Midterm Exam Solution

1. A microcontroller system has a 12 bit AD converter, 10,000 bytes of available RAM, and samples stereo audio signal with sampling frequency of 20 KHz. How many seconds of audio signal can be stored in memory?

T = 10000 bytes / (2 channels * 2 bytes/sample * 20000 samples/s) = 0.125 s (125ms)

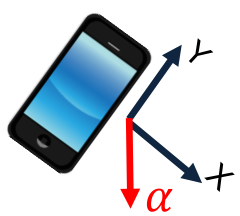
or optimized:

T = 10000 bytes / (2 channels * 1.5 bytes/sample * 20000 samples/s) = 0.166 s (166ms)

2. The output of a causal LTI system with the impulse response $h(t)$ to a causal input $x(t)$ is

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

3. Accelerometer (± 2g) with analog output and power supply of +3V is used in smartphone to determine orientation of the smartphone according to the figure below. What are the values of X and Y components of the accelerometer for α = 45°.



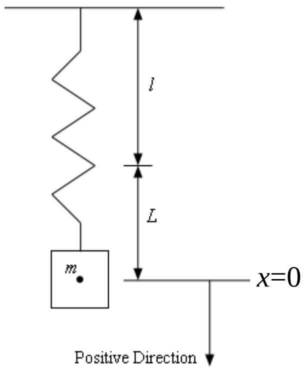
Sensitivity of the accelerometer 1*g* → *s = 3V / 4 g* = 0.75 [V / g]

A_0 (0 g) = 1.5V

$A_X = A_0 + 1g * \cos(\pi/4) = 2.03$ V

$A_Y = A_0 + -(1g * \sin(\pi/4)) = 0.97$ V

4. (20 points) A 2 kg weight is hung on the end of a vertically suspended spring, thereby stretching the spring L=20 cm. The weight is raised 5 cm above its equilibrium position and released from rest at time t=0. Dumping factor of the environment is d = 30, and initial speed is 0. Find the displacement x of the weight from its equilibrium position at time t. Use g=10m/s^2.



Since in equilibrium:

$$F = kL, \qquad k = \frac{F}{L} = \frac{mg}{L} = \frac{2[kg]\,10\left[\frac{m}{s^2}\right]}{0.2[m]} = 100\left[\frac{kg}{s^2}\right]$$

At any time, sum of all forces is equal to zero

$$m\ddot{x} + d\dot{x} + kx = 0$$

With initial conditions

$$x(0) = -0.05[m] \;\; \dot{x}(0) = 0$$

By using Laplace transform

$$\mathcal{L}(m\ddot{x} + d\dot{x} + kx) = ms^2X(s) - msx(0) - m\dot{x}(0) + dsX(s) - dx(0) + kX(s) = 0$$

$$2s^2X(s) - 2s(-0.05) - 0 + 30sX(s) - 30(-0.05) + 100X(s) = 0$$

$$(2s^2 + 30s + 100)X(s) = -0.1s - 1.5$$

$$X(s) = \frac{-0.1s - 1.5}{2s^2 + 30s + 100} = \frac{-0.1s - 1.5}{2(s+10)(s+5)} = \frac{A}{(s+10)} + \frac{B}{(s+5)}$$

$$A = X(s)(s+10)|_{s=-10} = \frac{1 - 1.5}{2(-5)} = 0.05$$

$$B = X(s)(s+5)|_{s=-5} = \frac{0.5 - 1.5}{2 \cdot 5} = -0.1$$

and

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = 0.05e^{-10t} - 0.1e^{-5t}$$

If m=1kg and L=10 cm → k = 100.

Using Laplace transform

$$\mathcal{L}(m\ddot{x} + d\dot{x} + kx) = ms^2X(s) - msx(0) - m\dot{x}(0) + dsX(s) - dx(0) + kX(s) = 0$$

$$s^2X(s) - s(-0.05) - 0 + 30sX(s) - 30(-0.05) + 100X(s) = 0$$

$$(s^2 + 30s + 100)X(s) = -0.05s - 1.5$$

$$X(s) = \frac{-0.05s - 1.5}{s^2 + 30s + 100} = \frac{-0.05s - 1.5}{(s + 26.2)(s + 3.8)} = \frac{A}{(s + 26.2)} + \frac{B}{(s + 3.8)}$$

