CS 453 Project 2

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Problem 1.

To find the critical points of the function $f(x, y) = x^3 - 6x^2y + 3xy^2 - y^3 - 3x + 3y$ we must first compute its gradient vector $\nabla f(x, y)$ and then solve for when $\nabla f(x, y) = \vec{\mathbf{0}}$.

First lets compute $\nabla f(x, y)$. We know that

$$\nabla f(x, y) =< \frac{\partial f}{\partial x} f(x, y), \frac{\partial f}{\partial y} f(x, y) >= \begin{bmatrix} \frac{\partial f}{\partial x} f(x, y) \\ \frac{\partial f}{\partial y} f(x, y) \end{bmatrix}$$

Now lets compute $\frac{\partial f}{\partial x} f(x, y)$

$$\begin{aligned} \frac{\partial f}{\partial x} f(x, y) &= \frac{\partial}{\partial x} x^3 - 6x^2y + 3xy^2 - y^3 - 3x + 3y \\ &= 3x^2 - 12yx + 3y^2 - 3 \end{aligned}$$

Next lets compute $\frac{\partial f}{\partial y} f(x, y)$

$$\begin{aligned} \frac{\partial f}{\partial y} f(x, y) &= \frac{\partial}{\partial y} x^3 - 6x^2y + 3xy^2 - y^3 - 3x + 3y \\ &= 6x^2 + 6xy - 3y^2 + 3 \end{aligned}$$

We now have our $\nabla f(x, y)$. We can then solve for when the $\nabla f(x, y) = \vec{\mathbf{0}}$ by solving for when each component of the vector is equal to 0.

Looking first at the x component of the gradient we can see

$$\begin{aligned} 3x^2 - 12yx + 3y^2 - 3 &= 0 \\ 3 &= 3x^2 - 12yx + 3y^2 \\ 3 &= 3(x^2 - 4yx - y^2) \\ 1 &= x^2 - 4yx - y^2 \end{aligned}$$

Then for the y component of the gradient we can get

$$\begin{aligned} 6x^2 + 6xy - 6y^2 + 3 &= 0 \\ -3 &= 6x^2 + 6xy - 6y^2 \\ -3 &= 6(x^2 - yx - y^2) \\ -\frac{1}{2} &= x^2 - yx - y^2 \\ 0 &= x^2 - yx - y^2 + \frac{1}{2} \end{aligned}$$

We can solve these two equations by setting them up as a linear combination of the

We can then reset these equations to when they equal 0 and solve. First we know we want to find all critical points of when $x = 0$ so we can then solve the equations

$$\begin{aligned} 0 &= x^2 - 4yx - y^2 - 1 \\ 0 &= 0^2 - 4y(0) - y^2 - 1 \\ 0 &= 0 - 0 - y^2 - 1 \\ 1 &= y^2 \\ \pm 1 &= y \end{aligned}$$

So we get our first to critical points of when $x = 0$ that $y = \pm 1$. So lets say C is the set of all critical points so, $\{(0,1),(0,-1)\} \in C$. We know there are a total of 4 critical points given from the hint sheet that Peter provided. We know that half of them are when $x = 0$. The other half must be when $x \neq 0$. If we take the y component of the gradient and divide it by x to prevent $x = 0$, this results in $x = -2y$.

Then evaluating $y(12y + y) = 1$ Thus we get $y = \pm\sqrt{\frac{1}{13}}$. Then we plug that back into x to get the critical points $\{(-2\sqrt{\frac{1}{13}}, \sqrt{\frac{1}{13}}),(2\sqrt{\frac{1}{13}}, -\sqrt{\frac{1}{13}})\}$.

This results in the critical points $\{(-2\sqrt{\frac{1}{13}}, \sqrt{\frac{1}{13}}),(2\sqrt{\frac{1}{13}}, -\sqrt{\frac{1}{13}}),(0,1),(0,-1)\}$

Then we can compute what type of points they are via the Hessian matrix. To compute the Hessian matrix we must find all the second partials then evaluate them at all critical points to check their sign.

