# Foundations of Machine Learning

## Support Vector Machines

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# Binary Classification Problem

- Training data: sample drawn i.i.d. from set $X \subseteq \mathbb{R}^N$ according to some distribution $D$,

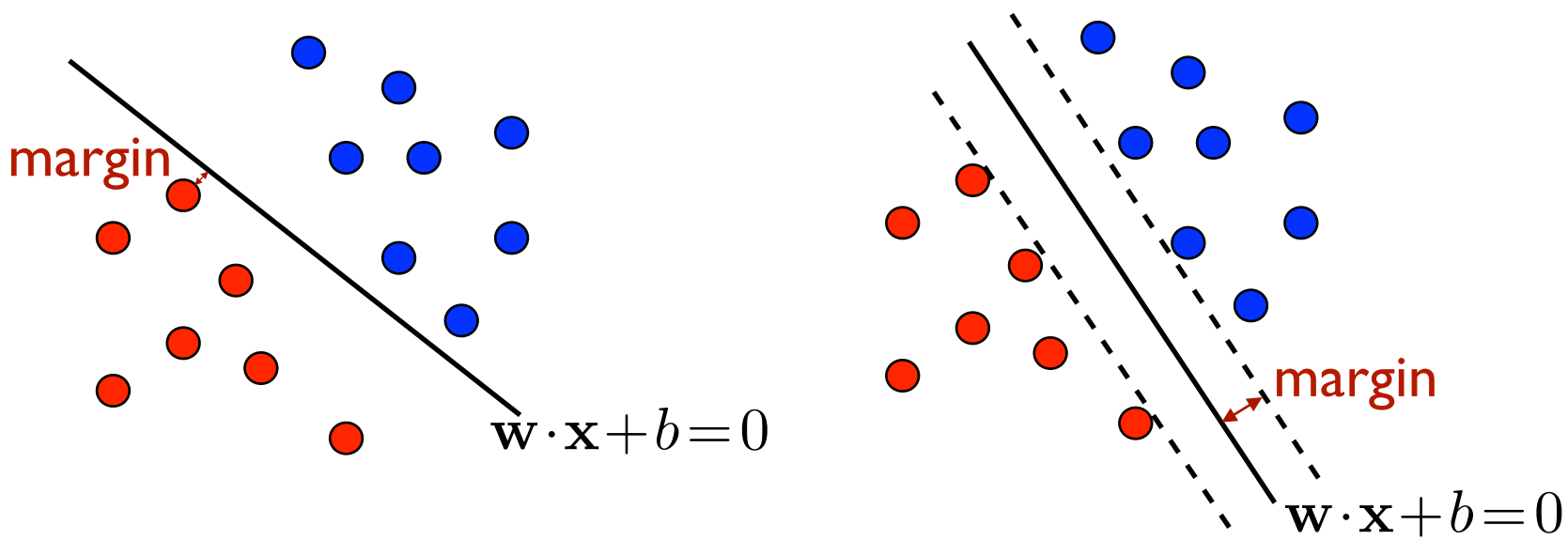
$$S = ((x_1, y_1), \ldots, (x_m, y_m)) \in X \times \{-1, +1\}.$$

- Problem: find hypothesis $h : X \mapsto \{-1, +1\}$ in $H$ (classifier) with small generalization error $R_D(h)$.
- Linear classification:
  - Hypotheses based on hyperplanes.
  - Linear separation in high-dimensional space.

# This Lecture

- Support Vector Machines - separable case
- Support Vector Machines - non-separable case
- Margin guarantees

