# 1. AXISYMMETRIC LAMINATED CYLINDERS

An exact solution for a laminated composite cylinder may be found using linear elastic relationships when the cylinder is subject to any of the following axisymmetric conditions:

a) a uniform interior pressure, $p_{in}$
b) a uniform exterior pressure, $p_{out}$
c)a uniform temperature change, $\Delta T$
d) an axial load applied at the ends, $P_x$ or a uniform axial strain, $\varepsilon_x^o$
e) a torque, $T_x$, or uniform angle of twist per unit length, $\gamma^o$

The geometry of the laminated cylinder along with the cylindrical coordinates and loading are shown in Fig 3.

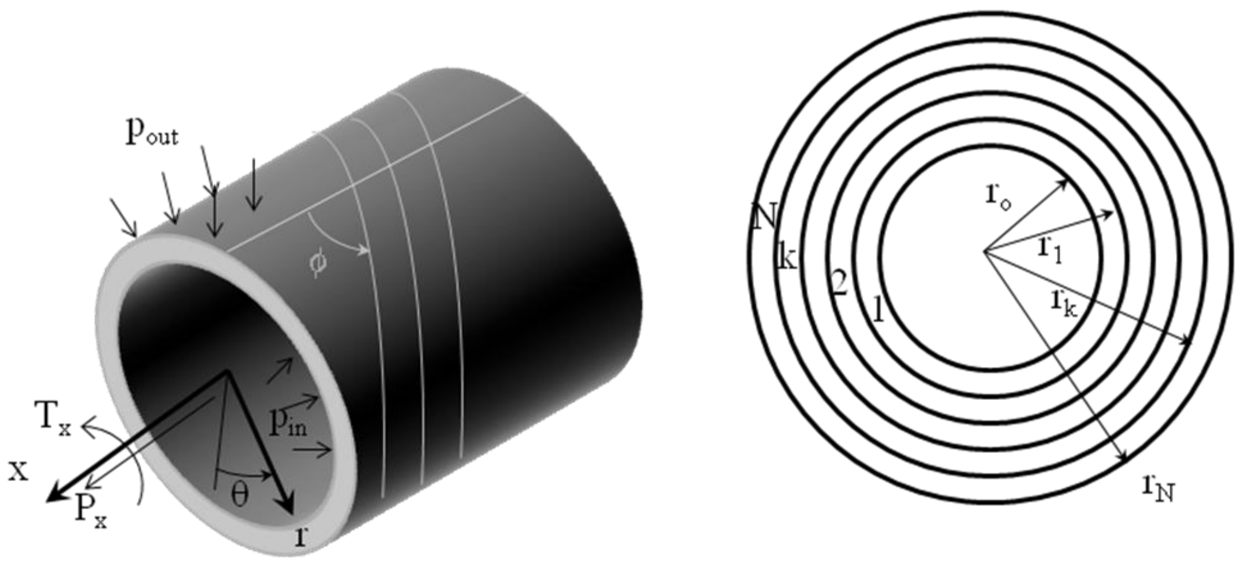


Figure 3: Geometry, loading directions, and layer numbering of an axisymmetric laminated cylinder.

It is assumed that each layer is homogeneous. Also each layer may be oriented off the cylinder axial direction at an angle, $\phi$, as shown. A complete and well-articulated summary of the solution is given by Herakovich [9] including an improvement in the solution method by Pindera. The pertinent expressions, including the governing equations are shown here.

