Kinematics, statics and stiffness analysis of n(4-SPS+SP) S-PM

Article · January 2012

DOI: 10.2316/Journal.206.2012.3.206-3664

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International Journal of Robotics and Automation, Vol. 27, No. 3, 2012

KINEMATICS, STATICS AND STIFFNESS ANALYSIS OF n(4-SPS+SP) S-PM

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Abstract

A novel n(4-SPS+SP) serial-parallel manipulator (S-PM) formed by n 4-SPS+SP PMs with four degrees of freedom (DOF) connected in series is proposed in this paper. The kinematics, statics and stiffness model of this serial-PM is established. First, the forward kinematics of 4-SPS+SP (spherical joint-prismatic joint-revolute+spherical joint-prismatic joint) PM is derived and the kinematics of n(4-SPS+SP) S-PM is studied combined with the kinematics result of single 4-SPS+SP PM. Second, the statics, deformation and stiffness of n(4-SPS+SP) S-PM are studied. Finally, an analytically solved example is given for 3(4-SPS+SP) formed by three 4-SPS+SP PMs. The method for this mechanism is fit for other S-PMs.

Key Words

Kinematics, serial-parallel manipulators, statics, deformation, stiffness

1. Introduction

Serial-parallel manipulators (S-PMs) have higher stiffness than serial robots and possess the advantages of larger workspace than PMs. This class of mechanisms has good research and application prospect. In this aspect, Romdhane [1] designed and analysed a hybrid S-PM formed by a pure translational PM that has a PPP-type passive leg and a pure rotational PM that has an S-type passive leg. Jaime *et al.* [2] studied the dynamic of 2(3-RPS) (revolute joint-prismatic joint-spherical joint) S-PM by using screw theory and principle of virtual work. Lu and Hu [3] solved driving forces of 2(3-SPR) (spherical joint-prismatic joint-revolute) S-PM by CAD variation geometry approach and solved the velocity, acceleration and statics by analytical method subsequently [4]. Zheng *et al.* [5] studied the kinematics of a hybrid S-PM formed by a pure translational 3-UPU (universal joint-prismatic joint-universal joint) PM and a pure rotational 3-UPU PM. The previous research of these type of mechanisms focused on the S-PMs formed by two PMs. The S-PMs formed by n PMs were seldom studied. However, establishing theory of this kind of S-PMs is significant.

4-DOF PMs have attracted a lot of attention because of their merits such as larger workspace, simple structure and easy controllability. Many 4-DOF PMs have been systemically synthesized by different means [6]–[9]. The kinematics, singularity, stiffness and dynamics of this class of PMs have been deeply investigated [10]–[12]. This paper proposes a novel n(4-SPS+SP) serial-PM formed by n 4-SPS+SP PMs connected in series. The 4-SPS+SP PM was proposed and studied by Lu and Hu [13], which has some merits and potential application prospects [14]. The inverse kinematics, velocity, acceleration and stiffness of this PM have been solved. However, the forward kinematics of this PM has not been studied [13], [14]. In this paper, the forward kinematics of 4-SPS+SP is solved in close form. Furthermore, the kinematics, statics and stiffness of n(4-SPS+SP) S-PM are established based on the results of 4-SPS+SP PM.

The n(4-SPS+SP) has some merits such as large workspace and high stiffness. It has some potential applications for the robot arms and the serial-PM machine tools. The kinematics, statics and stiffness model established for n(4-SPS+SP) serial-PM is fit for other serial-PMs.

2. Kinematics of the n(4-SPS+SP) S-PM

The n(4-SPS+SP) S-PM is formed by n identical 4-SPS+SP PMs connected in series. Let PM i be the ith 4-SPS+SP PM form bottom to top in n(4-SPS+SP) S-PM. The ith 4-SPS+SP PM includes a upper platform n_{i1}, a lower platform n_{i0}, four SPS active limbs r_{ij} ($i = 1, 2, \dots, n$; $j = 1, 2, 3, 4$) with linear actuators, and one SP passive limb r_{i0} (see Fig. 1(a)). n_{i1} is a rectangle quaternary with a length sideline l_{i1}, a width sideline l_{i2}, and n_{i0} is a square with sideline L_i. Each of the driving limbs r_{ij} connects n_{i1} with n_{i0} by using a spherical joint S at B_{ij} ($i = 1, 2, \dots, n$; $j = 1, 2, 3, 4$) on n_{i1}, a prismatic joint P along r_{ij}, and a spherical joint S at A_{ij} ($i = 1, 2, \dots, n$; $j = 1, 2, 3, 4$) on n_{i0}. The passive limb r_{i0} is perpendicular with n_{i1} and connects n_{i1} with n_{i0} by using a prismatic joint P on n_{i1} and spherical joint S at o_{i-1} on n_{i0}. The lower platform n_{i0} of PM i ($i = 2, \dots, n$) and the upper platform $n_{(i-1)1}$ of PM $i-1$ are coplanar, fixed connected and have common centre.

