Linear Sys

Line in xy-plane: ax+by = c, plane: ax+by+cz = d
Linear sys: finite set of linear eqn in variables $x_1, x_2 \dots x_n$
Solution set: set of all soln to linear sys, $\{(t,2t-1) | t \in \mathbb{R}\}$
Zero system: all constant are zero,
Inconsistent sys: sys has no soln
Every LS must either have no soln, only 1 soln, or infinitely many sol

In general: linear eqn in n vars $x_1, x_2 \dots x_n$ has form $a_1x_1 + \dots a_nx_n = b$, where $a_1 \dots a_n$ and b are constants
General soln: expression that gives all the soln, $\begin{cases} x = t \\ y = 2t-1 \end{cases}$
Nonzerosys: not zero sys
Consistent sys: ≥ 1 soln

ERO

System of linear eqn
$x_1 + x_2 + 2x_3 = 9$
$2x_1 + 4x_2 - 3x_3 = 1$
$3x_1 + 6x_2 - 5x_3 = 0$

Augmented matrix
$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix}$$

1. Multiply row by nonzero constant, aR_i
2. Interchange 2 rows, $R_i \leftrightarrow R_j$
3. Add multiple of 1 row to another row, $R_i + aR_j$

Row equivalent: If one augmented matrix can be obtained another by series of ERO
If 2 augmented matrix are row-equivalent ⇒ both have same set of soln

REF

REF: 1. Zero rows are all at bottom of matrix
2. In any 2 successive nonzero rows, leading entry/pivot point in lower row is to the right of leading entry of higher row

RREF: 3. Leading entry of every nonzero row is 1
4. In each pivot col, except pivot pt, all other entries = 0
Back-substitution: finding soln from REF/RREF

Pivot col: col containing pivot pt; Non-pivot col: col not containing pivot pt

Gaussian Elimination

Gaussian Elimination: get to REF.
RREF is unique but REF is not unique

Gauss-Jordan Elimination: get to RREF

LS is inconsistent it last col of REF is pivot col (row with nonzero last entry but 0 elsewhere)
LS has 1 soln if except last col, every col of REF is pivot col
LS has infinitely many soln if except for last col, REF has at least 1 more non-pivot col

For m x 3 matrix, REF ≤ 3 nonzero row
3 nonzero row (0 free parameter): intersect at pt
2 nonzero row (1 free parameter): intersect at line
1 nonzero row (2 free parameter): intersect at plane
3 zero row (3 free parameter): whole R^3 space

Homogeneous sys

Homogeneous sys if Ax = b, where b = 0
Trivial soln: $x_1 = \dots = x_n = 0$ is always a soln to homogeneous sys
Homogeneous sys has either only trivial soln or infinitely many soln (including trivial soln)
Homogeneous sys with more unknown that eqn has infinitely many soln

Non-homogeneous sys if not homogeneous
Always consistent

Matrix

Entries: nums in matrix, (i,j)-entry: num in i^{th} row, j^{th} col of matrix
Size: m x n (num of rows x num of cols)
Col matrix: only 1 column; Row matrix: only 1 row
Sq matrix: same num of rows and cols. Size of sq matrix = order n
Diagonal entries: i = j. Non-diagonal entries: i ≠ j
Diagonal matrix: sq matrix and non-diagonal entries = 0
Scalar matrix: diagonal matrix, diagonal entries have same val

Identity matrix, I: diagonal matrix, diagonal entries =1
Zero matrix: all entries = 0
Symmetric: square matrix and $a_{ij} = a_{ji}$ or $A = A^T$
Upper triangular: square matrix, entries = 0 below diagonal entries
Lower triangular: square matrix, entries = 0 above diagonal entries

Matrix ops

2 matrix are equal if they have same size and entries are equal
Ax = b
A: coefficient matrix, x: variable matrix, b: constant matrix

Matrix Multiplication only when num of cols of A = num of rows of B	
Let $A = (a_{ij})_{m \times p}$ and $B = (b_{ij})_{p \times n}$. (i,j)-entry of AB = "sum (row of A x col of B)"	
Matrix multiplication not commutative: pre-multiplication of A to B, AB ≠ BA, post-multiplication of A to B	
1. Associative Law, A(BC) = (AB)C 2. Distributive Law, $A(B_1 + B_2) = AB_1 + AB_2$	
3. AB = 0 ≠> A = 0 or B = 0	4. A0 = 0A = 0
5. c(AB) = (cA)B = A(cB)	6. AI = IA = A

Addition and scalar multiplication	
1. $A \pm B = (a_{ij} \pm b_{ij})_{m \times n}$	2. $cA = (ca_{ij})_{m \times n}$
Commutative Law: A+B = B+A	
Associative Law: A+(B+C) = (A+B)+C	
3. c(A+B) = cA + cB	4. (c+d)A = cA + dA
5. c(dA) = (cd)A = d(cA)	6. A+0 = 0+A = A
7. A-A = 0	8. 0A = 0

Let A be sq matrix and n a nonngeative int	
n = 0: $A^n = I$	n ≥ 1: $A^n = AA \dots A$ n times
$A^mA^n = A^{m+n}$	$(AB)^n \neq A^nB^n$

Transpose. rows of A = cols of A^T ($a_{ij} = a^T_{ji}$)	
$(A^T)^T = A$	$(A+B)^T = A^T + B^T$
$(cA)^T = cA^T$	$(AB)^T = B^TA^T$

