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| Linear eqn in 2 var  3 var  n var | | | line  plane | | ax + by = c, where a and b are not both 0  ax + by + cz = d  a1x1+ a2x2+...+anxn=b | |
|  | Solution set: {(t,2t-1)|t} | | | | General soln: expression representing all soln, | |
| Inconsistent sys  Consistent sys | no soln  1 soln  > 1 soln | | | | does not intersect  intersect at a pt  1 free param: line, 2 free param: plane | |
| Augmented matrix | x1 + x2 + 2x3 = 9  2x1 + 4x2 - 3x3 = 1  3x1 + 6x2 - 5x3 = 0 | | | |  | |
| Elementary row ops (ERO) | 1. Multiply row by nonzero constant, cRi  2. Interchange 2 rows, Ri Rj | | | | 3. Add multiple of one row to row i, Ri + cRj | |
| Row Equivalent | | 2 matrix are row equivalent if A --ERO--> B | | | Same set of soln for both matrices | |
| Row-echelon form (REF)  -not unique | 1. Rows consisting entirely of 0 must all be at bottom of matrix  2. The 1st nonzero num in lower row must be to the right of 1st nonzero num in higher row | | | | Reduced row-echelon form (RREF, unique)  3. Leading entry of every nonzero row is 1  4. In each pivot col, all other entry except pivot points/leading entry must be 0 | |
| Gaussian Elimination  -get REF | 1. Locate leftmost col not consisting entirely of zero  2. Interchange top row with another row to bring a nonzero entry to top of col found in step 1  3. For each row below top row, add multiple of top row so entry below leading entry of top row become 0  4. Cover top row and begin again with Step 1 | | | | | |
| Gauss-Jordan Elimination  -get RREF | 1. Use gaussian elimination  2. Multiply constant to each row so leading entries become 1  3. From last nonzero row, add multiple of this row to rows above to get 0 above leading entries | | | | | |
| REF | No soln  1 soln  >1 soln | | | -zero row have nonzero num at last col (last col is pivot col)  - num of var = num of nonzero rows (every col is pivot col except last)  - num of var > num of nonzero row (some col are non-pivot col) | | |
| Homogeneous sys | a11x1 + a12x2 + a13x3 = 0  a21x1 + a22x2 + a23x3 = 0  a31x1 + a32x2 + a33x3 = 0 | | | | all constant terms = 0  If some nonzero constant term: non-homogeneous | |
| Trivial soln: x1 = 0, x2 = 0,..., xn = 0 (pass through origin)  Non-trivial soln: soln other than trivial | | | | | -Homogeneous sys will at least have trivial soln (1 soln) |

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| Matrix | **A** = (aij)mxn, size of A is m x n, where aij is (i,j)-entry of A | | | | | | | m: num of rows | n: num of cols | | | | | | |
| Square matrix  Diagonal matrix  Identity matrix, **I**  Scalar matrix  Zero matrix, **0**  Symmetrix matrix  Upper triangular  Lower triangular | | | | | A is n x n matrix (a11,a22,...,ann are diagonal entries)  all non diagonal entries are zero, aij = 0 when i ≠ j  diagonal matrix but all diagonal entries = 1  diagonal matrix but all diagonal entries are same, aij =  all entries = 0  symmetric along diagonal, aij = aji, A = AT  entries below diagonal = 0, aij = 0 for i > j, (strictly upper => diagonal also 0)  entries above diagonal = 0, aij = 0 for i < j, (strictly lower => diagonal also 0) | | | | | | | | |
| Matrix Equality  Matrix addition and scalar multiplication obey 2 laws | | | | | | | | | | **A = B** if same size and same entries  Commutative Law: A + B = B + A  Associative Law: (A + B) + C = A + (B + C) | | | |
| Matrix multiplication (only if num of cols in A = num of cols in B)  **AB** = summation of (rowa multiply with colb)  - **A**n = **AA...A** (n times)  - **A**0 = **I**  - **A**r**A**s = **A**r+s (if both sq matrix)  - (***AB***)*n* ≠ ***A****n****B****n*  but (***AB***) (***AB***) ... (***AB***) n times | | | | | | | | | | Not commutative, **AB ≠ BA**  **A(BC) = (AB)C**, associative  **A(B1+B2) = AB1 + AB2**, distributive law  c**(AB) =** (C**A**)**B = A(**c**B)**  - **A0 = 0**  - **AI** = **A** | | | |
|  | zipping matrix  -basically just representing A by its "condensed" row/col  -same idea for zipping along col as that on RHS | | | | | | | | | | A = , where ai is row i of matrix A | | | |
|  | matrix eqn form:  Ax = b coeff mat var mat constant mat | | | | | | | | | vector eqn form: | | | | |
| Transpose & its properties | | Transpose, AT  (AT)T = A  (A+B)T = AT + BT | | | | | | | | interchange row and col of A, aij = aTji  (cA)T = cAT  (AB)T = BTAT | | | | |
| Inverse & its properties | | | Sq matrix A is invertible if B such that AB = I and BA = I  if inverse exist -> non-singular -> det ≠ 0 (inverse DNE -> singular)  -inverse are unique if A is sq matrix, if AB = I = AC, then B = C | | | | | | | | | | (aA)-1 = (1/a)A-1  (AT)-1 = (A-1)T  (A-1)-1 = A | (AB)-1 = B-1A-1  A-n = (A-1)n = (An)-1 |
| Inverse/ Reverse of ERO | | | | cRi, E.g. cR2, EA, E =  Ri Rj, E.g. R2 R3, E =  Ri + cRj, E.g. R3 + 2R1, E = | | | | | | (1/c)R2, E-1 =  R2 R3, E-1 = E  R3 - 2R1, E-1 = | | | | |
| Elementary matrix | | | -matrix obtained starting from Identity matrix and performing a single ERO (E above are e.g. of such matrix) | | | | | | | - All elementary matrix are invertible  - Their inverse are also elementary matrix | | | | |
| Row Equivalent | | | | A and B are row equivalent | | | | | | En...E2E1A = B, then A = E1-1E2-1...En-1B | | | | |