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Recommended by Prof. Z. Qu
(DOI: 10.2316/Journal.206.2012.3.206-3664)

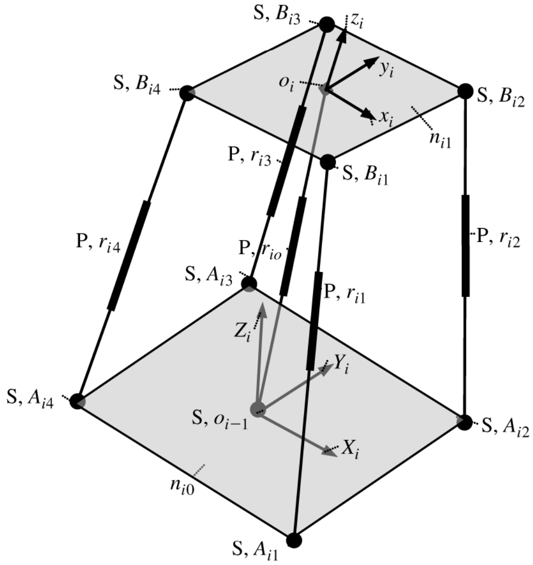

Figure 1. Sketch of 4SPS+SP PM.

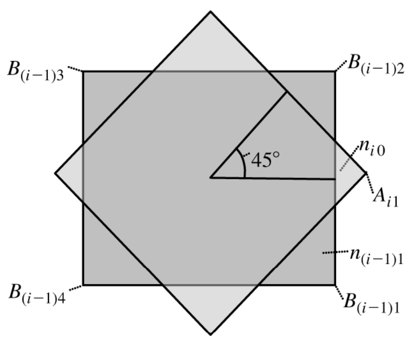

Figure 2. Relation between n_{i0} and $n_{(i-1)1}$.

An angle between $n_{(i-1)1}$ and n_{i0} being $\theta = 45° \times (-1)^i$ is satisfied (see Fig. 2). Based on the defined notation, n_{10} and n_{n1} represents the base and terminal platform of whole n(4-SPS+SP) S-PM, respectively.

2.1 Forward Position Modelling of n(4-SPS+SP) S-PM

The forward position kinematics problem solves the position and orientation of the terminal platform n_{n1} from a given set of actuated inputs r_{ij} $(i=1,2,\ldots,n;$ $j=1,2,3,4)$.

Let $\{n_{i0}\}$ be a coordinate system o_{i-1}-$X_iY_iZ_i$ $(i=1,2,\ldots,n)$ fixed on the centre of lower platform of ith PM with some conditions ($X_i \perp A_{i1}A_{i2}$, $Y \| A_{i1}A_{i2}$, Z$\perp X_i$, Z$\perp Y_i$) for its coordinate axes being satisfied. Here o_0 is the centre of base. Let $\{n_{i1}\}$ be a coordinate system o_i-x_iyz_i fixed on the centre of upper platform of PM i with some conditions ($x_i \perp B_{i1}B_{i2}$, $y \| B_{i1}B_{i2}$, z$\perp x_i$, z$\perp y_i$) for its coordinate axes being satisfied.

To solve the forward kinematics of n(4-SPS+SP) S-PM, the forward kinematics of 4-SPS+SP PM must be solved first. The Sylvester dialytic elimination method is applied to solve the forward position problem.

The position vector of A_{ij} $(i=1,2,\ldots,n;\ j=1,2,3,4)$ and o_i in $\{n_{i0}\}$ can be expressed as follows:

$$
{}^{n_{i0}}A_{i1} = \begin{bmatrix} L_i \\ -L_i \\ 0 \end{bmatrix}, \quad {}^{n_{i0}}A_{i2} = \begin{bmatrix} L_i \\ L_i \\ 0 \end{bmatrix}, \quad {}^{n_{i0}}A_{i3} = \begin{bmatrix} -L_i \\ L_i \\ 0 \end{bmatrix},
$$

$$
{}^{n_{i0}}A_{i4} = \begin{bmatrix} -L_i \\ -L_i \\ 0 \end{bmatrix}, \quad {}^{n_{i0}}o_i = \begin{bmatrix} X_{io} \\ Y_{io} \\ Z_{io} \end{bmatrix} \tag{1a}
$$