Lets call the Hessian matrix as $\mathcal{H}$.

$$\mathcal{H} = \begin{bmatrix} 6x - 12y & 12x + 6y \\ -12x - 6y & 6x - 12y \end{bmatrix}$$

First evaluating $\mathcal{H}$ for the simple critical points $(0,1)$ and $(0,-1)$.

$$\begin{aligned} \mathcal{H}_{(0,1)} &= \begin{bmatrix} 6(0) - 12(1) & 12(0) + 6(1) \\ -12(0) - 6(1) & 6(0) - 12(1) \end{bmatrix} \\ &= \begin{bmatrix} -12 & 6 \\ -6 & -12 \end{bmatrix} \end{aligned}$$

This matrix gives us eigenvalues of $\lambda_1 = -12 + 6i$ and $\lambda_2 = -12 - 6i$. Such the point is not a saddle point.

$$\begin{aligned} \mathcal{H}_{(0,-1)} &= \begin{bmatrix} 6(0) - 12(-1) & 12(0) + 6(-1) \\ -12(0) - 6(-1) & 6(0) - 12(-1) \end{bmatrix} \\ &= \begin{bmatrix} 12 & -6 \\ 6 & 12 \end{bmatrix} \end{aligned}$$

This matrix gives us eigenvalues of $\lambda_1 = 12 + 6i$ and $\lambda_2 = 12 - 6i$. This point is inconclusive

Then evaluating the other two points we see

$$\mathcal{H}_{(-2\sqrt{\frac{1}{13}},\sqrt{\frac{1}{13}})} = \begin{bmatrix} 6(-2\sqrt{\frac{1}{13}}) - 12\sqrt{\frac{1}{13}}) & 12(-2\sqrt{\frac{1}{13}}) + 6(\sqrt{\frac{1}{13}})) \\ -12(-2\sqrt{\frac{1}{13}}) - 6(\sqrt{\frac{1}{13}})) & 6(-2\sqrt{\frac{1}{13}}) - 12(\sqrt{\frac{1}{13}})) \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{24\sqrt{13}}{13} & -\frac{18\sqrt{13}}{13} \\ \frac{18\sqrt{13}}{13} & -\frac{24\sqrt{13}}{13} \end{bmatrix}$$

Thus we get the eigenvalues $\lambda_1 = -\frac{24-18i}{\sqrt{13}}$ and $\lambda_2 = -\frac{24+18i}{\sqrt{13}}$ this shows that this point is a saddle point because both eigenvalues are negative.

$$\mathcal{H}_{(2\sqrt{\frac{1}{13}},-\sqrt{\frac{1}{13}})} = \begin{bmatrix} 6(2\sqrt{\frac{1}{13}}) - 12 * -\sqrt{\frac{1}{13}})) & 12(2\sqrt{\frac{1}{13}}) + 6(-\sqrt{\frac{1}{13}})) \\ -12(2\sqrt{\frac{1}{13}}) - 6(-\sqrt{\frac{1}{13}})) & 6(2\sqrt{\frac{1}{13}}) - 12(-\sqrt{\frac{1}{13}})) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{24\sqrt{13}}{13} & \frac{18\sqrt{13}}{13} \\ -\frac{18\sqrt{13}}{13} & \frac{24\sqrt{13}}{13} \end{bmatrix}$$

Thus we get the eigenvalues $\lambda_1 = \frac{24+18i}{\sqrt{13}}$ and $\lambda_2 = \frac{24-18i}{\sqrt{13}}$ this shows that this point is a saddle point because both eigenvalues are positive.

Problem 2.

When taking a scalar field and creating a set of N contour curves that are evenly spaced throughout the shape and projecting them on the xy plane we get a better understanding of what the field looks like without having to see the height of it.