| Finding inverse | | | Ek...E2E1A = I, then Ek...E2E1I = A-1  -From A|I augmented matrix, do ERO until LHS becomes I, then RHS will be A-1  - Only square matrix have inverse  - Non-sq matrix can have at most one-sided inverse | | | | | | | - AB is invertible iff A and B are invertible  1. A is invertible  2. Ax = 0 only has trivial soln  3. RREF of A is identity matrix (no zero row)  4. A can be expressed as a pdt of elementary matrices | | | | |
| Proof of the 4 statements | | | 1 -> 2: If A is invertible, then Ax = 0 implies A-1Ax = A-10, then x = 0  2-> 3: Suppose Ax = 0 only has trivial soln. Since A is square matrix, RREF of (A|0) cannot have any zero rows and must be (I|0) | | | | | | | 3 -> 4: Since RREF of A is I, there exist E1,E2...Ek such that Ek...E2E1A = I.  Hence A = (Ek...E2E1)-1I = E1-1E2-1...En-1  4 -> 1: Suppose A is pdt of elementary matrices. Since all elementary matrices are invertible, A-1 exists | | | | |
| Using the 4 Statements | | | Let A,B be sq matrices of same size. If AB = I, prove BA = I  Consider Bx = 0. Bu = 0 => ABu = A0 => Iu = 0 => u = 0  Then Bx = 0 only has trivial soln => B is invertible | | | | | | | AB = I ABB-1 = IB-1  AI = B-1  BA = BB-1 BA = I | | | | |
| Proof certain property hold for a type of matrix | | | | | -Use mathematical Induction  1. Property hold for all 1x1 matrices, Base case  2. Show if property hold for all k x k matrices, then property hold for all (k+1) x (k+1) matrices | | | | | | | | | |
| Determinant  (only for sq matrix) | | | A = , det(A) = = ad - bc  A is invertible if and only if det(A) ≠ 0  - If A is a 1 x 1 matrix, det(A) = a11  -Cofactor expansion can be done along any row/col -> just choose row/col with most 0  - If A is a triangular matrix (upper/lower)/diagonal matrix, det(A) = pdt of diagonal entries | | | | | | | | B = ,  det(B) = a \*- b \* + c \*  M11 M12 M13  Process known as cofactor expansion, where  cofactor =Aij = (-1)i+j det(Mij) (Mij = matrix minor) | | | |
| Properties of det | | | det(cA) = cndet(A)  if A is invertible, then det(A-1) = (det(A))-1  det(A+B) ≠ det(A) + det(B)  det(A) = det(AT)  det(A) = 0 if A has [zero/2 same] row/col | | | | Proof det(A) = det(A)det(B)  Case 1: A is singular => det(A) = 0 => det(A)det(B) = 0  => AB is singular => det(AB) = 0 = det(A)det(B)  Case 2: A is invertible => A = E1E2...Ek, we know det(EB) = det(E)det(B)  det(AB) = det(E1E2...EkB) = det(E1E2...Ek)det(B) = det(A)det(B) | | | | | | | |
| Doing col ops also have same effect | | | |  |  |  | | --- | --- | --- | | A -> B | | | | det(E) | ero | Determinant | | k | kRi | det(B) = k det(A) = det(E)det(A) | | -1 | Ri <-> Rj | det(B) = - det(A) = det(E)det(A) | | 1 | Ri + kRj | det(B) = det(A) = det(E)det(A) |   Hence to find det, can just do ERO and find det of ref | | | | | | = ab-ab = 0 (Base case)  = -\* + \* - \*  (cofactor expansion along row 2). Proof by induction | | | | | |
|  | | | Proof Ri + kRj : det(B) = det(A)  A = -R2 + kR1-> = B  Note that the (2,j)-cofactor of A = (2,j)-cofactor of B (e.g. -), A21 = B21, A22 = B22, A23 = B23  det(B) = ()B21 + ()B22 + ()B23 (cofactor expansion along row 2 of B)  = ()A21 + ()A22 + ()A23  = (a21A21 + a22A22 + a23A23) + k(a11A21 + a12A22 + a13A23)  = det(A) + k(a11A21 + a12A22 + a13A23) = det(A)  And a11A21 + a12A22 + a13A23 = finding det of , which has 2 identical rows, so det = 0 | | | | | | | | | | | |
| Proof of square matrix A is invertible iff det(A) ≠ 0 | | | 1. A is invertible => det(A) ≠ 0  2. A is invertible <= det(A) ≠ 0  3. A is not invertible => det(A) = 0  4. A is not invertible <= det(A) = 0  1 & 4, 2 & 3 are contrapositive  1 & 2, 3 & 4 are converse | | | | Let Ek...E2E1A = B  det(Ek...E2E1A) = det(B) => det(Ek)...det(E2)det(E1)det(A) = det(B)  Note that det of elementary matix ≠ 0  A invertible => RREF of B = I => det(B) = 1 => det(A) ≠ 0  A not invertible => RREF of B has zero row  => det(B) = 0 => det(A) = 0 | | | | | | | |
| adjoint | | | adj(A) = =  where Aij is the (i,j)-cofactor of A = (-1)i+jdet(Mij)  For 2 x 2 matrix, = adj(A) | | | | | | | | | A[adj(A)] = det(A)I for all sq matrix  If A is invertible, then A-1 = adj(A)  A[adj(A)] : Basically diagonal entries = det(A)  : Non-diagonal entries = 0 (same logic as proof of ERO, 2 same row in matrix)  A[adj(A)] = A[adj(A)] = I | | |
| Cramer's rule | | | Suppose Ax = b, and A is an n x n invertible matrix  Let Ai be the matrix obtained from A by replacing ith col of A by b  Then sys has unique soln  x = | | | | | | | | Ax = b => x = A-1b = adj(A)b  =  xi = (b1A1i + b2A2i + ... + bnAni), for i = 1,2...n  b1A1i + b2A2i + ... + bnAni = det(Ai) (cofactor expansion of Ai along col i as we replace col i with b) | | | |

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| Vector | | | Geometrically, vector can represent point or arrow  u+v  au | | | | | | | | | | Algebraic, let v, u = (u1, u2) (v1, v2) or (u1, u2, u3) (v1, v2, v3)  (u1+v1, u2+v2) or (u1+v1, u2+v2, u3+v3)  (au1, au2) or (au1, au2, au3) | | | | | | | | |
|  | | | n-vector (u1, u2, ... un) ≠ {u1, u2, ... un }  n-vector is 1 object | | | | | | | If u u is an n-vector u = (u1, u2, ... un)  , Euclidean n-space denotes the set of all n-vector of real nums  , Euclidean 2-space, xy-plane; , Euclidean 3-space, xyz-space | | | | | | | | | | | |
