = 3333 weeks = 3333/52 = 64 years

10 ppl are tested for certain disease. Their blood samples are pooled and analysed tgt. If test = -ve, only 1 test required. If test = +ve, all 10 ppl need to be individually tested. Assume P(disease) = p, let T = num of test needed for 10 ppl

$P(T = 1) = (1-p)^{10}$. $P(T = 11) = 1 - (1-p)^{10}$
$E(T) = 1(1-p)^{10} + 11[1-(1-p)^{10}]$
If p = 0.5, E(T) = 10.99, (in this case dont pool blood tgt)

St petersburg paradox. Toss a fair coin until tail appears. If tail appear on n^{th} flip, then win $\$2^n$
Let X = winnings of a player. Need to pay $1000000 to play once. Let P = profit. P = X - 10000

$E(X) = \sum_{n=1}^{\infty} 2^nP(tail\ on\ nth\ flip) = \sum_{n=1}^{\infty} 2^n\left(\frac{1}{2}\right)^n = \infty$
E(P) = E(X) - 10000000 = ∞

Coupon problem. How many coupons need to obtain a complete set? Suppose N = 8. Let X = num of coupons collected until complete set is obtained. X_i = num of additional coupons needed after i distinct types have been obtained in order to obtain another distinct type, i = 0,1,... 7. $X = X_0 + X_1 + X_2 + \ldots + X_7$

X_0 ~ Geometric(1), X_1 ~ Geometric(7/8), X_2 ~ Geometric(6/8), ... X_7 ~ Geometric(1/8)
$E(X) = E(X_0) + E(X_1) + E(X_2) + \ldots + E(X_7) = 1 + 8/7 + 8/6 + \ldots + 8/1 = 21.7$

If X ~ Binomial(n, p) and Y ~ Binomial(n-1, p), then $E(X^k) = np * E[(Y + 1)^{k-1}]$

Let k = 2, then $E(X^2) = npE[Y+1] = np[E(Y) + 1] = np[(n-1)(p) + 1]$

Proof. $E(X^k) = \sum_{i=0}^{n} i^k\binom{n}{i}p^i(1-p)^{n-i} = \sum_{i=1}^{n} i^k\binom{n}{i}p^i(1-p)^{n-i} = \sum_{i=1}^{n} i^k\frac{n(n-1)\ldots(n-i+1)}{i!}p^i(1-p)^{n-i} = np\sum_{i=1}^{n} i^{k-1}\frac{(n-1)\ldots(n-i+1)}{(i-1)!}p^{i-1}(1-p)^{n-i} = np\sum_{i=1}^{n} i^{k-1}\binom{n-1}{i-1}p^{i-1}(1-p)^{n-i} = np\sum_{j=0}^{n-1}(j+1)^{k-1}\binom{n-1}{j}p^j(1-p)^{n-1-j} = np * E[(Y + 1)^{k-1}]$

If X ~ Binomial(n,p), $P(X = k) = \frac{\boldsymbol{(n-k+1)p}}{\boldsymbol{k(1-p)}}P(X = k-1)$, k = 1,2,..., n

Proof. $\frac{P(X=k)}{P(X=k-1)} = \frac{\frac{n!}{(n-k)!k!}p^k(1-p)^{n-k}}{\frac{n!}{(n-k+1)!(k-1)!}p^{k-1}(1-p)^{n-k+1}} = \frac{n-k+1}{k}\frac{p}{1-p}$

Poisson r.v

Errors on a page has Poisson dist w $\boldsymbol{\lambda} = 0.25$. Let X = num of errors on given page. X ~ Poisson(0.25)

$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-0.25} = 0.22$

500000 ppl spend $1 each on a 4D num of their choice. Expected num of ppl win. 1st prize? P(> N ppl win 1st prize)? Let X = num of ppl strike 1st prize. X ~ Binomial(500000, 1/10000) ≈ X ~ Poisson(500000/10000 = 50)

E(X) = 50. $P(X > N) = \sum_{k=N+1}^{500000}\frac{e^{-50}50^k}{k!}$

Hat matching problem. n men randomly select hat. P(none of the men select own hat)?
$P(X = 0) \approx e^{-1} \approx 0.37$ using inclusion-exclusion identity