Inverses

Let A be sq matrix of order n. A is invertible if ∃ sq matrix B of order n s.t. AB = I and BA = I. Then B is inverse of A
singular: no inverse
If A is invertible and $AB_1 = AB_2$, then $B_1 = B_2$ (opp is false)
If A is invertible and $C_1A = C_2A$, then $C_1 = C_2$
Inverse is unique
Product of invertible matrices will be invertible

For 2 x 2 matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\text{adj}(A)$

$(cA)^{-1} = (1/c)A^{-1}$	$(A^T)^{-1} = (A^{-1})^T$
$(A^{-1})^{-1} = A$	$(AB)^{-1} = B^{-1}A^{-1}$
$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \dots A^{-1}$ n times	A^n is invertible
$A^rA^s = A^{r+s}$ for any int r,s	$(A^n)^{-1} = A^{-n}$

(A|I) -rref-> $(I|A^{-1})$

Elementary matrices

cR_i, E.g. cR_2, EA, $E = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$	$(1/c)R_2$, $E^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/c \end{pmatrix}$	det(E) = c
$R_i \leftrightarrow R_j$, E.g. $R_2 \leftrightarrow R_3$, $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$R_2 \leftrightarrow R_3$, $E^{-1} = E$	det(E) = -1
$R_i + cR_j$, E.g. $R_3 + 2R_1$, $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$	$R_3 - 2R_1$, $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$	det(E) = 1

Elementary matrix: sq matrix obtained from **I** by performing a single ERO (all E defined are elementary matrices)

All elementary matrices are invertible and their inverse are also elementary matrices
$E_n...E_2E_1A = B$, then $A = E_1^{-1}E_2^{-1}...E_n^{-1}B$

Following are equivalent where A is n x n matrix:	8. det(A) ≠ 0
1. A is invertible (not singular)	9. Rows/cols of A spans $\mathbb{R}^n$
2. A has a left inverse	10. Rows/cols of A are LI (9 & 10 combine to become basis of $\mathbb{R}^n$)
3. A has a right inverse	11. rank(A) = n (full rank)
4. Ax = 0 has only trivial soln	12. nullity(A) = 0
5. RREF of A is I	13. 0 is not an eigenvalue of A
6. A is a product of elementary matrices	14. LT T is injective (Ker(T) = {0})
7. For any b, Ax = b has a unique soln	15. LT T is surjective (R(T) = $\mathbb{R}^n$)

Let A,B be sq matrices of same size. If AB = I, then A,B are both invertible
Let A,B be sq matrices of same order. If A is singular, then AB and BA are singular
Post multiplying A by elementary matrix, E: performing elementary column operations (ECO)
Opp of pt 4 true: not invertible = Ax = 0 has infinite soln

Determi-nants

Let A = (a_{ij}) be n x n matrix. Let M_{ij} be (n-1) x (n-1) matrix obtained from A by deleting i^{th} row and j^{th} col.

det(A) = $\begin{cases} a_{11} & if\ n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} & if\ n > 1 \end{cases}$, where $A_{ij} = (-1)^{i+j}\det(M_{ij})$ = (i,j)-cofactor of A (det(A) = A[adj(A)])

This mtd of finding det = cofactor expansion. Cofactor expansion can be done along any row/col

If A is a triangular matrix, det(A) = product of diagonal entries	If sq matrix has 2 same rows/cols, then det = 0
det(A) = det(A^T)	det(cA) = c^ndet(A)
det(AB) = det(A)det(B)	If A is invertible, det(A^{-1}) = 1/det(A)

So det(A) = det(E_n)*...*det(E_2)*det(E_1)*det(rref(A)) = det(E_n)*...*det(E_2)*det(E_1)*prd of diagonal entries of rref(A)

adj(A) = $\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$, where A_{ij} is the (i,j)-cofactor of A = $(-1)^{i+j}\det(M_{ij})$

A[adj(A)] = det(A)I. If A is invertible, then A[$\frac{1}{det(A)}$adj(A)] = I and $A^{-1} = \frac{1}{det(A)}$adj(A)

Cramer's Rule: Ax = b. If A invertible, sys only has 1 soln. Let A_i be matrix obtained from A by replacing i^{th} col of A by b	$x = \frac{1}{det(A)}\begin{pmatrix} det(A_1) \\ det(A_2) \\ \vdots \\ det(A_n) \end{pmatrix}$ $x_1 = \frac{det(A_1)}{det(A)}$ $x_2 = \frac{det(A_2)}{det(A)}$ $x_n = \frac{det(A_n)}{det(A)}$

Euclidean n-Space

n-vector/ordered n-tuple of real numbers has form (u_1, u_2, ... u_n), where u_i is i^{th} coordinate of n-vector n-vector can be represented as row or column vector Euclidean n-space = set of all n-vectors = $\mathbb{R}^n$ Let v be an n-vector ⇒ v ∈ $\mathbb{R}^n$	Implicit form of subset: {(u_1, u_2, u_3, u_4) | u_1 = 0 and u_2 = u_4} OR {(x,y,z)|x - 2y + z = 1} Explicit form of subset: {(0, a, b, a) | a, b ∈ ℝ} OR {(0,0,1) + s(2,1,0)| s ∈ ℝ} |S| = num of elems in S

Linear Combi & Linear Span

Let S = { $u_1...u_k$}. Linear combi of $u_1...u_k$ = c_1u_1 + ... + c_ku_k. Linear Span of S = {c_1u_1 + ... + c_ku_k |$c_1...c_k$ ∈ ℝ} = span(S)	rref(u_1 u_2 ... u_k |v). If sys consistent ⇒ v is LC of u_1 ... u_k all LC of $u_1...u_k$
rref(u_1 u_2 ... u_k). If sys has no zero row ⇒ sys is consistent regardless of values of x, y, z... and span(S) = $\mathbb{R}^n$	Conversely, if sys has zero row ⇒ sys not always consistent and span(S) ≠ $\mathbb{R}^n$

