Waves on the Water Surface – Mathematical Models – Part 2

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Abstract
This is Part 2 of the paper titled: "Waves on water surface – mathematical models". It deals with the wave transformation above a non-uniform bed surface, treating it as a linear as well as a nonlinear problem.

3. WAVE TRANSFORMATION ABOVE A NON-UNIFORM BOTTOM

3.1. Linear problem

Let us assume, that the wave amplitude is negligible in comparison to the wave length. Then the problem is simplified, and the potential wave movement can be described as the linear system of equations (Kononkova and Pokazeev, 1985)

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0,$$

$$\frac{1}{g}\frac{\partial^2\phi}{\partial t^2}+\frac{\partial\phi}{\partial z}=0 \qquad \text{by } z=0,$$

$$\frac{\partial\phi}{\partial x}\frac{\partial d}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial d}{\partial y}+\frac{\partial\varphi}{\partial z}=0 \qquad \text{by } z=-d, \tag{1}$$

$$\frac{\partial\phi}{\partial t}=f_1(x,y,0) \qquad \text{by } t=0, \quad z=0,$$

$$\phi=f_2(x,y,0) \qquad \text{by } t=0, \quad z=0,$$

Where t = time; x, y, z = Cartesian coordinates; d = depth; g = gravitational acceleration; ϕ = velocity potential.

However, even by the linear problem definition, the problem still remains rather complex. For small bed slopes $\left|\frac{\partial d}{\partial x}\right|,\left|\frac{\partial d}{\partial y}\right| << 1$, asymptotic ray methods are applied, Aleshkov (1981), Munk and Arthur (1952), Krylov et al. (1976), Massel et al. (1990), which allows a considerable simplification of the problem. For example, by a frontal approach of waves to the shore, the solution for the progressive non-breaking wave is as below (Mei, 1983), (Meyer, 1979).

$$\eta=\sqrt{\frac{C_{g0}}{C_g}}\,a_0\cos\left[\omega t-\int_{x_0}^{x}k(x)\,dx\right], \tag{2}$$

where η = wave profile; a = wave amplitude; C_g = group speed; $\omega = \dfrac{2\pi}{T}$ = angular frequency; T = wave period; $k = \dfrac{2\pi}{\lambda}$ = wave number; λ = wave length; $\omega^2 = gk \tanh kd$;

the index «0» refers to the data in the entry range $x = x_0$. From eqn. (2) the expression for the wave height H is obtained as below:

$$H(x) = \sqrt{\frac{C_{g0}}{C_g}}\, H_0. \tag{3}$$

By solving plane problems, first of all, the rays are calculated, along which the wave energy is propagating, and after that – using the law of energy flux conservation - the wave parameters. This approach is assumed as a basis for the standard method of calculating the wave transformation in the coastal zone of the sea, (Lappo et al., 1990). However, by the practical realization of this method, an intersection of the wave rays is possible, which makes the calculations much more complex, since there arises a necessity to apply various smoothing methods.

In case the sea bed contains local underwater pits or elevations, the application of the ray method becomes incorrect, because even small inaccuracies by calculating the wave rays can result in a significant distortion of the calculated waves parameters. If the bottom is characterized by a high degree of heterogeneity, then by describing the wave transformation it is necessary to take into account the diffraction effects.

The equation for the wave calculation adjusted for the effects of refraction and diffraction can be obtained from the equations (1) and is as below (Berkhoff, 1976)

$$\frac{\partial}{\partial x} C C_g \frac{\partial \phi_0}{\partial x} + \frac{\partial}{\partial y} C C_g \frac{\partial \phi_0}{\partial y} + \frac{\omega^2 C_g}{C} \phi_0 = 0, \tag{4}$$

where $\phi_0 = \phi_0\ (x, y)$; $C = \left[\dfrac{g}{k} \tanh kd\right]^{0.5}$ = phase speed; $C_g = n\, C$ = group speed;

$$n = \frac{1}{2}\left(1 + \frac{2kd}{\sinh 2kd}\right).$$

In case of the deep water or by a constant fluid depth from the equation (4) results the diffraction equation of Helmholtz.

The required velocity potential is determined by the relation

$$\phi = \mathrm{Re}\left\{\phi_0 \cdot e^{i\omega t} \cdot \cosh k(d+z) / \cosh kd\right\}. \tag{5}$$

And by the relation

$$\eta = \frac{1}{g} \frac{\partial \phi}{\partial t}\ \Big|_{z=0} \tag{6}$$

the ordinate of the free surface is calculated.

It is known, (Ivanov et al., 1992), that by such a statement, the solution of particular practical problems represents a substantial difficulty. That is why more widespread are the above-mentioned ray-tracing method and the method of parabolic equation (Radder, 1979), which assumes that the alteration of the diffracted wave amplitude in the direction of wave propagation occurs slower, than in the direction, which is perpendicular to it.

The application of the parabolic equation considerably simplifies the problem, since the methods of solving the equations like the heat transfer equation, are well enough developed. However, by this, the limitations of this method should also be taken into account:

– the wave height along the ray should alter slowly,
– the method does not take into consideration the reflection of waves from the contour of the water area.

The description of the wave movement in non-viscous incompressible fluid with a small amplitude can be done also on the basis of the continuity equation and Euler equations to the linear approximation, (Landau and Lifshitz, 1988)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{7}$$

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{8}$$

$$\frac{\partial v}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{9}$$

$$\frac{\partial w}{\partial t}=-g-\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{10}$$

where u, v, w = velocity components; p = pressure; ρ = fluid density.

Let us suggest, that the waves are regular, the bottom alters smoothly and the pressure is described by the relation

$$p=-\rho g z+\rho g\frac{\cosh k\,(d+z)}{\cosh kd}\eta. \tag{11}$$

In view of these suggestions and the dispersion relation

$$k=\frac{\omega^2}{g}\coth kd, \tag{12}$$

let us integrate the equations (7)–(9) by depth.

The linear equation of continuity, averaged by depth, can be recorded, (Kononkova and Pokazeev, 1985)

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}d\cdot U+\frac{\partial}{\partial y}d\cdot V=0, \tag{13}$$

where U, V = averaged by depth velocity components u, v.

The integration of the equation (8) by depth is carried out in the following sequence:

$$\int_{-d}^{\eta}\frac{\partial u}{\partial t}dz=-\int_{-d}^{\eta}\frac{1}{\rho}\frac{\partial p}{\partial x}dz. \tag{14}$$

Since

$$\int_{-d}^{\eta}\frac{\partial f}{\partial \xi}\,dz=\frac{\partial}{\partial \xi}\int_{-d}^{\eta} f\,dz-f(\eta)\frac{\partial \eta}{\partial \xi}+f(-d)\frac{\partial(-d)}{\partial \xi},$$

where f – certain function; ξ – variable, then

$$\int_{-d}^{\eta}\frac{\partial u}{\partial t}\,dz=\frac{\partial}{\partial t}\int_{-d}^{\eta} u\,dz-u(\eta)\frac{\partial \eta}{\partial t}+u(-d)\frac{\partial(-d)}{\partial t}, \tag{15}$$

$$\int_{-d}^{\eta}\frac{1}{\rho}\frac{\partial p}{\partial x}\,dz=\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-d}^{\eta} p\,dz-\frac{1}{\rho}p(\eta)\frac{\partial \eta}{\partial x}+\frac{1}{\rho}p(-d)\frac{\partial(-d)}{\partial x}. \tag{16}$$

Since the linear problem is considered and $\frac{\partial d}{\partial t}=0$, we can record

$$\int_{-d}^{\eta}\frac{\partial u}{\partial t}\,dz=d\frac{\partial U}{\partial t}.$$

Let us consider the right part of the relation eqn. (16). By placing the relation eqn. (11) to the first member, we obtain:

$$\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-d}^{\eta} p\,dz=g\frac{\partial}{\partial x}\int_{-d}^{\eta}\left[-z+\frac{\cosh k(d+z)}{\cosh kd}\eta\right]dz=gd\frac{\partial d}{\partial x}+g\frac{\partial}{\partial x}\left(\frac{\tanh kd}{k}\eta\right).$$

Also here only the members of the first order of smallness are considered. The second member of the right part of the relation eqn. (16) is equal to zero, since $p(\eta)=0$, and the last member is equal to

$$\frac{1}{\rho}p(-d)\frac{\partial(-d)}{\partial x}=-g(d+\frac{\eta}{\cosh kd})\frac{\partial d}{\partial x}.$$