Note that $X = I_{E_1} + I_{E_2} + \ldots + I_{E_n}$, where $I_E = \begin{cases} 1 \; if \; E \; happens \\ 0 \; otherwise \end{cases}$ and E_i = i^{th} man select own hat
$P(E_i) = 1/n$. $P(E_i|E_j) = 1/(n-1) \neq P(E_i)$. Thus $E_1, E_2, \ldots, E_n$ not indep. But their dependence is weak for large n. So X ~ Poisson(λ) where $\lambda = nP(E_i) = 1$. $P(X = i) = \frac{1^i}{i!}e^{-1} = e^{-1}/i!$, i = 0,1,2,.... And $P(X = 0) = e^{-1}$

Birthday problem. n ppl in a room. P(no two of them have same bd) $\frac{(365)(364)(363)\ldots(365-n+1)}{365^n}$. If n ≥ 23, prob is less than 1/2

Consider $\binom{n}{2}$ pairs of person i and j, i ≠ j. E_{ij} = {person i and j have same bd}
Let X = num of pairs with same bd = $\sum_{i<j} I_{E_{ij}}$ where $I_{E_{ij}} = \begin{cases} 1 \; if \; E_{ij} \; happens \\ 0 \; otherwise \end{cases}$
$P(E_{ij}) = 1/365$, $P(E_{ij}|E_{jk}) = 1/365$. But $P(E_{ij}|E_{jk}E_{ik}) = 1$. So E_{ij} only pair-wise indep but dependence is weak. So
X~Poisson(λ) where $\lambda = \frac{\binom{n}{2}}{365} = \frac{n(n-1)}{730}$. $P(X = i) \approx \frac{e^{-\frac{n(n-1)}{730}}\left(\frac{n(n-1)}{730}\right)^i}{i!}$. $P(X = 0) = e^{-\frac{n(n-1)}{730}}$. And P(X = 0) ≤ 0.5 when n ≥ 23

Road accidents happen at rate of 5 per day. Let Y = num of accidents that occur in next 2 days. Y ~ Poisson(5*2 = 10)
Let W = num of accidents in next 3 days. W ~Poisson(15)
Find dist of time starting from now, until next accident

$P(Y \geq 10) = \sum_{k=10}^{\infty} \frac{e^{-10}(10)^k}{k!} = 1 - \sum_{k=0}^{9} \frac{e^{-10}(10)^k}{k!}$. $P(W = 0) = e^{-15}$
Let X = time (in days) until next accident. V = num of accidents in interval [0,t], V~Poisson(5t)
P(X > t) = P(no accident in interval [0,t]) = $P(V = 0) = e^{-5t}$. $P(X \leq t) = 1 - e^{-5t}$

Poisson approximation of binomial: If X ~ Binomial(n,p), n is large and p is small, then X~Poisson($\boldsymbol{\lambda}$) approximately, where $\boldsymbol{\lambda}$ = np

Proof. $P(X = i) = \frac{n!}{(n-i)!i!}p^i(1-p)^{n-i} = \frac{n!}{(n-i)!i!}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i} = \frac{n(n-1)\ldots(n-i+1)}{i!}\frac{\lambda^i}{n^i}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i} = \frac{n(n-1)\ldots(n-i+1)}{n^i}\frac{\lambda^i}{i!}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$
For large n, small p, $\left(1-\frac{\lambda}{n}\right)^i \approx 1$, $\frac{n(n-1)\ldots(n-i+1)}{n^i} = 1(1-\frac{1}{n})\ldots(1-\frac{i-1}{n}) \approx 1$
$\left(1-\frac{\lambda}{n}\right)^n = 1 + n\left(-\frac{\lambda}{n}\right) + \frac{n!}{(n-2)!2!}\left(-\frac{\lambda}{n}\right)^2 + \ldots$ (Binomial theorem) $= 1 - \lambda + \frac{\lambda^2}{2!}\frac{n!}{(n-2)!n^2} + \ldots = 1 - \lambda + \frac{\lambda^2}{2!}\frac{n-1}{n} + \ldots \approx e^{-\lambda}$.
$P(X = i) \approx \frac{\lambda^i}{i!}e^{-\lambda}$

Binomial(n,p), n is large and p is small, then X~Poisson($\boldsymbol{\lambda}$) approximately, where $\boldsymbol{\lambda}$ = np
E(X) = $\boldsymbol{\lambda}$. Var(X) = $\boldsymbol{\lambda}$