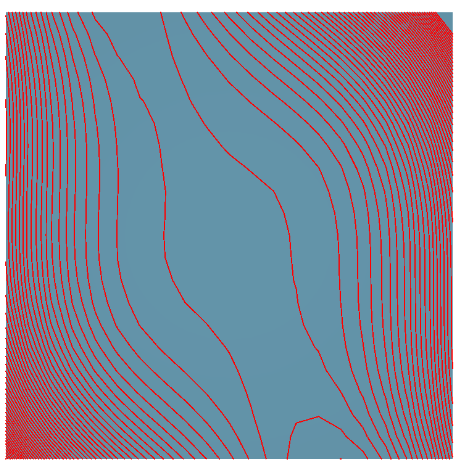


Figure 1: The contour curves of R14 with $N = 50$

In this visualization I was able to find the contour points and predict what the three dimensional object which the scalar field represented looked like. However, there are some drawbacks with this approach. You are not able to clearly see if there would be any critical points which would allow the viewer to make better inferences about the scalar field.

We can apply the same visualization but adding a gradient shift of the contour colors between the height of the contour level. In this example, you can see I have a gradient shift from the color purple to yellow which gives more insight to the viewer. The viewer is able to better understanding of how the height is changing over time especially with the color changing more dramatically at different points.

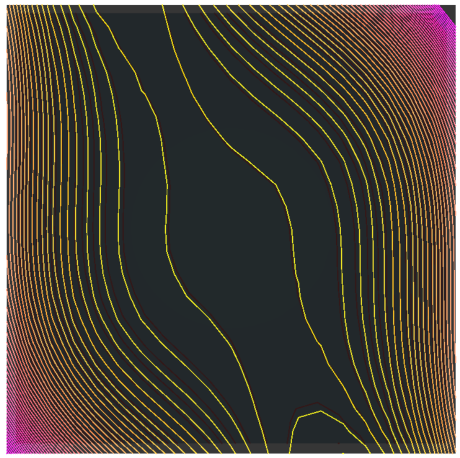
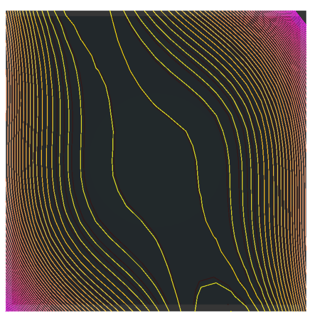
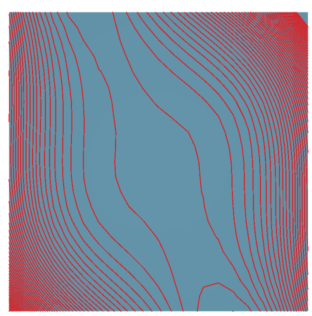


Figure 2: The bi-color visualization of contour curves of R14 with $N = 50$

If we were to compare the bicolor and the single color contour field visualizations side by side we can clearly see that the bicolor scalar field gives a better advantage by conveying the height changes with the color versus the contour curves being only the same color and the viewer not being able to extract as accurate information.



(a) Bi Color



(b) Single Color

Figure 3: Comparison of bicolor contour curves with single color contour curves in R14

Next if we combine a height map in combination with just the single color contour fields we get a better representation of how the scalar fields construct a shape in $\mathbb{R}^3$. With this new set of dimensionality we are able to see how the contour curves better fit the scalar fields. This will also give us a better inference of where the critical points are in the scalar field.

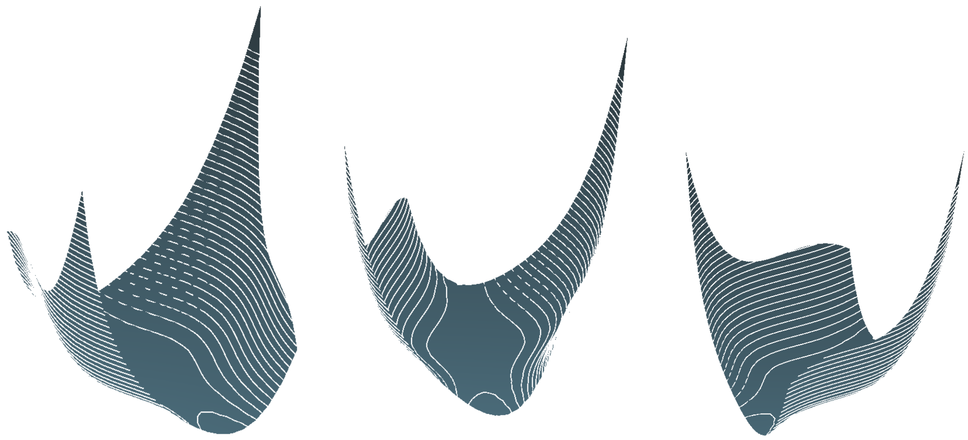


Figure 4: Three images of R14 with $N = 50$ contour curves with a height field scaled between 0 and 30