The position vectors of B_{ij} $(i=1,2,\ldots,n;\ j=1,2,3,4)$ in $\{n_{i1}\}$ can be expressed as follows:

$$
{}^{n_{i1}}B_{i1} = \begin{bmatrix} l_{i1} \\ -l_{i2} \\ 0 \end{bmatrix}, \quad {}^{n_{i1}}B_{i2} = \begin{bmatrix} l_{i1} \\ l_{i2} \\ 0 \end{bmatrix}, \quad {}^{n_{i1}}B_{i3} = \begin{bmatrix} -l_{i1} \\ l_{i2} \\ 0 \end{bmatrix},
$$

$$
{}^{n_{i1}}B_{i4} = \begin{bmatrix} -l_{i1} \\ -l_{i2} \\ 0 \end{bmatrix} \tag{1b}
$$

The position vectors of B_{ij} $(i=1,2,3,4)$ in $\{n_{i0}\}$ can be expressed as follows:

$$
{}^{n_{i0}}B_{ij} = {}^{n_{i0}}_{n_{i1}}\mathbf{R}\ {}^{n_{i1}}B_{ij} + {}^{n_{i0}}o_i \ (j=1,2,3,4) \tag{2a}
$$

where ${}^{n_{i0}}o_i$ is the position vectors for the centre of upper platform of PM i relative to its lower platform. ${}^{n_{i0}}_{n_{i1}}\mathbf{R}$ is the rotational matrix from upper platform to lower platform for PM i. ${}^{n_{i0}}_{n_{i1}}\mathbf{R}$ and ${}^{n_{i0}}o_i$ have been solved in Ref. [14] as follows:

$$
\begin{aligned}
{}^{n_{i0}}_{n_{i1}}\mathbf{R} &= \begin{bmatrix} c_{\alpha_i}c_{\beta_i}c_{\lambda_i} - s_{\alpha_i}s_{\lambda_i} & -c_{\alpha_i}c_{\beta_i}s_{\lambda_i} - s_{\alpha_i}c_{\lambda_i} & c_{\alpha_i}s_{\beta_i} \\ s_{\alpha_i}c_{\beta_i}c_{\lambda_i} + c_{\alpha_i}s_{\lambda_i} & -s_{\alpha_i}c_{\beta_i}s_{\lambda_i} + c_{\alpha_i}c_{\lambda_i} & s_{\alpha_i}s_{\beta_i} \\ -s_{\beta_i}c_{\lambda_i} & s_{\beta_i}s_{\lambda_i} & c_{\beta_i} \end{bmatrix}, \\
X_{io} &= r_{i0}c_{\alpha_i}s_{\beta_i}, \quad Y_{io} = r_{i0}s_{\alpha_i}s_{\beta_i}, \quad Z_{oi} = r_{i0}c_{\beta_i}
\end{aligned} \tag{2b}
$$

where X_{io}, Y_{io}, Z_{io} are three components of position vector of o_i in $\{n_{i0}\}$ and α_i, β_i and λ_i are three Euler angles that are used for representing the orientation of n_{i1} to n_{i0}.

The inverse kinematics of ith 4SPS+SP PM has been solved as follows [14]:

$$
\begin{aligned}
r_{i1}^2 &= Q_i + r_{i0}^2 - [c_{\beta_i}(c_{\alpha_i} - s_{\alpha_i})(l_{i1}c_{\lambda_i} + l_{i2}s_{\lambda_i}) \\
&\quad - (s_{\alpha_i} + c_{\alpha_i})(l_{i1}s_{\lambda_i} - l_{i2}c_{\lambda_i}) \\
&\quad + 2r_{i0}s_{\beta_i}(c_{\alpha_i} - s_{\alpha_i})]L_i/2 \\
r_{i2}^2 &= Q_i + r_{i0}^2 - [c_{\beta_i}(c_{\alpha_i} + s_{\alpha_i})(l_{i1}c_{\lambda_i} - l_{i2}s_{\lambda_i}) \\
&\quad - (s_{\alpha_i} - c_{\alpha_i})(l_{i1}s_{\lambda_i} + l_{i2}c_{\lambda_i}) \\
&\quad + 2r_{i0}s_{\beta_i}(c_{\alpha_i} + s_{\alpha_i})]L_i/2
\end{aligned}
$$