Proof. $E(X) = \sum_{i=0}^{\infty} iP(X=i) = \sum_{i=0}^{\infty} i\frac{e^{-\lambda}\lambda^i}{i!} = \lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!} = \lambda e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda$
$E(X^2) = \sum_{i=0}^{\infty} i^2P(X=i) = \sum_{i=0}^{\infty} i^2\frac{e^{-\lambda}\lambda^i}{i!} = \lambda\sum_{i=1}^{\infty}\frac{ie^{-\lambda}\lambda^{i-1}}{(i-1)!} = \lambda\sum_{j=0}^{\infty}\frac{(j+1)e^{-\lambda}\lambda^j}{j!} = \lambda[E(X+1)] = \lambda(\lambda+1)$
$Var(X) = E(X^2) - [E(X)]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$

Let N(t) = num of events occuring in time interval [0,t]. If the 3 assumptions are true, then N(t) ~ Poisson($\boldsymbol{\lambda}$t), where $\boldsymbol{\lambda}$ is rate of occurrences of events per unit time.
Divide timeline into n subintervals of length h, then nh = $\boldsymbol{\lambda}$t
Note $n\left[\frac{\lambda t}{n} + o\left(\frac{t}{n}\right)\right] = \lambda t + t\left[\frac{o(t/n)}{t/n}\right] \to \lambda t$ as $n \to \infty$
So, $\frac{\lambda t}{n} + o\left(\frac{t}{n}\right) = \frac{\lambda t}{n}$ for large n and
$1 - \frac{\lambda t}{n} - o\left(\frac{t}{n}\right) = 1 - \frac{\lambda t}{n}$

Proof. Let {N(t) = k} = A ∪ B. A = {k of n subintervals contain exactly 1 event and other n-k subintervals contain 0 events}. B = {N(t) = k and at least 1 subinterval contain ≥ 2 events}
P(B) = P({N(t) = k} ∩ {at least 1 subinterval contain ≥ 2 events}) ≤ P(at least 1 subinterval contain ≥ 2 events) = $P(\bigcup_{i=1}^{n}\{i^{th} \; subinterval \; contains \geq 2 \; events\}) \leq \sum_{i=1}^{n} P\{i^{th} \; subinterval \; contains \geq 2 \; events\} = \sum_{i=1}^{n} o\left(\frac{t}{n}\right)$ (from assumption 2) $= n \, o\left(\frac{t}{n}\right) = t\frac{o(t/n)}{t/n} \to 0$ as $n \to \infty$ for a fixed t.
P(0 events in interval of length h) = 1 - [λh + o(h)] - o(h) #from assumption 1 & 2 = 1 - λh - o(h) # since o(h) + o(h) = o(h)
$P(A) = \binom{n}{k}\left[\frac{\lambda t}{n} + o\left(\frac{t}{n}\right)\right]^k\left[1 - \frac{\lambda t}{n} - o\left(\frac{t}{n}\right)\right]^{n-k}$. So this become like a binomial problem where n is large, so $P(A) \to e^{-\lambda t}\frac{(\lambda t)^k}{k!}$ as $n \to \infty$. Thus, $P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$, k = 0,1,2,...

Expected Value of Sum of r.v.

$E(X) = \sum_i x_i P(X = x_i) = \sum_{s \in S} X(s)p(s)$, where s_i = {s: X(s) = x_i}

Proof. Suppose that distinct values of X are x_i, i ≥ 1. For each i, let S_i be event X = x_i, i.e. S_i = {s: X(s) = x_i}
$E(X) = \sum_i x_i P(S_i) = \sum_i x_i \sum_{s \in S_i} p(s) = \sum_i \sum_{s \in S_i} x_i p(s) = \sum_i \sum_{s \in S_i} X(s)p(s) = \sum_{s \in S} X(s)p(s)$

2 indep flips of a coin with prob p. Let X = num of heads obtained.

x	0	1	2
P(X = x)	$(1-p)^2$	2p(1-p)	p^2

S = {(t,t), (h,t), (t,h), (h,h)}. X(t,t) = 0. $P(X(t,t)) = (1-p)^2$. X(h,t) = 1. P(X(h,t)) = p(1-p).
X(t,h) = 1. P(X(t,h)) = (1-p)p. X(h,h) = 2. $P(X(h,h)) = p^2$
$E(X) = \sum_i x_i P(X = x_i) = 0(1-p)^2 + 1(2p)(1-p) + 2p^2$
$\sum_{s \in S} X(s)p(s) = 0(1-p)^2 + 1p(1-p) + 1(1-p)p + 2p^2$