Therefore, the equation (14) can be presented as follows:

$$\begin{aligned}\frac{\partial U}{\partial t}&=-g\frac{\partial d}{\partial x}-\frac{g}{d}\frac{\partial}{\partial x}\left(\frac{\tanh kd}{k}\eta\right)+g\frac{\partial d}{\partial x}+\frac{g\eta}{d\cosh kd}\frac{\partial d}{\partial x}=\\&=-\frac{g}{d}\left[\frac{\partial}{\partial x}\left(\frac{\tanh kd}{k}\eta\right)-\frac{\eta}{\cosh kd}\frac{\partial d}{\partial x}\right]=\\&=-\frac{c^2}{d}\frac{\partial \eta}{\partial x}-\frac{\eta\omega^2}{d}\frac{\partial}{\partial x}\frac{1}{k^2}+\frac{g\eta}{d\cosh kd}\frac{\partial d}{\partial x}=-\frac{C^2}{d}\frac{\partial \eta}{\partial x}+\frac{2\eta\omega^2}{dk^3}\frac{\partial k}{\partial x}+\frac{g\eta}{d\cosh kd}\frac{\partial d}{\partial x}.\end{aligned} \tag{17}$$

Let us differentiate the equation (12) by x:

$$\frac{\partial k}{\partial x}=\frac{\omega^2}{g}\frac{\partial}{\partial x}\coth kd=-\frac{\omega^2}{g\sinh^2 kd}\left(k\frac{\partial d}{\partial x}+d\frac{\partial k}{\partial x}\right)$$

or

$$\frac{\partial k}{\partial x} = -\frac{\omega^2 k}{g\sinh^2 kd(1+\dfrac{\omega^2 d}{g\sinh^2 kd})}\frac{\partial d}{\partial x} = -\frac{\omega k}{C_g \sinh 2kd}\frac{\partial d}{\partial x}\,. \tag{18}$$

Taking into account eqn. (18) the equation (17) can be recorded as follows:

$$\frac{\partial U}{\partial t} + \frac{C^2}{d}\frac{\partial \eta}{\partial x} + \frac{2\eta\omega^2}{dk^3}\frac{\omega k}{C_g \sinh 2kd}\frac{\partial d}{\partial x} - \frac{g\eta}{d\cosh kd}\frac{\partial d}{\partial x} = 0.$$

After manipulations we obtain:

$$\frac{\partial U}{\partial t} + \frac{C^2}{d}\frac{\partial \eta}{\partial x} + \frac{2\omega C(C - C_g \cosh kd)}{dC_g \sinh 2kd}\eta\frac{\partial d}{\partial x} = 0. \tag{19}$$

Respectively, after integrating by depth the equation (9) we obtain:

$$\frac{\partial V}{\partial t} + \frac{C^2}{d}\frac{\partial \eta}{\partial y} + \frac{2\omega C(C - C_g \cosh kd)}{dC_g \sinh 2kd}\eta\frac{\partial d}{\partial y} = 0. \tag{20}$$

The systems of equations (13), (19), (20) describe both refraction and diffraction of the surface waves by a heterogeneous relief of the bottom, (Shakhin and Shakhina, 2001). For solving this system of equations it is necessary to set the initial and the boundary conditions. If the calculation range has the form of a rectangle with impenetrable side walls and a shore boundary, then these conditions can be set as below:

$$\begin{array}{llll} \text{by} & t = 0: & U = V = 0, & \eta = 0; \\ \text{by} & x = 0, x = M: & U = 0, & \dfrac{\partial \eta}{\partial x} = 0; \\ \text{by} & y = 0: & U = 0, & \eta = a\sin \omega t; \\ \text{by} & y = N: & \partial U/\partial x = 0, & V = 0. \end{array} \tag{21}$$

Here $x = 0$ and $x = M$ the left and the right boundaries of the calculation range respectively; $y = 0$ entry range; $y = N$ shore boundary; a = wave amplitude. If there exist impenetrable structures inside the calculation range, then at their limits the conditions of non-passing are set. The hydraulic resistance can be taken into account by introducing in equation (19), (20) the additional members μU and μV respectively, where $\mu(x, y)$ = linearized (dimensional) coefficient of resistance.

In the above formulation a number of problems on the wave interaction with a heterogeneous bottom and structures were solved. The solution was done numerically by method of finite differences, applying the two-layer explicit-implicit scheme with conversion as in Marchuk et al. (1983).

The Figure 1 shows the derived lines of the wave fronts by wave interaction with a uniform under water slope, limited by side walls.

The calculations were implemented with the following data

- range width b = 500 m;
- range length l = 650 m;
- water depth by the slope end $d_{\max}$ = 40 m;
- angle of the wave approach to the slope α = 20°;
- slope i = 0,14;
- wave period T = 6 s.

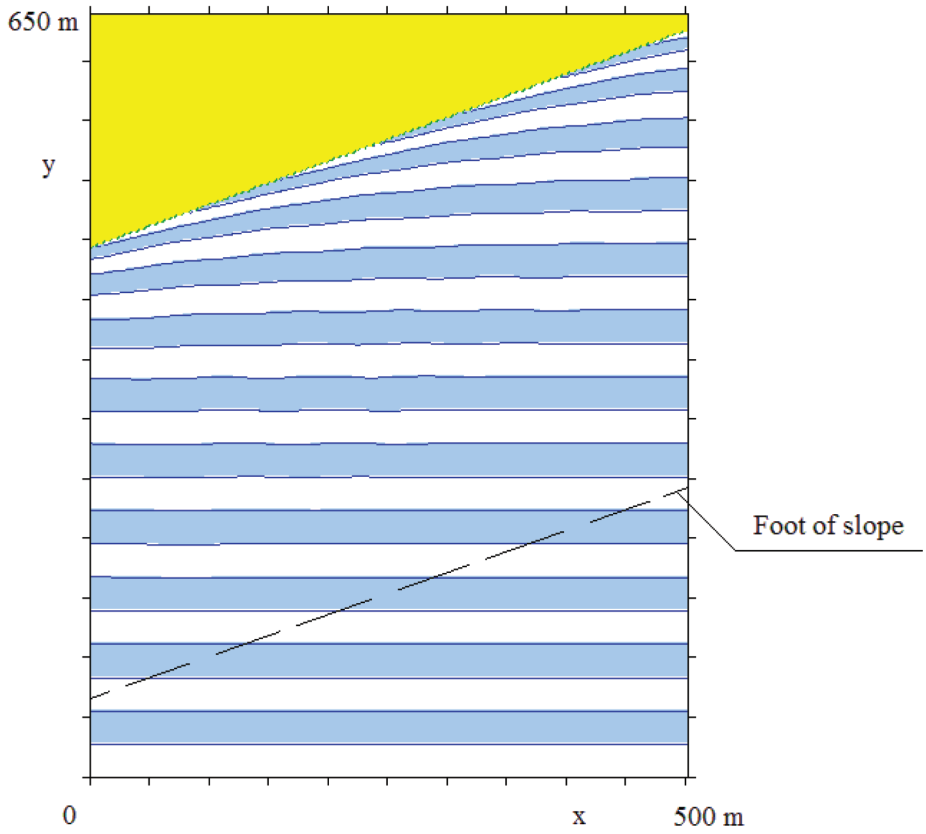

Figure 1. Lines of waves' fronts

The computations encompass all the characteristic fields: the deep water, the intermediate depth, the shallow zone. In order to eliminate the wave reflection, a mathematical wave absorber was "set" by the shore. It can be noted, that should the ray method be used, then by the left side wall there would be a concentration of the wave rays with an unlimited wave amplitude increase. It doesn't take place in this case, since the effects of wave diffraction are taken into account.

The Fig. 2 presents a calculated pattern of wave crests at the moment of passing the underwater elevation in the form of a sphere segment, situated at the horizontal bottom d = 35 m. The waves propagate along the axis y. The wave period is assumed to be T = 6 s. The position of the sphere segment in the calculation zone and the depth curves are depicted in Fig. 3. The minimal depth of water above the top of the sphere segment is equal to d_{min} = 3 m. In this example the effects of refraction and diffraction are also significant. The calculation results correspond satisfactorily to the experimental data, mentioned in Ivanov et al. (1992).

An example of solving a practical problem concerning the research of wave movement in the water area of the Sochi maritime port is given at the Fig. 4. It was assumed that the height of the waves, approaching the port, H = 3.5 m, wave period T = 7.7 s. In the protected water area one can clearly see zones with a greater wave height. These peculiarities of the wave regime have to be taken into account by the operation of the seaport.