$$\begin{aligned} r_{i3}^2 &= Q_i + r_{i0}^2 - [c_{\beta_i}(c_{\alpha_i} - s_{\alpha_i})(l_{i1}c_{\lambda_i} + l_{i2}s_{\lambda_i}) \\ &\quad - (s_{\alpha_i} + c_{\alpha_i})(l_{i1}s_{\lambda_i} - l_{i2}c_{\lambda_i}) \\ &\quad - 2r_{i0}s_{\beta_i}(c_{\alpha_i} - s_{\alpha_i})]L_i/2 \\ r_{i4}^2 &= Q_i + r_{i0}^2 - [c_{\beta_i}(c_{\alpha_i} + s_{\alpha_i})(l_{i1}c_{\lambda_i} - l_{i2}s_{\lambda_i}) \\ &\quad - (s_{\alpha_i} - c_{\alpha_i})(l_{i1}s_{\lambda_i} + l_{i2}c_{\lambda_i}) \\ &\quad -2r_{i0}s_{\beta_i}(c_{\alpha_i} + s_{\alpha_i})]L_i/2 \end{aligned} \tag{3}$$

where $Q_i = (2L_i^2 + l_{i1}^2 + l_{i2}^2)/4$ and $r_{i0}^2 = X_{io}^2 + Y_{io}^2 + Z_{io}^2$.

Equation (3) leads to:

$$r_{i3}^2 + r_{i4}^2 - r_{i1}^2 - r_{i2}^2 = 4L_iX_{io}, \quad X_{io} = \frac{r_{i3}^2 + r_{i4}^2 - r_{i1}^2 - r_{i2}^2}{4L_i} \tag{4a}$$

$$r_{i2}^2 + r_{i3}^2 - r_{i1}^2 - r_{i4}^2 = -4L_iY_{io}, \quad Y_{io} = \frac{r_{i1}^2 + r_{i4}^2 - r_{i2}^2 - r_{i3}^2}{4L_i} \tag{4b}$$

Equations (4a) and (4b) lead to:

$$\begin{aligned} tg_{\alpha_i} &= \frac{Y_{io}}{X_{io}} = \frac{r_{i1}^2 + r_{i4}^2 - r_{i2}^2 - r_{i3}^2}{r_{i3}^2 + r_{i4}^2 - r_{i1}^2 - r_{i2}^2}, \\ \alpha_i &= arctg\left(\frac{r_{i1}^2 + r_{i4}^2 - r_{i2}^2 - r_{i3}^2}{r_{i3}^2 + r_{i4}^2 - r_{i1}^2 - r_{i2}^2}\right) \end{aligned} \tag{5}$$

Equation (3) leads to:

$$\begin{aligned} &2L_i[l_{i1}(s_{\alpha_i}c_{\beta_i}c_{\lambda_i} + c_{\alpha_i}s_{\lambda_i}) - l_{i2}(c_{\alpha_i}c_{\beta_i}s_{\lambda_i} + s_{\alpha_i}c_{\lambda_i})] \\ &\quad = r_{i1}^2 - r_{i2}^2 + r_{i3}^2 - r_{i4}^2 \end{aligned} \tag{6a}$$

Equations (3) and (2b) lead to:

$$\begin{aligned} &2[l_{i1}(c_{\alpha_i}c_{\beta_i}c_{\lambda_i} - s_{\alpha_i}s_{\lambda_i}) + l_{i2}(-s_{\alpha_i}c_{\beta_i}s_{\lambda_i} + c_{\alpha_i}c_{\lambda_i})]L_i \\ &\quad + r_{i1}^2 + r_{i2}^2 + r_{i3}^2 + r_{i4}^2 = 4Q_i + 4(X_{io}/c_{\alpha_i})^2/s_{\beta_i}^2 \end{aligned} \tag{6b}$$

Let $t_{i1} = tg\frac{\beta_i}{2}$, $t_{i2} = tg\frac{\lambda_i}{2}$, there must be:

$$s_{\beta_i} = \frac{2t_{i1}}{1+t_{i1}^2}, \quad c_\beta = \frac{1-t_{i1}^2}{1+t_{i1}^2}, \quad s_{\lambda_i} = \frac{2t_{i2}}{1+t_{i2}^2}, \quad c_\lambda = \frac{1-t_{i2}^2}{1+t_{i2}^2} \tag{7}$$

Substituting (7) into (6a) leads to:

$$s_{i15}t_{i1}^2t_{i2}^2 + s_{i14}t_{i2}t_{i1}^2 + s_{i13}t_{i1}^2 + s_{i12}t_{i2}^2 + s_{i11}t_{i2} + s_{i10} = 0 \tag{8}$$