For r.v. $X_1, X_2, \ldots, X_n$, $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$

Let $Z = \sum_{i=1}^{n} X_i = X_1 + X_2 + \ldots + X_n$. $E(Z) = \sum_{s \in S} Z(s)p(s) = \sum_{s \in S}(X_1(s) + X_2(s) + \ldots + X_n(s))p(s) = \sum_{s \in S} X_1(s)p(s) + \ldots + \sum_{s \in S} X_n(s)p(s) = E(X_1) + \ldots + E(X_n)$

X~Binomial(n, p)
E(X) = np
Var(X) = np(1-p)

Let $X = X_1 + X_2 + \ldots + X_n$ where $x_i = \begin{cases} 1, if \; trial \; i \; is \; success \\ 0, if \; trial \; i \; is \; failure \end{cases}$. P(success) = p. P(failure) = 1-p
$E(X_i) = 1p + 0(1-p) = p$. Then $E(X) = E(X_1) + E(X_2) + \ldots + E(X_n) = p + p + \ldots + p = np$
$E(X^2) = E\left[(\sum_{i=1}^{n} X_i)(\sum_{j=1}^{n} X_i)\right] = E\left[\sum_{i=1}^{n} X_i^2 + \sum_{i \neq j}^{n} X_iX_j\right] = E\left[\sum_{i=1}^{n} X_i^2\right] + E\left[\sum_{i \neq j}^{n} X_iX_j\right] = \sum_{i=1}^{n} E(X_i^2) + \sum_{i \neq j}^{n} E(X_iX_j)$
$E(X_i^2) = E(X_i) = p$ ($\because X_i^2 = X_i$). $X_iX_j = \begin{cases} 1, if \; X_i = 1 \; and \; X_j = 1 \\ 0, otherwise \end{cases}$. $E(X_iX_j) = 1P(X_i = 1, X_j = 1) + 0 = 1P(X_i = 1)P(X_j = 1)$ ($\because$ indep) $= p*p = p^2$
$E(X^2) = \sum_{i=1}^{n} p + \sum_{i \neq j}^{n} p^2 = np + (n^2 - n)p^2$. $Var(X) = E(X^2) - [E(X)]^2 = np + n(n-1)p^2 - [np]^2 = np(1-p)$

Extra

Given 10 pokeballs to catch pokemon. P(catch) = 0.2. Let X = num of balls to catch pokemon. X ~Geo(0.2)

$P(X = 3) = 0.8^2(0.2) = 0.128$. P(X = 10|1st 7 balls fail to catch) = P(X = 3) = 0.128. (since balls are thrown independently). P(catch within 10 balls) = $1 - 0.8^{10} \approx 0.89$.

2 balls chosen randomly from urn containing 8W, 4B, 2O balls. Suppose we win $2 for each B ball selected and lose $1 for each W ball selected. Let X denote our winnings.

X can take values -2, -1, 0, 1 , 2, 4. pmf of X:

O	0	1	2	0	1	0
B	0	0	0	1	1	2
W	2	1	0	1	0	0
X = x	-2	-1	0	1	2	4
P(X = x)	$\frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$	$\frac{\binom{2}{1}\binom{8}{1}}{\binom{14}{2}} = \frac{16}{91}$	$\frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$	$\frac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \frac{32}{91}$	$\frac{\binom{2}{1}\binom{4}{1}}{\binom{14}{2}} = \frac{8}{91}$	$\frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$

Let X = diff btw num of heads and tails when coin tossed n times. What are the possible values of X. If coin is fair, what is the pmf of X?