Thus, the conducted research suggests a conclusion about the effectiveness of the method, proposed in Shakhin and Shakhina (2001) for calculating wave refraction and diffraction. The advantages of this method, when compared to the known methods, include:

– implementation of the method in the physical field;
– possibility of taking into account full or partial wave reflection, friction on the bottom and hydraulic resistance of structures;

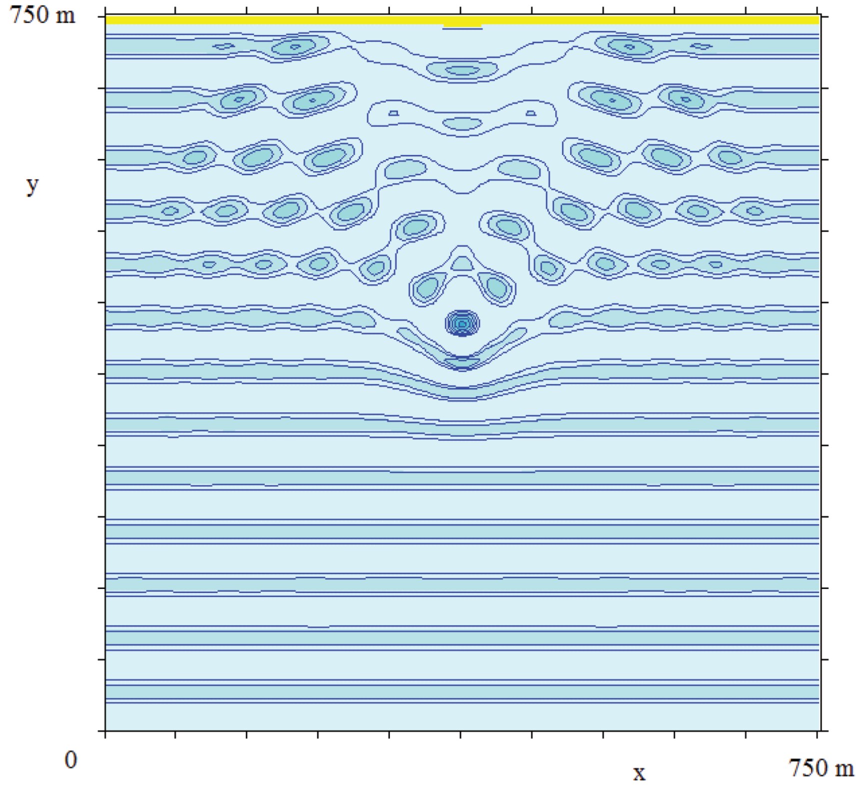

Figure 2. Position of the wave crests by the propagation above the under water elevation

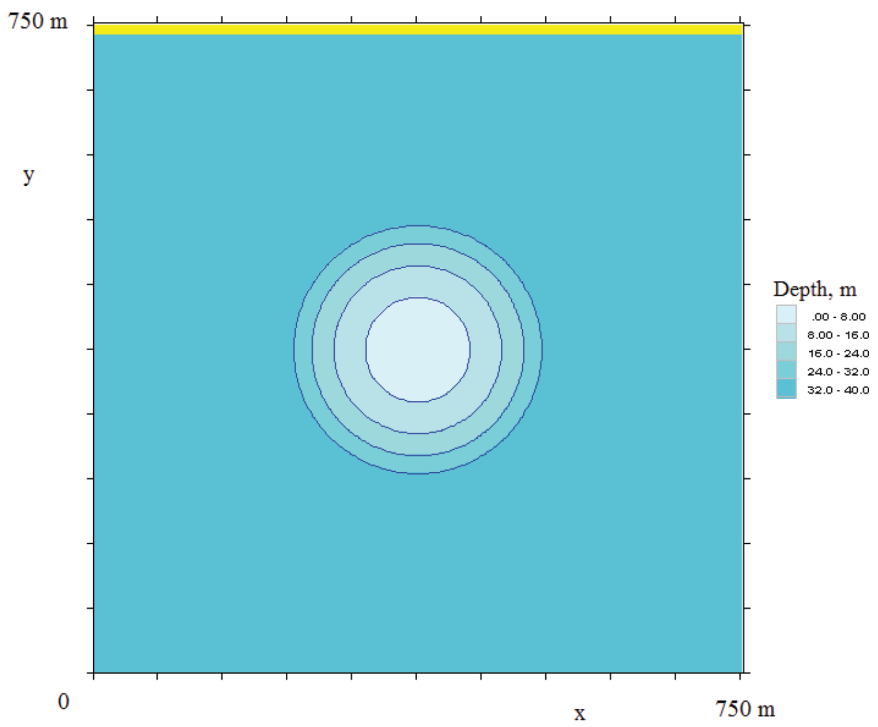

Figure 3. Bottom relief

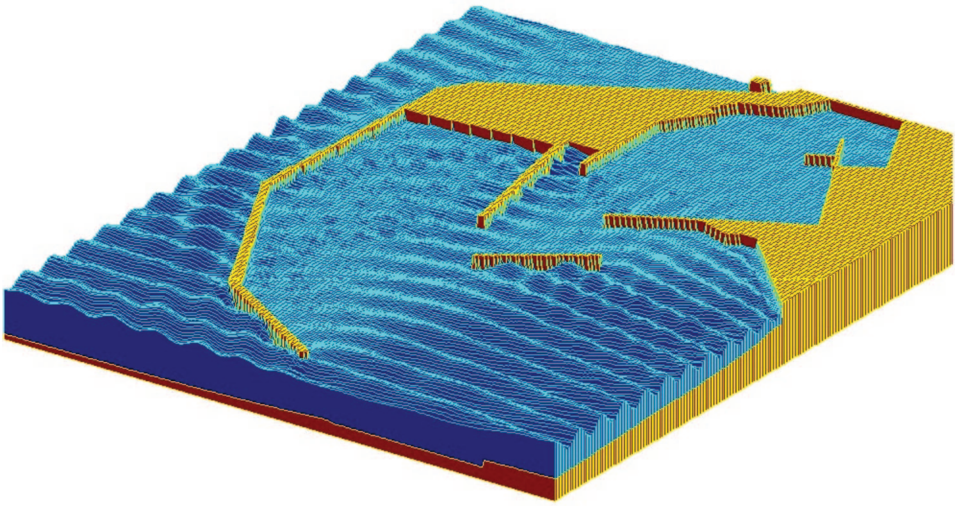

Figure 4. View of waves in the seaport of Sochi

– a simple solution of the "emission" problem: it is necessary to preset the entry range at such a distance from the object, that the waves reflected from the structures would not have enough time to pass the distance "object-entry range-object" in course of the calculation process.

3.2. Non-linear problem

3.2.1. Shallow water. Boussinesq model

At present it is generally recognized as appropriate to use the so called "shallow water of the second approximation" theory for studying various hydraulic phenomena – river floods, non-stationary processes in irrigation and ship transit canals, wind tides in river mouths, movement of solitary and periodical waves, tsunami and others. The basic suggestions of this theory are that the horizontal velocity components do not depend on the depth, and the vertical velocity component is small in relation to the horizontal. At the same time, the contribution of the vertical acceleration to the pressure is taken into account. These suggestions are correct for the waves, the length of which is considerably greater than the depth. The first mathematical model basing on these assumptions for describing solitary waves in the fluid with a constant depth was formulated by Boussinesq (1871). The model also allows for non-linear and dispersion effects.

Peregrine (1967) evolved the model of Boussinesq for the case of the variable water depth. The system of equations for describing the transformation of finite amplitude waves in the water area with a heterogeneous bottom relief can be recorded as:

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+g\frac{\partial \eta}{\partial x}=\frac{1}{3}d^2\left(\frac{\partial^3 U}{\partial t\partial x^2}+\frac{\partial^3 V}{\partial t\partial x\partial y}\right)+\frac{1}{2}d\left(2\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\partial x}+\frac{\partial^2 V}{\partial t\partial x}\frac{\partial d}{\partial y}+\frac{\partial d}{\partial x}\frac{\partial^2 V}{\partial t\partial y}\right), \quad (22)$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+g\frac{\partial \eta}{\partial y}=\frac{1}{3}d^2\left(\frac{\partial^3 U}{\partial t\partial x\partial y}+\frac{\partial^3 V}{\partial t\partial y^2}\right)+\frac{1}{2}d\left(\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\partial y}+2\frac{\partial^2 V}{\partial t\partial y}\frac{\partial d}{\partial y}+\frac{\partial d}{\partial y}\frac{\partial^2 U}{\partial t\partial x}\right), \quad (23)$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}(\eta+d)U+\frac{\partial}{\partial y}(\eta+d)V=0. \quad (24)$$

Here x, y = plane coordinates; U and V = averaged by depth velocity components on the axes x, y, equal to the local values of u and v, since in the given statement the horizontal velocity components do not depend on the vertical coordinate z.

The friction on the bottom can be taken into account parametrically, by introducing additional members to the left parts of the equations (22), (23) respectively

$$f_w \frac{|W|\cdot U}{(d+\eta)} \text{ and } f_w \frac{|W|\cdot V}{(d+\eta)},$$

where f_w = hydraulic resistance coefficient; W = vector of the velocity averaged by depth.