where,

$$\begin{aligned} s_{i15} &= l_{i1}s_{\alpha_i} + l_{i2}s_{\alpha_i} - \frac{(r_{i1}^2 - r_{i2}^2 + r_{i3}^2 - r_{i4}^2)}{2L}, \\ s_{i14} &= 2l_{i1}c_{\alpha_i} + 2l_{i2}c_{\alpha_i}, \\ s_{i13} &= -l_{i1}s_{\alpha_i} - l_{i2}s_{\alpha_i} - \frac{(r_{i1}^2 - r_{i2}^2 + r_{i3}^2 - r_{i4}^2)}{2L_i}, \\ s_{i12} &= -l_{i1}s_{\alpha_i} + l_{i2}s_{\alpha_i} - \frac{(r_{i1}^2 - r_{i2}^2 + r_{i3}^2 - r_{i4}^2)}{2L}, \\ s_{i11} &= 2l_{i1}c_{\alpha_i} - 2l_{i2}c_{\alpha_i}, \\ s_{i10} &= l_{i1}s_{\alpha_i} - l_{i2}s_{\alpha_i} - \frac{(r_{i1}^2 - r_{i2}^2 + r_{i3}^2 - r_{i4}^2)}{2L_i} \end{aligned}$$

Substituting (7) into (6b) leads to:

$$\begin{aligned} &s_{i29}t_{i1}^6t_{i2}^2 + s_{i28}t_{i1}^4t_{i2}^2 + s_{i27}t_{i1}^6 + s_{i26}t_{i2}t_{i1}^4 + s_{i25}t_{i1}^2t_{i2}^2 \\ &\quad + s_{i24}t_{i1}^4 + s_{i23}t_{i1}^2t_{i2} + s_{i22}t_{i1}^2 + s_{i21}t_{i2}^2 + s_{i20} = 0 \end{aligned} \tag{9}$$

where,

$$\begin{aligned} s_{i29} &= -4(X_{io}/c_{\alpha_i})^2, \\ s_{i28} &= -16Q_i - 12(X_{io}/c_{\alpha_i})^2 + 4(r_{i1}^2 + r_{i2}^2 + r_{i3}^2 + r_{i4}^2) \\ &\quad + 8c_{\alpha_i}L_i(l_{i1} - l_{i2}), \\ s_{i27} &= -4(X_{io}/c_{\alpha_i})^2, \\ s_{i26} &= -16s_{\alpha_i}L_i(l_{i1} - l_{i2}), \\ s_{i25} &= -16Q_i - 12(X_{io}/c_{\alpha_i})^2 + 4(r_{i1}^2 + r_{i2}^2 + r_{i3}^2 + r_{i4}^2) \\ &\quad - 8c_{\alpha_i}L_i(l_{i1} + l_{i2}), \\ s_{i24} &= -16Q_i - 12(X_{io}/c_{\alpha_i})^2 + 4(r_{i1}^2 + r_{i2}^2 + r_{i3}^2 + r_{i4}^2) \\ &\quad + 8c_{\alpha_i}L_i(-l_{i1} + l_{i2}), \\ s_{i23} &= -16s_{\alpha_i}(l_{i1} + l_{i2}), \\ s_{i22} &= -16Q_i - 12(X_{io}/c_{\alpha_i})^2 + 4(r_{i1}^2 + r_{i2}^2 + r_{i3}^2 + r_{i4}^2) \\ &\quad + 8c_{\alpha_i}L_i(l_{i1} + l_{i2}), \\ s_{i21} &= -4(X_{io}/c_{\alpha_i})^2, \\ s_{i20} &= -4(X_{io}/c_{\alpha_i})^2 \end{aligned}$$

Equations (8) and (9) can be expressed as follows:

$$p_{i12}t_{i2}^2 + p_{i11}t_{i2} + p_{i10} = 0 \tag{10a}$$

$$p_{i22}t_{i2}^2 + p_{i21}t_{i2} + p_{i20} = 0 \tag{10b}$$

where,

$$\begin{aligned} p_{i12} &= s_{i15}t_{i1}^2 + s_{i12}, \\ p_{i11} &= s_{i14}t_{i1}^2 + s_{i11}, \\ p_{i10} &= s_{i13}t_{i1}^2 + s_{i10}, \\ p_{i22} &= s_{i29}t_{i1}^6 + s_{i28}t_{i1}^4 + s_{i25}t_{i1}^2 + s_{i21}, \\ p_{i21} &= s_{i26}t_{i1}^4 + s_{i23}t_{i1}^2, \\ p_{i20} &= s_{i27}t_{i1}^6 + s_{i24}t_{i1}^4 + s_{i22}t_{i1}^2 + s_{i20} \end{aligned}$$