If k < n ⇒ S cannot span $\mathbb{R}^n$	0 ∈ span(S)	For any $v_1...v_r$ ∈ span(S) and $c_1...c_r$ ∈ ℝ, then c_1v_1 +... + c_rv_r ∈ span(S)

Let S_1 = {$u_1...u_k$} and S_2 = {$v_1...v_m$} be subsets of $\mathbb{R}^n$. span(S_1) ⊆ span(S_2) iff each u_i is LC of v_1, ...v_m	span(S_1) = span(S_2) iff span(S_1) ⊆ span(S_2) and span(S_2) ⊆ span(S_1)

If u_k is LC of $u_1...u_{k-1}$, then span{$u_1...u_{k-1}$} = span{$u_1...u_{k-1}$, u_k}. u_k is a redundant vector

			$\mathbb{R}^2$	$\mathbb{R}^3$
span{u}	{cu|c ∈ ℝ}	line through origin	{(x,y)|u_2x-u_1y = 0}	{(cu_1, cu_2, cu_3)|c ∈ ℝ}
span{u,v}, where u,v not parallel	{su + tv|s,t ∈ ℝ}	plane containing origin	$\mathbb{R}^2$	{(x,y,z)|ax+by+cz = 0}

Subspaces

Let V be subset of $\mathbb{R}^n$. V is a subspace of $\mathbb{R}^n$ if V = span(S) OR V is a subspace of $\mathbb{R}^n$ if it contains 0 and closure under addition and scalar multiplication, i.e. ∀ u,v ∈ V and a,b ∈ ℝ, au+bv ∈ V	Let 0 be zero vector in $\mathbb{R}^n$. span{0} is subspace of $\mathbb{R}^n$ and aka zero space $\mathbb{R}^n$ is also a subspace of $\mathbb{R}^n$ Soln set of homog sys is a subspace of $\mathbb{R}^n$ = soln space $\mathbb{R}^m$! ⊆ $\mathbb{R}^n$ (R^m: mx1 vectors but R^n: nx1)

Linear Indepen-dence

Let S = {u_1,...u_k} be set of vectors in $\mathbb{R}^n$ S is LI set iff c_1u_1 + c_2u_2 +...+c_ku_k = 0 (Ax = 0) only has trivial soln If c_1u_1 +...+c_ku_k = 0 has non trivial soln, then S is a linear dependent set	rref([u_1 u_2 ... u_k]): If no non-pivot col ⇒ trivial soln ⇒ LI If have non-pivot col ⇒ infinite sol^n ⇒ not LI
S = {u}. If u = 0, then S is linearly dependent S = {u,v}. If u = cv, then S is linearly dependent	As long as 0 is in a set, set would be linearly dependent ∅ is LI
S is linearly dependent iff at least 1 vector u_i in S is a LC of other vectors in S, i.e. $u_i = a_1u_1 + ... + a_{i-1}u_{i-1} + a_{i+1}u_{i+1} + ... + a_ku_k$	S is LI iff no vector in S is a LC of other vectors in S If S is LI, there is no redundant vector in S
If k > n, then S is linearly dependent	k unknowns, n eqn, then sys has non-trivial soln
2 vectors are linearly dependent if on same line	3 vectors are linearly dependent if on same line/same plane

Let u_1,...u_k be LI vectors in $\mathbb{R}^n$. If u_{k+1} is a vector in $\mathbb{R}^n$ and not LC of u_1,...u_k. Then u_1,...u_k, u_{k+1} is also LI