For solving the system of equations (22) – (24) it is necessary to preset the initial and the boundary conditions. If the calculation field is in the form of a rectangular with impenetrable side walls and an exit range, we can write,

$$\begin{aligned}
&by\; t=0: && U(x,y)=V(x,y)=0, \quad \eta(x,y)=0;\\
&by\; x=0: && U=0, \quad \partial V/\partial x=0;\\
&by\; x=M: && U=0, \quad \partial V/\partial x=0;\\
&by\; y=0: && U=0, \quad \eta=f(t,x);\\
&by\; y=N: && \frac{\partial U}{\partial x}=0, \quad V=0.
\end{aligned} \tag{25}$$

Here x = 0 and x = M left and right boundaries of the calculation field, y = 0, y = N entry and exit ranges, $f(t, x)$ = preset function.

By solving the one-dimensional problem, the momentum and the continuity equations are as below

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+g\frac{\partial \eta}{\partial x}=\frac{1}{3}d^2\frac{\partial^3 U}{\partial t \partial x^2}+d\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t \partial x} \tag{26}$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}(\eta+d)U=0 \tag{27}$$

Some examples of the wave transformation calculations on the basis of the presented mathematical model are illustrated in Fig. 5–8. These figures show fragments of the calculated field, the waves propagate from left to right, the vertical scale is approximately 15 times larger than the horizontal.

The Figure 5 depicts the free surface profile by the progressive waves propagation above the horizontal bottom, with the wave height of H = 1.75 m and the wave period of T = 11 s. The water depth

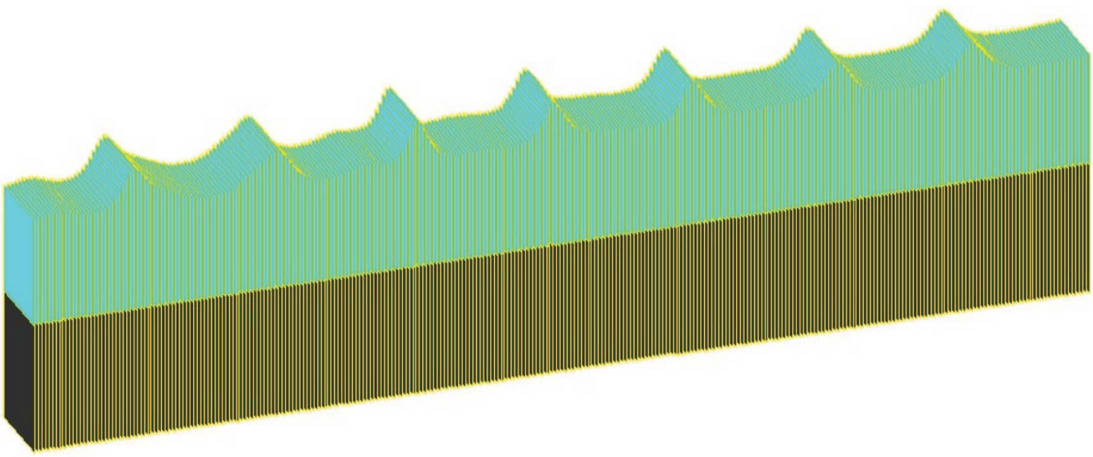

Figure 5. Wave propagation above the horizontal bottom

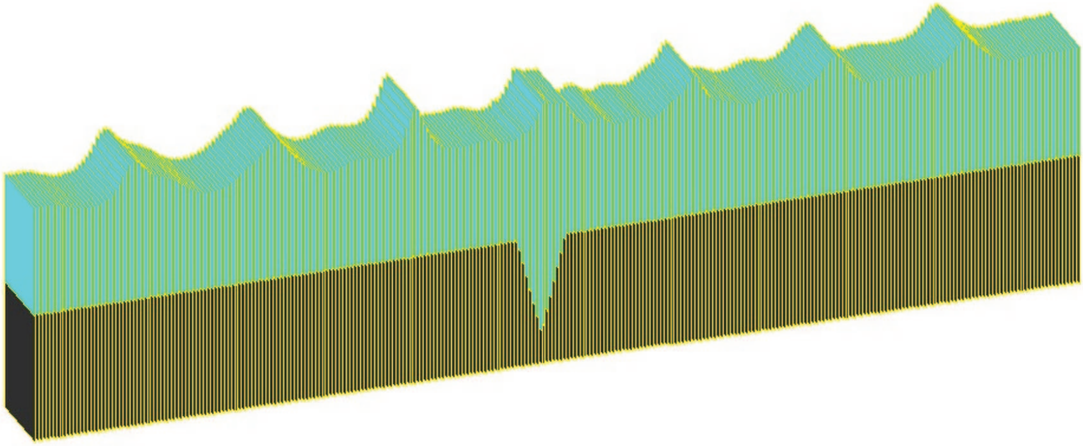

Figure 6. Interaction of waves and the under water trench

is assumed to be d = 5 m. It is clear that characteristic profiles of long waves are formed in the calculated field, these waves are relatively high, with steep and short crests and superficial long troughs. The Fig. 6 presents the result concerning the interaction of waves with the above parameters and the under water trench. The width of the trench on the top b = 26 m, the depth d_{tr} = 4 m. It is seen that above the trench the wave crests transform significantly.

Figures 7 and 8 show the calculation results for the interaction of waves with an altering period. The waves' parameters in the entry range were preset in the following way:

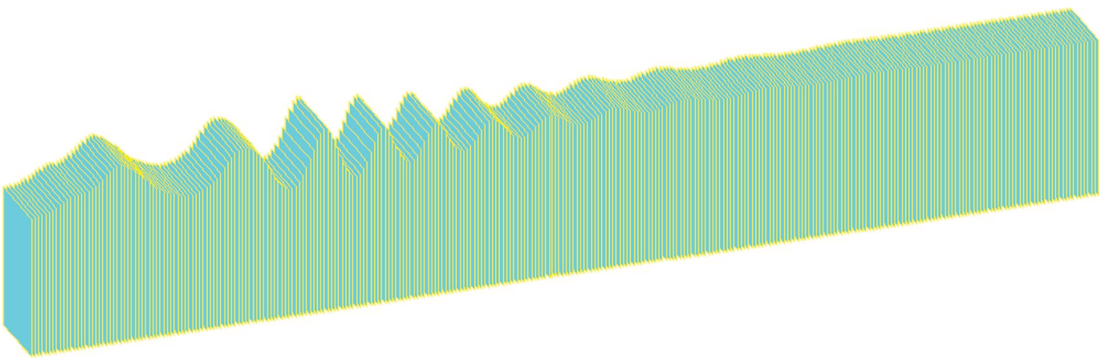

Figure 7. View of the free surface at the time point $t = 80\,\mathrm{s}$

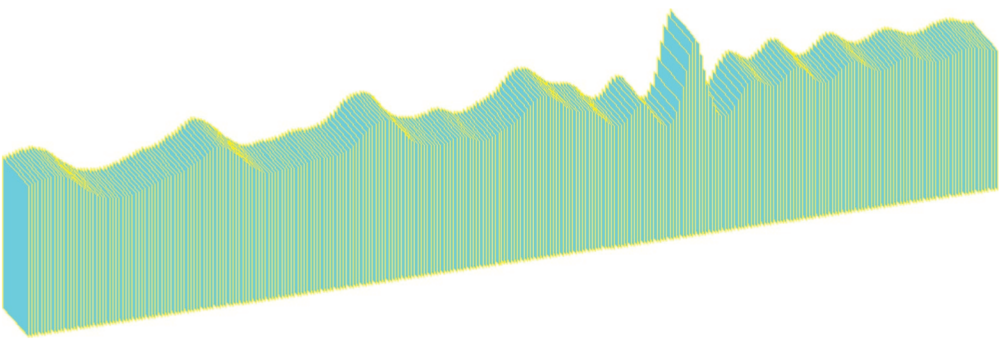

Figure 8. View of the free surface at the time point $t = 130\,\mathrm{s}$

– during the time interval from 0 to t_1 regular waves with the period T_1 = 5s were generated;
– during the time interval $t_1 < t \leq t_2$ the wave period was increasing according to the relation $T = T_1 + b\,(t - t_1)$;
– from the time point t_2 regular waves with the period T_2 = 13.75 s were "generated".

The wave height in the entry range did not change and was equal to H = 1 m, the water depth was constant and equal to d = 6 m. It was also assumed: t_1 = 30 s; t_2 = 70 s; b = 0.219. The appearance of the free surface at the time point t = 80 s is shown in Fig. 7, and at the time point t = 130 s in Fig. 8. It can be noted that in process of propagation, long waves catch up with short waves, and at a certain moment and in a certain place the concentration of the wave energy reaches maximum, which induces a wave with a considerably greater height in relation to the height of the initial waves. In the given example the maximum height of the induced wave was 3 times greater than the height of the initial waves.