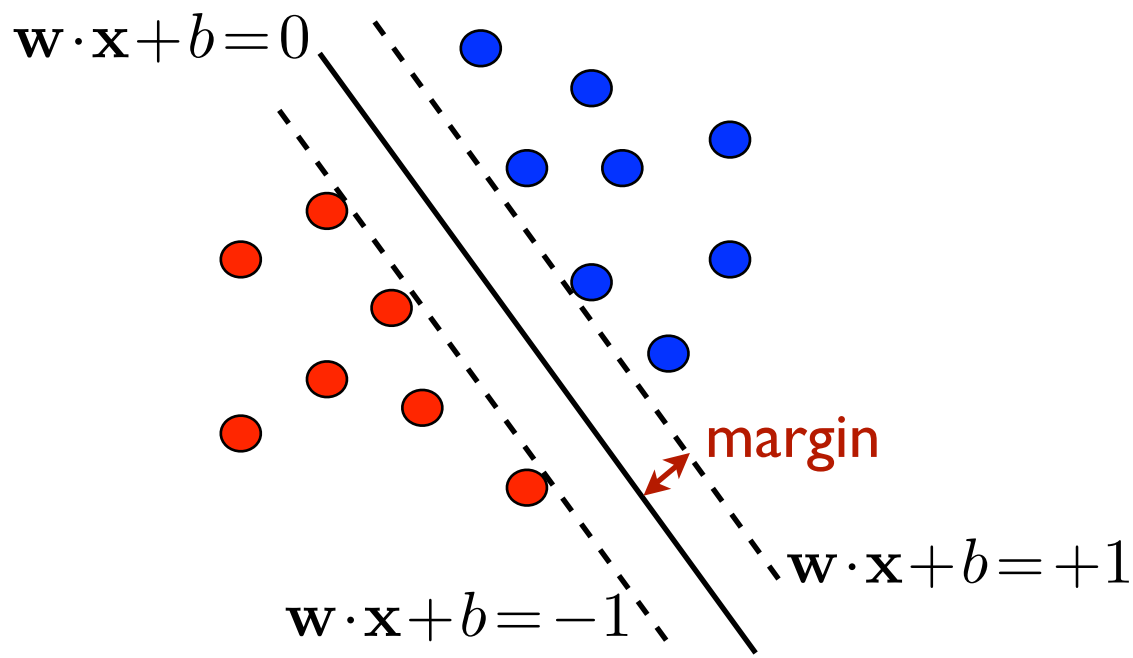
# Linear Separation




- Classifiers: $H = \{\mathbf{x} \mapsto \operatorname{sgn}(\mathbf{w} \cdot \mathbf{x} + b) : \mathbf{w} \in \mathbb{R}^N, b \in \mathbb{R}\}$.
- Geometric margin: $\rho = \min_{i \in [1,m]} \frac{|\mathbf{w} \cdot \mathbf{x}_i + b|}{\|\mathbf{w}\|}$.

# Optimal Hyperplane: Max. Margin

(Vapnik and Chervonenkis, 1965)




$$\rho = \max_{\mathbf{w},b\,:\,y_i(\mathbf{w}\cdot\mathbf{x}_i+b)\geq 0} \; \min_{i\in[1,m]} \frac{|\mathbf{w}\cdot\mathbf{x}_i+b|}{\|\mathbf{w}\|}.$$

# Optimal Hyperplane: Max. Margin

$$\begin{aligned}
\rho &= \max_{\mathbf{w},b:\ y_i(\mathbf{w}\cdot\mathbf{x}_i+b)\geq 0} \min_{i\in[1,m]} \frac{|\mathbf{w}\cdot\mathbf{x}_i+b|}{\|\mathbf{w}\|} \\
&= \max_{\substack{\mathbf{w},b:\ y_i(\mathbf{w}\cdot\mathbf{x}_i+b)\geq 0 \\ \min_{i\in[1,m]}|\mathbf{w}\cdot\mathbf{x}_i+b|=1}} \min_{i\in[1,m]} \frac{|\mathbf{w}\cdot\mathbf{x}_i+b|}{\|\mathbf{w}\|} \\
&= \max_{\substack{\mathbf{w},b:\ y_i(\mathbf{w}\cdot\mathbf{x}_i+b)\geq 0 \\ \min_{i\in[1,m]}|\mathbf{w}\cdot\mathbf{x}_i+b|=1}} \frac{1}{\|\mathbf{w}\|} \\
&= \max_{\mathbf{w},b:\ y_i(\mathbf{w}\cdot\mathbf{x}_i+b)\geq 1} \frac{1}{\|\mathbf{w}\|}. \qquad (\text{min. reached})
\end{aligned}$$

# Optimization Problem

- Constrained optimization:

$$\min_{\mathbf{w},b} \quad \frac{1}{2}\|\mathbf{w}\|^2$$
$$\text{subject to} \quad y_i(\mathbf{w}\cdot\mathbf{x}_i + b) \geq 1, i \in [1, m].$$

- Properties:
  - Convex optimization.
  - Unique solution for linearly separable sample.

# Optimal Hyperplane Equations

- **Lagrangian**: for all $\mathbf{w}, b, \alpha_i \geq 0$,

$$L(\mathbf{w}, b, \alpha) = \frac{1}{2}\|\mathbf{w}\|^2 - \sum_{i=1}^{m} \alpha_i [y_i(\mathbf{w} \cdot \mathbf{x}_i + b) - 1].$$

- **KKT conditions**:

$$\nabla_{\mathbf{w}} L = \mathbf{w} - \sum_{i=1}^{m} \alpha_i y_i \mathbf{x}_i = 0 \iff \mathbf{w} = \sum_{i=1}^{m} \alpha_i y_i \mathbf{x}_i.$$

$$\nabla_b L = -\sum_{i=1}^{m} \alpha_i y_i = 0 \iff \sum_{i=1}^{m} \alpha_i y_i = 0.$$

$$\forall i \in [1, m],\ \alpha_i [y_i(\mathbf{w} \cdot \mathbf{x}_i + b) - 1] = 0.$$

# Support Vectors

- Complementarity conditions:

$$\alpha_i [y_i (\mathbf{w} \cdot \mathbf{x}_i + b) - 1] = 0 \implies \alpha_i = 0 \vee y_i (\mathbf{w} \cdot \mathbf{x}_i + b) = 1.$$

- Support vectors: vectors $\mathbf{x}_i$ such that

$$\alpha_i \neq 0 \wedge y_i (\mathbf{w} \cdot \mathbf{x}_i + b) = 1.$$

  - Note: support vectors are not unique.