Bases

V is vector space if V = $\mathbb{R}^n$ or V is a subspace of $\mathbb{R}^n$

	Let W be a vector space. V is also a subspace of W if V is a vector space contained in W
	Let $S = \{u_1, \dots u_k\}$ be a subset of vector space V. S is basis for V if 1. S is LI 2. V = span(S) S is basis for R^n iff 1. k = n and 2. A is invertible (i.e. RREF of A = I)
	Except zero space, any vector space has infinitely many diff bases (not unique) | Basis for {0} is $\emptyset$
	If $S = \{u_1, \dots u_k\}$ is basis for vector space V, and v is vector in V. Then, there is unique values for c_i s.t. $v = c_1u_1 + \dots + c_ku_k$ Coefficients c_i are the coordinates of v relative to basis S. $(v)_S = (c_1, \dots c_k)$. $[v]_S$ is col form of $(v)_S$ Standard basis for $\mathbb{R}^n$. $e_1 = (1,0,\dots,0)$, $e_2 = (0,1,\dots,0)\dots$, $e_n = (0,\dots,0,1)$
	Let S be basis for vector space V | For any $u,v \in V$, $u = v$ iff $(u)_S = (v)_S$ For any $v_1, v_2, \dots v_r \in V$, $(c_1v_1 + \dots + c_rv_r)_S = c_1(v_1)_S + \dots + c_r(v_r)_S$
	Let S be basis for vector space V and $\|S\| = k$. Let $v_1, v_2, \dots v_r$ be vectors in V 1. $v_1, \dots v_r$ are linearly dependent/indep iff $(v_1)_S, \dots (v_r)_S$ are linearly dependent/indep vectors in $\mathbb{R}^k$ 2. $\text{span}\{v_1, \dots v_r\} = V$ iff $\text{span}\{(v_1)_S, \dots (v_r)_S\} = \mathbb{R}^k$
	If S and T are bases for subspace V, then $\|S\| = \|T\|$
Dimensions	Let V be vector space with basis with k vectors dim(V) = num of vectors in basis for V (dim({0}) = 0) | 1. Any subset of V with > k vectors is always LD 2. any subset of V with < k vectors cannot span V
	dim = 1 $\Rightarrow$ line through origin | dim = 2 $\Rightarrow$ plane containing origin
	dim of soln space = num of parameters needed for soln of homogeneous sys = num of non-pivot cols
	S is basis for V if 1. $\|S\|$ = dim(V) 2. S is subset of V 3. S is LI | S is basis for V if 1. $V \subseteq \text{span}(S)$ (textbook: =) 2. $\|S\|$ = dim(V)
	Let U be subspace of vector space V. Then $\dim(U) \le \dim(V)$ | U = V iff dim(U) = dim(V)
	Let V, W be subspace of $\mathbb{R}^n$. | $\dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W)$
Transition Matrices	Let $S = \{u_1, \dots u_k\}$, $T = \{v_1, \dots v_k\}$ be 2 bases for vector space V. Let w be a vector in V $[w]_T = P[w]_S$ where $P = ([u_1]_T, [u_2]_T, \dots, [u_k]_T)$ = transition matrix from S to T | $\text{rref}([v_1\ v_2\ \dots\ v_k \mid u_1 \mid u_2 \mid \dots \mid u_k])$: P = RHS of rref (exclude zero row)
	Let S, T be 2 bases of vector space, and P be transition matrix from S to T. P must be sq matrix. Then 1. P is invertible and 2. P^{-1} is transition matrix from T to S ($P^{-1} = ([v_1]_S, [v_2]_S, \dots, [v_k]_S)$)
	$[u]_S = [2v + w]_S = 2[v]_S + [w]_S$
Row space & column space	Let A be m x n matrix row space of A = $\text{span}\{r_1, \dots, r_m\}$ a subspace of R^n = col space of A^T col space of A = $\text{span}\{c_1, \dots, c_n\}$ a subspace of R^m = row space of A^T Row /col space of 0 = zero space, Row /col space of I = R^n Let A and B be row equivalent matrices. row space of A = row space of B | Nonzero rows of REF of A will form basis for row space of A Row equivalent matrices preserve linear dependency of cols (i.e. from REF we can tell which columns of A are a LC of other cols) Cols in A corresponding to pivot cols of REF of A will form basis for col space of A
	Finding basis for linear span (use row space/ col space mtd if need original vectors) Ax = b has solution $\Leftrightarrow$ b is a LC of cols of A Col space of A = $\{Au \mid u \in R^n\}$ Nullspace of A = $\{u \in R^n \mid Au = 0\}$ | Extending set to basis (form matrix with vectors in row form, ref and create new vectors with leading entries at non-pivot cols) $\text{Col}(AB) \subseteq \text{Col}(A)$ Suppose AB = 0. Then $\text{Col}(B) \subseteq \text{Null}(A)$
Ranks	Let A be m x n matrix. rank(A) = dim of row/col space of A $\text{rank}(A) \le \min\{m, n\}$. Full rank: rank(A) = min{m, n} Sq matrix A is full rank iff $\det(A) \ne 0$ iff Col(A) / Row(A) = R^n iff rows/cols of A form basis for R^n | rank 0 = 0 matrix $\text{rank}(A) = \text{rank}(A^T)$ for any matrix A Ax = b consistent iff $\text{rank}(A) = \text{rank}(A \mid b)$ $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$ $\text{rank}(A + B) \le \text{rank}(A) + \text{rank}(B)$ If A invertible, rank(AB) = rank(B)
Nullspace & Nullities	Let A be m x n matrix. Nullspace of A: soln space of Ax = 0, is subspace of R^n nullity(A) = dim of nullspace of A, is $\le n$, = num of free params in general soln (non-pivot cols) Dimension thm: rank(A) + nullity(A) = n | General soln of Ax = b = [(general soln of Ax = 0) + particular soln for Ax = b] Soln set of Ax = b = $\{u+v \mid u \in$ nullspace of A$\}$, and v is particular soln for Ax = b If Ax = b is consistent, sys has only 1 soln iff nullspace of A = {0}
	Solution set of homogeneous linear sys Ax = 0 is always a subspace of R^n, where A is m x n matrix = nullspace of A
	Let A be a m x n matrix | If m > n, A cannot have a right inverse If n > m, A cannot have a left inverse | If rank(A) = n, A has a left inverse If rank(A) = m, A has a right inverse
Inner/ dot/ scalar product	$\|\|u\|\|$ = length/norm of u = $\sqrt{u_1^2 + \dots + u_n^2}$ $\cos\theta = \frac{u \cdot v}{\|\|u\|\|\|\|v\|\|}$ (derived from cosine rule) ($u \cdot v = u_1v_1 + u_2v_2 + \dots + u_kv_k$) And $u \cdot u = u_1^2 + u_2^2 + \dots + u_n^2$ Unit vectors: $\|\|u\|\| = 1$ If u, v are row vectors, $u \cdot v = uv^T$ If u, v are col vectors, $u \cdot v = u^Tv$ | 1. $u \cdot v = v \cdot u$ (commutative law) 2. $(u + v) \cdot w = u \cdot w + v \cdot w$. $w \cdot (u + v) = w \cdot u + w \cdot v$ 3. $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$ 4. $\|\|cu\|\| = \|c\|\ \|\|u\|\|$ ($\|c\|$ is abs value) 5. $u \cdot u \ge 0$. $u \cdot u = 0$ iff u = 0 - Av = 0 iff $A^TAv = 0$ 6. Cauchy-Schwarz inequality: $\|u \cdot v\| \le \|\|u\|\|\ \|\|v\|\|$
Orthogonal/ Orthonormal set	1. 2 vectors u, v are orthogonal if $u \cdot v = 0$ (perpendicular) 2. Set S of vectors is orthogonal is every pairs of vectors in S are orthogonal (i.e., $u_1 \cdot u_2 = 0$, $u_1 \cdot u_3 = 0$, ..., $u_{k-1} \cdot v_k = 0$) | Orthogornal set: $\{u_1, u_2, \dots u_k\}$ Orthonormal set: $\{\frac{1}{\|\|u_1\|\|}u_1, \frac{1}{\|\|u_2\|\|}u_2, \dots, \frac{1}{\|\|u_k\|\|}u_k\}$ If S is orthogonal set of nonzero vectors in vector space, then S is linearly independent