Obviously, the mechanism of the disperse wave amplification must be taken into account by forecasting the forming of anomalously high waves in maritime water areas.

3.2.2. Wave transformation in the surf zone

When approaching to the shore, the waves transform considerably with the diminishing depth. They now have an asymmetrical view – the crests become shorter than the troughs, the height of the crests increases together with the incline of the front slope. After reaching the critical point of the slope incline, the waves start to break. This is accompanied by the wave energy dissipation, water flood, an intensive drawing-in of the bottom sediment into the flow and other phenomena, in particular, the formation of circulation and alongshore currents. The area adjacent to the shore, which is situated between the limit where the waves start to break and the limit of the wave climb, is called the surf zone. Generally, in this zone the wave energy dissipation and the sediment transfer take place, which can cause water logging of territories as a result of storm floods, coastal erosion and collapse of engineering structures. That is why the task of studying the processes in the surf zone attracts attention of many specialists. It would seem, that in this zone, where the wave length considerably exceeds the depth, the classical shallow water or the first approximation theory, which does not take into account effects of dispersion, should "work" fine. This theory describes with a satisfactory accuracy the movement of the long waves, in particular, their breaking in form of a hydraulic jump (Stoker (1957), Whitham (1974), Lighthill (1978)). However, in dealing with wind waves, this theory should be applied with caution. Firstly, it does not allow to determine correctly the place, where the waves start to break (if the calculations are carried out taking into account the non-linear members then regardless of the water depth, all moving waves break), and secondly, the speed of the energy dissipation in the hydraulic jumpis relatively high, and in calculations the wind waves dampen fast. In reality, in smooth shore slopes, a breaking wave can overcome a big distance, before its height diminishes significantly.

The type of the wave breaking on the shore slope is determined by their steepness and the bottom incline, or Iribarren (Battjes) parameter, (Zheleznyak and Pelinovsky, 1985), (Leontiev, 1989, 2001):

$$I_r = i\sqrt{\frac{gT_0^{\,2}}{2\pi H_0}},$$

where H_0, T_0 = wave height and wave period in the deep water, respectively; i = bottom incline.

It is established, that the type of the wave breaking depends on this parameter, (Leontiev, 2001): $I_r < 0.5$ – scattering, $0.5 < I_r < 3$ – breaking.

By solving the problem of the wave transformation in the surf zone, most commonly applied is the method, suggested by Mehaute (1962). This method is based on solving the wave energy balance equation, (Leontiev, 2001)

$$\frac{d}{dx}(E \cdot C_g) = -D, \tag{28}$$

where E = wave energy per unit area; C_g = group speed; D = speed of energy dissipation, conditioned by the wave breaking and the friction on the bottom; x = horizontal coordinate directed to the shore. In the surf zone the friction on the bottom is usually neglected, and the energy dissipation speed is determined by the relation, which is similar to that for the dissipation speed in a hydraulic jump:

$$D=\frac{1}{4}\rho g\frac{(d_2-d_1)^3}{d_2 d_1}q_w\cdot\beta, \tag{29}$$

Where d_1, d_2 = depths before and after the jump; $q_w \approx d/T$ = specific flow rate; β = coefficient, taking into account the size of the roller (β = 0 no breaking; β = 1 fully developed roller). If assumed that $E=\frac{1}{8}\rho gH^2$ and $C_g=\sqrt{gd}$, then, having determined β, it is possible to evaluate the wave parameters in the breaking zone. The method of estimating β is suggested in various works. For example, Massel (1990), Leontiev (1989), which take into account both the bottom relief and the irregularity of the waves.

The breaking waves create additional stress in the direction of the wave propagation. This results in the elevation of an average water level (wave surge) and in alongshore currents in the surf zone. Theoretical research of these phenomena started to gain momentum after the work of Longuet-Higgins and Stewart (1964), in which the authors introduced the concept of radiation stress S_{ij}. Thus, by a frontal approach of the waves to the shore,

$$S_{xx}=\frac{3}{2}E. \tag{30}$$

The increase of an average water level in the surf zone can be determined from the equation

$$\frac{d\xi}{dx}=-\frac{1}{\rho g(d+\xi)}\frac{dS_{xx}}{dx}. \tag{31}$$

By applying the relation of the linear wave theory for the wave energy density and suggesting, that the relation of the wave height to the depth in the surf zone is constant $\gamma = (H/d)_{br} = const$, it can be recorded, Leontiev (1989)

$$\frac{d\xi}{dx}=\frac{3\gamma^2}{3\gamma^2+8}\cdot i, \tag{32}$$

where i = under water shore slope.

From eqn. (32), in particular, it is clear that by a constant bottom incline, the incline of the water flood level in the surf zone will also be constant. It can be noted that, by irregular along shore wave parameters, the wave surge will also be irregular. This can cause along shore, circulating or riptide currents, (Leontiev, 2001).

By a diagonal wave approach, the additional along shore momentum flow in the surf zone is determined by the relation:

$$S_{xy}=E\frac{C_g}{C}\cos\theta\cdot\sin\theta, \tag{33}$$

where θ = angle between the wave direction and the perpendicular to the shore. By including the expression (33) to the 'averaged by depth' momentum equation in the direction of the shore line, it is possible to calculate the distribution of the along shore current velocity.

The advantages of the above method of calculating wave transformation and currents in the surf zone are its relative simplicity and, in many cases, a fair compliance of the calculation results to the laboratory and natural data. The disadvantages are the neglect of non-linear effects, which are rather significant in the surf zone, failure to take into account the interaction of waves with different lengths, as well as the interaction of waves and currents.

In the last years for calculating the transformation of the breaking waves and the currents, generated by them, the non-linear dispersion model is widely applied. It is known that Boussinesq model does not describe the wave breaking. That is why it needs to be modified for the calculations in the surf zone. In the work of Madsen et al. (1997) it is proposed to take into account the wave breaking effect with the help of additional members, introduced to the momentum equations, which describe the impact of the surface roller. For example, for the one-dimensional problem, to the left part of the momentum equation (26) an additional member is introduced, which can be recorded as follows $\frac{1}{(d+\eta)}\frac{\partial}{\partial x}R_{xx}$, where $R_{xx}=\frac{\sigma}{1-\sigma/(d+\eta)}(C-U)^2$; σ = thickness of the surface roller, depending on the shape of the front slope of a breaking wave crest; C = roller speed. In works of Kennedy et al. (2000), Kazolea et al. (2014), Tonelly and Petti (2011) for describing the wave transformation in the surf zone, the momentum equations are supplemented with the additional members, which take into account turbulent stress caused by the breaking waves.

In the work of Shakhin and Shakhina (2000) it is suggested to take into account an additional impulse in the area of the wave breaking. After reaching a critical height the waves start to break. Consequently, a part of the water volume is thrown down on the front slope of the wave crests, which results in the increase of momentum in the breaking range. To the first approximation this effect can be taken into account as follows. Let us consider a volume element $\Delta\tau = (d+\eta)\cdot\Delta x\cdot\Delta y$. We assume, that the relative speed of the breaking stream is equal to $w=\sqrt{g(d+\eta)}-|W|$ and that during the calculated time interval Δt the stream is spreading to the depth $\Delta d = m(\sqrt{g(d+\eta)}-|W|)\Delta t$ (m – empirical coefficient). Taking into account these assumptions, in process of breaking the additional momentum ΔI in the volume element $\Delta\tau$ can be determined by the relation. $\Delta I = \rho\,(\Delta x\Delta y\Delta d)w$.

In the mathematical model the effect of breaking is taken into account by introducing additional members to the right parts of the equations (22), (23) respectively

$$\frac{m(\sqrt{g(d+\eta)}-|W|)^2\cdot U}{(d+\eta)|W|}, \tag{34}$$

$$\frac{m(\sqrt{g(d+\eta)}-|W|)^2\cdot V}{(d+\eta)|W|}. \tag{35}$$

As an example, the Fig. 9, 10 show the results for the disturbed surface and an averaged field of currents on the under water shore slope basing on the mathematical model eqns. (22) – (24) including the additional members eqns. (34), (35). It was assumed: $m = 0$, if $\eta < 0.4\,d$, $m = 0.025$, if $\eta \geq 0.4\,d$.

The bottom configuration is shown on the Fig. 11. The water depth in the area from the entry range to the end of the under water slope $d = 10$ m. The slope of bottom $i = 1/15$. The under water bar with the width $b = 70$ m and the height $d = 2.7$ m is situated at the depth of $d = 6$ m. The wave height in the entry range is preset to be $H = 3$ m, the period $T = 7.7$ s, the wave direction in relation to the perpendicular to the shore line $\theta = 20°$. The derived results witness a significant wave transformation in the coastal zone: their height, length, shape and direction change. The breaking waves generate an alongshore current, in particular, a current above the submarine bar. These results correspond to the existing notions and observation data.