# Moving to The Dual

- Plugging in the expression of $\mathbf{w}$ in $L$ gives:

$$L = \underbrace{\frac{1}{2}\Big\|\sum_{i=1}^{m}\alpha_i y_i \mathbf{x}_i\Big\|^2 - \sum_{i,j=1}^{m}\alpha_i\alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)}_{-\frac{1}{2}\sum_{i,j=1}^{m}\alpha_i\alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)} - \underbrace{\sum_{i=1}^{m}\alpha_i y_i b}_{0} + \sum_{i=1}^{m}\alpha_i.$$

- Thus,

$$L = \sum_{i=1}^{m}\alpha_i - \frac{1}{2}\sum_{i,j=1}^{m}\alpha_i\alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j).$$

# Equivalent Dual Opt. Problem

- Constrained optimization:

$$\max_{\alpha} \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)$$

$$\text{subject to: } \alpha_i \geq 0 \wedge \sum_{i=1}^{m} \alpha_i y_i = 0, i \in [1, m].$$

- Solution:

$$h(x) = \operatorname{sgn}\Big( \sum_{i=1}^{m} \alpha_i y_i (\mathbf{x}_i \cdot \mathbf{x}) + b \Big),$$

with $b = y_i - \sum_{j=1}^{m} \alpha_j y_j (\mathbf{x}_j \cdot \mathbf{x}_i)$ for any SV $\mathbf{x}_i$.

# Leave-One-Out Error

- **Definition**: let $h_S$ be the hypothesis output by learning algorithm $L$ after receiving sample $S$ of size $m$. Then, the leave-one-out error of $L$ over $S$ is:

$$\widehat{R}_{\text{loo}}(L) = \frac{1}{m}\sum_{i=1}^{m} 1_{h_{S-\{x_i\}}(x_i)\neq f(x_i)}.$$

- **Property**: unbiased estimate of expected error of hypothesis trained on sample of size $m-1$,

$$\boxed{\mathop{\mathrm{E}}_{S\sim D^m}[\widehat{R}_{\text{loo}}(L)]} = \frac{1}{m}\sum_{i=1}^{m} \mathop{\mathrm{E}}_{S}[1_{h_{S-\{x_i\}}(x_i)\neq f(x_i)}] = \mathop{\mathrm{E}}_{S}[1_{h_{S-\{x\}}(x)\neq f(x)}]$$

$$= \mathop{\mathrm{E}}_{S'\sim D^{m-1}}[\mathop{\mathrm{E}}_{x\sim D}[1_{h_{S'}(x)\neq f(x)}]] = \boxed{\mathop{\mathrm{E}}_{S'\sim D^{m-1}}[R(h_{S'})].}$$

# Leave-One-Out Analysis

- **Theorem**: let $h_S$ be the optimal hyperplane for a sample $S$ and let $N_{\mathrm{SV}}(S)$ be the number of support vectors defining $h_S$. Then,

$$\mathop{\mathrm{E}}_{S \sim D^m}[R(h_S)] \leq \mathop{\mathrm{E}}_{S \sim D^{m+1}}\left[\frac{N_{\mathrm{SV}}(S)}{m+1}\right].$$

- **Proof**: Let $S \sim D^{m+1}$ be a sample linearly separable and let $x \in S$. If $h_{S-\{x\}}$ misclassifies $x$, then $x$ must be a SV for $h_S$. Thus,

$$\widehat{R}_{\mathrm{loo}}(\text{opt.-hyp.}) \leq \frac{N_{\mathrm{SV}}(S)}{m+1}.$$

# Notes

- Bound on expectation of error only, not the probability of error.
- Argument based on sparsity (number of support vectors). We will see later other arguments in support of the optimal hyperplanes based on the concept of margin.