3. Set S of vectors is orthonormal if S is orthogonal and every vector in S is a unit vector
Standard basis is orthogonal and orthonormal set

1. Orthogonal basis: S is orthogonal & |S| = dim(V)
2. Orthonormal basis: S is orthonormal & |S| = dim(V)

To check if set is orthogonal basis: Let S be set of nonzero vectors in vector space V	
(i) S is orthonormal and (ii) span(S) = V	(i) S is orthonormal and (ii) |S| = dim V

Let S $\{u_1, u_2, \ldots u_k\}$ be orthogonal basis for V.
For any vector w in V, $w = c_1u_1 + c_2u_2 + \ldots + c_ku_k$
$(w)_S = (c_1\ c_2 \ldots c_k) = \left(\frac{w \cdot u_1}{||u_1||^2}\ \frac{w \cdot u_2}{||u_2||^2} \ldots \frac{w \cdot u_k}{||u_k||^2}\right)$
If S is orthonormal basis, $||u_i||^2 = 1$ for all i

Orthogonal set can contain 0 vector ⇒ set won't be LI and won't be basis
Let $S = \{u_1, \ldots, u_k\}$. $A = (u_1 \ldots u_k)$. S is orthogonal set iff A^TA is a diag matrix
S is orthonormal set iff $A^TA = I_k$

Find normal to subspace

Let V be subspace of R^n. Vector n is orthogonal (normal) to subspace V if u is orthogonal to all vectors in V
V has eqn $ax + by + cz = 0$ ⇒ normal vector = n = (a, b, c)
For any vector v (x_0, y_0, z_0) in V, $n \cdot v = ax_0 + by_0 + cz_0 = 0$

Let subspace V = span$\{u_1, u_2, \ldots u_k\}$ in R^n
1. Let $v = (x_1, x_2, \ldots x_n)$ = normal
2. Convert $v \cdot u_1 = 0$, $v \cdot u_k = 0$ into homogeneous sys
3. Solve LS

Let V be subspace of R^n and w a vector in R^n.
w can be decomposed uniquely as $w = w_p + w_n$ where w_p = projection ∈ V and $v_n \in V^\perp$
p = the projection of vector w onto subspace V ⇔ w - p is orthogonal to V (p is unique)

	1. $S = \{u_1, u_2, \ldots, u_k\}$: an orthogonal basis for V	2. $T = \{v_1, v_2, \ldots, v_k\}$: an orthonormal basis for V
p	$\frac{w \cdot u_1}{\|\|u_1\|\|^2}u_1 + \frac{w \cdot u_2}{\|\|u_2\|\|^2}u_2 + \ldots + \frac{w \cdot u_k}{\|\|u_k\|\|^2}u_k$	$(w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \ldots + (w \cdot v_k)v_k$
	$\frac{w \cdot u_1}{\|\|u_1\|\|^2}u_1 + \frac{w \cdot u_2}{\|\|u_2\|\|^2}u_2 + \ldots + \frac{w \cdot u_k}{\|\|u_k\|\|^2}u_k = \begin{cases} w \text{ if } w \in V, & \text{coordinate vector} \\ p \text{ if } w \notin V, & \text{projection of w} \end{cases}$	

Convert a basis to orthogonal basis

Use Gram-Schmidt Process (project vector to subspace)

	v_i	w_i
u_1	$v_1 = u_1$	$w_1 = \frac{1}{\|\|v_1\|\|}v_1$
u_2	$v_2 = u_2 - \frac{u_2 \cdot v_1}{\|\|v_1\|\|^2}v_1$ (orthogonal to v_1)	$w_2 = \frac{1}{\|\|v_2\|\|}v_2$
u_3	$v_3 = u_3 - \frac{u_3 \cdot v_1}{\|\|v_1\|\|^2}v_1 - \frac{u_3 \cdot v_2}{\|\|v_2\|\|^2}v_2$ (orthogonal to v_1 and v_2)	
u_k	$v_k = u_k - \frac{u_k \cdot v_1}{\|\|v_1\|\|^2}v_1 - \frac{u_k \cdot v_2}{\|\|v_2\|\|^2}v_2 - \ldots - \frac{u_k \cdot v_{k-1}}{\|\|v_{k-1}\|\|^2}v_{k-1}$	$w_k = \frac{1}{\|\|v_k\|\|}v_k$
$\{u_1, u_2, \ldots, u_k\}$ basis for vector space V	$\{v_1, v_2, \ldots, v_k\}$ is orthogonal basis for V	$\{w_1, w_2, \ldots, w_k\}$ orthonormal basis for V

