Probability in λ-rings and representation stability for configuration spaces of algebraic varieties

Xi He [1] Sean Howe [2] Austin Myers [3]

[1]University of California, Los Angeles [2]University of Utah [3]Florida State University

Computing Cohomology

The central problem of our work is to compute the cohomology of certain spaces associated to spaces of smooth hypersurface sections of algebraic varieties over $\mathbb{C}$. If we restrict ourselves to curves, these spaces can be identified with configuration spaces on the curve Y:

$$\mathrm{Conf}_n(Y) = \{\{y_1, \dots, y_n\} : y_i \in Y, y_i \neq y_j \ \forall i, j\}.$$

In general, this computation is very difficult. However, the space of hypersurface sections can be viewed as an algebraic variety, and we can use Artin's Comparison Theorem, which in our case produces an isomorphism between singular and étale cohomology. In essence, our work exploits this to use point-counting information over finite fields for the topological problem of computing cohomology over $\mathbb{C}$.

In particular, we wish to compute the *stable* cohomology, taking $n \to \infty$, if such stability phenomena occurs.

Λ-distributions and Motivic Probability

A *symmetric function* can be thought of as a limit of symmetric polynomials, for example

$$p_k(x_1, \cdots, x_n) = x_1^k + x_2^k + \cdots + x_n^k.$$

To be more precise, symmetric polynomials in n variables form a ring $\Lambda_n := \mathbb{Z}[x_1, \cdots, x_n]^{S_n}$, and taking a direct limit

$$\Lambda := \varinjlim \Lambda_n,$$

taken with respect to the obvious inclusions.

A λ*-ring*, roughly speaking, "acts like" the ring Λ of symmetric functions. In a certain sense, symmetric functions are a model for *moments* in classical probability theory, so a λ-ring structure can be viewed as the necessary structure for doing probability on a ring.

In particular, for a group G, the Grothendieck ring of the category of G-representations, and the Grothendieck ring of varieties are λ-rings. These form the basis for a notion of "motivic probability."

Point-Counting and Cohomology

Let X be a smooth projective $\mathbb{F}_q$-variety, and let $\mathcal{F}$ be an ℓ-adic sheaf on X. Then the *Grothendieck-Lefschetz formula* applied to the Frobenius morphism on X gives points counts over $\mathbb{F}_q$ in terms of étale cohomology with coefficients in $\mathcal{F}$:

$$|X(\mathbb{F}_q)| = |\mathrm{Fix}(\mathrm{Frob}_q)| = \sum_{i \geq 0} (-1)^i \mathrm{tr}(\mathrm{Frob}_q : H^i_t(X; \mathbb{Q}_\ell)).$$

This then gives a way to connect (étale) cohomology with point-counts, which gives us information about singular cohomology via the comparison morphism.

Representation Stability

Now let C be a smooth projective curve. There is a natural surjection $\pi_1(\mathrm{Conf}_n(C)_{\mathbb{C}}) \to S_n$, any finite-dimensional complex representation V of S_n extends to a monodromy representation $\pi_1(\mathrm{Conf}_n(C)_{\mathbb{C}}) \to GL(V)$. There is a natural correspondence between such representations V and certain sheaves $\mathcal{V}$ on Conf_n called local systems. The important point here is that

$$\dim_{\mathbb{Q}_\ell} H^i_{et}(\mathrm{Conf}_n; \mathcal{V}) = \langle V, H^i(\mathrm{PConf}_n(C)_{\mathbb{C}}; \mathbb{C}) \rangle_{S_n},$$

where the inner product is the standard one on S_n-representations.

A similar story holds for higher-dimensional varieties, although we are no longer working with configuration spaces, and now the correspondence between representations and local systems is given by different different groups.

An interesting point here is that (when Y is a curve), the multiplicities of irreducible S_n-representations in the ith cohomology stabilizes as n goes to ∞, and taking n to ∞, one can view Grothendieck-Lefschetz as a generating function for the multiplicity of an irreducible representation on the cohomologies.

Weights and Asymptotic Formulae

In general, the coefficients of the limit just keep track of the weight contributions of the action of Frobenius on the ith cohomology. In essence, these can be viewed as limits of certain Euler characteristics associated to the weights. Roughly, these weights are the eigenvalues of Frobenius acting on cohomology.

Results

We note that the Λ-distribution perspective gives an alternate way to compute these generating functions. Let Y be a smooth projective variety, and let $U_n(Y_{\overline{\mathbb{F}}_q})$ be the space of degree-n hypersurface sections of Y. One can get a λ-probability space $R = C(G, W(\mathbb{C}))$, with a random variable $X \in R$ sending $g \mapsto \sum_i (-1)^i [H^i_t(U_n(Y_{\overline{\mathbb{F}}_q}))]$, where as a representation, $[V]$ is viewed as a function $G \to W(\mathbb{C})$ sending g to its multiset of eigenvalues under the monodromy representation. This random variable also gives point-counts of functions on Y. The point-counting random variable sends a variety to its zeta function as a Witt vector, i.e. an element of $W(\mathbb{C})$. This can be expressed in ghost coordinates as $[Y] = (\#Y/\mathbb{F}_q, \#Y/\mathbb{F}_{q^2}, \#Y/\mathbb{F}_{q^3}, \dots)$.