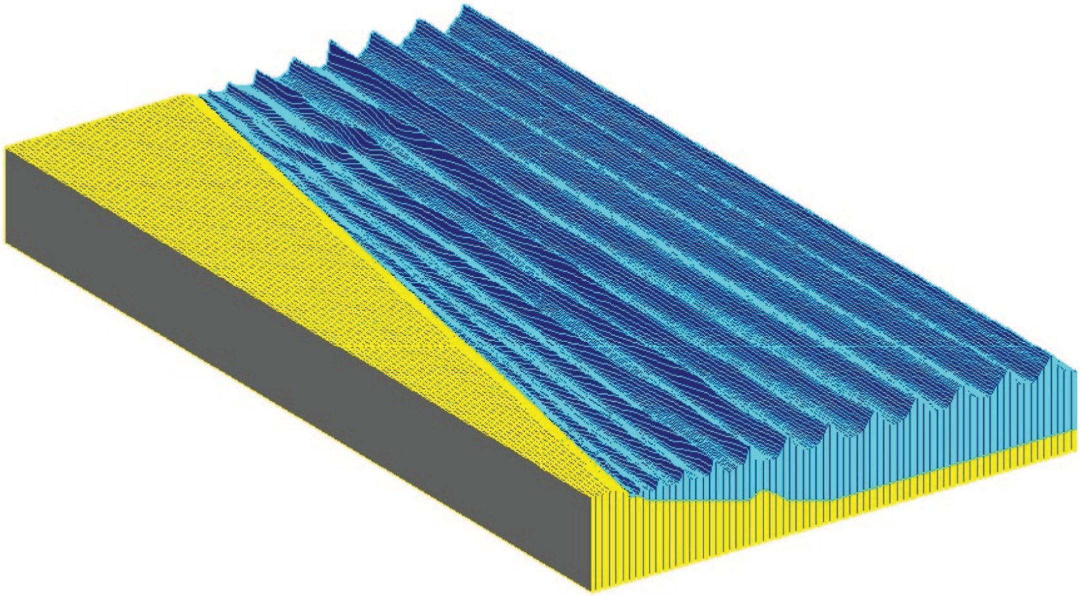

Figure 9. View of the free surface

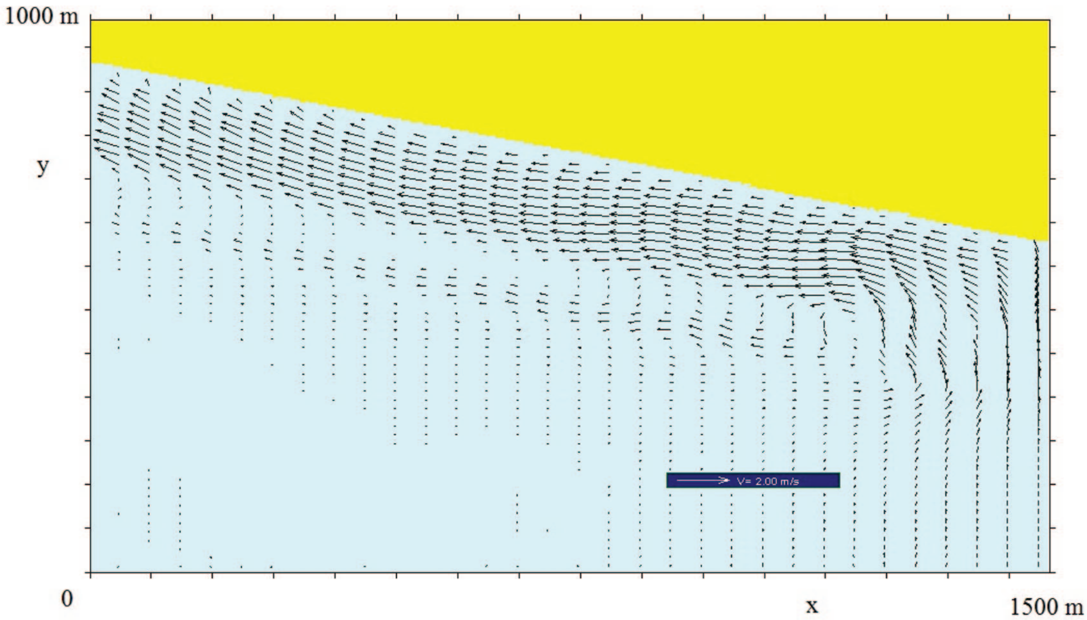

Figure 10. An averaged field of currents

3.2.3. Non-linear dispersion models with refined dispersion characteristics

It is known that the field of application of Boussinesq model (22)–(24) is limited by relatively small depths $d/\lambda \leq 0.12$, (Madsen and Sorensen, 1992). When the water depth is greater or the waves are shorter, inaccuracies in calculations according to this model become rather significant. However, often by solving practical problems such as, for example, problems of forecasting of wave climate in port areas or harbors, it is necessary to calculate the transformation of waves, which are propagation from

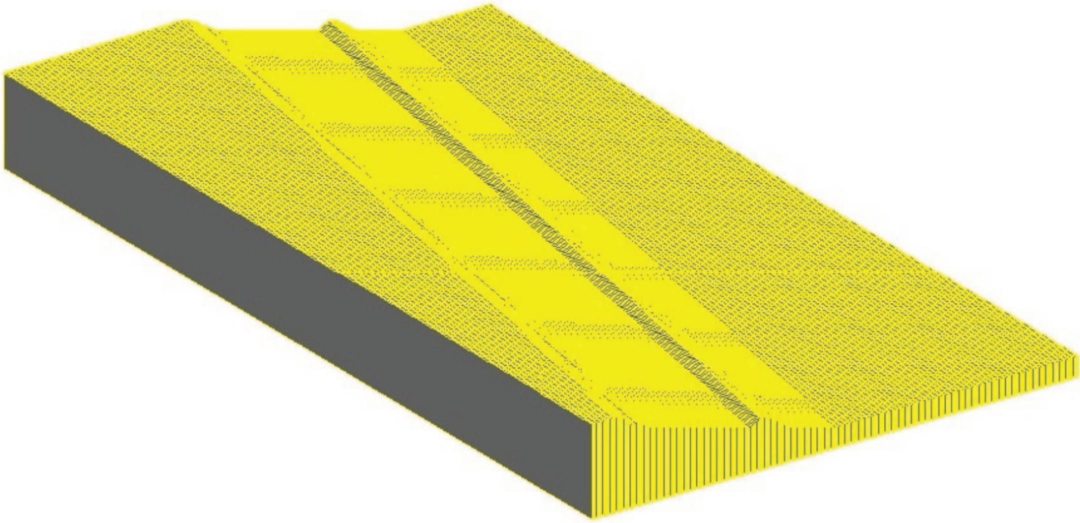

Figure 11. The bottom configuration

the area of deep water to the shallow zone. The mathematical model for solving such problems, which takes into account a smooth alteration of the bottom configuration, was for the first introduced in the work of Madsen and Sorensen (1992). Specifically, for the one-dimensional problem, the general equations of this model for the functions η and U can be recorded as:

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}(\eta+d)\,U=0 \tag{36}$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+g\frac{\partial \eta}{\partial x}=d^2\left(\frac{1}{3}+B\right)\frac{\partial^3 U}{\partial t\,\partial x^2}+d(1+B)\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\,\partial x}+Bgd^2\frac{\partial^3 \eta}{\partial x^3}+Bgd\frac{\partial d}{\partial x}\frac{\partial^2 \eta}{\partial x^2} \tag{37}$$

where B = 1/15 – disperse coefficient.

Let us present the derivation of equations for the non-linear dispersion mathematical model with refined dispersion characteristics. In the general view, the movement of the non-viscous incompressible fluid can be described by Euler equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{38}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{39}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-g-\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{40}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{41}$$

In these equations x, y = plane coordinates, lying in the plane of the undisturbed free surface; z = vertical coordinate.