# This Lecture

- Support Vector Machines - separable case
- Support Vector Machines - non-separable case
- Margin guarantees

# Support Vector Machines

(Cortes and Vapnik, 1995)

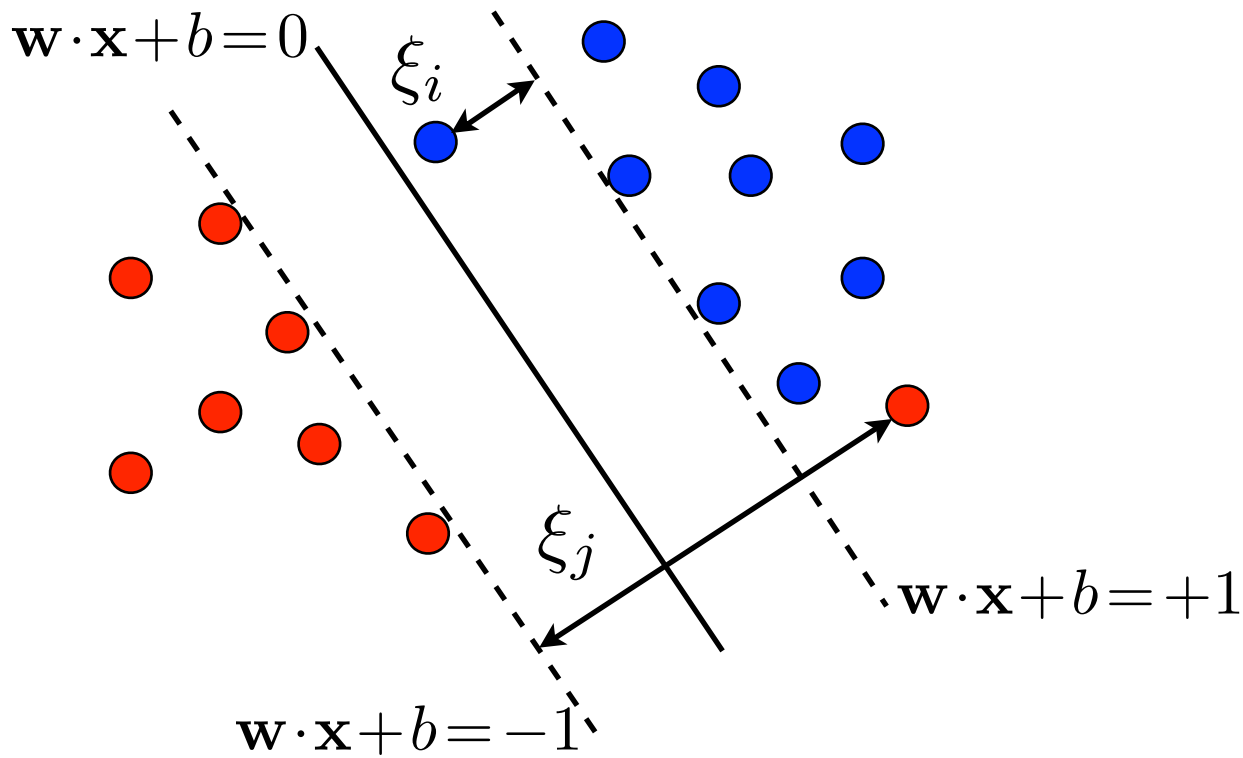
- Problem: data often not linearly separable in practice. For any hyperplane, there exists $\mathbf{x}_i$ such that

$$y_i \left[\mathbf{w} \cdot \mathbf{x}_i + b\right] \not\geq 1.$$

- Idea: relax constraints using slack variables $\xi_i \geq 0$

$$y_i \left[\mathbf{w} \cdot \mathbf{x}_i + b\right] \geq 1 - \xi_i.$$

# Soft-Margin Hyperplanes




- Support vectors: points along the margin or outliers.
- Soft margin: $\rho = 1/\|\mathbf{w}\|$.

# Optimization Problem

(Cortes and Vapnik, 1995)

- Constrained optimization:

$$\min_{\mathbf{w},b,\xi} \quad \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_{i=1}^{m}\xi_i$$

$$\text{subject to} \quad y_i(\mathbf{w}\cdot\mathbf{x}_i + b) \geq 1 - \xi_i \quad \wedge \quad \xi_i \geq 0, i \in [1, m].$$

- Properties:
  - $C \geq 0$ trade-off parameter.
  - Convex optimization.
  - Unique solution.

# Notes

- Parameter $C$: trade-off between maximizing margin and minimizing training error. How do we determine $C$ ?
- The general problem of determining a hyperplane minimizing the error on the training set is NP-complete (as a function of dimension).
- Other convex functions of the slack variables could be used: this choice and a similar one with squared slack variables lead to a convenient formulation and solution.