| Expressing subsets of | | | implicit form  explicit form | | | | | | | | | | | | | | | S = {( u1, u2, u3, u4) | u1 = 0 and u2 = u4} (4-vector)  S = {( 0, a, b, a) | a,b } | | | |
| Num of elems in set | | | | | |S| = num of elems of S, S1 = {1,2,3,4}, S2 = {(1,2,3,4)} | | | | | | | | | | | | | |S1| = 4, |S2| = 1 | | | |
| Linear Combination | | | u1, u2, ..., uk : set of vectors in Rn  c1, c2, ..., ck: real nums  Linear combination: c1u1 + c2u2 + ... + ckuk = v | | | | | | | | | | | | | | | u1 = (2,1,0), u2 = (-3,0,1)  Specific linear combi: 1(2,1,0) + 1(-3,0,1)  General linear combi: s(2,1,0) + t(-3,0,1) | | | |
| To find specific a,b,c | | | (3,3,4) = a(2,1,3) + b(1,-1,2) + c(3,0,5)  = , so just RREF | | | | | | | | | If sys is consistent, v is a linear combination of u1, u2, u3  i.e. there is specific values for a, b, c  If sys is inconsistent, v is not a linear combination of u1, u2, u3 | | | | | | | | | |
| Standard basis | | | | e1 = (1,0,0,...,0), e2 = (0,1,0,...,0), en = (0,0,...,1) | | | | | | | | | | | | | |  | | | |
| Span | | | span{u1, u2} = linear span of u1and u2 = {su1+ tu2| s,t } = set of all linear combination of u1 and u2  If v is a linear combination of u1, u2, then v span{u1, u2}  If w not linear combination of u1, u2, then w span{u1, u2} | | | | | | | | | | | | | | If S = {u1, u2, ..., uk}  Linear span of S = span{ u1, u2, ..., uk } = span(S)  u1, u2, ..., uk , S , span(S), S span(S)  span(S) can be equal to but not always | | | | |
| E.g. | | | S = {(1,0,0,-1), (0,1,1,0)} , span(S)  span(S) = span{(1,0,0,-1), (0,1,1,0)} (linear span form)  = {a(1,0,0,-1) + b(0,1,1,0) | a,b } (set notation)  = {(a,b,b,-a)|a,b } (explicit form) | | | | | | | | | | | | | | V = span{(2,1,-1), (1,0,3)}  V is spanned by (2,1,-1), (1,0,3)  (2,1,-1), (1,0,3) spans V | | | | |
| To show linear span equal to | | | Show every vector (x,y,z) in can be written as linear combination of the vectors  E.g. u1 = (1,0,1), u2 = (1,1,0), u3 = (0,1,1)  (x,y,z) = au1 + bu2 + cu3  -If bottom row is zero row with non-zero constant, then there exists some x,y,z that sys is inconsistent | | | | | | | | | | ->  Sys is consistent regardless of x,y,z. So span{u1, u2, u3} =  -If num of vectors = k < n = num of var, then S cannot span Rn, span(S) ≠ Rn | | | | | | | | |
|  | | | 0 vector always span(S) (c1 = c2 =...= ck = 0)  If u and v span(S), then u+v span(S) (closure under addition)  If u span(S) and c , then cu span(S) (closure under multiplication) | | | | | | | | | | | | | | | | | | Let S = {u1, u2,..., uk}  If v1, v2,...vk span(S), c1, c2,..., ck  then c1v1 + c2v2 + ... + ckvk span(S) |
| To show 2 linear spans related | | | Given 2 sets, A, B, to show A B.  Show every vector in A can be written as a linear combination of vectors in B  To show A = B, need show A B and B A | | | | | | E.g. span{(1,0,1), (1,1,2), (-1,2,1)} span{(1,2,3),(2,-1,1)}  ->  sys is consistent, every ui can be av1 + bv2 for some real num a,b  So span{u1,u2,u3} span{v1,v2} | | | | | | | | | | | | |
| Redundant vector | | | Suppose u1, u2,..., uk are vectors in  If uk is a linear combi of u1, u2,...uk-1,  i.e. uk = c1u1 + c2u2 + ... + ck-1uk-1 | | | | | | | | | | | | | | Then span{u1, u2, ..., uk-1} = span { u1, u2, ..., uk-1, uk}  - uk is a redundant vector in span { u1, u2, ..., uk-1, uk}  - If u span(S), then span(S) = span(S u) | | | | |
| Geometrical meaning of linear span | | | In and , let S = {u}, where u is a non-zero vector  Then span(S) = span{u} = {cu|c }  span(S) represents a line through origin  x + span{u} = {x + cu|c } represent line not through origin | | | | | | | | | | | | In & , let S = {u, v}, where u,v is a non-zero vector  Then span(S) = span{u, v} = {su + tv|s, t }  span(S) represents a plane through origin (u≠kv)  x + span{u,v} = {x + su + tv|s,t } represent plane not through origin | | | | | | |
| Subspace | | | Let V be a subset of  V is a subspace of provided there is a set S = { u1, u2,..., uk} of such that V = span(S)  i.e. V can be expressed in linear span form  Every subspace of is a subset of , but not every subset of is a subspace of | | | | | | | | | | | | | | {0} is a subspace of . Take S = {0}.  Then {0} = span(S)  is a subspace of . Take S to be standard basis vectors for .  e1 = (1,0,...,0), e2 = (0,1,...,0)..., en = (0,...,0,1).  Then = span{e1, e2, ..., en} | | | | |
| To show given subset is a subspace | | | V1 = {(a+4b, a)|a,b }  V1 = span{(1,1), (4,0)}  Thus, V1 is a subspace of , In fact, V1 = | | | | | | | OR do  1. check subset V contains 0 vector  2. take 2 vector u,v V, c,d , show cu+dv V | | | | | | | | | | | |
| To show subset not a subspace | | | V2 = {(1,a)|a }. (1,a) = (1,0) + a(0,1)  V2 is not a linear span of "any" set of vectors V2 not a subspace of  OR (0,0) V2 = {(1,a)|a } V2 not subspace of | | | | | | | | | | | | | | | | | V4 = {(x,y,z)|x2 ≤ y2 ≤ z2}  e.g. (1,1,2), (1,1,-2), (0,0,0) V4  But (1,1,2) + (1,1,-2) = (2,2,0) V4 (violate closure under addition) V4 not subspace of | |
| Geometrical interpretation | | | All subspaces of  a. {0}, spanned by zero vector 0  b. line through the origin, spanned by a non-zero vector u  c. , spanned by 2 non-parallel vectors, u,v | | | | | | | | All subspaces of  a. {0}, spanned by zero vector 0  b. line through origin, spanned by a non-zero vector u  c. plane containing origin, spanned by 2 non-parallel vectors, u,v  d. , spanned by 3 non-coplanar vectors, u,v,w | | | | | | | | | | |