$X = n_H - n_T = (n - n_T) - n_T = n - 2n_T$, where $0 \leq n_T \leq n$.
If coin is fair, $P(X = n-2i) = P(n_T = i) = \frac{1}{2^n}\binom{n}{i}$ for 0 ≤ i ≤ n

5 distinct nums are randomly distributed to players numbered 1 to 5. When 2 players compare their nums, one with higher num wins. Player 1 and 2 compare; winner compare with player 3 and so on. Let X = num of times 1 is winner. Find P(X = i), i = 0,1,2,3,4

P(X = 0) = P(1 loses to 2) = 1/2. (of the 2 cards btw 1 and 2, 1 has the smaller one)
P(X = 1) = P(of 1,2,3: 3 has largest, then 1, then 2) = $\frac{1*1*1}{3!} = \frac{1}{6}$
P(X = 2) = P(of 1,2,3,4: 4 has largest, 1 has next largest) = $\frac{1*1*2!}{4!} = \frac{1}{12}$
P(X = 3) = P(of 1,2,3,4,5: 5 has largest then 1) = $\frac{1*1*3!}{5!} = \frac{1}{20}$. P(X = 4) = P(1 has largest) = $\frac{1*4!}{5!} = \frac{1}{5}$

F(x) = $\begin{cases} 0, & x < 0 \\ x/4, & 0 \le x < 1 \\ .5 + (x-1)/4, & 1 \le x < 2 \\ 11/12, & 2 \le x < 3 \\ 1, & 3 \le x \end{cases}$

P(X = 1) = P(X ≤ 1) - P(X < 1) = 1/2 - 1/4
P(X = 2) = P(X ≤ 2) - P(X < 2) = 11/12 - [1/2 + 1/4] = 1/6
P(X = 3) = P(X ≤ 3) - P(X < 3) = 1 - 11/12 = 1/12
P(1/2 < X < 3/2) = P(X < 3/2) - P(X < 1/2) = [1/2 + 1/8] - 1/8 = 1/2

4 buses carrying total of 148 students arrives. Buses carry 40,33,25,50 students. One of the student is randomly selected. Let X = num of students on bus carrying randomly selected student. 1 of the 4 bus drivers also randomly selected. Let Y = num of students on selected bus driver bus.

E(X) or E(Y) is bigger? Prob of selecting student on bus is proportional to num of students on that bus but prob a selecting bus driver is 1/4. So E(X) should be larger)
Calculate. P(X = i) = i/148 for i = 40,33,25,50. P(Y = i) = 1/4
E(X) = $\sum_x xP(X=x)$ = $[40^2 + 33^2 + 25^2 + 50^2]/148$ ≈ 39.28
E(y) = [40+33+25+50]/4 = 37

Sample of 3 items selected at random from box containing 20 items of which 4 is defective. Find expected num of defective items in sample.

num of defective items D is a hypergeometric dist with (n,N,m) = (3,20,4). E(D) = nm/N = 3/5
OR $\sum_x xP(X=x) = 0\frac{\binom{4}{0}\binom{16}{3}}{\binom{20}{3}} + 1\frac{\binom{4}{1}\binom{16}{2}}{\binom{20}{3}} + 2\frac{\binom{4}{2}\binom{16}{1}}{\binom{20}{3}} + 3\frac{\binom{4}{3}\binom{16}{0}}{\binom{20}{3}}$ = 3/5

Newsboy purchase newspapers at 10cents and sell at 15. He is not allowed to return unsold papers. If daily demand is binomial r.v. w n = 10, p = 1/3. How many papers should he purchase to maximise expected profit?

OR just purchase num as close as possible to expected num sold = 10(1/3) = 3

Let m be num of copies ordered. 1 ≤ m ≤ 10.
Let X be r.v. for demand and G_m be r.v. of profit when his order is m copies. (k is num sold)
E(G_m) = $\sum_{k=0}^{m}[15k - 10m]P(X=k) + 5mP(X \ge m+1) = 15\sum_{k=0}^{m} kP(X=k) - 10mP(X \le m) + 5m - 5mP(X \le m)$ = $15\sum_{k=0}^{m} kP(X=k) - 15mP(X \le m) + 5m$
For 1 ≤ m ≤ 9, E(G_{m+1}) - E(G_m) = $\{15\sum_{k=0}^{m+1} kP(X=k) - 15(m+1\_P(X \le m+1) + 5(m+1)\} - \{15\sum_{k=0}^{m} kP(X=k) - 15mP(X \le m) + 5m\}$ = 15(m+1)P(X = m+1) - 15(m+1)[P(X ≤ m) + P(X = m+1)] + 15mP(X ≤ m) + 5 = 5 - 15P(X ≤ m)
Thus E(G_{m+1}) - E(G_m) iff P(X , m) ≤ 1/3

m	0	1	2	3
P(X = m)	0.017342	0.086708	0.195092	0.260123
P(X ≤ m)	0.017342	0.104049	0.299141	0.559264

So optimal order is 3 copies.