## 1.1 Governing Equations

From the axisymmetric assumption, the strain-displacement equations may be written as:

$$\varepsilon_x = \frac{du}{dx} \qquad \varepsilon_\theta = \frac{w}{r} \qquad \varepsilon_r = \frac{dw}{dr} \qquad [35,36.37]$$

$$\gamma_{r\theta} = \frac{dv}{dr} - \frac{v}{r} \qquad \gamma_{xr} = \frac{du}{dr} \qquad \gamma_{\theta x} = \frac{dv}{dx} \qquad [38,39,40]$$

and the nontrivial compatibility equations are:

$$\frac{d^2\varepsilon_x}{\partial r^2} = 0 \qquad \frac{1}{r}\frac{d\varepsilon_x}{\partial r} = 0 \qquad \frac{1}{2}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{\partial r}(r\gamma_{x\theta})\right] = 0 \qquad [41,42,43]$$

It is assumed that the cylinder is long and all loading and geometry is axisymmetric. Therefore, the stresses are assumed to be independent of the axial direction, *x*, and the hoop direction, *θ*. The three-dimensional equilibrium equations then reduce to:

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \qquad \frac{d\tau_{r\theta}}{dr} + \frac{2}{r}\tau_{r\theta} = 0 \qquad \frac{d\tau_{xr}}{dr} + \frac{1}{r}\tau_{xr} = 0 \qquad [44.45,46]$$

The lamina constitutive equations, including the free thermal strains, may expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{\theta r} \\ \tau_{rx} \\ \tau_{x\theta} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \Delta T\alpha_x \\ \varepsilon_\theta - \Delta T\alpha_\theta \\ \varepsilon_r - \Delta T\alpha_r \\ \gamma_{\theta r} \\ \gamma_{rx} \\ \gamma_{x\theta} - \Delta T\alpha_{x\theta} \end{Bmatrix} \qquad [47]$$

where $\overline{C}_{ij}$ are the transformed stiffness terms and the off-axis coefficients of thermal expansion in terms of the on-axis coefficients of thermal expansion are:

$$\alpha_x = \alpha_1\cos^2\phi + \alpha_2\sin^2\phi \;, \quad \alpha_\theta = \alpha_1\sin^2\phi + \alpha_2\cos^2\phi \;, \quad \alpha_r = \alpha_3 \qquad [48,49,50]$$

$$\alpha_{x\theta} = 2\sin\phi\cos\phi(\alpha_1 - \alpha_2) \qquad [51]$$

The displacements in an individual lamina (the $k^{th}$ lamina) are found through clever manipulation of the stated equations and are found to be:

$$u_k(x) = \varepsilon_x^o x \qquad [52]$$

$$v_k(x,r) = \gamma^o xr \qquad [53]$$

$$w_k(r) = A_1^{(k)} r^{\lambda_k} + A_2^{(k)} r^{-\lambda_k} + \Gamma_k \varepsilon_x^o r + \Omega_k \gamma^o r^2 + \Psi_k r \Delta T \qquad [54]$$

where,

$$\Gamma = \left( \frac{\overline{C}_{12} - \overline{C}_{13}}{\overline{C}_{33} - \overline{C}_{22}} \right) \quad \Omega = \left( \frac{\overline{C}_{26} - 2\overline{C}_{36}}{4\overline{C}_{33} - \overline{C}_{22}} \right) \quad \Psi = \left( \frac{\tilde{\Sigma}}{\overline{C}_{33} - \overline{C}_{22}} \right) \qquad [55\text{-}57]$$

,

$$\tilde{\Sigma} = \left(\overline{C}_{13} - \overline{C}_{12}\right)\alpha_x + \left(\overline{C}_{23} - \overline{C}_{22}\right)\alpha_\theta + \left(\overline{C}_{33} - \overline{C}_{32}\right)\alpha_r + \left(\overline{C}_{63} - \overline{C}_{62}\right)\alpha_{x\theta} \qquad [58]$$

and

$$\lambda_k = \sqrt{\frac{\overline{C}_{22}}{\overline{C}_{33}}} \qquad [59]$$

The values of $\Gamma_k, \Omega_k$, and $\lambda_k$ are based on the lamina stiffness terms and are all unit-less. The terms $\tilde{\Sigma}_k$, and $\Psi_k$ are found from the lamina stiffnesses and coefficients of thermal expansion and $\Psi_k$ has the units of the inverse of the change in temperature, $\frac{1}{\Delta \mathrm{C}^{\mathrm{o}}}$. None of the lamina constants have any physical meaning and are simply used to consolidate the expressions.