# SVM - Equivalent Problem

- Optimization:

$$\min_{\mathbf{w},b} \quad \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_{i=1}^{m}\Big(1 - y_i(\mathbf{w}\cdot\mathbf{x}_i + b)\Big)_+.$$

- Loss functions:
  - hinge loss:

$$L(h(x), y) = (1 - yh(x))_+.$$

  - quadratic hinge loss:

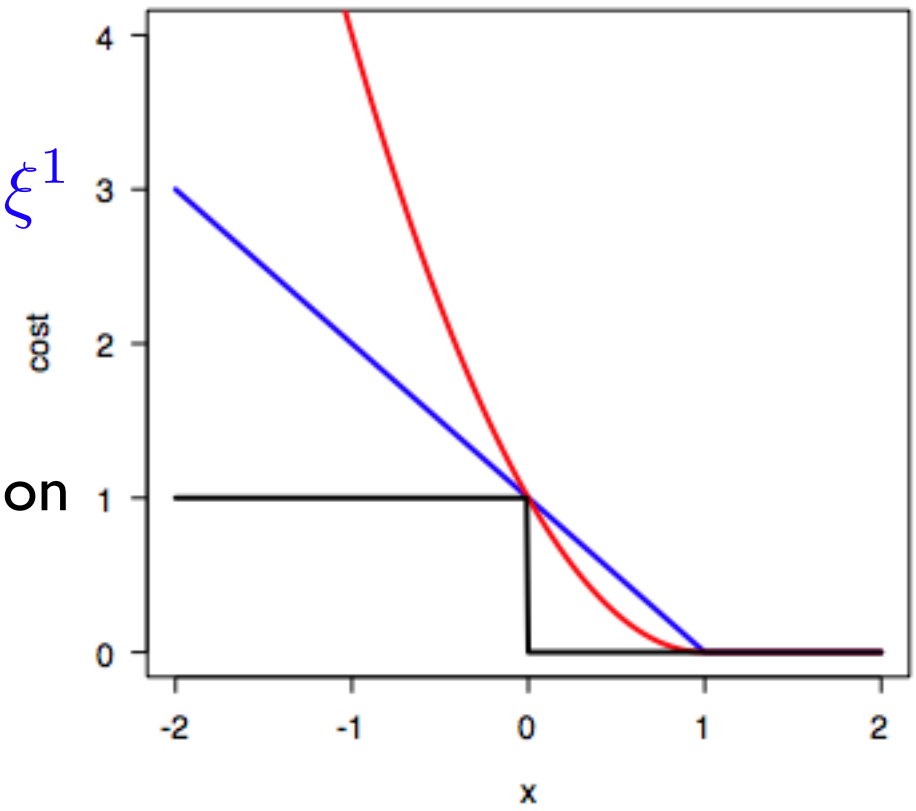
$$L(h(x), y) = (1 - yh(x))_+^2.$$

# Hinge Loss

'Quadratic' hinge loss $\xi^2$

Hinge loss $\xi^1$

0/1 loss function



# SVMs Equations

- **Lagrangian**: for all $\mathbf{w}, b, \alpha_i \geq 0, \beta_i \geq 0,$

$$L(\mathbf{w}, b, \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_{i=1}^{m}\xi_i - \sum_{i=1}^{m}\alpha_i[y_i(\mathbf{w}\cdot\mathbf{x}_i + b) - 1 + \xi_i] - \sum_{i=1}^{m}\beta_i\xi_i\,.$$

- **KKT conditions**:

$$\nabla_w L = \mathbf{w} - \sum_{i=1}^{m}\alpha_i y_i \mathbf{x}_i = 0 \iff \mathbf{w} = \sum_{i=1}^{m}\alpha_i y_i \mathbf{x}_i.$$

$$\nabla_b L = -\sum_{i=1}^{m}\alpha_i y_i = 0 \iff \sum_{i=1}^{m}\alpha_i y_i = 0.$$

$$\nabla_{\xi_i} L = C - \alpha_i - \beta_i = 0 \iff \alpha_i + \beta_i = C.$$

$$\forall i \in [1, m],\ \alpha_i[y_i(\mathbf{w}\cdot\mathbf{x}_i + b) - 1 + \xi_i] = 0$$

$$\beta_i\xi_i = 0.$$

# Support Vectors

- Complementarity conditions:

$$\alpha_i[y_i(\mathbf{w} \cdot \mathbf{x}_i + b) - 1 + \xi_i] = 0 \implies \alpha_i = 0 \vee y_i(\mathbf{w} \cdot \mathbf{x}_i + b) = 1 - \xi_i.$$

- Support vectors: vectors $\mathbf{x}_i$ such that

$$\alpha_i \neq 0 \wedge y_i(\mathbf{w} \cdot \mathbf{x}_i + b) = 1 - \xi_i.$$

  - Note: support vectors are not unique.

