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| Sets  A set is a collection of objects.  - denoted by capital letters A, B, C,  - The objects a,b,c,... in A are called the elements of the set A. | | | - A = {a,b,c,...}. - A = {x | properties of x}. E.g. {x | x2 = 1}, {x | x is a prime number}.  A = B if all elem are same, regardless of order  = empty set(no elem) | | | | | |
| elem of: | | subset (all elem in A also elem of B): | | | = integers = {0, } | | | [a,b]: a ≤ x ≤ b |
| not elem of: | | union: A B = {x|x or x } | | | = natural nums = = {1,2,3,4,...} | | | (a,b): a < x < b |
| not subset: | | intersect: A B = {x|x and x } | | | = rational = {m/n|m,n} | | | (a,b]: a < x ≤ b |
| diff: A \ B = {x|x and x } | | product: A x B = {(x,y)|x and y } | | | = real nums | | | (a,: a < x |
| Fn | f: A B (assign elem in A to unique elem in B) | | | the unique elem in B is called image of a, f(a) | | | | |
| existence: for each a A, f(a) B  uniqueness: each a A only map to 1 image in B | | | A is domain of f  B is codomain of f, , range = {f(x) | x A}, actual output | | | | |
| (f+g)(x) = f(x) + g(x), (f-g)(x) = f(x) - g(x), (fg)(x) = f(x) \* g(x), not composite  (f/g)(x) = f(x) / g(x)  (fg)(x) = f(g(x)), composite | | | | | | domain: A B  A B {x|g(x)}  {x| x and f(x) } | |
| absolute: f(x) = |x|  polynomial: P(x) =  rational: R(x) = where Q(x)  algebraic: fn constructed from polynomials using algebraic ops  trigo | | | | | domain: , range: {x|x } = = [0,)  if , then degree of P(x) = n  Every polynomial is rational fn (set Q(x) = 1)  f(x) = , g(x) =  sin x, cos x, tan x, cot x, sec x, csc x | | |
| fn is increasing if a < b f(a) < f(b) for any a, b | | | | | fn is decreasing if a < b f(a) > f(b) for any a, b | | |
| even fn: f(-x) = f(x)  odd fn: f(-x) = -f(x) (if not odd/even, proof by counter e.g.)  power fn: , n | | | | | symmetric about y-axis  symmetric about origin  if n is odd, fn is odd | if n is even, fn is even | | |

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| Limits | slope = gradient = m = | | | | | | | | | | tangent line at : ( | | |
|  | Intuitive defn of limit  only depends on values of f(x) for x near a, (not at a) | | | | | | | | | | , if value of f(x) is arbitrarily close to L by taking x sufficiently close to a (intuitive definition) | | |
|  | If and  1.  2.  3. | | | | 4.  5.  6. , n  7. , n and if n is even, f(x) for all x near a | | | | | | | | |
|  | Let f be a polynomial/rational fn  If a is in domain of f, then  If a not in domain, try simplifying/rationalise fraction | | | | | | | | | | | | direct substitution property aka continuity |
|  | Heaviside fn: | | | | | | | | = 0, left-hand limit, = 1, right-hand limit  Hence if does not exist | | | | |
|  | Infinite limits  If taking x sufficiently close to a, value of f(x) is arbitrarily large  Infinite limits **make sense**, but **do not exist** | | | | | | | | | | | |  |
|  | Squeeze Theorem  If f(x) ≤ g(x) ≤ h(x) for all x near a (except at a) and  = L, then exists and = L  Usually for sin cos fn as | Suppose f(x) ≤ g(x) ≤ h(x) x in an open interval containing a, except possibly at a  If and , proof exists and equals L  There exists a num r > 0 such that 0<|x-a|<r => f(x) ≤ g(x) ≤ h(x)  Let > 0 , > 0 s.t. 0<|x-a|< => |f(x)-L| < - < f(x)-L <  0<|x-a|<min{r,} => - < f(x)-L ≤ g(x)-L (original)  There exists > 0 s.t.0<|x-a|< => |h(x)-L| < - < h(x)-L <  0<|x-a|<min{r,} => g(x)-L < h(x)-L ≤ (original)  Choosing = min{r,}, 0<|x-a|<=> - < g(x)-L < (x)-L| < | | | | | | | | | | | |
|  | Proof limits DNE e.g. =  Proof by contradiction. Suppose lim f(x) exists and = L  =  -1 = L 0 = 0  Thus lim f(x) DNE | | | | | | | Proof infinite limits. E.g. = -  x 0  x 0 < 0 | | | | | |
|  | Is there a real num a s.t. exists?  Suppose exists and equals L  = ()  Take limit . Then a + a2- 2 = L 0 = 0 (a = 1 or -2) | | | | | | | Let a = 1. Then = =  Since = 1 ≠ 0 and = 0, so DNE  Let a = -2. Then = =  So lim exists and = -2/3 if a = -2 | | | | | |
|  | , if for every > 0, there exists a num > 0 such that |f(x) - L| < whenever 0 < |x-a| < | | | | | | | | | | | | Precise definition  binds x, binds y |
|  | To prove that , we just need to find a > 0 for the given > 0,  such that 0 < |x-a| < |f(x) - L| <  E.g. f(x) = 4x-5, show  Let > 0. We need to find a > 0 such that  0 < |x-3| < |(4x-5) - 7| <  = |4(x-3)| <  = |x-3| < /4  Proof.  Let > 0. Choose = /4. Then  0 < |x-3| < |(4x-5) - 7| = 4|x-3| < 4 = | | | | | | E.g. show  Let > 0. Find a > 0 such that 0 < |x-3| < |x2-9| < .  |x2-9| = |x-3||x+3| = |x-3||(x-3)+6|  ≤ |x-3|(|x-3|+6)  < (+6) (triangle inequality)  For (+6) , (+6) s  s-6 and /s, since > 0  s can be any value as long as s-6 > 0 for > 0. E.g. choose s = 7  Proof.  Let > 0. Choose = min{1, /7}  0<|x−3|< δ ⇒|x2 −9| = |x-3||x+3| = |x-3||(x-3)+6|  ≤ |x-3|(|x-3|+6) (triangle inequality)  < (+6)  ≤ δ · 7 (δ ≤ 1)  ≤ (δ ≤ /7) | | | | | | |
|  | Triangle Inequality  For any a,b , |a|-|b| |a+b| |a|+|b|  Result 1  Result 2, combine both result to get above | | | | | | | | | | | ab |ab| 2ab 2|ab|  a2+b2+2ab a2+b2+2|ab| = |a|2+|b|2+2|a||b|  (a+b)2 (|a|+|b|)2  => |a+b| |a|+|b|  Hence, |a| = |(a+b)+(-b)| |a+b|+|-b| = |a+b|+|b|  and |a|-|b| |a+b| | |
|  | Suppose .  Prove  If c = 0, the conclusion becomes , which is proven.  Suppose c ≠ 0. Let > 0. Our aim is to find δ > 0 such that 0<|x−a|<δ ⇒ |cf(x)−cL|<  |f(x) − L| < /|c|  Let > 0. There exists δ > 0 such that  0 < |x − a| < δ ⇒ |f(x) − L| < /|c|  ⇒ |c||f (x) − L| = |cf (x) − cL| <  Therefore, | | | Suppose = L & = M. Prove = L + M.  Let > 0. Our aim is to choose a proper δ > 0 such that  0<|x−a|<δ ⇒ |(f(x)+g(x))−(L+M)|< ⇒|f(x) − L| + |g(x) − M| < .  It suffices to find> 0 and> 0 with = such that |f(x)−L|<and|g(x)−M|<. E.g. one may take= /2 and = /2  Proof. Let > 0  - Since = L, a δ1 > 0 such that 0 <|x−a|<δ1 ⇒|f(x)−L|</2.  - Since = M, a δ2 > 0 such that 0 <|x−a|<δ2 ⇒|g(x)−M|</2  - Choose δ = min{δ1,δ2}. If 0 <|x−a|< δ,then  |(f(x) + g(x)) − (L + M)| ≤ |f(x) − L| + |g(x) − M| ( inequality)  < /2 + /2 = . | | | | | | | | | |