Then, we compute the multiplicities $d_i(V)$ for a family $V = (V_1, V_2, \dots)$ of irreducible representation of G. For a $k-$dimensional smooth, closed subvariety Y of $\mathbb{P}^m$, with Z_n the $W(\mathbb{C})$-valued random variable sending $f \in U_n(Y_{\overline{\mathbb{F}}_q})$ (the space of degree-n hypersurface sections of Y), to $[V(f) \cap Y]$, the formula is

$$\lim_{n \to \infty} \mathbb{E}[\mathrm{Exp}_\sigma(Z_n h_1)] = \left(1 + \frac{[q]^k - 1}{[q]^{k+1} - 1}(h_1 + h_1 + \dots)\right)^{[Y]}.$$

For P a symmetric function coinciding with the character of an irreducible representation of G, we have the formula:

$$\lim_{n \to \infty} \zeta_{Y,\mathrm{Kap}}(\mathrm{dim(Y)+1}) \langle \mathbb{E}[\mathrm{Exp}_\sigma(Z_n h_1)], P \rangle = \sum_{i=0}^{\infty} (-1)^i \frac{\langle P, \chi_{H^i(U(Y)_{\mathbb{F}_q})} \rangle G}{q^i},$$

where $\zeta_{Y,\mathrm{Kap}}(s) = Z_{Y,\mathrm{Kap}}([q^{-s}])$ is the *Kapranov motivic zeta function.* We also find that for the family of representations corresponding to the partition (1), the stable multiplicity in the cohomology is 0.

Examples and Computations

Previously, the representation stability computations were only known for the case $Y = \mathbb{A}^1$, which was computed by Emil Geisler. Our computational methods allow us to compute these for a much wider class of algebraic varieties, including elliptic curves and projective spaces. Unlike other computational techniques, our results compute the generating series as a rational function in q, which can then be expanded as a Laurent series at ∞ to obtain multiplicities.

The Affine line

Our results agree with the previous results for the affine line:

$Y = \mathbb{A}^1$, the result for the family of representations given by the partition (1,1,1):

$$\frac{-3q^5 + 3q^4 - 3q^3 + 2q^2 + 1}{q^8 + 2q^7 + 2q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + q}$$
$$= -3q^{-3} + 9q^{-4} - 15q^{-5} + 23q^{-6} - 34q^{-7} + 47q^{-8} + \cdots$$

The Projective line

For $Y = \mathbb{P}^1$, and the family of representations corresponding to (2,1):

$$\frac{q^4 - 2q^3 + 2q^2 - 2q + 1}{q^6 + q^3}$$
$$= 1q^{-2} - 2q^{-3} + 2q^{-4} - 3q^{-5} + 3q^{-6} - 2q^7 + 3q^{-8} + \cdots$$

Elliptic Curves

For $Y = E$ an elliptic curve, $\alpha, \overline{\alpha}$ the roots of $t^2 + (q + 1 - \#E(\mathbb{F}_q))t + q$, the result for the family of "standard representations" is

$$-\frac{\alpha + \overline{\alpha}}{q} + 2\frac{\alpha + \overline{\alpha}}{q^2} - \frac{(\alpha + \overline{\alpha})^2 + \alpha + \overline{\alpha}}{q^3} + 2\frac{(\alpha + \overline{\alpha})^2}{q^4} - \cdots$$

Conclusion

If a range of values is known where the multiplicity is 0 in a particular degree i of cohomology, then the stable cohomology can be computed as a representation of the appropriate group. Such computations were done by Geisler for $\mathbb{A}^1$. The significance of our work is that it allows the possibility of such computations for a wider class of algebraic varieties. Some of our computations can be found at *https://representationstability.github.io*

References

[1] Thomas Church, Jordan S. Ellenberg, and Benson Farb. Representation stability in cohomology and asymptotics for families of varieties over finite fields. *arXiv e-prints*, arXiv:1309.6038, 2013. doi:10.48550/arXiv.1309.6038. URL `https://arxiv.org/abs/1309.6038`. Published on arXiv, September 2013.

[2] Emil Geisler. Computations of stable multiplicities in the cohomology of configuration space. *arXiv e-prints*, arXiv:2411.11337, 2024. doi:10.48550/arXiv.2411.11337. URL `https://arxiv.org/abs/2411.11337`. Submitted on arXiv November 18, 2024; revised September 18, 2025.

[3] Sean Howe. Motivic random variables and representation stability i: Configuration spaces. *arXiv e-prints*, arXiv:1610.05723, 2016. doi:10.48550/arXiv.1610.05723. URL `https://arxiv.org/abs/1610.05723`. Published on arXiv, October 2016.

[4] Sean Howe. Motivic random variables and representation stability ii: Hypersurface sections. *arXiv e-prints*, arXiv:1610.05720, 2016. doi:10.48550/arXiv.1610.05720. URL `https://arxiv.org/abs/1610.05720`. Published on arXiv, October 2016.

[5] Sean Howe. Random matrix statistics and zeroes of l-functions via probability in λ-rings. *arXiv e-prints*, arXiv:2412.19295, 2024. doi:10.48550/arXiv.2412.19295. URL `https://arxiv.org/abs/2412.19295`. Submitted on arXiv December 2024, revised June 2025.