# Moving to The Dual

- Plugging in the expression of $w$ in $L$ gives:

$$L = \underbrace{\frac{1}{2}\|\sum_{i=1}^{m}\alpha_i y_i \mathbf{x}_i\|^2 - \sum_{i,j=1}^{m}\alpha_i\alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)}_{-\frac{1}{2}\sum_{i,j=1}^{m}\alpha_i\alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)} - \underbrace{\sum_{i=1}^{m}\alpha_i y_i b}_{0} + \sum_{i=1}^{m}\alpha_i.$$

- Thus,

$$L = \sum_{i=1}^{m}\alpha_i - \frac{1}{2}\sum_{i,j=1}^{m}\alpha_i\alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j).$$

- The condition $\beta_i \geq 0$ is equivalent to $\alpha_i \leq C$.

# Dual Optimization Problem

- Constrained optimization:

$$\max_{\alpha} \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)$$

$$\text{subject to: } 0 \leq \alpha_i \leq C \wedge \sum_{i=1}^{m} \alpha_i y_i = 0, i \in [1, m].$$

- Solution:

$$h(x) = \operatorname{sgn}\Big(\sum_{i=1}^{m} \alpha_i y_i (\mathbf{x}_i \cdot \mathbf{x}) + b\Big),$$

with $b = y_i - \sum_{j=1}^{m} \alpha_j y_j (\mathbf{x}_j \cdot \mathbf{x}_i)$ for any $\mathbf{x}_i$ with $0 < \alpha_i < C.$

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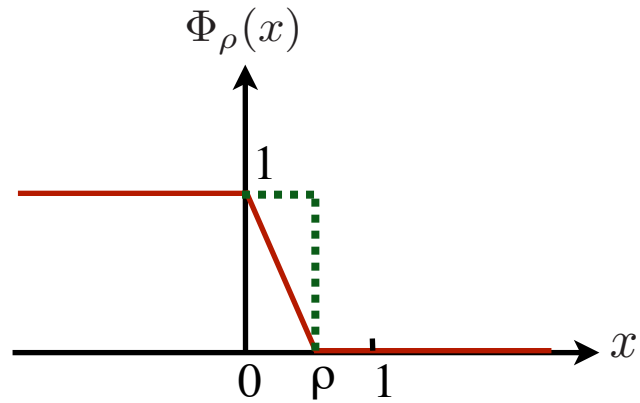
# Confidence Margin

- Definition: for any confidence margin $\rho > 0$, the $\rho$-margin function is defined by




- For a sample $S = (x_1, \ldots, x_m)$ and hypothesis $h$, the empirical margin loss is

$$\widehat{R}_\rho(h) = \frac{1}{m} \sum_{i=1}^{m} \Phi_\rho(y_i h(x_i)) \leq \frac{1}{m} \sum_{i=1}^{m} 1_{y_i h(x_i) < \rho}.$$

# General Margin Bound

- **Theorem**: Let $H$ be a set of real-valued functions. Fix $\rho > 0$. For any $\delta > 0$, with probability at least $1-\delta$, the following holds for all $h \in H$:

$$R(h) \leq \widehat{R}_\rho(h) + \frac{2}{\rho}\mathfrak{R}_m(H) + \sqrt{\frac{\log \frac{1}{\delta}}{2m}}$$

$$R(h) \leq \widehat{R}_\rho(h) + \frac{2}{\rho}\widehat{\mathfrak{R}}_S(H) + 3\sqrt{\frac{\log \frac{2}{\delta}}{2m}}.$$

- **Proof**: Let $\widetilde{H} = \{z = (x, y) \mapsto yh(x) : h \in H\}$. Consider the family of functions taking values in $[0, 1]$:

$$\widetilde{\mathcal{H}} = \{\Phi_\rho \circ f : f \in \widetilde{H}\}.$$

- By the theorem of Lecture 3, with probability at least $1-\delta$, for all $g \in \widetilde{\mathcal{H}}$,

$$\mathrm{E}[g(z)] \leq \frac{1}{m}\sum_{i=1}^{m} g(z_i) + 2\mathfrak{R}_m(\widetilde{\mathcal{H}}) + \sqrt{\frac{\log\frac{1}{\delta}}{2m}}.$$

- Thus,

$$\mathrm{E}[\Phi_\rho(yh(x))] \leq \widehat{R}_\rho(h) + 2\mathfrak{R}_m\big(\Phi_\rho \circ \widetilde{H}\big) + \sqrt{\frac{\log\frac{1}{\delta}}{2m}}.$$

- Since $\Phi_\rho$ is $\frac{1}{\rho}$ - Lipschitz, by Talagrand's lemma,

$$\mathfrak{R}_m\big(\Phi_\rho \circ \widetilde{H}\big) \leq \frac{1}{\rho}\,\mathfrak{R}_m(\widetilde{H}) = \frac{1}{\rho m}\,\mathop{\mathrm{E}}_{\sigma,S}\left[\sup_{h\in H}\sum_{i=1}^{m}\sigma_i y_i h(x_i)\right] = \frac{1}{\rho}\,\mathfrak{R}_m\big(H\big).$$

- Since $1_{yh(x)<0} \leq \Phi_\rho(yh(x))$, this shows the first statement, and similarly the second one.