|  | Suppose = L and = M  Prove .  =  = + (Sum law)  = + (-1)(Scalar pdt law)  = L + (-1)M = L-M | | Prove .  Find δ > 0 such that 0<|x−a|<δ ⇒ |(f(x)-g(x))−(L-M)|<  |(f(x)-L) + (M-g(x))|<  |f(x) − L| + |g(x) - M|< ( inequality)  It suffices to find>0 &>0 with + = s.t. |f(x)−L|<and|g(x)-M|< Let > 0  - Since = L, a δ1 > 0 such that 0 <|x−a|<δ1 ⇒|f(x)−L|</2.  - Since = M, a δ2 > 0 such that 0 <|x−a|<δ2 ⇒|g(x)−M|</2.  - Let δ = min{δ1,δ2}. If 0 <|x−a|< δ,then |(f(x) - g(x)) − (L - M)|  ≤ |f(x) − L| + |g(x) − M| ( inequality)  < /2 + /2 = . | | | | | | | | | | |
|  | Suppose . Prove  Let > 0. There exists δ > 0 such that  0 < |x − a| < δ ⇒ |f(x) − 0| <  ⇒ |(f(x))2−0| = |(f(x))2|< ()2 = | | | | | Suppose . Prove  Let g(x) = f(x) - L. Then = L - L = 0  Thus, (proven as on LHS)  = + +  = 0 + 2L + = | | | | | | | |
|  | Suppose and  Prove  Let A and B be real numbers.  (A+B)2 −(A−B)2 =4AB -> AB = [(A+B)2 −(A−B)2]  =  =[(L+M)2 - (L-M)2] = LM | | | | | | | | | If fn not symmetric about a, take nearer endpoint as δ  Right hand limit: for every > 0, there exists a num > 0 such that 0 < x-a < => |f(x) - L| <  Left hand limit: for every > 0, there exists a num > 0 such that 0 < a-x < => |f(x) - L| <  + limit: for every > 0, there exists a num > 0 such that 0 < |x-a| < => f(x) > M  limit: for every < 0, there exists a num > 0 such that 0 < |x-a| < => f(x) < M | | | |
|  | Proving limits qn  (working) To find > 0 such that  0 < |x-a| < => |f(x)-L| <  1. Let > 0  2. Choose a (found in working) to prove that by constraining x, y would be less than | Suppose f(x) ≥ 0 x in an open interval containing a. If = L, proof L ≥ 0.  There exists a num r > 0 such that 0<|x-a|<r => f(x) ≥ 0  For any > 0 , > 0 s.t. 0<|x-a|< => |f(x)-L| < => L-e < f(x) < L+  0<|x-a|< min{r,} => 0 ≤ f(x) < L+  If L ≥ 0, then L+ ≥ 0+ > 0 (proven true for L ≥ 0)  If L < 0, then L+ > 0 <=> > -L  Assume L < 0. Suppose = -L > 0, there exists > 0 such that  0<|x-a|< => |f(x)-L| < = -L  L < f(x)-L < -L => 2L < f(x) < 0  0<|x-a|< min{r,} => 0 ≤ f(x) < 0 (contradiction, so L cannot be < 0) | | | | | | | | | | | |

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| Continuous Fn | | Let f be a polynomial or rational fn  If f has the direct substitution property at a, i.e. , then f is continuous at a  Opp of continuous is discontinuous | Defn of continuity:  1. f(a) is well-defined, i.e. a is in domain of f; and  2. exists, i.e. it is a real number; and  3. , precise defn of limits apply here as well | | | | | | | |
| a) Let f be a fn such that exists  Suppose f(a) is undefined or  f(a) is well-defined but f(a) ≠  Then if f1(x) =  Then f1 is the *continuous extension* of f at a  Also, discontinuity of f at a can be removed by redefining f(a) => f is said to have a *removable discontinuity* at a | | | | | b) Suppose f has at least 1 1-sided infinite limit at a:  or  Then vertical line x = a is an asymptote of y = f(x) and f is said to have an *infinite discontinuity* at a  ⌊x⌋ is the floor of x, floor fn: f(x) = ⌊x⌋  c) Suppose and exists,  but ≠  Then f is said to have a *jump discontinuity* at a | | | |
| One-sided continuity | | = f(a). Then f is continuous from the left at a  = f(a). Then f is continuous from the right at a  f is continuous at a iff f is continuous from the left at a and from the right at a | | | | The floor fn, f(x) = ⌊x⌋ (e.g. f(4.1) = 4 but f(3.9) = 3)  = a = ⌊a⌋. f is continuous from the right at a  = a - 1 ≠ ⌊a⌋. f is not continuous from the left at a | | | | |
| Continuity on intervals | | a) f is continuous on (a,b)  if f is continuous at every x (a,b)  b) f is continuous on [a,b]  if f is continuous at every x (a,b), and  f is continuous from the right at a, and  f is continuous from the left at b | c) f is continuous on [a,b)  if f is continuous at every x (a,b), and  f is continuous from the right at a  d) f is continuous on (a,b]  if f is continuous at every x (a,b), and  f is continuous from the left at b | | | | | | | |
| Properties | | Let f and g be fn continuous at a. Then  cf is continuous at a, where c is a constant  f+g, f-g, fg, is continuous at a | f/g is continuous at a if g(a) ≠ 0  - As long as f and g are continuous at a, can apply direct sub to limits laws | | | | | | | |
|  | | Let c . Then = c  For any > 0 , choose = 1. Then  0<|x-a|<=> |c-c| = 0 <  any constant fn f(x) = c is continuous on | = a  For any > 0 , choose = . Then  0<|x-a|<=> |x-a| <  any constant fn f(x) = x is continuous on | | | | | | | |
| Power fn, xn, where n and  Monomial, cxn, where c and  Polynomial, cnxn + cn-1xn-1 +... + c1x + c0 are all continuous on | | | | | | | Rational fn, P(x)/Q(x) is continuous on its domain (where Q(x) ≠ 0) | |
| Substi-tution in limits  &  Comp-osite fn | Suppose = b and = c. Let y = f(x) and z = g(y).  (x ≠ a) y and y (y ≠ b) z. Can combine the 2 statements to get = c if | | | | | | | | | |
| 1. 1st statement change to (x ≠ a) y (y ≠ b)  i.e. = b and f(x) ≠ b x in an open interval containing a except at a  Let > 0  Since = c, a δ2 > 0 s.t. 0 <|y−b|<δ2 ⇒|g(y)−c|<  Since = b, a δ> 0 s.t. 0 <|x−a|<δ⇒|f(x)−b|< δ2  Hence, 0 <|x−a|< δ => 0 < |(f(x) - b| < δ2 => |g(f(x)) - c| <  Then = c =  - = , where x = a + h  - f is continuous at a = f(a) | | | | 2. 2nd statement change to z  i.e. g is continuous at b  Let > 0  Since g is continuous at b, δ2 > 0 s.t.|y−b|<δ2 ⇒ |g(y)−g(b)|<  Since = b, δ> 0 s.t. 0 <|x−a|<δ⇒|f(x)−b|< δ2  Hence, 0 <|x−a|< δ => |(f(x) - b| < δ2 => |g(f(x)) - g(b)| <  Then = c = g(b) = g  - Suppose f is continuous at a and g is continuous at f(a). Then gf is continuous at a  ( = = g(f(a)) ) | | | | | |
| Root fn | | Let n . root function x1/n =  If n is odd, for any x , is the unique value y s.t. yn = x  If n is even, for any x ≥ 0, is the unique y ≥ 0 s.t. yn = x  Root fn, is continuous on  Any power fn xr with r is continuous on its domain | | | | | | Proof is continuous at a > 0  Let > 0. Choose δ = min{a, }  Suppose 0 < |x-a| < δ => Looking at RHS |x-a| < δ  => a-δ < x < a+δ. Combine this and initial, 0 ≤ a- δ < x  Hence | - | = < < ≤ | | |