## 1.2 Determining Lamina Constants, $A_1^k$ and $A_2^k$, $\varepsilon_x^o$ or $P_x$, and $\gamma^\circ$ or $T_x$

The problem is now reduced to finding the constants $A_1^{(k)}$ and $A_2^{(k)}$ for each lamina and either the axial strain, $\varepsilon_1^\circ$ or axial force, $P_x$ and either the angle of twist, $\gamma^\circ$ or torque, $T_x$. For a laminate of $N$ laminae, there are *2N+2* unknowns. Thus, *2N+2* equations containing the unknowns are constructed as,

$$\begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1,2N+2} \\ K_{21} & K_{22} & & K_{2,2N+2} \\ \vdots & & \ddots & \vdots \\ K_{2N+2,1} & K_{2N+2,2} & \cdots & K_{2N+2,2N+2} \end{bmatrix} \begin{Bmatrix} A_1^{(1)} \\ A_2^{(1)} \\ \vdots \\ A_1^{(k)} \\ A_2^{(k)} \\ \vdots \\ A_1^{(N)} \\ A_2^{(N)} \\ P_x \text{ or } \varepsilon_x^{\circ} \\ T_x \text{ or } \gamma_{x\theta}^{\circ} \end{Bmatrix} = \begin{Bmatrix} F_1 \\ \vdots \\ F_i \\ \vdots \\ F_{2N+2} \end{Bmatrix} \qquad [60]$$

The equations can be found from displacement and stress equilibrium at the laminae boundaries as follows:

### *1.2.1 Row 1; Radial Stress Equilibrium at r=r$_o$*

The first row of Equation (60) can be found from the radial stress equilibrium on the inside of the laminated cylinder.

$$\sigma_r(r_o) = -p_i \qquad [61]$$

The non-zero constants of the first row of the [K] matrix can now be expressed as,

$$K_{11} = \left\{\overline{C}_{23}^{(1)} + \lambda_1 \overline{C}_{33}^{(1)}\right\} r_o^{\lambda_1 - 1} \qquad [62]$$

$$K_{12} = \left\{\overline{C}_{23}^{(1)} - \lambda_1 \overline{C}_{33}^{(1)}\right\} r_o^{-\lambda_1 - 1} \qquad [63]$$

$$K_{1,2N+1} = \begin{cases} \overline{C}_{13}^{(1)} + \left\{\overline{C}_{23}^{(1)} + \overline{C}_{33}^{(1)}\right\}\Gamma_1 & \text{if } P_{\mathrm{x}} \text{ is given} \\ 0 & \text{if } \varepsilon_{\mathrm{x}}^{\mathrm{o}} \text{ is given} \end{cases} \qquad [64]$$

and

$$K_{1,2N+2} = \begin{cases} \left[\overline{C}_{36}^{(1)} + \left\{\overline{C}_{23}^{(1)} + 2\overline{C}_{33}^{(1)}\right\}\Omega_1\right] r_o & \text{if } T_{\mathrm{x}} \text{ is given} \\ 0 & \text{if } \gamma^{o} \text{ is given} \end{cases} \qquad [65]$$

And the term on the right-hand side of the first equation, $F_1$, is

$$F_1 = \begin{array}{ll} -p_{in} - \left[\left\{\overline{C}_{23}^{(1)} + \overline{C}_{33}^{(1)}\right\}\Psi_1 - \overline{C}_{i3}^{(k)}\alpha_i^{(k)}\right]\Delta T & \\ -\left[\overline{C}_{13}^{(1)} + \left\{\overline{C}_{23}^{(1)} + \overline{C}_{33}^{(1)}\right\}\Gamma_1\right]\varepsilon_x^o & \text{include if } \varepsilon_x^o \text{ is given} \\ -\left[\overline{C}_{36}^{(1)} + \left\{\overline{C}_{23}^{(1)} + 2\overline{C}_{33}^{(1)}\right\}\Omega_1\right]\gamma^o r_o & \text{include if } \gamma^o \text{ is given} \end{array} \quad [66]$$