Let X be r.v. taking values 1 and -1 with P(X = 1) = p = 1 - P(X = -1). Find c ≠ 1 s.t. E[c^X] = 1

E(c^X) = c^1P(X=1) + c^{-1}P(X = -1) = cp + (1-p)/c = $\frac{c^2p+(1-p)}{c}$. Since E(c^X) = 1. c^2p - c + (1-p) = 0. c = (1-p)/p

Suppose 4 fair dice are rolled. Let M = min of 4 numbers. What are the possible values of M? Find E(M)

Find P(M ≥ k), where k = 1,2,..., 6. Let X_i = num on die i. P(M ≥ k) = P(X_1 ≥ k)P(X_2 ≥ k)P(X_3 ≥ k)P(X_4 ≥ k) (by independence) = $\frac{6-(k-1)}{6}\frac{6-(k-1)}{6}\frac{6-(k-1)}{6}\frac{6-(k-1)}{6} = \left(\frac{7-k}{6}\right)^4$
E(M) = $\sum_{k=1}^{6} P(M \ge k) = \sum_{k=1}^{6}\left(\frac{7-k}{6}\right)^4 = \frac{1^4+2^4+\ldots+6^4}{6^4}$ = 1.755

Let X~Geo(p), P(X = k) = pq^{k-1}. Show P(X > n) = q^n for n ≥ 1 and P(X > n+k | X > n) = P(X > k) for all n,k ≥ 1

P(X > n) = first n trials all failures = q^n
P(X > n+k | X > n) = $\frac{P(X>n+k)}{P(X>n)} = \frac{q^{n+k}}{q^n}$ = q^k = P(X > k)

Ball is drawn from urn containing 3W, 3B balls with replacement.

P(of 1st 4 balls drawn, exactly 2 are W) = $\binom{4}{2}(3/6)^2(3/6)^2$ = 3/8

Suppose airplane engines will fail w prob 1-p. If airplane needs majority of engines to operate to fly, for what values of p is a 5-engine plane preferable to 3-engine plane?

For 3-engine: P(X ≥ 2) = $3p^2(1-p) + p^3 = p^2(3-2p)$
For 5-engine: P(X ≥ 3) = $\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5 = p^3(10-15p+6p^2)$
We need $p^3(10-15p+6p^2) > p^2(3-2p)$....simplify... p > 1/2

Suppose biased coin lands on heads w prob p is flipped 10 times. Given that there are 6 heads.
Find conditional p that first 3 outcomes are h,t,t.
Find conditional p that first 3 outcomes are t,h,t.

P(h,t,t | 6 heads) = $\frac{P(6\ heads|h,t,t)P(h,t,t)}{P(6\ heads)} = \frac{pq^2\binom{7}{5}p^5q^2}{\binom{10}{6}p^6q^4}$ = 1/10
P(t,h,t | 6 heads) = $\frac{P(6\ heads|t,h,t)P(t,h,t)}{P(6\ heads)} = \frac{pq^2\binom{7}{5}p^5q^2}{\binom{10}{6}p^6q^4}$ = 1/10

Integer N selected at random from {1,2,..., 10^3}. What happens when 10^3 replaced by 10^k as k → ∞

P(N divisible by 3) = 333/1000 → 1/3
P(N divisible by 7) = 142/1000 → 1/7
P(N divisible by 15) = 66/1000 → 1/15

Roulette (0, 00: green; 1,2,...36: 1/2 red, 1/2 black). Gambler bet $1 on red. If red appears, take $1 profit and quit. If lost, make additional $1 bet on red for next 2 spins and then quit. Let X = winnings

P(X > 0) = 18/38 + 20/38 * 18/38 * 18/38 ≈ 0.5918
E(X) = 1(0.5918) - 1(2*20/38*20/38*18/38) -3(20/38*20/38*20/38) ≈-0.108. So strategy is bad

You have $1000 and a good sells for $2 per ounce. After 1 week, the good will either be $1 or $4 an ounce, with equal prob.

Maximise expected amt of money at end of week: Buy 500 ounces now and then sell. E(money) = (1/2)(500) + (1/2)(2000) = 1250
Maximise expected amt of good at end of week: Buy after 1 week. E(amt) = (1/2)(1000) + (1/2)(250) = 625

$A must be paid if event E occur with prob p. How much to charge customer so that expected profit is 10% of A?