# Rademacher Complexity of Linear Hypotheses

- **Theorem**: Let $S \subseteq \{x : \|\mathbf{x}\| \leq R\}$ be a sample of size $m$ and let $H = \{\mathbf{x} \mapsto \mathbf{w} \cdot \mathbf{x} : \|\mathbf{w}\| \leq \Lambda\}$. Then,

$$\widehat{\mathfrak{R}}_S(H) \leq \sqrt{\frac{R^2\Lambda^2}{m}}.$$

- **Proof**:

$$\begin{aligned}
\widehat{\mathfrak{R}}_S(H) &= \frac{1}{m} \mathop{\mathrm{E}}_{\sigma}\left[\sup_{\|\mathbf{w}\| \leq \Lambda} \sum_{i=1}^{m} \sigma_i \mathbf{w} \cdot \mathbf{x}_i\right] = \frac{1}{m} \mathop{\mathrm{E}}_{\sigma}\left[\sup_{\|\mathbf{w}\| \leq \Lambda} \mathbf{w} \cdot \sum_{i=1}^{m} \sigma_i \mathbf{x}_i\right] \\
&\leq \frac{\Lambda}{m} \mathop{\mathrm{E}}_{\sigma}\left[\left\|\sum_{i=1}^{m} \sigma_i \mathbf{x}_i\right\|\right] \leq \frac{\Lambda}{m}\left[\mathop{\mathrm{E}}_{\sigma}\left[\left\|\sum_{i=1}^{m} \sigma_i \mathbf{x}_i\right\|^2\right]\right]^{1/2} \\
&\leq \frac{\Lambda}{m}\left[\mathop{\mathrm{E}}_{\sigma}\left[\sum_{i=1}^{m} \|\mathbf{x}_i\|^2\right]\right]^{1/2} \leq \frac{\Lambda\sqrt{mR^2}}{m} = \sqrt{\frac{R^2\Lambda^2}{m}}.
\end{aligned}$$

# Margin Bound - Linear Classifiers

- **Corollary**: Let $\rho > 0$ and $H = \{\mathbf{x} \mapsto \mathbf{w} \cdot \mathbf{x} : \|\mathbf{w}\| \leq \Lambda\}$. Assume that $X \subseteq \{\mathbf{x} : \|\mathbf{x}\| \leq R\}$. Then, for any $\delta > 0$, with probability at least $1 - \delta$, for any $h \in H$,

$$R(h) \leq \widehat{R}_\rho(h) + 2\sqrt{\frac{R^2\Lambda^2/\rho^2}{m}} + 3\sqrt{\frac{\log \frac{2}{\delta}}{2m}}.$$

- **Proof**: Follows directly general margin bound and bound on $\widehat{\mathfrak{R}}_S(H)$ for linear classifiers.

# High-Dimensional Feature Space

- Observations:
  - generalization bound does not depend on the dimension but on the margin.
  - this suggests seeking a large-margin separating hyperplane in a higher-dimensional feature space.
- Computational problems:
  - taking dot products in a high-dimensional feature space can be very costly.
  - solution based on kernels (next lecture).

# References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# Appendix

# Saddle Point

- Let $(\mathbf{w}^*, b^*, \alpha^*)$ be the saddle point of the Langrangian. Multiplying both sides of the equation giving $b^*$ by $\alpha_i^* y_i$ and taking the sum leads to:

$$\sum_{i=1}^{m} \alpha_i^* y_i b = \sum_{i=1}^{m} \alpha_i^* y_i^2 - \sum_{i,j=1}^{m} \alpha_i^* \alpha_j^* y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j).$$

- Using $y_i^2 = 1, \sum_{i=1}^{m} \alpha_i^* y_i = 0$, and $\mathbf{w}^* = \sum_{i=1}^{m} \alpha_i^* y_i \mathbf{x}_i$ yields

$$0 = \sum_{i=1}^{m} \alpha_i^* - \|\mathbf{w}^*\|^2.$$

- Thus, the margin is also given by:

$$\boxed{\rho^2 = \frac{1}{\|\mathbf{w}^*\|_2^2} = \frac{1}{\|\alpha^*\|_1}.}$$

# Talagrand's Contraction Lemma

(Ledoux and Talagrand, 1991; pp. 112-114)