| Trigo fn | | sin x, cos x, tan x, csc x, sec x, cot x  Consider sine fn  Let x (0, π/2). Then 0 < sin x < x  = 0 and = 0  By Squeeze Theorem, exists and = 0 | Let x (π/2, 0). Then 0 < sin (-x) < -x, i.e. x < sin x < 0  = 0 and = 0  By Squeeze Theorem, exists and = 0  Since =, then exists and = sin 0 = 0  Thus, sin x is continuous at 0 | | | | | | | |
| Let x (-π/2, π/2). We know sin2x + cos2x = 1  |sin x| ≤ |x| => sin2x ≤ x2  0 ≤ cos x ≤ 1 => RHS cos2x ≤ cos x  1 = sin2x + cos2x ≤ x2 + cos x => 1 - x2 ≤ cos x | For x (-π/2, π/2), 1 - x2 ≤ cos x ≤ 1, = 1 and = 1  By Squeeze Theorem, exists and = 1  Since = 1 = cos 0. cos x is continuous at 0 | | | | | | | |
| Let a and b be real nums  A(cos a, sin a), B(cos b, -sin b), A'(cos(a+b), sin(a+b)), B'(1,0)  Line from A to B = Line from A' to B', |AB|=|A'B'|, i.e. |AB|2 = |A'B'|2  (cos a - cos b)2 + (sin a + sin b)2 = [cos(a+b) - 1]2 + sin2(a+b)  2 - 2(cos a cos b - sin a sin b) = 2 - 2cos(a+b)  cos(a+b) = cos a cos b - sin a sin b | | | | | | | | Note that sin x = cos(π/2 - x) for any x  sin(a+b) = cos(π/2 - (a+b)) = cos[(π/2-a) + (-b)]  = cos(π/2 - a)cos(-b) - sin(π/2 - a)sin(-b)  = sin a cos b + cos a sin b |
| Let a . =  =  = sin a + cos a  = sin a 1 + cos a 0 = sin a | | | | | | =  =  = cos a + sin a  = cos a 1 + sin a 0 = cos a | | |
| sin x and cos x are continuous on  tan x = (sin x / cos x) and sec x = (1 / cos x) are continuous whenever cos x ≠ 0 \ {±π/2, ±3π/2, ±5π/2,...} | | | | | | cot x = cos x / sin x and  csc x = 1 / sin x are continuous whenever sin x ≠ 0  \ {0, ±π, ±2π, ±3π,...} | | |
| IVT | | Intermediate Value Theorem  Let f be a continuous fn on [a,b]  Suppose f(a) < 0 and f(b) > 0 or f(a)> 0 and f(b)<0  then c (a,b) s.t. f(c) = 0 | | -There can be ≥1 root  More generally, Let f be a continuous fn on [a,b]  Suppose f(a) ≠ f(b) and N is btw f(a) and f(b)  Then c (a,b) s.t. f(c) = N | | | | | | |

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| Derivative | | | | Slope/gradient = m = = = f'(a) = = = f(a)  Suppose f'(a) exists, then tangent line at x = a: y = f'(a)(x-a) + f(a)  Note that = So f'(a) = | | | | | | | | | | | | | | | is the differentiation operator  f is differentiable on an open interval I if f is differentiable at every point in I |
| Differentiability implies continuity | | | | | If f is differentiable at a, then f is continuous at a  Suppose f'(a) = L, i.e. = L | | | | | | | | | f(x) - f(a) = (x-a) => f(x) = (x-a) + f(a)  = +  = L 0 + f(a) = f(a), i.e. f is continuous at a | | | | | |
| Differentiation formulas | | | | | (cf)' = cf'  (f ± g)' = f' ± g'  (f2)' = 2f f'  (fg)' = f'g + fg'  (xn) = nxn-1, if n < 0, x cannot be 0 | | | | | | All can be proven using lim defn of derivative  (g-1)' = (1/g)' = -g'/g2  = , assuming g(x) ≠ 0  Use ab = 1/4 [(a+b)2 - (a-b)2]  Using lim defn and binomial theorem to expand | | | | | | | | |
|  | | | | | (sin x) = cos x  (cos x) = -sin x  (tan x) = sec2x  (cot x) = -csc2x  (sec x) = sec x tan x  (csc x) = -csc x cot x | (sin x) = =  = + cos x = sin x + cos x  Let 0 < h < π/2. Using triangle and circular sector, h < tan h  0 < h < tan h = sin h/cos h => cos h < (sin h)/h && 0 < sin h < h => (sin h)/h < 1  Let -π/2 < h < 0. Then 0 < -h < π/2 and cos(-h) < [sin(-h)]/(-h) < 1 => cos h < (sin h)/h < 1  Thus for any h (-π/2, π/2) \ {0}, cos h < (sin h)/h < 1  = 1 & = 1. By Squeeze theorem, exists and = 1  Same process for . cos h - 1 > cos2h - 1 = -sin2h ...by squeeze theorem, lim = 0  (sin x) = sin x + cos x = sin x 0 + cos x 1 = cos x | | | | | | | | | | | | | |
| Chain rule | =  If f is differentiable at x and g is differentiable at f(x), then gf is differentiable at x and (gf)'(x) = g'(f(x))f'(x)  - Can apply to many terms, dw/dx = dy/dx \* dz/dy \* dw/dz  Proof. (Only works if condition below is satisfied)  Let y = f(x) be differentiable at a, z = g(y) differentiable at b = f(a)  Let y = f(a + x) - f(a) and z = g(b + y) - g(b)  = = =  Since y = f(x) differentiable at a continuous at a x , (where can be 0)  Only If x (x ≠ 0) ( ≠ 0), (meaning f(x) ≠ f(a) for x near a except at a), then we can say  = = . And =  Proof on right is better as works even if condition not satisfied | | | | | | | | | | | | | | | x (x ≠ 0) = f'(a)  Let = - f'(a). Then x (x ≠ 0)  = (f'(a) + )x (x ≠ 0)  Similarly, (y ≠ 0) = g'(b)  Let = - g'(b). Then y (y ≠ 0)  Let = 0 at y = 0. Then y  z = (g'(b) + )y (eqn still holds regardless if y = 0)  = (g'(b) + ) (f'(a) + )x (x ≠ 0)  = (g'(b) + ) (f'(a) + )  x (x ≠ 0) and y  y  (gf)'(a) = = = = g'(b) f'(a) = g'(f(a)) f'(a) | | | |
| Implicit differen-tiation | | Let f(x,y) = 0 be an eqn in x and y  Suppose y can be expressed in x near a point on f(x,y) = 0, then y is an implicit fn of x near the point  To use implicit dy/dx, need assume dy/dx exists (cannot prove differentiability) | | | | | | | | | | | | | | | | | E.g. x3 + y3 = 3xy  3x2 + 3y2  = 3y + 3x  (3x - 3y2) = 3x2 - 3y = |
| 2nd order derivative | | | f is twice differentiable at a if f''(a) exists  2nd order derivative = f'' = | | | | | | | | | | | zeroth derivative = f(0)  nth derivative = f(n) = (f(n-1))' = | | | | | |
| Extreme Values | | Let f be fn with domain D  (Absolute/global) max at c if f(c) ≥ f(x) x D  (Absolute/global) min at c if f(c) ≤ f(x) x D  Extreme values: absolute max and min | | | | | | | Local/relative max at c if f(c) ≥ f(x) x in an open interval containing c  Local/relative min at c if f(c) ≤ f(x) x in an open interval containing c  Local extreme values: local max and min  Endpoint cannot be local extreme values | | | | | | | | | | |
| Extreme Value Theorem | | | Suppose f is continuous on [a,b]  c, d [a,b] s.t. f(c) ≤ f(x) ≤ f(d) x [a,b] | | | | | | Must be close interval, continuous fn  Extreme values may be attained more than once  EVT only shows existence of extreme values | | | | | | | | | | |
| Fermat's Theorem | | | Let f has local extreme value at c, and f is differentiable at c. Then, f'(c) = 0  This c is called a stationary point (also a critical pt)  If f has local extreme value at c, then either f'(c) don't exist or f'(c) exist and = 0  This c is called a critical point | | | | | | | | | Suppose f has local min at c  f(c) ≤ f(x) x in an open interval containing c  If x > c, then ≥ 0. f'(c) = = ≥ 0  If x < c, then ≤ 0. f'(c) = = ≤ 0  So for lim at c to exist, only way is is lim = f'(c) = 0 | | | | | | | |