Let us integrate the continuity equation (41) by depth

$$\int_{-d}^{z}\frac{\partial u}{\partial x}dz+\int_{-d}^{z}\frac{\partial v}{\partial y}dz+\int_{-d}^{z}\frac{\partial w}{\partial z}dz=0.$$

Consequently

$$w(z)=-\int_{-d}^{z}\frac{\partial u}{\partial x}dz-\int_{-d}^{z}\frac{\partial v}{\partial y}dz+w(-d), \tag{42}$$

with the condition of non-passing at the bottom we derive

$$w(-d)=-u(-d)\frac{\partial d}{\partial x}-v(-d)\frac{\partial d}{\partial y}. \tag{43}$$

Taking into account eqn. (43) let us record the equation (42) as

$$w(z)=-\frac{\partial}{\partial x}\int_{-d}^{z}u\,dz-\frac{\partial}{\partial y}\int_{-d}^{z}v\,dz. \tag{44}$$

Let us assume, that for the velocity components u and v the following relations are correct

$$u=kd\frac{\cosh k(d+z)}{\sinh kd}U, \tag{45}$$

$$v=kd\frac{\cosh k(d+z)}{\sinh kd}V, \tag{46}$$

where U, V – averaged by depth velocity components u, v. By introducing eqns. (45), (46) into eqn. (44) we obtain

$$\begin{aligned} w(z)=&-\frac{\partial}{\partial x}\int_{-d}^{z}kd\frac{\cosh k(d+z)}{\sinh kd}U\,dz-\frac{\partial}{\partial y}\int_{-d}^{z}kd\frac{\cosh k(d+z)}{\sinh kd}V\,dz=-\frac{d\sinh k(d+z)}{\sinh kd}\frac{\partial U}{\partial x}-\\ &-U\frac{\partial}{\partial x}\left[\frac{d\sinh k(d+z)}{\sinh kd}\right]-\frac{d\sinh k(d+z)}{\sinh kd}\frac{\partial V}{\partial y}-V\frac{\partial}{\partial y}\left[\frac{d\sinh k(d+z)}{\sinh kd}\right]. \end{aligned} \tag{47}$$

Let us assume, that $dw/dt \approx \partial w/\partial t$. Then the momentum equation along the axis z eqn. (40) will be recorded as

$$\frac{\partial w}{\partial t}=-g-\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{48}$$

Taking into account eqn. (47) we can record

$$\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial z}=&\,g-\frac{d\sinh k(d+z)}{\sinh kd}\frac{\partial^2 U}{\partial t\,\partial x}-\\ &-\frac{\partial}{\partial x}\left[\frac{d\sinh k(d+z)}{\sinh kd}\right]\frac{\partial U}{\partial t}-\frac{d\sinh k(d+z)}{\sinh kd}\frac{\partial^2 V}{\partial t\,\partial y}-\frac{\partial}{\partial y}\left[\frac{d\sinh k(d+z)}{\sinh kd}\right]\frac{\partial V}{\partial t}. \end{aligned} \tag{49}$$

Let us integrate the equation (49) by depth

$$-\int_z^\eta \frac{1}{\rho}\frac{\partial p}{\partial z}dz = \int_z^\eta g\,dz - \int_z^\eta \frac{d\sinh k(d+z)}{\sinh kd}\frac{\partial^2 U}{\partial t\,\partial x}dz -$$
$$-\int_z^\eta \frac{\partial}{\partial x}\left[\frac{d\sinh k(d+z)}{\sinh kd}\right]\frac{\partial U}{\partial t}dz - \int_z^\eta \frac{d\sinh k(d+z)}{\sinh kd}\frac{\partial^2 V}{\partial t\,\partial y}dz - \int_z^\eta \frac{\partial}{\partial y}\left[\frac{d\sinh k(d+z)}{\sinh kd}\right]\frac{\partial V}{\partial t}dz.$$

$$-\frac{1}{\rho}p(\eta)+\frac{1}{\rho}p(z) = g(\eta - z) - \frac{d}{\sinh kd}\frac{\partial^2 U}{\partial t\,\partial x}\frac{[\cosh k(d+\eta)] - \cosh k(d+z)}{k} -$$
$$-\frac{\partial U}{\partial t}\left[\frac{\partial}{\partial x}\int_z^\eta \frac{d\sinh k(d+z)}{\sinh kd}dz - \frac{d\sinh k(d+\eta)}{\sinh kd}\frac{\partial\eta}{\partial x}\right] - \frac{d}{\sinh kd}\frac{\partial^2 V}{\partial t\,\partial y}\frac{[\cosh k(d+\eta)] - \cosh k(d+z)}{k} -$$
$$-\frac{\partial V}{\partial t}\left[\frac{\partial}{\partial y}\int_z^\eta \frac{d\sinh k(d+z)}{\sinh kd}dz - \frac{d\sinh k(d+\eta)}{\sinh kd}\frac{\partial\eta}{\partial y}\right].$$

Let us assume that $p(\eta) = 0$. By confining to the members of the first order of smallness we derive

$$-\frac{1}{\rho}p(z) = g(\eta - z) - \frac{gd}{\omega^2}\frac{\partial^2 U}{\partial t\,\partial x} + \frac{d\cdot\cosh k(d+z)}{k\sinh kd}\frac{\partial^2 U}{\partial t\,\partial x} - \frac{g}{\omega^2}\frac{\partial U}{\partial t}\frac{\partial d}{\partial x} + \frac{\partial U}{\partial t}\frac{\partial}{\partial x}\left[\frac{d\sinh k(d+z)}{k\sinh kd}\right] -$$
$$-\frac{gd}{\omega^2}\frac{\partial^2 V}{\partial t\,\partial y} + \frac{d\cdot\cosh k(d+z)}{k\sinh kd}\frac{\partial^2 V}{\partial t\,\partial y} - \frac{g}{\omega^2}\frac{\partial V}{\partial t}\frac{\partial d}{\partial y} + \frac{\partial V}{\partial t}\frac{\partial}{\partial y}\left[\frac{d\sinh k(d+z)}{k\sinh kd}\right]. \tag{50}$$

Knowing the dependence of the functions u, v, p from the vertical coordinate z (eqns. (45), (46), (50)), the equations (38) and (39) can be integrated by depth. After accomplishing the integration procedures, suggesting that the bottom relief is altering smoothly and confining to the members up to the second order of smallness, we obtain

$$\frac{\partial U}{\partial t} + (2\alpha-1)U\frac{\partial U}{\partial x} + (2\alpha-1)V\frac{\partial U}{\partial y} + g\frac{\partial\eta}{\partial x} = \frac{d}{k}\left(\coth kd - \frac{1}{kd}\right)\left(\frac{\partial^3 U}{\partial t\,\partial x^2} + \frac{\partial^3 V}{\partial t\,\partial x\,\partial y}\right) +$$
$$+\frac{1}{k}\left[\coth kd - \frac{1}{kd} + \frac{1}{\sinh kd} - \frac{4}{(\sinh 2kd + 2kd)}\right]\left(2\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\,\partial x} + \frac{\partial d}{\partial x}\frac{\partial^2 V}{\partial t\,\partial y} + \frac{\partial d}{\partial y}\frac{\partial^2 V}{\partial t\,\partial x}\right), \tag{52}$$

$$\frac{\partial V}{\partial t} + (2\alpha-1)U\frac{\partial V}{\partial x} + (2\alpha-1)V\frac{\partial V}{\partial y} + g\frac{\partial\eta}{\partial y} = \frac{d}{k}\left(\coth kd - \frac{1}{kd}\right)\left(\frac{\partial^3 V}{\partial t\,\partial y^2} + \frac{\partial^3 U}{\partial t\,\partial x\,\partial y}\right) +$$
$$+\frac{1}{k}\left[\coth kd - \frac{1}{kd} + \frac{1}{\sinh kd} - \frac{4}{(\sinh 2kd + 2kd)}\right]\left(2\frac{\partial d}{\partial y}\frac{\partial^2 V}{\partial t\,\partial y} + \frac{\partial d}{\partial y}\frac{\partial^2 U}{\partial t\,\partial x} + \frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\,\partial y}\right), \tag{53}$$

where $\alpha = \dfrac{kd\cosh kd}{2\sinh kd} + \dfrac{k^2 d^2}{2\sinh^2 kd} =$ momentum coefficient. After averaging the continuity equation (41) by depth we obtain

$$\frac{\partial\eta}{\partial t} + \frac{\partial(d+\eta)U}{\partial x} + \frac{\partial(d+\eta)V}{\partial y} = 0. \tag{54}$$

The numerical experiments established, that the coefficient α can be assumed equal to "one". In this case the equations (52) and (53) will have the view

$$\begin{aligned}&\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+g\frac{\partial \eta}{\partial x}=\frac{d}{k}\left(\coth kd-\frac{1}{kd}\right)\left(\frac{\partial^3 U}{\partial t\,\partial x^2}+\frac{\partial^3 V}{\partial t\,\partial x\,\partial y}\right)+\\&+\frac{1}{k}\left[\coth kd-\frac{1}{kd}+\frac{1}{\sinh kd}-\frac{4}{(\sinh 2kd+2kd)}\right]\left(2\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\,\partial x}+\frac{\partial d}{\partial x}\frac{\partial^2 V}{\partial t\,\partial y}+\frac{\partial d}{\partial y}\frac{\partial^2 V}{\partial t\,\partial x}\right),\end{aligned} \tag{52*}$$

$$\begin{aligned}&\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+g\frac{\partial \eta}{\partial y}=\frac{d}{k}\left(\coth kd-\frac{1}{kd}\right)\left(\frac{\partial^3 V}{\partial t\,\partial y^2}+\frac{\partial^3 U}{\partial t\,\partial x\,\partial y}\right)+\\&+\frac{1}{k}\left[\coth kd-\frac{1}{kd}+\frac{1}{\sinh kd}-\frac{4}{(\sinh 2kd+2kd)}\right]\left(2\frac{\partial d}{\partial y}\frac{\partial^2 V}{\partial t\,\partial y}+\frac{\partial d}{\partial y}\frac{\partial^2 U}{\partial t\,\partial x}+\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\,\partial y}\right),\end{aligned} \tag{53*}$$