W = span$\{u_1, u_2, \ldots u_k\}$. $A = (u_1, u_2, \ldots, u_k)$
$v \in W^\perp$ iff $v \cdot u_i = 0\ \forall\ i = 1 \ldots k$
$v \in W^\perp$ iff $v \in \text{Null}(A^T)$. $W^\perp = \text{null}(A^T)$ / null(A) and W = col(A) / row(A)

$u \perp \text{Row}(A)$ iff $u \in \text{Null}(A^T)$
$\text{null}(A) = \text{null}(A^TA)$
$\text{null}(A^T) = \text{null}(AA^T)$

Best Approximations

Let V be subspace in R^n and $u \in R^n$
p: projection of u onto V = p is best approximation of u in V
dist(u, p) ≤ dist(u, v) for any v in V

Suppose Ax = b is inconsistent ⇒ Ax - b ≠ 0
Least sq soln of Ax = b is a vector u in R^n that minimise ||b-Ax||, i.e. $||b-Au|| \le ||b-Av||\ \forall$ v in R^n
iff $A^TAu = A^Tb \Rightarrow p = Au = A(A^TA)^{-1}A^Tb$

Suppose u is least sq soln (Au = projection p of b onto col space of A)
iff Au = p (always consistent since p lies on col space of A)
Suppose $A = (u_1\ u_2\ u_3) \Rightarrow Ax = cu_1 + du_2 + eu_3$ (LC of cols of A)
⇒ All Ax belongs to col space of A

A(least sq soln) = projection
least sq soln may not be unique
u might not be in col space of A

u is least sq soln of Ax = b ($A = (a_1\ a_2\ a_3)$) iff u is soln to $A^TAx = A^Tb$
⇔ Au is projection of b onto V (V = col space of A)
⇔ b - Au is orthogonal to V
⇔ b - Au is orthogonal to a_1, a_2, a_3

⇔ $A^T(b - Au) = \begin{pmatrix} a_1^T \\ a_2^T \\ a_3^T \end{pmatrix}(b - Au) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (dot product)
⇔ $A^TAu = A^Tb$ ⇔ u is soln to Ax = p (p = projection of b onto col space of A)

Orthogonal Matrices

Sq matrix A is orthogonal matrix if $A^{-1} = A^T \Leftrightarrow AA^T = I$ or $A^TA = I$
(i.e. all orthogonal matrices are invertible)
Product of 2 orthogonal matrix also orthogonal

Let A be sq matrix of order n
1. A is orthogonal matrix ⇔
2. Rows of A forms orthonormal basis for R^n ⇔
3. Cols of A form orthonormal basis for R^n

Transition matrix btw orthonormal bases

$S = \{u_1, u_2, \ldots, u_k\}$. $T = \{v_1, v_2, \ldots, v_k\}$. $P = ([u_1]_T\ [u_2]_T \ldots [u_k]_T)$.
Then $[w]_T = P[w]_S$
Suppose S and T are 2 orthonormal bases for a vector space
Then transition matrix P from S to T is orthogonal.
So P^T is transition matrix from T to S

$P = \begin{pmatrix} u_1 \cdot v_1 & u_2 \cdot v_1 & \ldots & u_k \cdot v_1 \\ u_1 \cdot v_2 & u_2 \cdot v_2 & \ldots & u_k \cdot v_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_1 \cdot v_k & u_2 \cdot v_k & \ldots & u_k \cdot v_k \end{pmatrix}$

$Q = P^{-1} = \begin{pmatrix} v_1 \cdot u_1 & v_2 \cdot u_1 & \ldots & v_k \cdot u_1 \\ v_1 \cdot u_2 & v_2 \cdot u_2 & \ldots & v_k \cdot u_2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1 \cdot u_k & v_2 \cdot u_k & \ldots & v_k \cdot u_k \end{pmatrix} = P^T$

Rotation of xy-coordinates

$S = \{(1, 0), (0, 1)\}$, $T = \{u_1, u_2\} = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$
$v = [v]_S$ (since standard basis)
$[v]_T = P^T[v]_S$ where $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $P^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

T = new coordinate sys
$[v]_T$ = rotating xy-coordinate anticlockwise by θ
= rotate vector clockwise by θ

Eigenvalues, Eigenvectors, Eigenspace

Diagonalizing a sq matrix: $A = PDP^{-1}$ (D is diag matrix). $A^n = PD^nP^{-1}$
Let A be sq matrix of order n, x be nonzero col vector in R^n
If $Ax = \lambda x$ for some scalar λ, x is an eigenvector of A

If A is a triangular matrix, eigenvalues of A = diag entries
characteristic polynomial of A =

