# Bias-Variance Decomposition for squared error

**Notation:** $f = f(x)$, $\hat{f} = \hat{f}(x)$

**Identities:**

1a. $Var[x] = E[x^2] - E[x]^2 = E[(x - E[x])^2]$

1b. $E[x^2] = Var[x] + E[x]^2$

2. $E[x - E[x]] = E[x] - E[E[x]] = E[x] - E[x] = 0$

3. Since $f$ is deterministic, $E[f] = f$

4. $E[\epsilon] = 0$ (zero-mean irreducible error)

5. $Y = f + \epsilon$

6. $E[Y] = E[f] + E[\epsilon] = f \quad \therefore E[Y] = f$

Expected value of real model $(f(x)+\epsilon)$ is Expected value of just $f(x)$

7. $Var[\epsilon] = \sigma^2$

8. $Var[Y] = E[(Y - E[Y])^2] = E[(Y-f)^2] = E[(f + \epsilon - f)^2]$

$= E[\epsilon^2] = Var[\epsilon] + E[\epsilon]^2 = Var[\epsilon] = \sigma^2$

$\therefore Var[Y] = \sigma^2$

9. $\epsilon$ and $\hat{f}$ are independent

Therefore,

$$
\begin{aligned}
E[(Y - \hat{f})^2] &= E[Y^2 + \hat{f}^2 - 2Y\hat{f}] \\
&= E[Y^2] + E[\hat{f}^2] - 2E[Y\hat{f}] \\
&= Var[Y] + E[Y]^2 + Var[\hat{f}] + E[\hat{f}]^2 - 2E[Y\hat{f}] \\
&= Var[Y] + Var[\hat{f}] + (E[Y]^2 - 2E[Y\hat{f}] + E[\hat{f}]^2) \\
&= Var[Y] + Var[\hat{f}] + (f^2 - 2fE[\hat{f}] + E[\hat{f}]^2) \\
&= Var[Y] + Var[\hat{f}] + (f - E[\hat{f}])^2
\end{aligned}
$$

$$= \underbrace{\sigma^2}_{Var[\epsilon]} + Var[\hat{f}] + Bias[\hat{f}]^2$$