| Closed Interval Method | | | Let f be continuous on [a,b]  1. Evalute values of f at endpoints: f(a) and f(b)  2. Evaluate values of f at critical points on (a,b) s.t. f'(c) don't exist or s.t. f'(c) = 0  3. Compare values obtained in 1 and 2 (largest value = absolute max; smallest = absolute min) | | | | | | | | | | | | | | | | |
| Rolle's Theorem | | | Suppose fn f is continuous on [a,b]  and differentiable on (a,b)  and f(a) = f(b)  There must be c (a,b) s.t. f'(c) = 0 | | | | Case 1: Suppose f is a constant fn. Then f'(c) = 0 for any c (a,b)  Case 2: f is not a constant fn. By EVT, there must be abs max and min  f(a) = f(b) cannot be both max and min at same point (else just constant fn)  f must be extreme val at some c (a,b), and also local extreme val  By Fermat's Theorem, f'(c) = 0 | | | | | | | | | | | | |
| Mean Value Theorem (MVT) | | | Suppose fn f is continuous on [a,b]  and differentiable on (a,b)  c (a,b) s.t. f'(c) =  If f(a) = f(b), f'(c) = 0  Rolle's Theorem is a special case of MVT | | | | | | | Proof. Let h(x) = f(x) -  h is continuous on [a,b] and differentiable on (a,b) (since just made up of fn f)  h(a) = 0 and h(b) = 0. By Rolle's Theorem, c (a,b) s.t. h'(c) = 0  h'(x) = f'(x) - f'(c) = | | | | | | | | | |
| Let fn f be continuous on interval I and differentiable on interior of I and f'(x) = 0 for every x in interior of I  Then there is constant C s.t. f(x) = C x I  For any interval I, its interior is just removing its endpoints (if any) e.g. [a,b],[a,b),(a,b],(a,b) interior is (a,b) | | | | | | | | | | | | Suffices to prove that f(a) = f(b) for any a < b in I  f continuous on [a,b] and differentiable on (a,b)  By MVT, c (a,b) s.t. f'(c) =  Since f'(c) = 0 f(a) = f(b) (converse of Rolle's Theorem) | | | | |
| Let fn f, g be continuous on interval I and  differentiable on interior on I and  f'(x) = g'(x) for every x in interior of I  Then there is constant C s.t. f(x) = g(x) + C x I | | | | | | | Let h(x) = f(x) - g(x). So h is continuous on I, differentiable on interior of I and h'(x) = f'(x) - g'(x) = 0 for every x in interior of I  Then there is constant C s.t. h(x) = C x I  i.e. f(x) - g(x) = C f(x) = g(x) + C x I | | | | | | | | | |
| Increasing Test | | | Let fn f be continuous on interval I, differentiable on interior of I and f'(x) > 0 for every x in interior of I  Then, f is increasing on I | | | | | Proof. Let a,b I s.t. a < b. Suppose f is continuous on [a,b] and differentiable on (a,b). By MVT, c (a,b) s.t. f'(c) =  f'(c) > 0 f(b) > f(a) | | | | | | | | | | | |
|  | | | Let fn f be continuous on interval I, differentiable on interior of I and f'(x) < 0 for every x in interior of I  Then, f is decreasing on I | | | | | Proof. Let g = -f. Then g is continuous on I, differentiable on interior of I and g'(x) = -f'(x) > 0 for every x in interior of I.  By Increasing Test, g is increasing on I. f = -g is hence decreasing on I | | | | | | | | | | | |
|  | | | Suppose f is differentiable on open interval I  If f is increasing/non-decreasing on I, then f'(x) ≥ 0 x I  If f is decreasing on I, then f'(x) ≤ 0 x I | | | | | | Proof. Suppose f is differentiable and increasing on I. Let a I  Then f'(a) = exists.  For any x I, x ≠ a, If x > a f(x) > f(a) and if x < a f(x) < f(a).  Hence > 0. f'(a) = ≥ 0 | | | | | | | | | | |
| 1st derivative test | | | Let fn f be continuous at critical pt c, differentiable on open interval containing c except at c  If f' changes from -ve to +ve at c, local min at c  If f' change from +ve to -ve at c, local max  If f' don't change sign at c, no local extreme val at c | | | | | | | | | | Suppose f'(x) < 0 on (a,c) and f'(x) > 0 on (c,b)  f is decreasing on (a,c] and increasing on [c,b)  f(c) ≤ f(x) for any x (a,b)  Thus, f has local min val at c | | | | | | |
| 2nd derivative test | | | If exists and +ve, then f(x) > 0 x in an open interval containing a, except at a  Let L = > 0  For any > 0, > 0 s.t. 0 < |x-a| < |f(x) - L| < f(x) > L +  Choose = L > 0. Then > 0 s.t. 0 < |x-a| < |f(x) - L| < = L f(x) > L - L = 0  Opp also true, f(x) < 0 | | | | | | | | | | Suppose f'(c) = 0  If f''(c) > 0, f has local min at c. If f''(c) < 0, f has local max at c  Suppose f'(c) = 0 and 0 < f''(c) =  Let g(x) = = . Then = f''(c) > 0  g(x) = > 0 for x ≠ c in open interval I containing c  x I and x < c f'(x) < 0. x I and x > c f'(x) > 0.  f has local min at c  2nd derivative test is inconclusive if f'(c) = f''(c) = 0 | | | | | | |
| Concavity | | | Suppose f is differentiable on open interval I  f(x) - f(c) = f'(c)(x-c)  If graph of f lies above all its tangent lines on I, f concave up on I, f(b) - f(a) > f'(a)(b-a)  If graph of f lies below all its tangent lines on I, f concave down on I, f(b) - f(a) < f'(a)(b-a) | | | | | | | | | | If f concave up on I f' increasing on I  If f concave down on I f' decreasing on I  Let a,b I s.t. a < b. Suppose f concave up on I  f(b) > f'(a)(b-a) + f(a) f'(a) <  f(a) > f'(b)(a-b) + f(b) f'(b) >  f'(a) < f'(b). Thus f' increasing on I | | | | | | |
|  | | | Prove f' increasing on I concave up  a < b. By MVT, c (a,b) s. t. = f'(c) > f'(a)  a > b. By MVT, c (b,a) s. t. = f'(c) < f'(a)  In both case, f(b) > f'(a)(b-a) + f(a)  Thus concave up on I | | | | | | | | | | Concavity Test (f is twice differentiable on open interval I)  If f''(x) > 0 x I f concave up on I  If f''(x) < 0 x I f concave down on I  Proof. Suppose f''(x) > 0 x I  Since f'' = (f')' > 0. f' increasing on I (increasing test)  So concave up. (If f''(x) = 0, inconclusive) | | | | | | |
| Inflection point | | | If f continuous at c, and change concavity at c inflection pt at c  If f is twice differentiable at c, f''(c) = 0  Proof. Suppose f'(x) decreasing on (a,c) but increasing on (c,b)  f' differentiable at c f' continuous at c f'(c) = | | | | | | | | | | Let z (a,c). For any x (z,c), f'(z) > f'(x).  f'(z) ≥ = f'(c)  Let z (c,b). For any x (c,z), f'(z) > f'(x)  f'(z) ≥ = f'(c)  f'(z) ≥ f'(c) z (a,b). f' has local min at c (a,b)  Since f' differentiable at c, By Fermat's Theorem, (f')'(c) = 0  f''(c) = 0 | | | | | | |