With the images provided above we get a better understanding of the scalar field and what it looks like. We can clearly see there is a local minimum at the bottom of the valley where there is a large open spot in the contour curves. A viewer can make better assumptions between the different aspects of the shape and visualization of the fields. This visualization works the best in my opinion because you are able to rotate the shape and see how the shape is formed with the scalar fields.

Finally when combining all three different techniques, bi-color contour lines with a height field applied we get arguably the best visual of what the scalar field is doing. The bicolor gradient of contour lines gives us more information about how steep the height is actually in a point which the contour curves can only give a small proportion of.

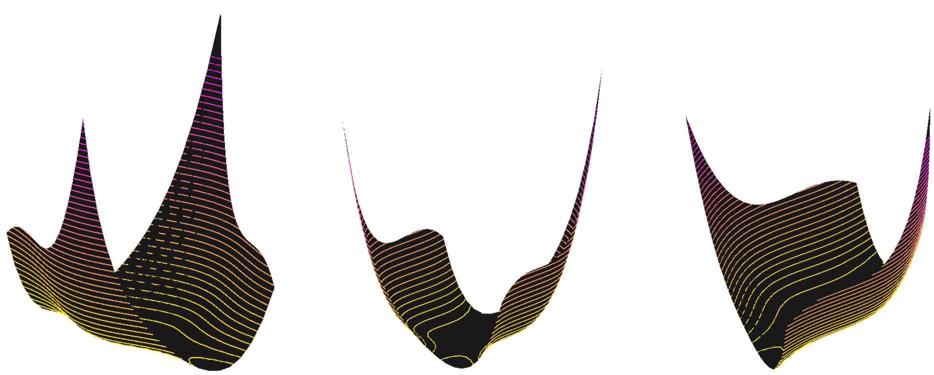
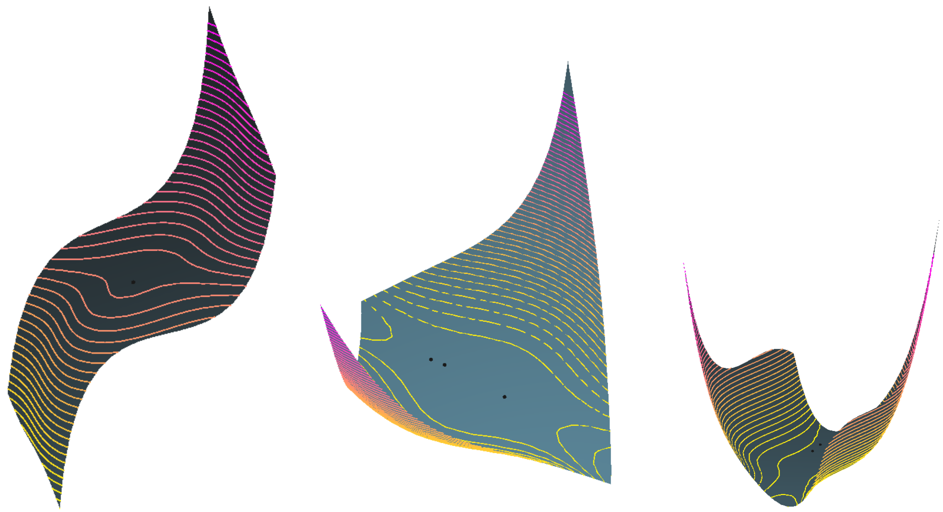


Figure 5: All visualization techniques applied to R14

Problem 3.

We then are able to extract the critical points (saddle only) of all the different visualizations which gives us the information about where the gradient of the scalar field is equal to the zero vector. This conveys to us that there is critical information we should most likely pay attention to at that point or there is something interesting happening with our data there.



(a) R2 (b) R9 (c) R14

Figure 6: Critical points of multiple scalar fields