If the hyperbolic functions in the right parts of the eqns. (52*), (53*) are expanded into series and confined to the three members of the series, we can record

$$\begin{aligned}&\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+g\frac{\partial \eta}{\partial x}=\frac{1}{3}d^2\left(1-\frac{k^2d^2}{15}\right)\left(\frac{\partial^3 U}{\partial t\,\partial x^2}+\frac{\partial^3 V}{\partial t\,\partial x\,\partial y}\right)+\\&+\frac{1}{2}d\left(1-\frac{17}{180}k^2d^2\right)\left(2\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\,\partial x}+\frac{\partial d}{\partial x}\frac{\partial^2 V}{\partial t\,\partial y}+\frac{\partial d}{\partial y}\frac{\partial^2 V}{\partial t\,\partial x}\right),\end{aligned} \tag{55}$$

$$\begin{aligned}&\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+g\frac{\partial \eta}{\partial y}=\frac{1}{3}d^2\left(1-\frac{k^2d^2}{15}\right)\left(\frac{\partial^3 V}{\partial t\,\partial y^2}+\frac{\partial^3 U}{\partial t\,\partial x\,\partial y}\right)+\\&+\frac{1}{2}d\left(1-\frac{17}{180}k^2d^2\right)\left(2\frac{\partial d}{\partial y}\frac{\partial^2 V}{\partial t\,\partial y}+\frac{\partial d}{\partial y}\frac{\partial^2 U}{\partial t\,\partial x}+\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\,\partial y}\right),\end{aligned} \tag{56}$$

Assuming that $\frac{\partial U}{\partial t}\approx-\frac{C^2}{d}\frac{\partial \eta}{\partial x}$ and $\frac{\partial V}{\partial t}\approx-\frac{C^2}{d}\frac{\partial \eta}{\partial y}$, the right part of the equations (55), (56) can be recorded in a different way

$$\begin{aligned}&\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial V}{\partial y}+g\frac{\partial \eta}{\partial x}=d^2\left(\frac{1}{3}+B\right)\left(\frac{\partial^3 U}{\partial t\,\partial x^2}+\frac{\partial^3 V}{\partial t\,\partial x\,\partial y}\right)+gd^2B\left(\frac{\partial^3 \eta}{\partial x^3}+\frac{\partial^3 \eta}{\partial x\,\partial y^2}\right)\\&+d(1+2B_1)\left[\frac{\partial d}{\partial x}\left(\frac{\partial^2 U}{\partial t\,\partial x}+\frac{1}{2}\frac{\partial^2 U}{\partial t\,\partial y}\right)+\frac{1}{2}\frac{\partial d}{\partial y}\frac{\partial^2 V}{\partial t\,\partial x}\right]+B_1gd\left[\frac{\partial d}{\partial x}\left(2\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 \eta}{\partial y^2}\right)+\frac{\partial d}{\partial y}\frac{\partial^2 \eta}{\partial x\,\partial y}\right],\end{aligned} \tag{57}$$

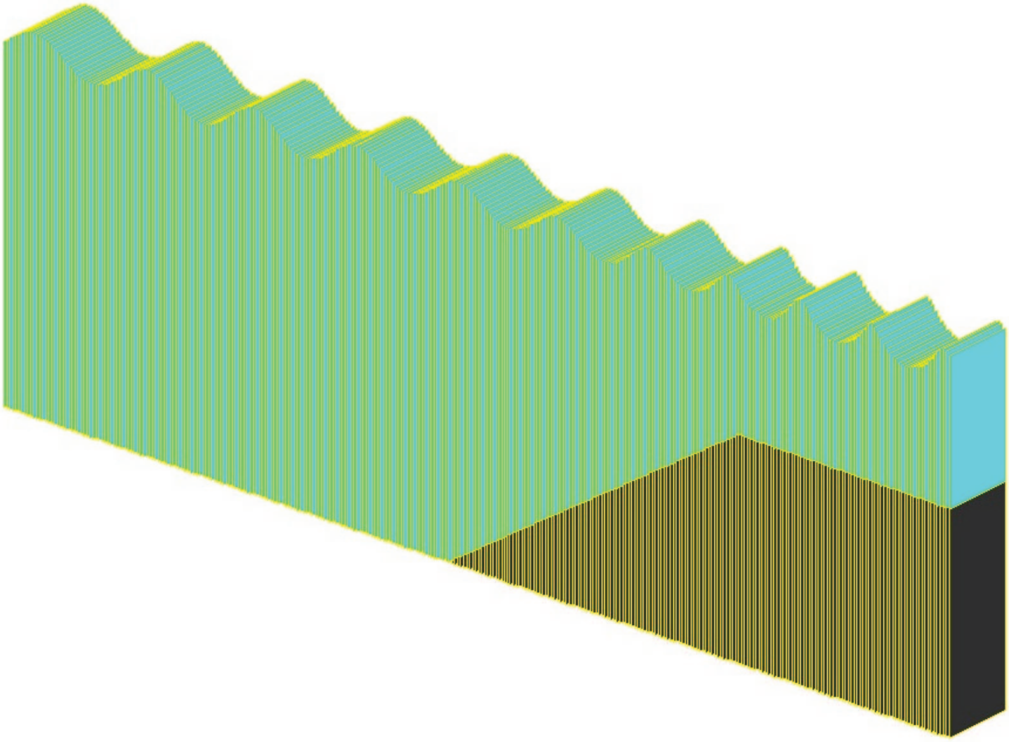

Figure 12. Wave transformation on the under water slope

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+g\frac{\partial \eta}{\partial y}=d^2\left(\frac{1}{3}+B\right)\left(\frac{\partial^3 V}{\partial t\,\partial y^2}+\frac{\partial^3 U}{\partial t\,\partial x\,\partial y}\right)+gd^2B\left(\frac{\partial^3 \eta}{\partial y^3}+\frac{\partial^3 \eta}{\partial y\,\partial x^2}\right)+$$
$$+d(1+2B_1)\left[\frac{\partial d}{\partial y}\left(\frac{\partial^2 V}{\partial t\,\partial y}+\frac{1}{2}\frac{\partial^2 U}{\partial t\,\partial x}\right)+\frac{1}{2}\frac{\partial d}{\partial x}\frac{\partial^2 U}{\partial t\,\partial y}\right]+B_1 gd\left[\frac{\partial d}{\partial y}\left(2\frac{\partial^2 \eta}{\partial y^2}+\frac{\partial^2 \eta}{\partial x^2}\right)+\frac{\partial d}{\partial x}\frac{\partial^2 \eta}{\partial x\,\partial y}\right], \qquad (58)$$

where B = 1/15; B_1 = 17/120. It should be noted that the eqns. (57), (58) nearly coincide with the momentum equations in Madsen and Sorensen (1992). Only the coefficient B_1 is different: in the work Madsen and Sorensen (1992) B_1 = B = 1/15.

The system of equations (52*), (53*), (54) allows to solve in the non-linear frame the problems of the wave transformation in water areas with random depth and relatively smoothly altering bottom configuration. As an example, the Fig. 12 shows the results of calculating the disturbed surface by the wave propagation above the slope from the depth $d_0 = \lambda_0$ to the depth $d_0 = 0.08\lambda_0$. The wave height in the entry range is equal to $H_0 = 0.02\lambda_0$. The incline of the slope is i = 0.05. The limitation of this model is that it describes regular waves.

For irregular waves the model eqns. (54), (57), (58) can be applied. Hereby it should be taken into account that the field of application of this model is limited by depth $d \le 0.5\lambda$, (Madsen and Sorensen, 1992).

4. CONCLUSION

The above discussed study considers a number of mathematical models of water waves of linear and non-linear form. New results based on Stokes and cnoidal wave theories are obtained. In particular it is established that by the propagation of the progressive waves with finite amplitude, in the bottom layers of the fluid, a compensatory counter-current is formed. A comparison of theoretical and experimental data is accomplished. The modified mathematical models for analyzing surface waves in water areas of any given shape and depth, taking into account the effects of refraction, diffraction and breaking, are presented. The examples of calculating the wave transformation in port and coastal areas with a non-uniform bottom configuration are given.

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