Multiplying t_{i2} on both sides of (10a) and (10b) lead to:

$$p_{i12}t_{i2}^3 + p_{i11}t_{i2}^2 + p_{i10}t_{i2} = 0 \tag{11a}$$

$$p_{i22}t_{i2}^3 + p_{i21}t_{i2}^2 + p_{i20}t_{i2} = 0 \tag{11b}$$

Equations (10a), (10b), (11a) and (11b) lead to:

$$\mathbf{D}_i \begin{bmatrix} t_{i2}^3 \\ t_{i2}^2 \\ t_{i2} \\ 1 \end{bmatrix} = 0, \quad \mathbf{D}_i = \begin{bmatrix} p_{i12} & p_{i11} & p_{i10} & 0 \\ p_{i22} & p_{i21} & p_{i20} & 0 \\ 0 & p_{i12} & p_{i11} & p_{i10} \\ 0 & p_{i22} & p_{i21} & p_{i20} \end{bmatrix} \tag{12}$$

The necessary condition for (12) to have non-trivial solutions is:

$$|\mathbf{D}_i| = 0 \tag{13}$$

Equation (13) is a non-linear equation with regard to t_{i1}. t_{i1} can be solved from (13), then t_{i2} can be solved from (10a), β_i and λ_i can be solved by (7) and Z_{io} can be solved from (2) subsequently.

As the lower platform of PM i $(i=2,\ldots,n)$ rotates anticlockwise with its perpendicular by $45\times(-1)^i$ degrees relative to the upper platform of PM $i-1$, it leads to:

$$^{n_{(i-1)1}}_{n_{i0}}\mathbf{R} = \begin{bmatrix} \cos[45^\circ\times(-1)^i] & -\sin[45^\circ\times(-1)^i] & 0 \\ \sin[45^\circ\times(-1)^i] & \cos[45^\circ\times(-1)^i] & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

where $^{n_{(i-1)1}}_{n_{i0}}\mathbf{R}$ is the rotational matrix from $\{n_{i0}\}$ to $\{n_{(i-1)1}\}$.

The centre of the terminal platform $^{n_{10}}o_n$ can be expressed as follows:

$$\begin{aligned} ^{n_0}o_n &= \sum_{i=1}^{n} {}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}o_i \\ ^{n_{10}}_{n_{10}}\mathbf{R} &= \mathbf{E}_{3\times3}, \quad {}^{n_{10}}_{n_{i0}}\mathbf{R} = {}^{n_{10}}_{n_{10}}\mathbf{R}({}^{n_{10}}_{n_{11}}\mathbf{R}\,{}^{n_{11}}_{n_{20}}\mathbf{R}) \\ &\quad\cdots({}^{n_{(i-1)0}}_{n_{(i-1)1}}\mathbf{R}\,{}^{n_{(i-1)1}}_{n_{i0}}\mathbf{R}) \end{aligned} \tag{15}$$

where $\mathbf{E}_{3\times3}$ is a 3×3 form identity matrix.

A composite rotational matrix $^{n_0}_{n_k}\mathbf{R}$ from $\{n_{n1}\}$ to $\{n_{10}\}$ can be expressed as follows:

$$\begin{aligned} ^{n_{10}}_{n_{n1}}\mathbf{R} &= {}^{n_{10}}_{n_{11}}\mathbf{R}({}^{n_{11}}_{n_{20}}\mathbf{R}\,{}^{n_{20}}_{n_{21}}\mathbf{R})\cdots({}^{n_{(i-1)1}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}_{n_{i1}}\mathbf{R}) \\ &\quad\cdots({}^{n_{(k-1)1}}_{n_{n0}}\mathbf{R}\,{}^{k_0}_{n_{n1}}\mathbf{R}) \end{aligned} \tag{16}$$

When given the extension of active limbs r_{ij} $(i=1,2,\ldots,n;\ j=1,2,3,4)$, the position vector $^{n_{10}}o_n$ and $^{n_{i0}}_{n_{i1}}\mathbf{R}$ can be solved from (2b), (7), (10a) and (13), and the position and orientation of terminal platform can be solved from (14), (15) and (16) subsequently.

2.2 Velocity and Acceleration Modelling of the n(4-SPS+SP) S-PM

The objective of the forward velocity and acceleration analysis of the n(4-SPS+SP) S-PM is to determine the velocity and acceleration of the terminal platform from a given set of velocities and accelerations of the actuators in a given pose.

Let $t=[t_x\, t_y\, t_z]^{\mathrm{T}}$, $s=[s_x\, s_y\, s_z]^{\mathrm{T}}$ be two arbitrary vectors, $S(t)$ be a skew-symmetric matrix. There must be:

$$S(t) = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}, \;\; S(t) = -S(t)^{\mathrm{T}}, \;\; t\times s = S(t)s \tag{17}$$