| Solution space | | | Ax = 0. Soln set of homogeneous linear sys in n var is a subspace of  Soln set of every homogeneous LS can be written as a linear span = solution space  Soln set of non-homogeneous LS is not a subspace of | | | | | | | | | | | | | | | | | | |
| Summary | | -Show S is subspace of  1. express S as linear span  2. Check 0 vector and closure under addition/multiplication  3. Show S is soln set of homogeneous sys  4. For show S represent line/plane through origin | | | | | | | | | | | | | | | - Show not a subspace of  1. Show 0 vector not in S  2. Find u,v S s.t. u+v S  3. Find v S and scalar c s.t. cu S  4. For show S is not a line/plane through origin | | | | |
| Linear Independence | | | Let S = {v1, v2,...vk} be subset of vectors in  S is linearly independent if c1v1 + c2v2 +... + ckvk = 0 has only the trivial soln c1 = c2 = ... = ck = 0 | | | | | | | | | | | | | | rref([v1 v2 ... vk]), if all pivot cols = linearly indep.  No vector in S can be written as linear combi of other vectors in S. Empty set is LI | | | | |
| Linear dependent | | | non-pivot cols = linearly dependent ( redundant vector) or vk is a linear combi of v1,...vk-1 or  homogeneous sys has non-trivial soln  S = {0} is linearly dependent as c10 = 0 (c1 )  Any S with 0 in it must be linearly dependent | | | | | | | | | | | | | | If S only contains 2 vector, if they are scalar multiples of one another = linearly dependent  Let S = {u1, u2,...,uk} be set of vectors in , if k > n more var than eqn sys has non-trivial soln non-pivot col linearly dependent | | | | |
| Extending linearly independent set | | | u1, u2, ..., uk are linearly independent -- (a)  If uk+1 not linear combination of u1, u2, ..., uk -- (b)  then u1, u2, ..., uk, uk+1 are linearly independent  Prove by contradiction | | | | | | | | | Suppose u1, u2, ..., uk, uk+1 are linearly dependent  Then c1u1 + c2u2+...+ck+1uk+1 = 0, for some c1, c2,..., ck+1 not all 0  If ck+1 = 0, c1u1 + c2u2 + ... ckuk = 0 contradicting (a)  If ck+1 ≠ 0, c1u1 + c2u2 + ... ckuk = - ck+1uk+1 contradicting (b) | | | | | | | | | |
| Vector Space | | | Set V is vector space if V = or V is subspace of | | | | | | | | | | | | | | , {0}, span{(1,2,3),(2,1,4)} all subspace of , are also vector space | | | | |
| Basis | | | Let T be a subset of vectors in . T is a basis for vector space V if  V = span(T) & T is linearly independent  T is the smallest possible subset of  Vectors in T are not unique. A = (u1, u2,...uk) is invertible | | | | | | | | | | | | | | Check T linearly independent (trivial soln)  Check span(T) = V (consistent sys)  So if k > n: linearly dependent, k < n: cannot span,  k = n: possible basis  Basis for zero space {0} is empty set | | | | |
| Coordinate vectors | | | Let S be basis for vector space V  Every vector v in V can be expressed in form v = c1u1 +...ckuk (span) in exactly one way (linear indep.)  i.e. v has unique linear combi of vectors in S  Proof (Suppose v can be expressed in 2 linear combi)  v = c1u1 +...ckuk, v = d1u1 + ... + dkuk  Subtract both: (c1-d1)u1 + ... + (ck-dk)uk = 0  Since S is linearly indep. (since basis), c1-d1 =...= ck-dk = 0  So c1 = d1,..., ck = dk and expression is unique | | | | | | | | | | | | | | coordinate vector of v relative to S = (v)s =  (c1, c2, ..., ck)  c1, c2, ..., ck, are the coordinates of v relative to basis S  For any u, v V, u = v iff (u)s = (v)s  For any v1, v2,...,vr V and c1, c2,..., cr  (c1v1 + c2v2 + ... + crvr)s = c1(v1)s + c2(v2)s + ... + cr(vr)s,  coordinate vector of linear combi = linear combi of coordinate vectors  - (v)s is row form. [v]s is column form | | | | |
| Properties of coordinate vectors | | | Let S be basis for vector space V with |S| = k. Let v1, v2, ..., vr V. Then  1. {v1, v2, ..., vr} is basis for V iff {(v1)s, (v2)s, ..., (vr)s} is basis for | | | | | | | | | | | | | 2. v1, v2, ..., vr are linearly indep/dependent in V iff (v1)s, (v2)s, ..., (vr)s are linearly indep/dependent in  3. span{v1, v2, ..., vr} = V iff span{(v1)s, (v2)s, ..., (vr)s} = | | | | | |
| Dimensions | | | Let V be vector space with basis S = {u1,... uk} with k vectors. dim(V) = k. dim({0}) = 0  1. Any subset of V with > k vectors is linearly dependent  2. Any subset of V with < k vectors cannot span V | | | | | | | | | | | | | | dim 0 = origin, dim 1 = line through origin  dim 2 = plane containing origin  For homogeneous sys -> reduced to REF:  # of non-pivot cols = # of params in general soln = # of vectors in basis for soln space = dim of solution space | | | | |
| Verifying basis | | | Let V be vector space of dim k and S be subset of V  Statement 1,2,3 are equivalent  1. S is basis for V  2. S is LI and |S| = k = dim(V)  3. S spans V and |S| = k = dim(V)  Proof. 1 2 and 1 3 as that is defn of basis | | | 2 1: Suppose S not a basis for V. Given S is LI and |S| = k  span(S) ≠ V. There is a vector u in V and u span(S)  Let S' = S {u} where S' is LI.  But, S' has k+1 vectors S' is not LI (contradiction). So S must be basis for V  3 1: Suppose S not a basis for V. Given S spans V and |S| = k  S is not LI. There is redundant vector v in S  Let S'' = S - {v} span(S'') = span(S) = V  But, S'' has k-1 vectors span(S'') ≠ V (contradiction). So S must be basis for V | | | | | | | | | | | | | | | |