#### 1.2.2 Even Rows (k=1, N-1); Radial Displacement Continuity at Laminae Interfaces

The even rows of Equation (60) represent the continuity of displacements at the lamina boundaries.

$$w(r_k) = w(r_{k+1}) \quad \text{for } k = 1, N-1 \quad [67]$$

The non-zero terms of the even rows of [K] may now be expressed as:

$$K_{2k,2k-1} = r_k^{\lambda_k} \quad K_{2k,2k} = r_k^{-\lambda_k} \quad [68,69]$$

$$K_{2k,2k+1} = -r_k^{\lambda_{k+1}} \quad K_{2k,2k+2} = -r_k^{-\lambda_{k+1}} \quad [70,71]$$

$$K_{2k,2N+1} = \begin{cases} \{\Gamma_k - \Gamma_{k+1}\}r_k & \text{if } P_x \text{ is given} \\ 0 & \text{if } \varepsilon_x^o \text{ is given} \end{cases} \quad [72]$$

$$K_{2k,2N+2} = \begin{cases} \{\Omega_k - \Omega_{k+1}\}r_k^2 & \text{if } T_x \text{ is given} \\ 0 & \text{if } \gamma^o \text{ is given} \end{cases} \quad [73]$$

for k=1, N-1

And the even numbered $F_{2k}$ terms are

$$F_{2k} = \begin{array}{ll} r_k\{\Psi_{k+1} - \Psi_k\}\Delta T & \\ -\{\Gamma_k - \Gamma_{k+1}\}r_k\,\varepsilon_x^o & \text{include if } \varepsilon_x^o \text{ is given} \\ -\{\Omega_k - \Omega_{k+1}\}r_k^2\,\gamma^o & \text{include if } \gamma^o \text{ is given} \end{array} \quad [74]$$

for k=1, N-1

#### 1.2.3 Odd Rows (k=1, N-1); Radial Normal Stress Equilibrium at Laminae Interfaces

The odd rows of Equation (60) represent the equilibrium of radial stress at the lamina boundaries.

$$\sigma_r^{(k)}(r_k) = \sigma_r^{(k+1)}(r_k) \quad \text{for } k = 1, N-1 \quad [75]$$

The non-zero terms of the odd rows of [K] may now be expressed as:

$$K_{2k+1,2k-1} = r_k^{(\lambda_k - 1)}\left\{\overline{C}_{23}^{(k)} + \lambda_k\overline{C}_{33}^{(k)}\right\}, \quad K_{2k+1,2k} = r_k^{(-\lambda_k - 1)}\left\{\overline{C}_{23}^{(k)} - \lambda_k\overline{C}_{33}^{(k)}\right\} \quad [76,77]$$

$$K_{2k+1,2k+1} = -r_k^{(\lambda_{k+1}-1)}\left\{\overline{C}_{23}^{(k+1)} + \lambda_{k+1}\overline{C}_{33}^{(k+1)}\right\} \quad K_{2k+1,2k+2} = -r_k^{(-\lambda_{k+1}-1)}\left\{\overline{C}_{23}^{(k+1)} - \lambda_{k+1}\overline{C}_{33}^{(k+1)}\right\} \qquad [78,79]$$

$$K_{2k+1,2N+1} = \begin{cases} \overline{C}_{13}^{(k)} + \left\{\overline{C}_{23}^{(k)} + \overline{C}_{33}^{(k)}\right\}\Gamma_k - \left[\overline{C}_{13}^{(k+1)} + \left\{\overline{C}_{23}^{(k+1)} + \overline{C}_{33}^{(k+1)}\right\}\Gamma_{k+1}\right] & \text{if } P_x \text{ is given} \\ 0 & \text{if } \varepsilon_x^o \text{ is given} \end{cases} \qquad [80]$$