- **Theorem**: Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be an $L$-Lipschitz function. Then, for any hypothesis set $H$ of real-valued functions,

$$\widehat{\mathfrak{R}}_S(\Phi \circ H) \leq L\, \widehat{\mathfrak{R}}_S(H).$$

- **Proof**: fix sample $S = (x_1, \ldots, x_m)$. By definition,

$$\begin{aligned}\mathfrak{R}_S(\Phi \circ H) &= \frac{1}{m} \mathop{\mathrm{E}}_{\boldsymbol{\sigma}}\left[\sup_{h \in H} \sum_{i=1}^{m} \sigma_i (\Phi \circ h)(x_i)\right] \\ &= \frac{1}{m} \mathop{\mathrm{E}}_{\sigma_1, \ldots, \sigma_{m-1}}\left[\mathop{\mathrm{E}}_{\sigma_m}\left[\sup_{h \in H} u_{m-1}(h) + \sigma_m (\Phi \circ h)(x_m)\right]\right],\end{aligned}$$

with $u_{m-1}(h) = \sum_{i=1}^{m-1} \sigma_i (\Phi \circ h)(x_i)$.

# Talagrand's Contraction Lemma

- Now, assuming that the suprema are reached, there exist $h_1, h_2 \in H$ such that

$$\begin{aligned}
&\mathop{\mathrm{E}}_{\sigma_m}\left[\sup_{h\in H} u_{m-1}(h) + \sigma_m(\Phi\circ h)(x_m)\right]\\
&= \frac{1}{2}[u_{m-1}(h_1) + (\Phi\circ h_1)(x_m)] + \frac{1}{2}[u_{m-1}(h_2) - (\Phi\circ h_2)(x_m)]\\
&\leq \frac{1}{2}[u_{m-1}(h_1) + u_{m-1}(h_2) + sL(h_1(x_m) - h_2(x_m))]\\
&= \frac{1}{2}[u_{m-1}(h_1) + sLh_1(x_m)] + \frac{1}{2}[u_{m-1}(h_2) - sLh_2(x_m)]\\
&\leq \mathop{\mathrm{E}}_{\sigma_m}\left[\sup_{h\in H} u_{m-1}(h) + \sigma_m Lh(x_m)\right],
\end{aligned}$$

where $s = \operatorname{sgn}(h_1(x_m) - h_2(x_m))$.

# Talagrand's Contraction Lemma

- When the suprema are not reached, the same can be shown modulo $\epsilon$, followed by $\epsilon \to 0$.
- Proceeding similarly for other $\sigma_i$s directly leads to the result.

# VC Dimension of Canonical Hyperplanes

- **Theorem**: Let $S \subseteq \{\mathbf{x} : \|\mathbf{x}\| \leq R\}$. Then, the VC dimension $d$ of the set of canonical hyperplanes $\{x \mapsto \operatorname{sgn}(\mathbf{w} \cdot \mathbf{x}) : \min_{x \in S} |\mathbf{w} \cdot \mathbf{x}| = 1 \wedge \|\mathbf{w}\| \leq \Lambda\}$ verifies

$$d \leq R^2 \Lambda^2.$$

- **Proof**: Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ be a set fully shattered. Then, for all $\mathbf{y} \in \{-1, +1\}^d$, there exists $\mathbf{w}$ such

$$\forall i \in [1, d], 1 \leq y_i (\mathbf{w} \cdot \mathbf{x}_i).$$

- Summing up the inequalities gives

$$d \leq \mathbf{w} \cdot \sum_{i=1}^{d} y_i \mathbf{x}_i \leq \|\mathbf{w}\| \| \sum_{i=1}^{d} y_i \mathbf{x}_i \| \leq \Lambda \| \sum_{i=1}^{d} y_i \mathbf{x}_i \|.$$

- Taking the expectation over $\mathbf{y} \sim U$ (uniform) yields

$$d \leq \Lambda \operatorname*{E}_{\mathbf{y} \sim U} [\| \sum_{i=1}^{d} y_i \mathbf{x}_i \|] \leq \Lambda \big[ \operatorname*{E}_{\mathbf{y} \sim U} [\| \sum_{i=1}^{d} y_i \mathbf{x}_i \|^2] \big]^{1/2}$$ (Jensen's ineq.)

$$= \Lambda \big[ \sum_{i,j=1}^{d} \operatorname{E}[y_i y_j] (\mathbf{x}_i \cdot \mathbf{x}_j) \big]^{1/2}$$

$$= \Lambda \big[ \sum_{i=1}^{d} (\mathbf{x}_i \cdot \mathbf{x}_i) \big]^{1/2} \leq \Lambda \big[ d R^2 \big]^{1/2} = \Lambda R \sqrt{d}.$$

- Thus, $\sqrt{d} \leq \Lambda R$.