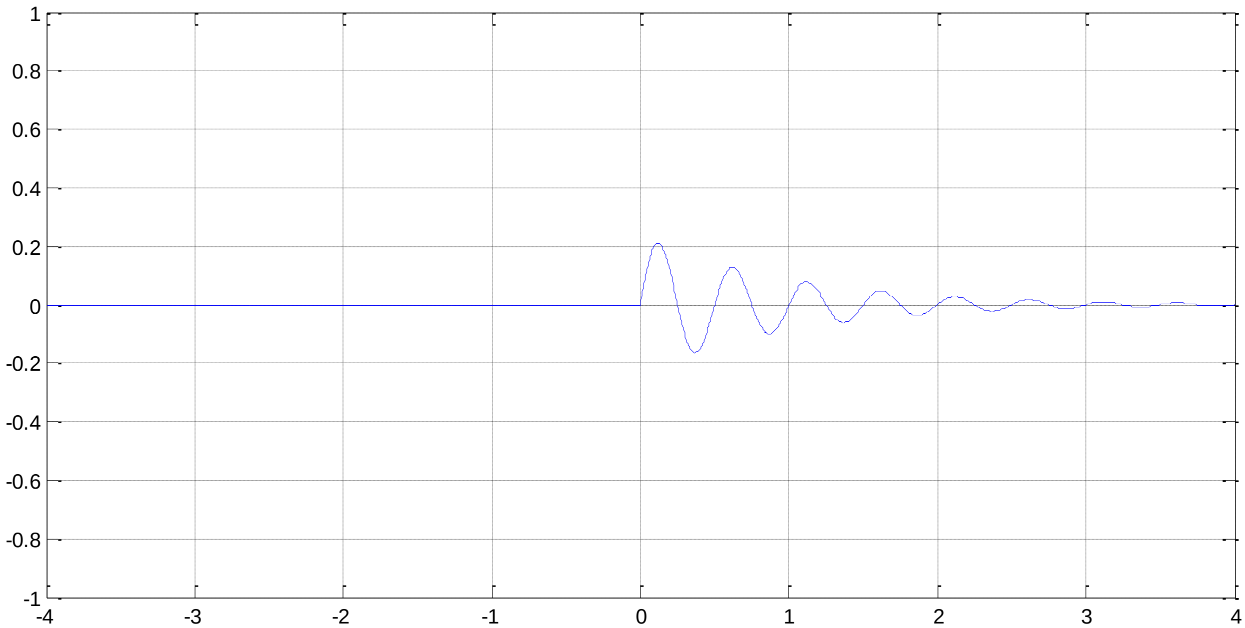
$$A = X(s)(s + 26.2)|_{s=-26.2} = \frac{0.05 * 26.2 - 1.5}{2(-26.2 + 3.8)} = 0.01$$

$$B = X(s)(s + 3.8)|_{s=-3.8} = \frac{0.05 * 3.8 - 1.5}{2 \cdot (-3.8 + 26.2)} = -0.06$$

and

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = 0.01e^{-26.2t} - 0.06e^{-3.8t}$$

5. $x(t) = \dfrac{3}{4\pi} e^{-t} \cdot \sin(4\pi t) \cdot u(t)$



6. Consider the periodic signal x(t) = cos(0.4π·*t*) + 3·cos(2 π·*t/7*), −∞ < t < ∞..

Is x(t) periodic? If it is, what is the period T_0 of $x(t)$? T_0 = 35 s

$x(t) = x_1(t) + x_2(t)$

$T_1 = 2\pi / (0.2\pi) = 5$ s

$T_2 = 2\pi / (2\pi/7) = 7$ s

$T_0 = N*T_1 = M*T_2$ → The least common multiple of 5 and 7 is 35, therefore 7N = 5M → $T_0 = 7*5 = 35$ s

What is the average power of $x(t)$?

$$\int_0^x \cos^2(x)dx = \int_0^x \frac{1}{2}(1+\cos(2x)) = \frac{1}{2}\int_0^x dx + \frac{1}{4}\int_0^x \cos(y)dy = \left(\frac{x}{2}+\frac{1}{4}\sin(2x)\right)\Big|_0^x$$

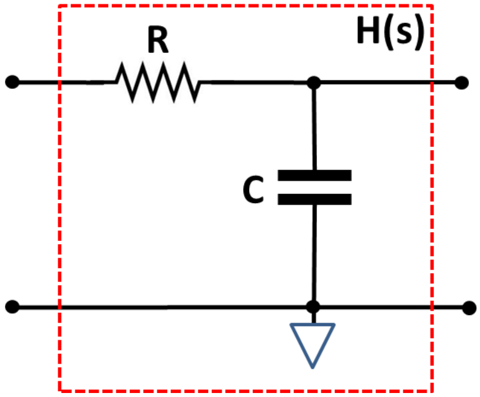
$$\int_0^t \cos^2(x)dx = \left(\frac{x}{2}+\frac{1}{4}\sin(2x)\right)\Big|_0^t = \frac{t}{2}+\frac{1}{4}\sin(2t) \Rightarrow \; for\; t = T \int_0^T \cos^2(x)dx = \frac{1}{2}$$