| Curve sketching & Optimiza-tion problems | | | Lesson 19: Curve Sketching  increasing and concave up | | | | | | | | | | Find critical point  Check f'(x) and f''(x) in intervals according to critical points  Find relations among vars. Express prob as finding absolute max of min of single-variable fn on interval I  If I = [a,b], use closed interval mtd  For arbitrary interval I, use increasing and decreasing test to find intervals where fn is increasing or decreasing | | | | | | |
| l'Hôpital's Rule | | | Assume fn f,g are differentiable at a and  = f(a) = 0 and = g(a) = 0  Then f,g are continuous at a  = = = = , provided g'(a) ≠ 0 (simple version)  Let fn f,g s.t. = 0 and = 0 and exists or = ±  Then = (0/0 version)  For exists or = ±, must be defined on open interval I containing a, except at a, f and g must be differentiable on I \ {a}, and g'(x) ≠ 0 for any x I \ {a}. | | | | | | | | | | | | | | Proof. Suppose = L, where L or L = ±  Assume f(a) = 0 and g(a) = 0  Let x > a be near a. Apply Cauchy's MVT on [a,x]  c (a,x) s.t. = =  So = = = L  Similarly, = = = L  So = = L.  (l'Hôpital's Rule also hold for one-sided limits)  lim is inclusive if not real or not infinity??? | | |
| Let fn f,g s.t. = 0 and = 0 and exists or ±  Then = (Limits at infinity)  For exists or = ±, must be defined on I = (a,), f and g must be differentiable on I, and g'(x) ≠ 0 for any x I.  Same idea if x tends to - | | | | | | | | | | | | | | Proof. Let f1(z) = f(1/z) for z > 0  Let x = 1/z. Then x z 0+  = = 0, f'1(z) = f'(1/z)(-1/z2)  Let g1(z) = 0 and g'1(z) = g'(1/z)(-1/z2)  = = = = = | | |
| Let fn f,g s.t. = and = and exists or ±  Then = (/ form) | | | | | | | | | | | | | | | Theorem holds for one-sided limits  Theorem holds for limits at infinity  Condition = not necessary | |
| Cauchy's MVT | | | Suppose f,g are continuous on [a,b] and differentiable on (a,b) and g'(x) ≠ 0 for any x (a,b)  Then c (a,b) s.t. =  Let g(x) = x. Then g'(x) = 1 x  Then c (a,b) s.t. f'(c) =  So Cauchy's MVT aka Generalized MVT (g(x) = x) | | | | | | | | | | Proof. Suppose g(a) = g(b).  By Rolle's Theorem, c (a,b) s.t. g'(c) = 0. (contradict g'(x) ≠ 0)  So g(a) ≠ g(b)  From proof of MVT, let h(x) = f(x) -  h is continuous on [a,b] and differentiable on (a,b),  and h(a) = h(b) = 0  By Rolle's Theorem, c (a,b) s.t h'(c) = 0  f'(c) - g'(c) = 0. i.e. = | | | | | | |

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| Area Under Graph | Let f be non-negative continuous fn on [a,b]  1. Divide [a,b] into n equal subintervals  2. Construct rectangle on each subinterval whose height is value of f at left/right endpoint of subinterval  3. Find total area of n rectangles, Ln / Rn­  4. Ln / Rn approaches actual area as n | | | | E.g. y = x2 . FInd area of region btw graph and x-axis on [0,1]  1. [0, 1/n], [1/n, 2/n]..., [n-1/n, n/n]  2. Using right endpoint: area = 1/n \* (i/n)2 = i2/n3  3. Total area of rectangles = + + ... + + = =  4. Let n , = = 1/3 | | | | | |
| Definite Integral | Let f be continuous fn on [a,b]  1. Divide [a,b] into n equal subintervals, x =  2. Choose sample point x1\*, x2\*, ..., xn\* where x1\* [x0, x1], x2\* [x1, x2], ..., xn\* [xn-1, xn]  3. Compute Riemann sum:  4. = | | | | Let f be a fn s.t. an = f(n) n  If , then  E.g. Let an = and f(x) =  = = = = = = | | | | | |
| Geometric properties | Let f be nonnegative continuous fn on [a,b]  Then = area btw y = f(x) and x-axis from a to b  Let f be a nonpositive continuous fn on [a,b]  Then -f is nonnegative and continuous on [a,b]  = area btw y = f(x) and x-axis from a to b  = -ve of area btw y = f(x) and x-axis from a to b | | | | | Let f be continuous fn on [a,b]. Let A1 represent area above x-axis and A2 area below x-axis  = A1 - A2 = net area of region btw y = f(x) and x-axis from a to b  = A1 + A2 = area of region | | | | |
|  | 1. = c(b-a)  Let f(x) = c be constant on [a,b]  = = c(b-a)  = = c(b-a)  5. = =  - = -  6. If f is defined at a, = = = 0  7. = = c= c | | 2. Let f, g be continuous fn on [a,b]. If f(x) ≥ g(x) x [a,b], then ≥  = ≥  Let m and M be min and max values of f on [a,b]. Then 3. m(b-a) ≤ ≤ M(b-a)  Let f be continuous fn on [a,b]. For any c (a,b),  4. + =  Can proof this is true even if b < a < c and any other permutation, will still get by changing signs and considering graph  8. = = += + | | | | | | | |
| Fundamental Theorem of Calculus (Part I) | | Suppose f is continuous on [a,b]. Let g(x) = dt  Then g is continuous on [a,b], differentiable on (a,b) and g'(x) = f(x) x (a,b)  Proof (i). Let [x,x+h] [a,b]. By EVT, f attains max val M at some u [x,x+h] and min val m at some v [x,x+h]  Then f(v) ≤ f(t) ≤ f(u) for any t [x,x+h]  f(v)\*h ≤ dt ≤ f(u)\*h  g(x) + f(v)h ≤ g(x+h) ≤ g(x) + f(u)h  Let h 0+. u = x or u x+, and v = x or v x+,  then f(u) f(x) and f(v) f(x)  g(x) + f(v)h] = g(x) and g(x) + f(u)h] = g(x)  By Squeeze Theorem, (x) + h = g(x)  g is continuous at x from the right, x [a,b)  Similarly, Let [x+h,x] [a,b]. There exists u, v [x+h,x]  Then f(v) ≤ f(t) ≤ f(u) for any t [x+h,x] | | | | | f(v)\*(-h) ≤ dt ≤ f(u)\*(-h)  g(x) + f(u)h ≤ g(x+h) ≤ g(x) + f(v)h  Let h 0-. u x-, f(u) f(x) and v x-, f(v) f(x)  g(x) + f(v)h] = g(x) and g(x) + f(u)h] = g(x)  By Squeeze Theorem, = g(x)  g is continuous at x from the left, x (a,b]  Hence g is continuous on [a,b]  Proof (ii). Let [x,x+h] [a,b]. There exists u, v [x,x+h] s.t. f(v)h ≤ g(x+h) - g(x) ≤ f(u). f(v) ≤ ≤ f(u)  Let h 0+. u x+, f(u) f(x) and v x+, f(v) f(x)  By Squeeze Theorem, = f(x) x [a,b)  Similarly, = f(x) x (a,b]  For any x (a,b), f(x) = = g'(x) | | | |
| = f(x) for any constant a | | | | | OR = f(u) for any constant a | | | |
| Part II  and MVT for definite integrals | Suppose f is continuous on [a,b]. Let g(x) = dt  Then g is continuous on [a,b], differentiable on (a,b) and g'(x) = f(x)  By MVT, c (a,b) s.t. g'(c) =  i.e. f(c) = = dt, i.e. dt = (b-a)f(c) | | | | Suppose now there is another fn F s.t. F is continuous on [a,b], differentiable on (a,b), F'(x) = f(x)  Then C s.t. F(x) = g(x) + C x [a,b]  F(a) = dt + C = C and F(b) = dt + C  F(b) - F(a) = dt = F(x).  F is an anti-derivative of f | | | | | |