	λ is eigenvalue of A associated with eigenvector x A = $(x_1\ x_2)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}(x_1\ x_2)^{-1}$ (x_i is eigenvector associated with eigenvalue λ_i)	det(λI - A) Let A = PDP^{-1}. Then $A^n = PD^nP^{-1}$
Finding eigen-values	Let A be sq matrix of order n, λ is eigenvalue of A $\Leftrightarrow$ Ax = λx, for some nonzero col vector x $\Leftrightarrow$ λx - Ax = 0 $\Leftrightarrow$ (λI - A)x = 0 has non-trivial soln (sys always consistent since homog sys) $\Leftrightarrow$ det(λI - A) = 0	λ is an eigenvalue of A $\Leftrightarrow$ det(λI - A) = 0 $\Leftrightarrow$ λ is a root of the characteristic polynomial
Finding eigenvectors	det(A) ≠ 0 $\Leftrightarrow$ 0 is not an eigenvalue of A Proof. 0 is not an eigenvalue of A $\Leftrightarrow$ 0 not a root of char polynomial (det(λI - A) ≠ 0) $\Leftrightarrow$ det(0I - A) ≠ 0 $\Leftrightarrow$ det(-A) ≠ 0 $\Leftrightarrow$ $(-1)^n$det(A) ≠ 0 $\Leftrightarrow$ det(A) ≠ 0	Finding eigenvectors. Ax = λx, for some nonzero col vector x λx - Ax = 0 $\Rightarrow$ (λI - A)x = 0 Then just solve this homogeneous sys to find x
Eigenspace	E_λ = eigenspace of A associated with eigenvalue λ = soln space of LS (λI - A)x = 0 (has nontrivial soln) If u is a nonzero vector in E_λ, then u is an eigenvector in A associated with eigenvalue λ	Just find general soln of (λI - A)x = 0, then eigenspace is span by the vector Although 0 in eigenspace, 0 cannot be eigenvector as eigenvector always nonzero vector
Diagonal-ization	A square matrix A is diagonalizable if ∃ an invertible matrix P s.t. $P^{-1}AP$ is a diagonal matrix, i.e. A = PDP^{-1} or $P^{-1}AP$ = D Matrix P diagonalizes A	Let A be sq matrix of order n. A is diagonalizable $\Leftrightarrow$ A has n LI eigenvectors. A has n distinct eigenvalues (λ) $\Rightarrow$ A is diagonalizable Diagonal matrices are diagonalizable
	Note that BD = $(b_1\ b_2\ \dots\ b_n)$D = $(d_1b_1\ d_2b_2\ \dots\ d_nb_n)$ if D is a diagonal matrix with diagonal entries $d_1\ d_2 \dots\ d_n$	
Check if A is diagonali-zable	1. Solve det(λI - A) = 0 to find all eigenvalues 2. For each eigenvalues, find basis S_{λ_i} for eigenspace E_{λ_i} 3. Let S = $S_{\lambda_1} \cup S_{\lambda_2} \cup \dots \cup S_{\lambda_k}$. (S is always LI) a) If $\lvert S\rvert$ < n, A is not digonalizable b) If $\lvert S\rvert$ = n, A is diagonalizable	det(λI - A) = $(\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\dots(\lambda-\lambda_k)^{r_k}$, where r_i = multiplicity, then dim(E_{λ_i}) ≤ r_i A is diagonalizable iff dim(E_{λ_i}) = r_i for all λ_i A only has 1 eigenvalue and is a scalar matrix $\Rightarrow$ A is diagonalizable
	In general, a_0 = s, a_1 = t, $a_n = pa_{n-1} + qa_{n-1}$. Then recurrence matrix A = $\begin{pmatrix}0 & 1\\ q & p\end{pmatrix}$, $\begin{pmatrix}a_n\\ a_{n+1}\end{pmatrix} = A^n\begin{pmatrix}a_0\\ a_1\end{pmatrix} = PD^nP^{-1}\begin{pmatrix}s\\ t\end{pmatrix}$	
Orthogonal Diagonali-zation	A sq matrix A is orthogonally diagonalizable if ∃ an orthogonal matrix P s.t. P^TAP is a diagonal matrix	Matrix P orthogonally diagonalizes A Sq matrix is orthogonally diagonalizable iff it is symmetric
	1. Solve det(λI - A) = 0 to find all eigenvalues 2. For each λ, a) find basis S_{λ_i} for eigenspace E_{λ_i} b) Gram-Schmidt to transform S_{λ_i} into orthonormal basis T_{λ_i} 3. Let T = $T_{\lambda_1} \cup T_{\lambda_2} \cup \dots \cup T_{\lambda_k}$. T = $\{v_1, v_2, \dots, v_n\}$. (T is orthonormal) Then P = $(v_1\ v_2\ \dots\ v_n)$ is orthogonal matrix that diagonalizes A	Eigenvalues of symmetric matrix are always real nums Let A be a symmetrix matrix, and det(λI - A) = $(\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\dots(\lambda-\lambda_k)^{r_k}$, then dim($E_{\lambda_i}$) = r_i, i.e. A is always diagonalizable $r_1 + r_2 + \dots + r_k$ = order of A dimE_{λ_1}+ dimE_{λ_2}+...+ dimE_{λ_n} = num of LI eigenvectors
	If A is invertible and diagonalizable, then A^{-1} also diagonalizable If A diagonalizable, A^{-1} also diagonalizable	If A and B are orthogonally diagonalizable, then A+B also orthogonally diagonalizable (since sum of symmetric matrix still symmetric)
Linear Transform-ation	$T\left(\begin{pmatrix}x\\ y\end{pmatrix}\right) = \begin{pmatrix}-y\\ x\end{pmatrix}$ (formula) T: $R^n \to R^m$ is LT iff T(u) = Au ∀ u in R^n T is a linear transformation from R^n to R^m. A is the standard matrix of the linear transformation, A = m x n R^n: domain of T, R^m: codomain of T	I: $R^n \to R^n$: the identity transformation, i.e. I(u) = u, A = I O: $R^n \to R^m$: the zero transformation, i.e. O(u) = 0, A = $0_{m \times n}$ If T: $R^n \to R^m$ is a linear transformation, then 1. T(0) = 0, i.e. A0 = 0 2. $T(c_1u_1 + c_2u_2 + \dots + c_ku_k) = c_1T(u_1) + c_2T(u_2) + \dots + c_kT(u_k)$ OR T(au + bv) = aT(u) + bT(v)
	If linear transformation T: $R^n \to R^n$, i.e. domain = codomain, then T is a linear operator on R^n, and standard matrix for T is a sq matrix	If given $T(u_1) = v_1$, $T(u_2) = v_2$, $T(u_3) = v_3$ Can find image of any other vector if u_1, u_2, u_3 form basis for R^3. Find LC of $u_1\ u_2\ u_3 = u_4$. Then $T(u_4)$ = LC of $v_1\ v_2\ v_3$
Finding A	Given $T(u_1) = v_1$, $T(u_2) = v_2$, $T(u_3) = v_3$, to find formula for T, 1. Direct Gaussian elimination $\begin{pmatrix}x\\ y\\ z\end{pmatrix} = c_1u_1 + c_2u_2 + c_3u_3$, $\left(\begin{array}{ccc\vert c}u_1 & u_2 & u_3 & \begin{matrix}x\\ y\\ z\end{matrix}\end{array}\right)$ to find c_i Then $T\left(\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = c_1v_1 + c_2v_2 + c_3v_3$	2. Find $T(e_1)$, $T(e_2)$, $T(e_3)$ Note $T(e_i) = Ae_i$ = i^{th} col of A. So A = $(T(e_1)\ T(e_2)\ \dots\ T(e_n))$ Find e_1, e_2, e_3 in terms of u_1, u_2, u_3 $(u_1\quad u_2\quad u_3\vert e_1\quad \vert e_2\quad \vert e_3) \to \left(I\left\vert\begin{matrix}c_1\\ c_2\\ c_3\end{matrix}\right.\left\vert\begin{matrix}c_4\\ c_5\\ c_6\end{matrix}\right.\left\vert\begin{matrix}c_7\\ c_8\\ c_9\end{matrix}\right.\right)$ Then $T(e_1) = c_1v_1 + c_2v_2 + c_3v_3$
Finding A & Composite	3. Stack matrices So $A(u_1\quad u_2\quad u_3) = (v_1\quad v_2\quad v_3)$ Then A = $(v_1\quad v_2\quad v_3)(u_1\quad u_2\quad u_3)^{-1}$	Let S: $R^n \to R^m$ and T: $R^m \to R^k$ be LT. Then (T ∘ S)(u) = T(S(u)) ∀ u in R^n $(T\circ S)\left(\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = T\left(S\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right)$ to find final formula OR $(T\circ S)\left(\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = BA\begin{pmatrix}x\\ y\\ z\end{pmatrix}$
Range	T: $R^n \to R^m$ is linear transformation. Range = possible images Range of T = R(T) = set of images of T = $\{T(u) \mid u \in R^n\}$ (explicit set notation)	R(T) ⊆ R^m R(T) ⊆ codomain of T
	$R(T) = \left\{\begin{pmatrix}x+y\\ y\\ x\end{pmatrix} \middle\vert\ x, y \in \mathbb{R}\right\} = \text{span}\left\{\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}, \begin{pmatrix}1\\ 1\\ 0\end{pmatrix}\right\}$ explicit set notation linear span form A = $\begin{pmatrix}1 & 1\\ 0 & 1\\ 1 & 0\end{pmatrix}$,R(T) = col space of A. R(T) is subspace of R^m	Finding basis for range of T = finding basis for col space of A 1. If formula of T:$R^n \to R^m$ is given R(T) = {formula in $x_1, x_2, \dots, x_n \vert x_1, x_2, \dots, x_n \in R$} 2. If standard matrix A given, R(T) = span{cols of A} = span{$T(e_1), T(e_2), \dots, T(e_n)$} 3. If image of basis $\{u_1, u_2, \dots, u_n\}$ for R^n