|  | | | Let U and V be subspaces of Rn.  Let U be subspace of V  (i) If U V, then dim(U) ≤ dim(V)  (ii) If U V and U ≠ V, then dim(U) < dim(V)  (iii) If U V and dim(U) < dim(V), then U = V | | | | | | | Proof. Let basis of U = {u1...uk}  For (i), dim(U) = k, u1,...uk must also be in V. So dim(U) = k ≤ dim(V)  For (ii), suppose dim(U) = dim(V).Then dim(V) = k. Then V = span{u1...uk} = U (contradiction) | | | | | | | | | | | |
| Invertibility | | | 1. A is invertible  2. Ax = 0 has only trivial soln  5. det(A) ≠ 0  6. Rows of A form a basis for Rn  7. Columns of A form a basis for Rn | | | | Proof. 1,2 7. Suppose A = (u1, u2, u3), where ui represent column i. From 1 and 2, we know (u1, u2, u3)x = 0 xu1 + yu2 + zu3 = 0 x = y = z = 0  Then u1, u2, u3 are LI and must form basis for R3  1,5,7 6. Suppose A = (u1, u2, u3). We know det(A) ≠ 0. det(AT) ≠ 0  So columns in AT form basis for R3. But columns in AT = rows in A  So rows in A also form basis for R3 | | | | | | | | | | | | | | |
| Transition matrix | | | Let S = {u1...uk} and T = {v1...vk} be 2 bases for vector space V. Let w V. [w]T = P[w]S for some fixed k x k matrix P, where P = transition matrix from S to T  1. Express each ui as LC of {v1...vk}  2. P = ([u1]T, [u2]T, ... [uk]T)  Theorem: a. P is always invertible  b. P-1 is the transition matrix from T to S | | | | | Proof. (a) Since S is basis S is LI [u1]T, [u2]T, ... [uk]T are LI (property of coordinate vectors) P is invertible  (b) Let Q be transition matrix form T to S, Q = ([v1]S, ... [vk]S). Note [u1]S, [u2]S...[uk]S are just as u1 = 1u1 + 0u2 +...+ 0uk (since all LI)  Observe any m x n matrix A, A[ui]S = ith col of A  So ith col of QP = QP[ui]S = Q[ui]T = [ui]S which is just a standard basis  So QP = ([u1]S, [u2]S, ... [uk]S) = I P-1 = Q | | | | | | | | | | | | | |
| Row space & Column space | | Let A be m x n matrix  Row space of A: space span by rows of A, subspace of Rn  Column space of A: space span by cols of A, subspace of Rm  Row space of A = col space of AT  Col space of A = row space of AT  Row/col space of zero matrix 0 = zero space  Row/col space of n x n identity matrix I = Rn | | | | | | | | | | | | | | | Let A, B be row equivalent matrices,  then row space of A = row space of B  ERO do not change row space of matrix  Proof. Let a1, a2, ..., an be rows of matrix  1. span{a1, a2, ..., ai,..., an} = span{a1, a2, ..., cai,..., an }  2. span{a1, ..., ai, aj, ..., an} = span{a1, ..., aj, ai, ..., an }  3. span{a1, ..., ai,..., an} = span{a1, ..., ai + caj,..., an } | | | | |
| Finding basis for row/col space | | | From REF of A: non-zero rows will form basis for row space of A  However, this basis may not contain original rows of A  ERO may not preserve col space of matrix (ECO will)  Columns of A that correspond to pivot cols of REF of A form basis for col space of A | | | | | | | | | | | | Row operations preserve linear relations among columns  Idea of Proof. Suppose col1 + col2 = col3  1. 2.  3. | | | | | | |
| Finding basis from a spanning set | | | Suppose we need find basis for span{u1, u2, u3}  1. Row space mtd: Let A = . RREF(A) and choose non-zero rows for basis (might not get original rows) | | | | | | | | | | | | | | 2. Column space mtd. Let B = (u1 u2 u3). RREF(B) and choose col from B corresponding to pivot cols in RREF  Will get original vectors | | | | |
| Extending a set to a basis | | | -Adding non-redundant vectors to S to form basis for Rn  1. Form matrix A using vectors in S as rows  2. Reduce A to REF, R  3. Identify non-pivot cols of R  4. Form vectors with leading entries at non-pivot cols of R  5. Put basis for row space + newly formed vectors | | | | | | | | | | | | | | S = {(1,4,-2,5,1), (2,9,-1,8,2), (2,9,-1,9,3}  A = --> = R  basis for R5 = {(1,4,-2,5,1), (2,9,-1,8,2), (2,9,-1,9,3), (0,0,1,0,0), (0,0,0,0,1)} | | | | |
| SLE | Ax = b. Sys has no sol? unique soln? infinite soln?z   |  |  | | --- | --- | | 1. Form (A|b) | Look at REF | | 2. If A is sq matrix | A is invertible => unique soln  A is singular => no or infinite soln | | 3. A is any matrix | b belongs to col space of A => unique or infinite soln  b don't belong to col space of A => no soln | | | | | | | | | | | | | | | | | | | Ax = b has soln b cab be written as LC of cols of A b belongs to col space of A  Let A be m x n matrix. Col space of A = {Au|u Rn} (C1|C2|...|Cn) = xC1 + yC2 + .. + zCn span{C1, C2, ..., Cn} = {all LC of col vectors in A}  Ax = b consistent iff b lies in col space of A | | |
| Ranks | | | Row space and col space of matrix have same dim  dim of row space of A = num of nonzero rows of REF = num of leading entries of REF = num of pivot cols of REF = dim of col space of A  Rank of matrix: dim of row space / col space  rank(A) = largest num of LI rows in A = largest num of LI cols  rank(0) = 0, rank(I) = n  For m x n matrix A, rank(A) ≤ min{m, n}  A is full rank iff rank(A) = min{m, n}  Sq matrix A is full rank iff det(A) ≠ 0 A is invertible RREF of A is I  rank(A) = rank(AT) for any matrix A  Ax = b is consistent iff rank(A) = rank(A|b) col b not pivot col | | | | | | | | | | | | | | rank(AB) ≤ min{rank(A), rank(B)}  Proof. Let A be m x n matrix, B be n x p matrix  Let B = (b1 b2 ... bp). AB = (Ab1 Ab2 ... Abp) where Abi is ith col of AB  Abi col space of A (just LC with bi as coefficients)  span{Ab1, Ab2, ..., Abp} col space of A  col space of AB col space of A  dim(col space of AB) ≤ dim(col space of A)  So rank(AB) ≤ rank(A)  From rank(XY) ≤ rank(X) rank(BTAT) ≤ rank (BT)  rank(AB) = rank((AB)T) = rank(BTAT) ≤ rank (BT)  Thus rank(AB) ≤ min{rank(A), rank(B)}  - col space of AB col space of A  col space of (AB)T = col space of BTAT col space of BT  row space of AB row space of B | | | | |