The velocity of 4SPS+SP PM has been solved in Ref. [13] as follows:

$$v_{ri} = \mathbf{J}_i \begin{bmatrix} ^{n_{i0}}v_{oi} \\ ^{n_{i0}}\omega \end{bmatrix}, \begin{bmatrix} ^{n_{i0}}v_{oi} \\ ^{n_{i0}}\omega \end{bmatrix} = \mathbf{J}^{-1}v_{ri}, \quad v_{ri} = \begin{bmatrix} v_{ri1} \\ v_{ri2} \\ v_{ri3} \\ v_{ri4} \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{J}_i = \begin{bmatrix} \delta_{i1}^{\mathrm{T}} & (e_{i1}\times\delta_{i1})^{\mathrm{T}} \\ \delta_{i2}^{\mathrm{T}} & (e_{i2}\times\delta_{i2})^{\mathrm{T}} \\ \delta_{i3}^{\mathrm{T}} & (e_{i3}\times\delta_{i3})^{\mathrm{T}} \\ \delta_{i4}^{\mathrm{T}} & (e_{i4}\times\delta_{i4})^{\mathrm{T}} \\ f_{i1}^{\mathrm{T}} & (d_{i1}\times f_{i1})^{\mathrm{T}} \\ f_{i2}^{\mathrm{T}} & (d_{i2}\times f_{i2})^{\mathrm{T}} \end{bmatrix} \tag{18}$$

$$\delta_{ij} = \frac{B_i - A_i}{|B_i - A_i|}, \quad e_i = B_{ij} - o, \quad d_{ij} = A_{ij} - o_i,$$

$$f_{i1} = x_i, \quad f_{i2} = y_i$$

where $^{n_{i0}}v_{oi}$ and $^{n_{i0}}_{n_{i1}}\omega$ are the linear velocity and angular velocity of upper platform relative to lower platform for PM i, respectively. v_{rij} $(i=1,2,\ldots,n;\ j=1,2,3,4)$ is the velocity of r_{ij} and $\mathbf{J}_i$ is the inverse Jacobian matrix of PM i.

Let $^{n_{10}}_{n_{n1}}\omega$ and $^{n_{10}}v_{on}$ be the angular velocity and linear velocity of terminal platform n_{n1} relative to $\{n_{10}\}$.

$^{n_{10}}_{n_{n1}}\omega$ can be expressed as follows:

$$\begin{aligned} ^{n_{10}}_{n_{n1}}\omega &= \sum_{i=1}^{k} {}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}_{n_{i1}}\omega, \quad {}^{n_{10}}_{n_{i1}}\mathbf{R} = {}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}_{n_{i1}}\mathbf{R}, \\ ^{n_{10}}_{n_{i1}}\mathbf{R} &= {}^{n_{10}}_{n_{11}}\mathbf{R}({}^{n_{11}}_{n_{20}}\mathbf{R}\,{}^{n_{20}}_{n_{21}}\mathbf{R})\cdots({}^{n_{(i-1)1}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}_{n_{i1}}\mathbf{R}) \end{aligned} \tag{19}$$

By differentiating both sides of (15), $^{n_{10}}v_{on}$ can be expressed as follows:

$$^{n_{10}}v_{on} = \sum_{i=1}^{n} \left[{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}v_{oi} + \left({}^{n_{10}}_{n_{i0}}\omega \times {}^{n_{10}}_{n_{i0}}\mathbf{R}\right)^{n_{i0}}o_i\right] \tag{20}$$

The acceleration of i-th 4SPS+SP PM has been solved in Ref. [13] as follows:

$$\begin{aligned} a_{rij} &= \mathbf{J}_i \begin{bmatrix} ^{n_{i0}}a_{oi} \\ ^{n_{i0}}\varepsilon \end{bmatrix} + \begin{bmatrix} ^{n_{i0}}v_{oi} & ^{n_{i0}}\omega \end{bmatrix}^{\mathrm{T}} \mathbf{H} \begin{bmatrix} ^{n_{i0}}v_{oi} \\ ^{n_{i0}}\omega \end{bmatrix}, \begin{bmatrix} ^{n_{i0}}a_{oi} \\ ^{n_{i0}}\varepsilon \end{bmatrix} \\ &= \mathbf{J}_i^{-1}\left(a_{ri} - \begin{bmatrix} ^{n_{i0}}v_{oi} & ^{n_{i0}}\omega \end{bmatrix}^{\mathrm{T}} \mathbf{H} \begin{bmatrix} ^{n_{i0}}v_{oi} \\ ^{n_{i0}}\omega \end{bmatrix}\right) \end{aligned}$$