Let C = amt charged to customer. E(profit) = C - [Ap + 0(1-p)] = C - Ap
For E(profit) = A/10, C-Ap = A/10. C = A(p + 1/10)

E(X) = 1 and Var(X) = 5.

E[$(2+X)^2$] = Var(2+X) + $[E(2+X)]^2$ = Var(X) + $[2+E(X)]^2$ = 14. Var(4+3X) = 9Var(X) = 45

4-sided fair die numbered 1,2,3,4 tossed 2 times. Let X = 1st num, Y = 2nd num

P(X - Y = 0) = P(X = y) = 4/16 = 1/4. E(X - Y) = 0
E[$(X-Y)^2$] = $(-3)^2(1/16) + (-2)^2(2/16) + (-1)^2(3/16) + 1^2(3/16) + 2^2(2/16) + 3^2(1/16)$ = 5/2
Var(X - Y) = 5/2 - 0^2 = 5/2

Approx 80,000 marriages took place. Estimate prob that for at least 1 of these couples
a) both partners born on April 30
b) both partners celebrated bd on same day

Assuming independence of bd and equal chance of being born on any data,
P(both born on April 30) = $1/365^2$. Let X = num of couples born on this date. X ~ Binomial(80,000, $1/365^2$) ≈ Poisson($80000/365^2$ ≈ 0.6). P(X ≥ 1) = 1 - P(X = 0) = 1 - $e^{-0.6}$
P(same day) = 1/365. Y ~ ≈ Poisson(80000/365 ≈ 219.18. P(Y . 1) = 1 - $e^{-219.18}$ ≈ 1

Suppose average num of typos on page document is 1. P(no typos on a page). P(no typos in 5-page document)

Let Z = num of typos on page. Z ~ Poisson(1). P(Z = 0)
Let Y = num of typos on 5 pages. Y ~ Poisson(5). P(Y = 0) = e^{-5} OR P(Y = 0) = P(Z = 0)^5 = $(e^{-1})^5$ = e^{-5}

Num of times person contracts cold in a year is a Poisson r.v. with λ = 5. Suppose a new drug is marketed as reducing λ to 3 for 75% of the pop. For the other 25%, no effect. If an individual tries the drug for a year and has 2 colds, how likely is it that the drug is beneficial for him?

P(beneficial | 2 cold) = $\frac{P(2\ cold|beneficial)P(beneficial)}{P(2\ cold|beneficial)P(beneficial) + P(2\ cold|not\ B)P(B)}$ = $\frac{e^{-3}3^2/2! * 0.75}{e^{-3}3^2/2! * 0.75 + e^{-5}5^2/2! * 0.25}$ ≈ 0.8886

P(full house) ≈ 0.0014. Find an approximation for prob that in 1000 hands of poker you will be dealt at least 2 full houses.

Use poisson approx to binomial. X ~ Poisson(1000*0.0014 = 1.4). P(X ≥ 2) = 1 - P(X = 0) - P(X = 1) = 1 - $e^{-1.4}$ - $e^{-1.4}$(1.4) ≈ 0.4082

There are 3 highways. Num of daily accident on these highways are Poisson r.v. w λ 0.3, 0.5 and 0.7. Find expected num of accidents that will happend on any of these highways today.

Let X_i = num of accidents on highway i. E[$X_1 + X_2 + X_3$] = E[X_1] + E[X_2] + E[X_3] = .3 + .5 + .7 = 1.5

Suppose 10 balls are put into 5 boxes, w ea ball indep being put in box i w prob p_i. Find expected num of boxes that do not have any balls. Find expected num of boxes that have exactly 1 ball.	Let X_i = 1 if box i don't have any balls, 0 otherwise. Then $E\left[\sum_{i=1}^{5} X_i\right] = \sum_{i=1}^{5} E[X_i] = \sum_{i=1}^{5} P(X_i = 1)$ $= \sum_{i=1}^{5}(1-p_i)^{10}$. (ball not in i^{th} box)^{10} Let Y_i = 1 if box i have exactly 1 ball and 0 otherwise. Then $E\left[\sum_{i=1}^{5} Y_i\right] = \sum_{i=1}^{5} E[Y_i] =$ $\sum_{i=1}^{5} P(Y_i = 1) = \sum_{i=1}^{5} 10p_i(1-p_i)^9$. (10 balls, only 1 in box i)