| Nullspaces & Nullities | | | Let A be m x n matrix  Nullspace of A = sol space of Ax = 0, and is subspace of Rn  nullity(A) = dim of nullspace of A = num of non-pivot cols = num of parameters in general soln ≤ n  Express general soln of Ax = 0, in vector eqn form to get basis for nullspace, num of vectors = nullity  Dimension Theorem, rank(A) + nullity(A) = n  rank(A) = pivot cols + nullity(A) = non-pivot cols = n | | | | | | | | | | | Suppose Ax = b has a particular soln v  Soln set of Ax = b = {u+v|u nullspace of A}  General soln of Ax = b = [(general soln of Ax = 0) + v]  Proof. Let T = sol set of Ax = b. S = {u+v|u nullspace of A}  Since A(u+v) = Au + Av = 0 + Av = b, S T  Let w T. We know Aw = b and Av = b. So A(w-v) = Aw - Av = b - b = 0. So w-v nullspace of A  i.e. w-v = u w = u+v w S T S  Combining, T = S | | | | | | | |
| Let Ax = b be consistent linear sys. Ax = b has only 1 soln iff nullspace of A = {0} | | | | | | | | | | | | | | | | | | |
| Solution set of homogeneous linear sys Ax = 0 is always a subspace of Rn, where A is m x n matrix = nullspace of A | | | | | | | | | | | | | | | | | | |

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| Inner/ dot/ scalar product | | ||u|| = length of u =  cos = (derived from cosine rule)  (uv = u1v1 + u2v2 +... + ukvk)  And uu = u12 + u22 + ... + un2  Unit vectors: vectors with norm/ length = 1, i.e. ||u|| = 1  If u, v are row vectors, uv = uvT  If u, v are col vectors, uv = uTv | | | | | | 1. uv = vu (commutative law)  2. (u + v)w = uw + vw. w(u + v) = wu + wv  3. (cu)v = u(cv) = c(uv)  4. ||cu|| = |c| ||u|| (|c| is abs value)  5. uu ≥ 0. uu = 0 iff u = 0  - Av = 0 iff ATAv = 0  Proof. Av = 0 ATAv = AT0 = 0 ATAv = 0 (LHS - RHS)  ATAv = 0 vTATAv = vT0 (Av)T(Av) = 0  (Av)(Av) = 0 Av = 0. (RHS - LHS) |
| Orthogonal/ Orthonormal set | | 1. 2 vectors u, v are orthogonal if uv = 0 (  cos-1() = cos-1(0) = π/2 = perpendicular)  2. Set S of vectors is orthogonal is every pairs of vectors in S are orthogonal (i.e., u1u2 = 0, u1u3 = 0, ..., uk-1vk = 0)  3. Set S of vectors is orthonormal if S is orthogonal and every vector in S is a unit vector  Standard basis if orthogonal and orthonormal set  Orthogornal set: {u1, u2, ... uk}  For i ≠ j, vivj = (ui)(uj) = (uiuj) = 0  Orthonormal set: {u1, u2, ...,uk} | | | | | | Let S be orthogonal set of nonzero vectors in vector space, then S is linearly independent  Proof. Let S = {u1, u2, ... uk} orthogonal set  c1u1 + c2u2 + ... + ckuk = 0  (c1u1 + c2u2 + ... + ckuk) u1 = 0 u1  c1(u1 u1) + c2(u2 u1) +...+ ck(uk u1) = 0  c1(u1 u1) = 0 c1 = 0 (since u1 ≠ 0 so ||u1||2 > 0)  Similarly, can show c2 = ... = ck = 0  1. Orthogonal basis: basis S for vector space is orthogonal  2. Orthonormal basis: basis that is orthonormal |
| Check if set if orthogonal basis | | To check if set is orthogonal basis:  Let S be set of nonzero vectors in vector space V  (i) S is orthonormal and  (ii) span(S) = V OR  (i) S is orthonormal and  (ii) |S| = dim V | Let S {u1, u2,... uk} be orthogonal basis for V.  For any vector w in V, w = c­1u1 + c2u2 + ... + ckuk  (w)s = (c1 c2 ... ck) = ( ... )  Let w = c1u1 + c2u2 + ... + ckuk  w u1 = (c1u1 + c2u2 + ... + ckuk) u1 = c1(u1 u1) + c2(u2 u1) +...+ ck(uk u1)  = c1(u1 u1) = c1||u1||2  So c1 = . Same for c2... ck  If S is orthonormal basis, ||ui||2 = 1 for all i | | | | | |
| Finding normal to subspace | | Let V be subspace of Rn. Vector n is orthogonal (normal) to subspace V if u is orthogonal to all vectors in V  V has eqn ax + by + cz = 0  normal vector = n = (a, b, c)  For any vector v (x0, y0, z0) in V, nv = ax0 + by0 + cz0 = 0  So n is orthogonal to every vector v in V | | | | | | To find vector v that is orthogonal to subspace V = span{u1, u2, ... uk} in Rn  1. Let v = (x1, x2, ... xn)  2. Convert v u1 = 0, v u2 = 0,..., v uk = 0 into homogeneous sys  3. Solve LS |
| Projection of vector onto subspace | | Let V be subspace of Rn and w a vector in Rn  p = the projection of vector w onto subspace V  w - p is orthogonal to V  Projection of w onto V is unique, i.e. p is unique  1. S = {u1, u2, ..., uk}: an orthogonal basis for V  p = u1 + u2 + ...+ uk  2. T = {v1, v2, ..., vk}: an orthonormal basis for V  p = (w v1)v1 + (w v2)v2 + ... + (w vk)vk | | | Proof. Let {u1, u2, ..., uk} be orthogonal basis for V  Let p = u1 + u2 + ...+ uk  (w - p) u1 = w u1 - p u1  = w u1 - (u1 + u2 + ...+ uk) u1  = w u1 - u1 u1 = w u1 - ||u1||2 = 0  u1 + u2 + ...+ uk = | | | |
| Convert a basis to orthogonal basis | | Use Gram-Schmidt Process (project vector to subspace)  Let {u1, u2, ..., uk} be basis for vector space V  v1 = u1, v2 = u2 - v1 (orthogonal to v1)  v3 = u3 - v1 - v2 (orthogonal to v1 and v2) | | | | | | vk = uk - v1 - v2 - ... - vk-1  Then {v1, v2, ..., vk} is orthogonal basis for V  w1 = v1, w2 = v2, ... wk = vk,  Then {w1, w2, ..., wk} is an orthonormal basis for V |