| Indefinite Integral | F = an anti-derivative of f = an indefinite integral of f = dx  Suppose G is another anti-derivative of f on interval,  G(x) = F(x) + C, C  So, the indefinite integral = entire family of antiderivatives = dx = F(x) + C, C : arbitrary constant | | | | | | | | Let dx = F(x) + C and dx = G(x) + C  + g(x)) dx = dx + dx =  F(x) + G(x) + C  dx = a dx = aF(x) + C | |
| Substi-tuition Rule | Suppose u = g(x) is differentiable, whose range is interval I. Suppose g' is continuous and f is continuous on I. Then(g(x))g'(x) dx = du  Proof. Let du = F(u) + C  F(u) = F(u) = g'(x)f(u) = g'(x)f(g(x))  (g(x))g'(x) dx = F(u) + C = du | | | | | | | | | E.g. dx. Let u = 1-x2. = -2x  dx = dx = - du = - + C = - (1-x2)3/2 + C |
| Substituition Rule (Definite Integral) | | Let g' be continuous on [a,b], f continuous on range of g,  then (g(x))g'(x) dx = du | | Proof. Let u = g(x). Then (g(x))g'(x) dx = du  Let du = F(u) + C = F(g(x)) + C  (g(x))g'(x) dx = F(g(x))= F(u)= du | | | | | | |
| Odd / Even Fn | Let f be continuous fn on [-a,a]  If f is odd, then dx = 0 | | | | If f is even, then dx = 2dx | | | | | |
| Disconti-nuous Fn | Let f be continuous on [a,b), discontinuous at b from left, dx = dx  Let f be continuous on (a,b], discontinuous at a from right, dx = dx  dx is convergent is limit exist / divergent is limit DNE  E.g. dx = dx  dx = 2 = 2 - 2  dx = dx =[2 - 2] = 2 | | | | Suppose f is discontinuous at c (a,b), and  dx = dx + dx  integral is convergent if both dx and dx exists  divergent if at least one of dx and dx is divergent  Let f be fn s.t. it is continuous on (a,b) anddx and dx exist  Let f1 be the continuous extension of f, then dx = dx | | | | | |
| Infinity | If dx exists t ≥ a, then dx = dx  If dx exists t ≤ b, then dx = dx  Integral is convergent if limits exist else divergent | | | | | | | dx =dx dx  Convergent if both improper integrals on right are convergent, else divergent | | |

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| 1-1 and inverse fn | | Fn is 1-1 (one-to-one) if horizontal line intersects graph at most once  OR a ≠ b f(a) ≠ f(b) for any a,b in domain, D of f  OR f(a) = f(b) a = b for any a, b D  P Q is a conditional statement  Contrapositive of P Q is not Q not P  Both are logically equivalent  Let R be range of f. For each y R, there is unique x D s.t. f(x) = y  f-1(y) = x y = f(x). | | | | | | | | | | f : D R, f-1 : R D. R is domain of f-1 and D is range of f-1  (f -1)-1 = f  f-1(f(x)) = x for any x D  f(f-1(y)) = y for any y R  For any set S, let iS: S S be identity fn on S, iS(x) = x for any x S,  then f-1  f = iD and f f-1 = iR. Note f-1  f ≠ f f-1 | |
| Finding inverse | | Let f be 1-1 fn, to find inverse fn f-1  f and f-1are symmetric (reflection) w.r.t line y = x | | | | | | | 1. Let y = f(x)  2. Express x in terms of y, x = f-1(y)  3. Interchange x and y to express f-1 as fn in x, y = f-1(x) | | | | |
| f is 1-1 iff f is mono-tonic | Suppose f is increasing on set I, i.e., a < b f(a) < f(b)  Let a,b I s.t. f(a) = f(b)  a < b f(a) < f(b) f(a) ≠ f(b)  a > b f(a) > f(b) f(a) ≠ f(b)  So a must be = b f is 1-1  If fn is increasing/decreasing on a set I, fn must be 1-1 | | | Let f be continuous fn on interval I. f is 1-1 iff it is monotonic (i.e. incring or decring)  Proof. Suppose f is continuous and 1-1 on interval I  Take any a,b I with a < b. Then f(a) ≠ f(b) and f(a) < f(b)  Suppose f is not increasing, i.e. , ß I s.t. < ß but f() > f(ß)  Let g(x) = f(a + x( - a)) - f(b + x(ß - b)), [0, 1]. Then g is continuous on [0,1]  g(0) = f(a) - f(b) < 0 and g(1) = f() - f(ß) > 0  By IVT c (0,1) s.t. g(c) = 0 f(a + c( - a)) = f(b + c(ß - b))  Since f is 1-1a + c( - a) = b + c(ß - b), (1-c)a + c = (1-c)b + cß  Since a < b, < ß and 0 < c < 1, (1-c)a + c < (1-c)b + cß, contradicting f is 1-1. So f must be increasing. Similar idea for decreasing | | | | | | | | | |
| If f incring, f-1 also incring | | Suppose f is 1-1, continuous on interval I and increasing, i.e. for any a < b, f(a) < f(b)  then f-1 is also increasing  Similarly, if f is decreasing, f-1 is also decreasing | | | | | | | Let a < b in domain of f-1, i.e. range of f, , ß in range of f-1  a = f() and b = f(ß) for some , ß I  = ß f() = f(ß) a = b & > ß f() > f(ß) a > b  < ß f() < f(ß) f-1(a) < f-1(b), so f-1 is increasing | | | | |
| If f conti-nuous, f-1 also conti-nuous | | Suppose f is 1-1 and continuous on interval I, then inverse fn f-1 is also continuous  Similarly, f-1 is continuous at b from the left if b is not the left endpoint of domain of f-1. Thus f-1 is continuous on domain | | | | Proof. Suppose f is increasing. Then f-1 is also increasing  Let b be in domain of f-1. Let a = f-1(b) I  Suppose b is not the right endpoint of domain of f-1  Let > 0. Take 0 < ≤ s.t. [a, a + ] I. Choose = f(a +) - b > 0  0 < y-b < b < y < b + = f(a +) a < f-1(y) < a + ≤ a +  0 < f-1(y) - a <  So f-1 is continuous at b from the right. | | | | | | | |
| (f-1)'(b) = 1/f'(a) | Suppose f is 1-1 and continuous on interval I. If f is differentiable at interior pt a of I, and f'(a) ≠ 0,  then f-1 is differentiable at b = f(a) and (f-1)'(b) = 1/f'(a)  f'(x) = and f-1(y) = = | | | | | | Proof. Let y = f(x). Then x = f-1(y)  (f-1)'(b) = =  Since f-1 is continuous at b, y b f-1(y) f-1(b) x a  Since f -1 is 1-1, y ≠ b f-1(y) ≠ f-1(b) x ≠ a  (f-1)'(b) = = = | | | | | | |
| Inverse Trigo | Let f(x) = sin x. f is 1-1 if domain = [-π/2, π/2], range[-1,1]  y = arcsin x or y = sin-1 x is inverse sine fn  sin-1 : [-1,1] [-π/2, π/2]  Let y = sin-1 x. Then sin y = x. sin-1 x = = =  y [-π/2, π/2] cos y ≥ 0 cos y = =  sin-1 is continuous on [-1,1], differentiable on (-1,1) and  sin-1 x = , x (-1,1) | | | | | | | | Let f(x) = cos x. f is 1-1 if domain = [0, π], range[-1,1]  y = arccos x or y = cos-1 x is inverse cosine fn  cos-1 : [-1,1] [0, π]  Let y = cos-1 x. Then cos y = x and sin y =  cos-1 x = = = =  cos-1 is continuous on [-1,1], differentiable on (-1,1) and  cos-1 x = , x (-1,1)  sin-1 x + cos-1 x = π/2, x [-1,1] | | | | |