$$P_{x1} = \frac{1}{T_1}\int_0^{T_1} x_1^{\,2}(t)dt = \frac{1}{0.5}\cdot\left(\frac{x}{2}+\frac{1}{4}\sin(2x)\right)\Big|_0^{T_1} = \frac{1}{T_1}\left(\frac{T_1}{2}+\frac{1}{4}\sin\left(12\pi\cdot\frac{1}{6}\right)\right) = 1\cdot\frac{1}{2} = 0.5$$

$$P_{x2} = \frac{1}{T_2}\int_0^{T_2} x_2^{\,2}(t)dt = \frac{1}{T_2}\int_0^{T_2}(3\cos(16\pi t))^2 dt = 9\cdot\frac{1}{T_2}\int_0^{T_2}\cos^2(16\pi\, t)dt = 9\cdot\frac{1}{2} = 4.5$$

$P = P_{x1} + P_{x2} = 0.5 + 4.5 = 5$

7. (4 points)



$$H(s)=\frac{Y(s)}{X(s)}=\frac{\frac{1}{Cs}}{R+\frac{1}{Cs}}=\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

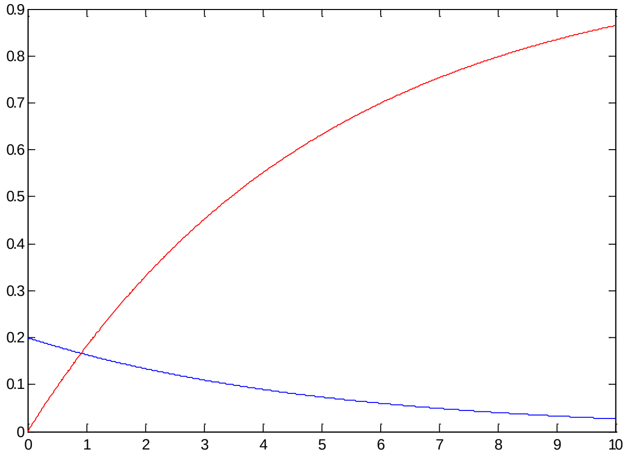
$$h(t)=\frac{1}{RC}\cdot e^{-\frac{t}{RC}}$$

b) (6 points) Step response:

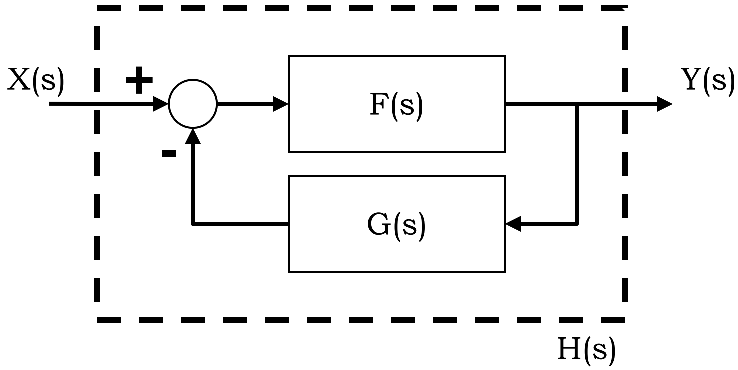
$$\frac{1}{RC}=\frac{1}{5M\Omega\cdot 1\mu F}=0.2$$

$$S(s)=\frac{1}{s}H(s)=\frac{1}{s}\frac{0.2}{s+0.2}=\frac{A}{s}+\frac{B}{s+0.2}=\frac{1}{s}-\frac{1}{s+0.2}$$

$$s(t)=\left(1-e^{-0.2\cdot t}\right)\cdot u(t)$$



8. (5 points) What is the transfer function H(s) of the system represented below?



Y(s) = F(s)*(X(s) - G(s)*Y(s)) = F(s)*X(s) - F(s)*G(s)*Y(s)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{F(s)}{1 + F(s) \cdot G(s)}$$

b) (10 points) Transfer function of the feedback block is

$$G(s) = \frac{Z_{R||C}}{Ls + Z_{R||C}}$$

$$Z_{R||C} = \frac{R\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{R}{RCs + 1}$$

$$G(s) = \frac{R}{R + Ls(RCs + 1)} = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Transfer function of the system is:

$$H(s) = \frac{F(s)}{1 + F(s) \cdot G(s)}$$

$F(s) = A$ and $G(s) = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$

$$H(s) = \frac{A}{1 + A\left(\frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}\right)} for\ A \to \infty\ H(s) = LCs^2 + \frac{L}{R}s + 1$$

9.