$$K_{2k+1,2N+2} = \begin{cases} r_k\left[\overline{C}_{36}^{(k)} + \left(\overline{C}_{23}^{(k)} + 2\overline{C}_{33}^{(k)}\right)\Omega_k - \left\{\overline{C}_{36}^{(k+1)} + \left(\overline{C}_{23}^{(k+1)} + 2\overline{C}_{33}^{(k+1)}\right)\Omega_{k+1}\right\}\right] & \text{if } T_{\mathrm{x}} \text{ is given} \\ 0 & \text{if } \gamma^o \text{ is given} \end{cases} \qquad [81]$$

for k=1,N-1.

And the odd numbered $F_{2k+1}$ terms are:

$$F_{2k+1} = \begin{array}{ll} \left[\left(\overline{C}_{23}^{(k+1)} + \overline{C}_{33}^{(k+1)}\right)\Psi_{k+1} - \overline{C}_{i3}^{(k+1)}\alpha_i^{(k+1)} - \left\{\left(\overline{C}_{23}^{(k)} + \overline{C}_{33}^{(k)}\right)\Psi_k - \overline{C}_{i3}^{(k)}\alpha_i^{(k)}\right\}\right]\Delta\mathrm{T} & \\ -\left[\overline{C}_{13}^{(k)} + \left\{\overline{C}_{23}^{(k)} + \overline{C}_{33}^{(k)}\right\}\Gamma_k - \left[\overline{C}_{13}^{(k+1)} + \left\{\overline{C}_{23}^{(k+1)} + \overline{C}_{33}^{(k+1)}\right\}\Gamma_{k+1}\right]\right]\varepsilon_{\mathrm{x}}^{\mathrm{o}} & \text{include if } \varepsilon_{\mathrm{x}}^{\mathrm{o}} \text{ is given} \\ -r_k\left[\overline{C}_{36}^{(k)} + \left(\overline{C}_{23}^{(k)} + 2\overline{C}_{33}^{(k)}\right)\Omega_k - \left\{\overline{C}_{36}^{(k+1)} + \left(\overline{C}_{23}^{(k+1)} + 2\overline{C}_{33}^{(k+1)}\right)\Omega_{k+1}\right\}\right]\gamma^o & \text{include if } \gamma^{\mathrm{o}} \text{ is given} \end{array} \qquad [82]$$

for k=1,N-1.

### *1.2.4 Row 2N; Radial Stress Equilibrium at r=r$_N$*

Row *2N* of Equation (60) can be found from the radial stress equilibrium on the outside of the laminated cylinder.

$$\sigma_r(r_N) = -p_{out} \qquad [83]$$

The non-zero constants of row 2N of the [K] matrix are expressed as:

$$K_{2N,2N-1} = \left\{\overline{C}_{23}^{(N)} + \lambda_N\overline{C}_{33}^{(N)}\right\}r_N^{\lambda_N-1}, \qquad K_{2N,2N} = \left\{\overline{C}_{23}^{(N)} - \lambda_N\overline{C}_{33}^{(N)}\right\}r_N^{-\lambda_N-1} \qquad [84]$$

$$K_{2N,2N+1} = \begin{cases} \overline{C}_{13}^{(N)} + \left\{\overline{C}_{23}^{(N)} + \overline{C}_{33}^{(N)}\right\}\Gamma_N & \text{if } P_x \text{ is given} \\ 0 & \text{if } \varepsilon_x^o \text{ is given} \end{cases} \qquad [85]$$

$$K_{2N,2N+2} = \begin{cases} \left[\overline{C}_{36}^{(N)} + \left\{\overline{C}_{23}^{(N)} + 2\overline{C}_{33}^{(N)}\right\}\Omega_N\right]r_N & \text{if } T_{\mathrm{x}} \text{ is given} \\ 0 & \text{if } \gamma^o \text{ is given} \end{cases} \qquad [86]$$