| Let f(x) = tan x. f is 1-1 if domain = (-π/2, π/2), range  y = arctan x or y = tan-1 x is inverse tangent fn  tan-1 : (-π/2, π/2)  Let g(x) = cot x. g is 1-1 if domain = (0, π), range  y = arccot x or y = cot-1 x is inverse cotangent fn  cot-1 : (0, π)  tan-1 and cot-1are differentiable on | | | | | | | | Let y = tan-1 x. Then tan y = x  tan-1 x = = = = =  Let y = cot-1 x. Then cot y = x  cot-1 x = = = = =  tan-1 x = , cot-1 x =  tan-1 x + cot-1 x = π/2 | | | | |
| Let f(x) = sec x. f is 1-1 if domain = [0, π/2) [π, 3π/2), range = (-∞, -1] [1, ∞)  y = arcsec x or y = sec-1 x is inverse secant fn  sec-1 : (-∞, -1] [1, ∞) [0, π/2) [π, 3π/2)  Let g(x) = csc x. g is 1-1 if domain = [0, π/2) [π, 3π/2), range = (-∞, -1] [1, ∞)  y = arccsc x or y = csc-1 x is inverse cosecant fn  csc-1 : (-∞, -1] [1, ∞) (0, π/2] (π, 3π/2]  sec-1 and csc-1are continuous on(-∞, -1] [1, ∞) | | | | | | | | Let y = sec-1 x. Then sec y = x and tan y ≥ 0  sec-1 x = = = = =  Let y = csc-1 x. Then csc y = x and cot y ≥ 0  csc-1 x = = = = =  sec-1 x = , csc-1 x = , |x| > 1  sec-1 x + csc-1 x = | | | | |
| Logarith-mic fn | | Natural logarithmic fn = ln x = (x > 0)  ln 1 = 0; ln x is continuous and differentiable on  ln x = 1/x and ln x = < 0  ln x is increasing and concave down on  = –∞, = ∞, so range of ln x is  Let x > 0 and a > 0. Then ln(ax) = ln a + ln x  Let f(x) = ln(ax) - ln x. f is differentiable on  f'(x) = a - = 0 f is constant on  f(x) = f(1) ln(ax) - ln x = ln a - ln 1 = ln a | | | | | | Let x > 0 and r . Then ln(xr) = r ln x  Let g(x) = ln(xr) -r ln x. Then g is differentiable on  g'(x) = rxr-1 – = 0 g is constant on  g(x) = g(1) ln(xr) - r ln x = ln 1 - r ln 1 = 0  If x > 0, ln x = . Then dx = ln x + C  If x < 0, then -x > 0 and ln(-x) = = . Then dx = ln(-x) + C  For any x ≠ 0, ln|x| = and dx = ln|x| + C | | | | | |
| Logarithmic differen-tiation | | | Let y = [f1(x)]r1[fn(x)]rn, r1, ..., rn and fi are nonzero differentiable fn  This mtd of finding dy/dx = logarithmic differentiation | | | | | | | | 1. Take abs value: |y| = |f1(x)|r1|fn(x)|rn  2. Take natural log: ln|y| = r1 ln|f1(x)|+ + rn ln|fn(x)|  3. Differentiate w.r.t x | | |
| Expo-nential fn | | Note ln 2 > ln 1 = 0 rational num r > 1/ln 2  Let f(x) = ln x. It is continuous on [1, 2r]  f(1) = 0 < 1 and f(2r) = r ln 2 > 1  By IVT c (1, 2r) s.t. f(c) = ln c = 1  Since f is increasing on , f is 1-1 a unique c s.t. ln c = 1. This c = Euler's num = e = 2.71828 and ln e = 1 | | | | | | | | e0= 1. Let n . Then en = e\*e...\*e and e-n = 1/en  Define e1/n = . Let r = m/n . er = em/n =  Let f(x) = ln x. f is increasing on with range  Let f-1: be inverse fn of f  Let x . Then f(ex) = ln(ex) = x ln e = x  So f-1(x) = ex = exp x = exponential fn | | | |
| = 0, = ∞ and ex = ex  Let a > 0. Then ar is well-defined for r  ln(ar) = r ln a ar = exp(r ln a)  Exponential fn of base a > 0: ax = exp(x ln a) | | | | Let x,y  1. ex \* ey = ex+y .ln(ex \* ey) = ln(ex) + ln(ey) = x + y ex \* ey = ex+y  2. e-x = 1/ex. e-x \* ex = e0 = 1 e-x = 1/ex  3. (ex)y = exy. Let a = ex. Then ln[(ex)y] = ln(ay) = y ln a = y ln(ex) = yx | | | | | | | |
| axay = ex ln aey ln a = ex ln a + y ln a = e(x+y)ln a = ax+y  a-x = e(-x)ln a = e-(x ln a) = 1/ex ln a = 1/ax  (ax)y = = exy ln a = axy  Let u = x ln a. ax = eu = eu = ln a \* eu = axln a | | | | | | | xa = axa-1 (x > 0). Proof. Let u = a ln x  xa = eu = eu = \*eu = axa-1  dx =  If a , then domain of f(x) = xa is [0, ∞] | | | | |
| e = . Proof. = = exp = exp = exp = exp(1) = e1 = e | | | | | | | | | | | To find f(x)g(x), where f(x) > 0  1. Express f(x)g(x) = exp[g(x)ln f(x)]  2. Interchange lim and exp function |
| Hyperbolic Trigo fns | | Hyperbolic sine fn: sinh x =  Hyperbolic cosine fn: cosh x =  Let x = cosh t and y = sinh t. Then x2 - y2 = 1, x > 0.  This represent the right branch of a hyperbola | | | | | | | sinh(x + y) = sinh x cosh y + cosh x sinh y  cosh(x + y) = cosh x cosh y + sinh x sinh y  sinh x = cosh x and cosh x = sinh x | | | | |
| sinh x = cosh x = ≥ 1  Then sinh is increasing on with range  sinh-1: is inverse hyperbolic sine fn  Let y = sinh-1 x. Then x = sinh y  = = = =  sinh-1: is differentiable on where sinh-1 x = | | | cosh x = sinh x > 0 for x > 0 and cosh x = cosh x > 0  Then cosh is increasing on [0,∞) and concave up on  Restrict cosh on [0,∞), then range = [1,∞)  cosh-1: [1,∞) [0,∞) is inverse hyperbolic cosine fn  Let y = cosh-1 x. Then x = cosh y  = = = =  cosh-1: [1,∞) [0,∞) is continuous on [1,∞) and cosh-1 x = for x >1 | | | | | | | | |

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| Inverse substi-tuition rule | Let f be continuous fn. Suppose x = g(t) is 1-1 and g' is continuous, then f(x) dx = f(g(t))g'(t) dt | E.g. , n . tan-1x =  Let x = tan t, t (-π/2, π/2). Then = sec2 t  = = = = | | |
| Integrat-ion by parts | (uv) = v + u ∫ (v + u) dx = uv + C  ∫ u dx = uv - ∫ v dx ∫ u dv = uv - ∫ v du | | ∫ ln x dx = x ln x - ∫ x \* 1/x dx = x ln x - x + C  ∫ (cos x)n dx = (cos x)n-1sin x + (n-1)∫ [(cos x)n-2 - (cos x)n] dx  n∫ (cos x)n dx = (cos x)n-1sin x + (n-1)∫ (cos x)n-2 dx | |
|  | Let x = tan t, t (-π/2, π/2). Then sin t = , cos t =  ∫ sec x dx = ln|sec x + tan x| | | | ∫ sec3 x dx = sec x tan x + ln|sec x + tan x| + C |
| Trigo sub | 1. (a > 0). Let x = a sin t, t [-π/2, π/2]  = = = a|cos t| = a cos t  2. (a > 0). Let x = a tan t, t (-π/2, π/2])  = = = a|sec t| = a sec t  3. (a > 0). Let x = a sec t, t [0, π/2) [π, 3π/2)  = = = a|tan t| = a tan t | | | E.g. ∫ dx = ∫ cos t \* cos t dt = ∫ dt =  t + sin t cos t + C = sin-1 x + x + C |
|  | dx =  = (Let x = tan t) | | = tan-1 x + C  = tan-1 x + + C | |
| Integrat-ion of Rational Fns | Every non-constant single var polynomial with real coefficients can be factorized as product of real linear factors and real irreducible quadratic factors  x6 - 1 = (x-1)(x+1)(x2 + x + 1)(x2 - x + 1)  Every rational fn can be uniquely expressed as the sum of partial fractions  f(x) = is proper rational fn if deg A(x) < deg B(x) | | E.g. ≤ ≤ 1 on [0, 1]  ≤ ≤  x4(1-x)4 = x8 - 4x7 + 6x6 - 4x5 + x4  = x6 - 4x5+ 5x4 - 4x2 + 4 -  ≤ - π ≤ | |