<table>
<tr><td></td><td>rank(T) = dim of R(T) = dim of col space of A = rank(A)</td><td>R(T) = span{T(u_1), T(u_2), ..., T(u_n)}. Find basis from this set</td></tr>
<tr><td>Kernel</td><td>Let T: $R^n \rightarrow R^m$. The kernel of T = ker(T) = set of vectors in R^n whose image is zero vector in R^m = { u ∈ R^n | T(u) = 0}
ker(T) ⊆ R^n</td><td>ker(T) = all u s.t T(u) = 0 = all u s.t. Au = 0 = soln space of Ax = 0 = nullspace of A, ker(T) is subapce of R^n</td></tr>
<tr><td></td><td>dim of ker(T) = nullity(T) = nullity(A)

rank(T) + nullity(T) = rank(A) + nullity(A) = n</td><td>Proving qns. Let T: $R^n \rightarrow R^m$ be linear transformation
<table>
<tr><td></td><td>ker(T) = { u ∈ R^n | T(u) = 0}</td><td>R(T) = {T(u) | u ∈ R^n}</td><td></td></tr>
<tr><td>Given</td><td>v ∈ ker(T)</td><td>v ∈ R(T)</td><td>WTS</td></tr>
<tr><td>Follow up with</td><td>T(u) = 0</td><td>v = T(u) for some u ∈ R^n</td><td>try to show</td></tr>
</table></td></tr>
<tr><td></td><td colspan="2">T: $R^n \rightarrow R^m$. T is injective if whenever T(u) = T(v), then u = v. iff Ker(T) = {0} iff nullity(T) = 0
T is surjective if for any w ∈ R^m, there is a u ∈ R^n s.t. T(u) = w. iff R(T) = R^m iff rank(T) = m
If n = m, T is injective iff T is surjective</td></tr>
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