| Best Approxi-mations | Let V be subspace in Rn and u Rn  p: projection of u onto V = p is best approximation of u in V  dist(u, p) ≤ dist(u, v) for any v in V (i.e. least dist from u to V)  dist2(u, v) = ||x||2 = ||n+w||2 = ||n||2 + ||w||2 ≥ ||n||2  So ||n|| ≤ ||x|| | | | | | | | Suppose Ax = b is inconsistent Ax - b ≠ 0  Then the best approximate soln = x0 where  ||Ax0 - b|| is the smallest  x0 is the least square soln to Ax = b  A least sq soln of Ax = b is a vector u in Rn that minimise ||b-Ax||, i.e. ||b-Au|| ≤ ||b-Av|| v in Rn  ATAu = ATb |
|  | Find least sq soln of Ax = b  Find u that minimise ||b-Ax|| (Ax = projection p of b onto col space of A)  Find u s.t. Au = p (always consistent since p lies on col space of A)  Suppose A = (u1 u2 u3) Ax = cu1 + du2 + eu3 (LC of cols of A)  All Ax belongs to col space of A | | | | | | | Suppose u is least sq soln of Ax = b  u is soln of Ax = p (p = projection of b onto col space of A)  Au = p  A(least sq soln) = projection  least sq soln may not be unique |
|  | Suppose u is least sq soln of Ax = b (A = (a1 a2 a3))  iff u is soln to ATAx = ATb  Au is projection of b onto V (V = col space of A)  b - Au is orthogonal to V  b - Au is orthogonal to a1, a2, a3 | | | | | AT(b - Au) = (b - Au) = (dot product)  ATb - ATAu = 0 ATAu = ATb  u is soln to Ax = p (p = projection of b onto col space of A) | | |
| Orthogonal Matrices | | Sq matrix A is orthogonal matrix if A-1 = AT AAT = I or ATA = I (i.e. all orthogonal matrices are invertible)  Let A be sq matrix of order n  1. A is orthogonal matrix  2. Rows of A forms orthonormal basis for Rn  3. Cols of A form orthonormal basis for Rn | | | | | Proof. A = , AT = (a1T, a2T, a3T)  AAT = = = I  = 1 for all i ||ai|| = 1  = 0 for i ≠ j ai and aj are orthogonal  So {a1, a2, ..., an} is orthonormal basis for Rn  Rows of AT = Cols of A will also form basis for RN | |
| Transition matrix btw orthonormal bases | | S = {u1, u2, ..., uk}. T = {v1, v2, ..., vk}  P = ([u1]T [u2]T ... [uk]T). Then [w]T = P[w]S  Suppose S and T are 2 orthonormal bases for a vector space  Then transition matrix P from S to T is orthogonal.  So PT is transition matrix from T to S | | Proof. S = {u1, u2, ..., uk}. T = {v1, v2, ..., vk} are orthonormal bases  u1 = (u1 v1)v1 + (u1 v2)v2 + ... + (u1  uk)vk  ...uk = (uk v1)v1 + (uk v2)v2 + ... + (uk  uk)vk  transition matrix from S to T = P =  Q = P-1 = = PT | | | | |
| Rotation of xy-coordinates | S = {(1, 0), (0, 1)}, T = {u1, u2} = {(cos , sin ), (-sin , cos )}  v = [v]S (since standard basis)  [v]T = PT[v]S where P = , PT = | | | | | | | T = new coordinate sys  [v]T = rotating xy-coordinate anticlockwise by  = rotate vector clockwise by |

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| Eigenvalues, Eigenvectors, Eigenspace | | Diagonalizing a sq matrix: A = PDP-1 (D is diag matrix)  An = PDnP-1  Let A be sq matrix of order n, x be nonzero col vector in Rn  If Ax = x for some scalar , x is an eigenvector of A  is eigenvalue of A associated with eigenvector x  A = (x1 x2)(x1 x2)-1 (where xi is eigenvector associated with eigenvalue i) | | If A is a triangular matrix, eigenvalues of A = diag entries  Proof. A = , I - A =  characteristic polynomial of A = det(I - A) = ( - a11)( - a22)...( - ann) | | | |
| Finding eigenvalues | | Let A be sq matrix of order n, is eigenvalue of A  Ax = x, for some nonzero col vector x  x - Ax = 0 (I - A)x = 0 (homog sys has non-trivial soln, x) det(I - A) = 0 | | | | Characteristic polynomial of A = det(I - A)  is an eigenvalue of A det(I - A) = 0  is a root of the characteristic polynomial | |
|  | | det(A) ≠ 0 0 is not an eigenvalue of A  Proof. 0 is not an eigenvalue of A  0 not a root of char polynomial (det(I - A) ≠ 0)  det(0I - A) ≠ 0 det(-A) ≠ 0 (-1)ndet(A) ≠ 0  det(A) ≠ 0 (not invertible) | | | | Finding eigenvectors.  Ax = x, for some nonzero col vector x  x - Ax = 0 (I - A)x = 0  Then just solve this homogeneous sys to find x | |
| Eigenspace | | = eigenspace of A associated with eigenvalue = soln space of LS (I - A)x = 0 (has nontrivial soln)  If u is a nonzero vector in , then u is an eigenvector in A associated with eigenvalue | | | | Just find general soln of (I - A)x = 0, then eigenspace is span by the vector  Although 0 in eigenspace, 0 cannot be eigenvector as eigenvector always nonzero vector | |
| Diagonal-ization | | A square matrix A is diagonalizable if an invertible matrix P s.t. P-1AP is a diagonal matrix, i.e.  A = PDP-1 or P-1AP = D  Matrix P diagonalizes A | | | | Let A be sq matrix of order n. A is diagonalizable iff A has n linearly independent eigenvectors.  Note that BD = (b1 b2 ... bn)D = (d1b1 d2b2 ... dnbn) if D is a diagonal matrix with diagonal entries d1 d2... dn | |
|  | | Proof. Suppose A has n LI eigenvectors, u1, u2, ..., un  with associated eigenvalues . Let P =  (u1 u2 ... un). AP = (Au1 Au2 ... Aun) = (u1 u2 ... un) =  (u1 u2 ... un) = P  Then P-1AP = D. So A is diagonalizable | | | | A is diagonalizable P-1AP = D. Let P = (u1 u2 ... un).  AP = PD A(u1 u2 ... un) = (u1 u2 ... un)  (Au1 Au2 ... Aun) = (u1 u2 ... un)  Comparing cols on LHS and RHS, Aui = ui for all i  ui are eigenvectors of A with eigenvalues | |