| Universal Trigo Sub | Let f be a rational expression in 2 var.  ∫ f(sin x, cos x) dx, -π < x < π, can be evaluated by  t = tan(x/2), i.e. x = 2tan-1 t. =  sin x = = =  cos x = = = | | ∫ f(sin x, cos x) dx = ∫ f(, ) dt : aka universal trigometric substitution  ∫ sec x dx = ∫ \* dt = ∫ dt = ∫ + dt =  ln|1+t| - ln|1-t| + C = ln|| + C = ln |sec x + tan x| + C  valid where sec x is continuous (i.e. x ≠ ± π/2) | |

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| Area Problem | Let f be a nonnegative continuous fn on [a, b]  area of region btw y = f(x) and x-axis is dx  Let A(x) be area of region on [a,x]. ∆A = f(x\*)∆x.  = = = f(x) | | Let f,g be cts fn with f ≥ g on [a, b]. f(a) = c, f(b) = d  Let A(x) be area bounded btw y = f(x) and y = g(x) on [a, x]  = = f(x\*) - g(x\*)] = f(x) - g(x). A = f(x) - g(x)] dx  If l(x) is length of f(x) - g(x) (upper y - lower y), A = (x) dx  If L(y) is length of f-1(y) - g-1(y) (right x - left x), A = (y) dy | | | | |
| Vol Problem | Let V(x) be vol on [a,x]. ∆V = A(x\*)∆x  Suppose solid is placed along the x-axis on [a,b]  Let A(x) be area of cross section at x (a,b)  = = = A(x), if A(x) is cts  V = (x) dx (planes perp to x-axis) | | Suppose solid is placed along the y-axis on [c,d]  Let A(y) be area of cross section at x (a,b)  (planes perp to y-axis)  V = (y) dy , if A(y) is cts | | | | |
| Solids of Revolution | Let f be a cts fn, y = f(x).  Solid is formed by rotating region about x-axis. Disk perp to x-axis  Disk mtd: V = [f(x)]2 dx  Solid formed by rotating about y-axis. Disk perp to y-axis, V = [f-1(y)]2 dy | | Rotating region abt x-axis, with vert line segment (washer mtd)  V = [f(x)2 - g(x)2] dx, f(x) ≥ g(x)  If one endpoint of segment is on x-axis, g(x) = 0  Solid, rotate disk abt y-axis  V = [f-1(y)2 - g-1(y)2] dy, f-1(y) ≥ g-1(y)  If one endpoint of segment is on y-axis, g(x) = 0 | | | | |
| Cylindrical Shell mtd | Let f be cts and nonnegative on [a,b], a ≥ 0  Solid formed by rotating region about y-axis. Disk perp to x-axis, V = x[f(x) - g(x)] dx, f(x) ≥ g(x) | | | Solid formed by rotating region about x-axis. Disk perp to y-axis, V = y[f-1(y) - g-1(y)] dy, f-1(y) ≥ g-1(y) | | | |
| Arc Length | Let f be cts on [a, b], y = f(x)  arc length, L = dx = dx  = dy | | | Let L(x) be length of curve on [a, x]  (∆L)2 = (∆x)2 + (∆y)2, =  = = = | | | |
| Surface Area of Revolution | Let f be cts and nonnegative on [a, b], y = f(x)  Let A(x) be surface area and L(x) be arc length  ∆A = π[f(x) + f(x+∆x)]\*∆L  = = [f(x) + f(x + ∆x)]\* =  π[f(x) + f(x)]\*= 2πf(x) | | | Surface area of region by rotating about x-axis is  dx = dx  Surface area of region by rotating about y-axis is  dy | | | |
| Ordinary Differ-ential Equations | Odinary differential eqn (ODE): F(x, y, , ..., ) = 0  Degree of ODE = highest order of derivative  1st order ODE: = F(x, y) | | | = f(x). Then just integrate, y = ∫ f(x) dx  = g(y) = x = ∫ dy, provided g(y) ≠ 0  General soln: set of all possible soln | | | |
| Separable & Homo-geneous ODE | 1st order ODE is separable if = f(x)g(y)  Suppose g(y) ≠ 0, f(x) =  ∫ f(x) dx = ∫ dx = ∫ dy | Let F(x1, ..., xm) be fn in m vars. It is homogeneous of degree n if F(tx1, ..., txm) = tnF(x1, ..., xm) for any t \ {0}  Linear fn: homogeneous of deg 1: F(x1,...,xm) = a1x1+... + amxm  1st order ODE is homogeneous if F(x, y) is homogeneous of deg 0, i.e. F(tx, ty) = F(x, y) for t \ {0}  1. Let z = . Then y = xz and = z + x. F(x, y) = F(x, xz) = F(1, z) for x ≠ 0  2. ODE becomes z + x = F(1, z), which is separable | | | | | |
| Linear ODE | 1st order ODE is linear if F(x, y) = f(x)y + g(x) is linear fn in y  Standard form of 1st order linear ODE is + p(x)y = q(x)  1. Evaluate ∫ p(x) dx = P(x) + C  Choice of C don't matter as final soln is same (easiest to just use C = 0) | | | | 2. Use integrating factor eP(x). Then eP(x) = p(x)eP(x)  3. Multiply integrating factor to eqn,  eP(x) + eP(x)p(x)y = eP(x)q(x) [eP(x)y] = eP(x)q(x)  4. y = ∫ eP(x)q(x) dx | | |
| Bernoulli's Eqn | Bernoulli's differential eqn has form, + p(x)y = q(x)yn  n = 0: 1st order linear ODE  n = 1: 1st order seperable ODE | | | For n ≠ 0, 1. Let z = y1-n. Then = (1-n)y-n  Multiply (1-n)y-n to DE,  (1-n)y-n + (1-n)y-np(x)y = (1-n)y-nq(x)yn  + (1-n)p(x)z = (1-n)q(x), (linear ODE) | | | |
| Initial Value Problem | Initial conditions are specified for DE  Solution to initial value problem = particular soln  Just solve per normal, then sub in values to find C | | | Model of exponential growth and decay. Let y(t) be value of qty y at time t. Suppose = ky, general soln: y = Cekt  If k > 0: law of natural growth  If k < 0: law of natural decay | | | |
| Real life problems | Continuously compounded interest, A(t), A0 = initial amt  Annually: A(t) = A0(1+r)t, r = interest per annum  n times per year: A(t) = A0(1+)nt, t = num of years  Continuously compounded: Let n ∞, A(t) = = A0 = A0 exp = A0 exp = A0 exp = A0ert | | | | | | Radiocarbon Dating, decay rate m'(t) remaining mass m(t), i.e. = km (separable), where k < 0  m(t) = Cekt, m(0) = C, so m(t) = m(0)ekt  half-life: t1/2 = time for half of qty to decay  m(t+t1/2) = m(t) for any t ≥ 0  m(0)exp[k(t + t1/2)] = m(0)exp(kt)  exp(kt1/2) = k = |
|  | Logistic Population Growth, Let P(t) be pop at time t, = kP, then P(t) = P0ekt. If k = 0: P(t) = P0   |  |  |  | | --- | --- | --- | | k < 0 | t ∞ | P(t) 0 | | k > 0 | t ∞ | P(t) ∞ |   M = max pop = limiting pop = carrying capacity   |  |  |  |  | | --- | --- | --- | --- | | P(t) > M | k < 0 |  |  | | P(t) < M | k > 0 | as P(t) M–, k decreases | |   Logistic growth model: = r(M-P)P, M > 0, r > 0 (constant) | | | | | = r(M-P)P (Bernoulli's DE)  + (-rM)P = (-r)P2. Let z = P1-2 = P-1. + (rM)z = r  Integrating factor: e∫ rM dt = eMrt  z = ∫ eMrt \* r dt = \* = \* = , P(t) = 1/z = (logistic fn)  P(0) = , = M and = 0  P = M/2 at t = ln(C)/(Mr) | |
|  | Newton's Law of Cooling. Let T(t) = temp of obj at time t, TS = surrounding temp, then = k(T-TS)  If T(t) > TS, then T'(t) < 0; if T(t) < TS, then T'(t) > 0  So k < 0, let k = -r, where r > 0  = -r(T-TS) (heat transfer model)  Let A(t) = T(t) - TS. = = -r(T - TS) = -rA  So A(t) = A(0)e-rt. T(t) - TS = (T(0) - TS)e-rt  T(t) = TS + (T0 - TS)e-rt. As t ∞, T(t) TS | | | Draining Tank Problem | | | |