And the term, $F_{2N}$, on the right-hand side of equation *2N* is

$$F_{2N} = \begin{array}{ll} -p_{out} - \left[\left\{\overline{C}_{23}^{(N)} + \overline{C}_{33}^{(N)}\right\}\Psi_N - \overline{C}_{i3}^{(k)}\alpha_i^{(k)}\right]\Delta T & \\ -\left[\overline{C}_{13}^{(N)} + \left\{\overline{C}_{23}^{(N)} + \overline{C}_{33}^{(N)}\right\}\Gamma_N\right]\varepsilon_{\mathrm{x}}^{\mathrm{o}} & \text{include if } \varepsilon_{\mathrm{x}}^{\mathrm{o}} \text{ is given} \\ -\left[\overline{C}_{36}^{(N)} + \left\{\overline{C}_{23}^{(N)} + 2\overline{C}_{33}^{(N)}\right\}\Omega_N\right]\gamma^o r_N & \text{include if } \gamma^{\mathrm{o}} \text{ is given} \end{array} \qquad [87]$$

### *1.2.5 Row 2N+1; Axial Load Equilibrium* $P_x = 2\pi \int \sigma_x r dr$

Two more equations are needed in order to be able to solve for all of the unknowns. Row *2N+1* of Equation (60) can be found from the axial load equilibrium.

$$P_x = 2\pi \int_{\mathrm{r_o}}^{\mathrm{r_N}} \sigma_x r dr \qquad [88]$$

The non-zero constants of the *2N+1* row of the [K] matrix become:

$$K_{2N+1,2k-1} = \frac{2\pi\left(\overline{C}_{12}^{(k)} + \lambda_k \overline{C}_{13}^{(k)}\right)\left(r_k^{\lambda_k+1} - r_{k-1}^{\lambda_k+1}\right)}{\lambda_k + 1}, \quad K_{2N+1,2k} = \frac{2\pi\left(\overline{C}_{12}^{(k)} - \lambda_k \overline{C}_{13}^{(k)}\right)\left(r_k^{1-\lambda_k} - r_{k-1}^{1-\lambda_k}\right)}{1-\lambda_k} \qquad [89]$$

$$K_{2N+1,2N+1} = \begin{cases} \pi \sum_{\mathrm{k=1}}^{\mathrm{N}} \left\{\overline{C}_{11}^{(k)} + \left(\overline{C}_{13}^{(k)} + \overline{C}_{12}^{(k)}\right)\Gamma_k\right\}\left(\mathrm{r_k^2 - r_{k-1}^2}\right) & \text{if } P_x \text{ is given} \\ -1 & \text{if } \varepsilon_x^o \text{ is given} \end{cases} \qquad [90]$$

$$K_{2N+1,2N+2} = \begin{cases} \dfrac{2\pi}{3} \sum_{\mathrm{k=1}}^{\mathrm{N}} \left\{\overline{C}_{16}^{(k)} + \left(\overline{C}_{12}^{(k)} + 2\overline{C}_{13}^{(k)}\right)\Omega_k\right\}\left(\mathrm{r_k^3 - r_{k-1}^3}\right) & \text{if } T_{\mathrm{x}} \text{ is given} \\ 0 & \text{if } \gamma^o \text{ is given} \end{cases} \qquad [91]$$

and the term, $F_{2N+1}$, on the right-hand side of the *2N+1* equation is:

$$F_{2N+1} = \begin{cases} P_x - \Delta T \pi \sum_{k=1}^{N} \left\{\left(\overline{C}_{12}^{(k)} + \overline{C}_{13}^{(k)}\right)\Psi_k - \overline{C}_{i1}^{(k)} \alpha_i^{(k)}\right\}\left(\mathrm{r_k^2 - r_{k-1}^2}\right) & \text{if } P_{\mathrm{x}} \text{ is given} \\ -\pi \sum_{\mathrm{k=1}}^{\mathrm{N}} \left[\varepsilon_{\mathrm{x}}^{\mathrm{o}}\left\{\overline{C}_{11}^{(k)} + \left(\overline{C}_{13}^{(k)} + \overline{C}_{12}^{(k)}\right)\Gamma_k\right\} + \Delta T\left\{\left(\overline{C}_{12}^{(k)} + \overline{C}_{13}^{(k)}\right)\Psi_k - \overline{C}_{i1}^{(k)}\alpha_i^{(k)}\right\}\right]\left(\mathrm{r_k^2 - r_{k-1}^2}\right) & \text{if } \varepsilon_{\mathrm{x}}^{\mathrm{o}} \text{ is given} \end{cases} \qquad [92]$$

$$-\gamma^o \frac{2\pi}{3} \sum_{\mathrm{k=1}}^{\mathrm{N}} \left\{\overline{C}_{16}^{(k)} + \left(\overline{C}_{12}^{(k)} + 2\overline{C}_{13}^{(k)}\right)\Omega_k\right\}\left(\mathrm{r_k^3 - r_{k-1}^3}\right) \quad \text{include if } \gamma^{\mathrm{o}} \text{ is given}$$

### *1.2.6 Row 2N+2; Equilibrium of Torsion and Shear* $T_x = 2\pi \int \tau_{x\theta} r^2 dr$

The final row (*2N+2)* of Equation (60) can be found from equilibrium between the applied torque, $T_x$, and the shear stress through the thickness of the laminate.

$$T_x = 2\pi \int_{\mathrm{r_o}}^{\mathrm{r_N}} \tau_{x\theta} r^2 dr \qquad [93]$$

The non-zero constants of the *2N+2* row of the [K] matrix can be shown to be,

$$K_{2N+2,2k-1} = \frac{2\pi\left(\overline{C}_{26}^{(k)} + \lambda_k \overline{C}_{36}^{(k)}\right)\left(r_k^{\lambda_k+2} - r_{k-1}^{\lambda_k+2}\right)}{\lambda_k + 2}, \quad K_{2N+2,2k} = \frac{2\pi\left(\overline{C}_{26}^{(k)} - \lambda_k \overline{C}_{36}^{(k)}\right)\left(r_k^{2-\lambda_k} - r_{k-1}^{2-\lambda_k}\right)}{2-\lambda_k} \qquad [94]$$

$$K_{2N+2,2N+1} = \begin{cases} \dfrac{2\pi}{3}\sum_{\mathrm{k=1}}^{\mathrm{N}}\left\{\overline{C}_{16}^{(k)} + \left(\overline{C}_{26}^{(k)} + \overline{C}_{36}^{(k)}\right)\Gamma_k\right\}\left(\mathrm{r_k^3 - r_{k-1}^3}\right) & \text{if } P_x \text{ is given} \\ 0 & \text{if } \varepsilon_x^o \text{ is given} \end{cases} \quad [95]$$

$$K_{2N+2,2N+2} = \begin{cases} \dfrac{\pi}{2}\sum_{\mathrm{k=1}}^{\mathrm{N}}\left\{\overline{C}_{66}^{(k)} + \left(\overline{C}_{26}^{(k)} + 2\overline{C}_{36}^{(k)}\right)\Omega_k\right\}\left(\mathrm{r_k^4 - r_{k-1}^4}\right) & \text{if } T_{\mathrm{x}} \text{ is given} \\ -1 & \text{if } \gamma^o \text{ is given} \end{cases} \quad [96]$$

and the term on the right-hand side of the *2N+2* row is:

$$F_{2N+2} = \begin{cases} T_x - \Delta T\dfrac{2\pi}{3}\sum_{k=1}^{N}\left\{\left(\overline{C}_{26}^{(k)} + \overline{C}_{36}^{(k)}\right)\Psi_k - \overline{C}_{i6}^{(k)}\alpha_i^k\right\}\left(\mathrm{r_k^3 - r_{k-1}^3}\right) & \text{if } T_{\mathrm{x}} \text{ is given} \\ -\gamma^o\dfrac{\pi}{2}\sum_{\mathrm{k=1}}^{\mathrm{N}}\left\{\overline{C}_{66}^{(k)} + \left(\overline{C}_{26}^{(k)} + 2\overline{C}_{36}^{(k)}\right)\Omega_k\right\}\left(\mathrm{r_k^4 - r_{k-1}^4}\right) - & \\ \qquad \Delta T\dfrac{2\pi}{3}\sum_{k=1}^{N}\left\{\left(\overline{C}_{26}^{(k)} + \overline{C}_{36}^{(k)}\right)\Psi_k - \overline{C}_{i6}^{(k)}\alpha_i^k\right\}\left(\mathrm{r_k^3 - r_{k-1}^3}\right) & \text{if } \gamma^{\mathrm{o}} \text{ is given} \end{cases} \quad [97]$$

$$-\varepsilon_x^o\frac{2\pi}{3}\sum_{\mathrm{k=1}}^{\mathrm{N}}\left\{\overline{C}_{16}^{(k)} + \left(\overline{C}_{26}^{(k)} + \overline{C}_{36}^{(k)}\right)\Gamma_k\right\}\left(\mathrm{r_k^3 - r_{k-1}^3}\right) \quad \text{include if } \varepsilon_{\mathrm{x}}^{\mathrm{o}} \text{ is given}$$

### 1.3 Determining Lamina Displacements

If the lamina is oriented such that the fibers are along the axis of the cylinder, (ϕ=0º), and the lamina is transversely isotropic, (the hoop direction and the radial direction have the same properties), the terms $\Gamma_k, \Psi_{\mathrm{k}}, \Omega_{\mathrm{k}}$ and $\tilde{\Sigma}_{\mathrm{k}}$ are all zero. However, $\Gamma_k$, and $\Psi_{\mathrm{k}}$ must be set to zero since their denominators are zero and will cause a fatal error in most codes.

Once the [K] matrix and the {F} vector are found all of the unknown terms may be solved using a linear equation solution technique. The displacements in each layer may now be found by substituting the values of $A_1^{(k)}$ and $A_2^{(k)}$, the axial strain, $\varepsilon_1^{\circ}$, and the angle of twist, $\gamma^{\circ}$, into Equations (52-54).

### 1.4 Determining Lamina Strains and Stresses

Once the displacements are found the non-zero, lamina strains are identified from Equations (35-40) as:

$$\varepsilon_x = \varepsilon_x^o \quad [98]$$

$$\varepsilon_\theta(r) = A_1^{(k)} r^{\lambda_k - 1} + A_2^{(k)} r^{-\lambda_k - 1} + \Gamma_k\varepsilon_x^o + \Omega_k\gamma^o r + \Psi_k\Delta T \quad [99]$$

$$\varepsilon_r(r) = A_1^{(k)}\lambda_k r^{\lambda_k - 1} - A_2^{(k)}\lambda_k r^{-\lambda_k - 1} + \Gamma_k \varepsilon_x^o + 2\Omega_k \gamma^o r + \Psi_k \Delta T \quad [100]$$

$$\gamma_{x\theta}(r) = \gamma^o r \quad [101]$$

the lamina stresses can be found from the constitutive relationships, Equation (47), put in a reduced form as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{x\theta} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & \overline{C}_{26} \\ \overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & \overline{C}_{36} \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{36} & \overline{C}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x - \Delta T \alpha_x \\ \varepsilon_\theta - \Delta T \alpha_\theta \\ \varepsilon_r - \Delta T \alpha_r \\ \gamma_{x\theta} - \Delta T \alpha_{x\theta} \end{Bmatrix} \quad [102]$$

## 2. REFERENCES


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