|  | | Check if A is diagonalizable  1. Solve det(I - A) = 0 to find all eigenvalues  2. For each eigenvalues, find basis for eigenspace  3. Let S = ... . (S is always LI)  a) If |S| < n, A is not digonalizable  b) If |S| = n, A is diagonalizable | | | | det(I - A) = ..., where ri is known as multiplicity, then dim() ≤ ri  i.e. num of basis vectors in each eigenspace cannot be more than multiplicity of the eigenvalue in the characteristic polynomial  A is diagonalizable iff dim() = ri for all | |
|  | | Let A be sq matrix of order n.  If A has n distinct eigenvalues, A is diagonalizable  Proof. We can find 1 eigenvector for each eigenvalue.  i.e. We have n eigenvectors, where set S containing all eigenvectors is LI and hence |S| = n, and A is diagonalizable | | | | | Diagonal matrices are diagonalizable  Converse may not be true. If A is n x n diagonalizable matrix, A need not have n distinct eigenvalues |
|  | | Let A = PDP-1. Then An = PDnP-1 | In general, a0 = s, a1 = t, an = pan-1 + qan-1  Then recurrence matrix A = , = An = PDnP-1 | | | | |
| Orthogonal Diagonali-zation | | A sq matrix A is orthogonally diagonalizable if an orthogonal matrix P s.t. PTAP is a diagonal matrix  Matrix P orthogonally diagonalizes A | | Sq matrix is orthogonally diagonalizable iff it is symmetric  Proof. Let A be orthogonally diagonalizable  PTAP = D A = PDPT AT = (PDPT)T AT = (PT)T(DT)(PT) AT = PDPT = A. So A is symettirc | | | |
|  | 1. Solve det(I - A) = 0 to find all eigenvalues  2. For each , a) find basis for eigenspace  b) Gram-Schmidt to transform into orthonormal basis  3. Let T = ... . T = {v1, v2, ..., vn}. (T is orthonormal)  Then P = (v1 v2 ... vn) is orthogonal matrix that diagonalizes A | | | | Eigenvalues of symmetric matrix are always real nums  Let A be a symmetrix matrix, and det(I - A) = ..., then  dim() = ri, i.e. A is always diagonalizable  r1 + r2 + ... + rk = order of A  dim+ dim+ ... + dim = num of LI eigenvectors | | |

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| Linear Transform-ation | T: U V (mapping from U to V)  U: Domain of T, V: Codomain of T (actual images under T)  E.g. T: R2 R2 defined by T(u) = Au u in R2  T = for all R2  T: Rn Rm defined by T(u) = Au u in Rn  T is a linear transformation from Rn to Rm. A is the standard matrix of the linear transformation | | | I: Rn Rn: the identity transformation, i.e. I(u) = u, standard matrix is identity matrix  O: Rn Rm: the zero transformation, i.e. O(u) = 0, standard matrix is 0m x n  If T: Rn Rm is a linear transformation, then  1. T(0) = 0, i.e. A0 = 0  2. T(c1u1 + c2u2 + ... + ckuk) = A(c1u1 + c2u2 + ... + ckuk) = c1Au1 + c2Au2 +...+ ckAuk = c1T(u1) + c2T(u2) +...+ ckT(uk)  OR T(au + bv) = aT(u) + bT(v) |
|  | If linear transformation T: Rn Rn, i.e. domain = codomain, then T is a linear operator on Rn, and standard matrix for T is a sq matrix | | | If given T(u1) = v1, T(u2) = v2, T(u3) = v3  Can find image of any other vector if u1, u2, u3 form basis for R3. E.g. w = u1 + 2u2 + 3u3. Then T(w) = T(u1) + 2T(u2) + 3T(u3) = v1 + 2v2 + 3v3 |
|  | Given T(u1) = v1, T(u2) = v2, T(u3) = v3, to find formula for T,  1. Direct Gaussian elimination  = c1u1 + c2u2 + c3u3, to find ci  Then T = c1v1 + c2v2 + c3v3 | | | 2. Find T(e1), T(e2) , T(e3)  Note T(ei) = Aei = ith col of A. So A = (T(e1) T(e2) ... T(en))  Find e1, e2, e3 in terms of u1, u2, u3  Then T(e1) = c1v1 + c2v2 + c3v3 |
|  | 3. Stack matrices  We know Au1 = v1, Au2 = v2, Au3 = v3  So A =  Then A = | | | Let S: Rn Rm and T: Rm Rk be linear transforamtions  Composition of T with S, (T S)(u) = T(S(u)) u in Rn  (T S) = T to find final formula  OR (T S)(u) = T(S(u)) = T(Au) = B(Au) = (BA)u  (T S) = BA, BA is standard matrix of T S |
| Range | T: R2 R2 linear transformation  Range of T are whole R2, line or origin  S: R3 R3 linear transformation  Range of S are whole R3, plane, line or origin | | | T: Rn Rmis linear transformation  Range of T = R(T) = set of images of T = {T(u) |u Rn} (explicit set notation), R(T) Rm and R(T) codomain of T |
|  | E.g. T = = x + y  Then R(T) = = span  explicit set notation linear span form  Note that standard matrix A = , so R(T) = col space of A. So R(T) is subspace of Rm | | rank(T) = dim of R(T) = dim of col space of A = rank(A)  So finding a basis for range of T = finding basis for col space of A  R(T) = span{cols of A} = span{T(e1), T(e2), ..., T(en)}  If {u1, u2, ..., un} is basis for Rn, R(T) = span{T(u1), T(u2), ..., T(un)}  1. If formula of T:Rn Rm is given  R(T) = {formula in x1, x2, ..., xn|x1, x2, ..., xn R}  2. If standard matrix A given  R(T) = span{cols of A} or derive formula of T  Find basis for col space of A for 1. and 2.  3. If image of basis {u1, u2, ..., un} for Rn  R(T) = span{T(u1), T(u2), ..., T(un)}  Find basis by throwing out redundant vectors | |
| Kernel | Let T: Rn Rm. The kernel of T = ker(T) = set of vectors in Rn whose image is zero vector in Rm = { u Rn| T(u) = 0}  ker(T) Rn | | E.g. T = . To find kernel, let =  Find all that satisfy this homogeneous sys  ker(T) = all u s.t T(u) = 0 = all u s.t. Au = 0 = soln space of Ax = 0 = nullspace of A, ker(T) is subapce of Rn | |
|  | dim of ker(T) = nullity(T) = nullity(A)  rank(T) + nullity(T) = rank(A) + nullity(A) = n | Proving qns. Let T: Rn Rm be linear transoformation   |  |  |  |  | | --- | --- | --- | --- | |  | ker(T) = { u Rn| T(u) = 0} | R(T) = {T(u) |u Rn} |  | | Given | v ker(T) | v R(T) | WTS | | Follow up with | T(u) = 0 | v = T(u) for some u Rn | try to show | | | |