$$
\begin{aligned}
a_{ri} &= [a_{r1} \quad a_{r2} \quad a_{r3} \quad a_{r4} \quad 0 \quad 0]^{\mathrm{T}}, \\
\mathbf{H} &= [h_1 \quad h_2 \quad h_3 \quad h_4 \quad h_5 \quad h_6] \\
h_i &= \frac{1}{r_i}\begin{bmatrix} -S(\delta_i)^2 & S(\delta_i)^2 S(e_i) \\ -S(e_i)S(\delta_i)^2 & \begin{matrix} r_i S(e_i)S(\delta_i) \\ +S(e_i)S(\delta_i)^2 S(e_i) \end{matrix} \end{bmatrix}_{6\times 6}, \\
i &= 1,2,3,4 \\
h_j &= \begin{bmatrix} 0_{3\times 3} & -S(f_i) \\ S(f_i) & -S(f_i)S(d_i) \end{bmatrix}_{6\times 6}, \quad j=5,6; \quad i=1,2
\end{aligned} \tag{21}
$$

where a_{rij} $(i=1,2,\ldots,n;\ j=1,2,3,4)$ is the acceleration along r_{ij}, and $\mathbf{H}_i$ is a $6\times 6\times 6$ Hessian matrix of PM i.

Let ${}^{n_{10}}_{n_{n0}}\varepsilon$ and ${}^{n_{10}}a_{on}$ be the angular acceleration and linear acceleration of terminal platform relative to $\{n_{10}\}$. By differentiating both sides of (19), ${}^{n_{10}}_{n_{n0}}\varepsilon$ can be expressed as follows:

$$
{}^{n_{10}}_{n_{n1}}\varepsilon = \sum_{i=1}^{n}\left[{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}_{n_{i1}}\varepsilon + \left({}^{n_{10}}_{n_{i0}}\omega \times {}^{n_{10}}_{n_{i0}}\mathbf{R}\right)^{n_{i0}}_{n_{i1}}\omega\right] \tag{22}
$$

By differentiating both sides of (20), ${}^{n_{10}}a_{on}$ can be expressed as follows:

$$
\begin{aligned}
{}^{n_{10}}a_{on} &= \sum_{i=1}^{n}\left\{{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}a_{oi} + 2\left({}^{n_{10}}_{n_{i0}}\omega \times {}^{n_{10}}_{n_{i0}}\mathbf{R}\right)\cdot {}^{n_{i0}}v_{oi}\right. \\
&\qquad \left. + \left[{}^{n_{10}}_{n_{i0}}\varepsilon \times {}^{n_{10}}_{n_{i0}}\mathbf{R} + {}^{n_{10}}_{n_{i0}}\omega \times \left({}^{n_{10}}_{n_{i0}}\omega \times {}^{n_{10}}_{n_{i0}}\mathbf{R}\right)\right]\cdot {}^{n_{i0}}o_i\right\} \\
&= \sum_{i=1}^{n}\left\{{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}a_{oi} - 2S\left({}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}v_{oi}\right)\cdot {}^{n_{10}}_{n_{i0}}\omega\right. \\
&\qquad \left. - S\left({}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}o_i\right){}^{n_{10}}_{n_{i0}}\varepsilon + S\left({}^{n_{10}}_{n_{i0}}\omega\right)^2 {}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}o_i\right\}
\end{aligned} \tag{23}
$$

From the analytical model above, the kinematics of an n (4-SPS+SP) S-PM built with an optional number of 4-SPS+SP PMs assembled in series connection can be solved directly.

3. Statics and Stiffness of the n(4-SPS+SP) S-PM

The objective of statics analysis of the n(4-SPS+SP) S-PM is to determine the active forces applied on actuators when given external force and torque applied on terminal platform in a given pose. The objective of stiffness model is to determine a mapping matrix, which reveals the relation between compliant deformations of the terminal platform and the given external wrench applied on terminal platform.

Let F and T be external force and torque applied on terminal platform n_{n1}, respectively.

Equation (20) leads to:

$$
\begin{aligned}
{}^{n_{10}}v_{on} &= \sum_{i=1}^{n}\left[{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}v_{oi} + \left({}^{n_{10}}_{n_{i0}}\omega \times {}^{n_{10}}_{n_{i0}}\mathbf{R}\right)^{n_{i0}} o_i\right] \\
&= \sum_{i=1}^{n}\left[{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}v_{oi} - S\left({}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}o_i\right){}^{n_{10}}_{n_{i0}}\omega\right] \\
&= \sum_{i=1}^{n}\left[{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}v_{oi} - S\left({}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}o_i\right)\sum_{j=1}^{i-1}{}^{n_{10}}_{n_{j0}}\mathbf{R}\,{}^{n_{j0}}_{n_{j1}}\omega\right] \\
&= \sum_{i=1}^{n}{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}v_{oi} - \sum_{i=2}^{n}S\left({}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}o_i\right)\sum_{j=1}^{i-1}{}^{n_{10}}_{n_{j0}}\mathbf{R}\,{}^{n_{j0}}_{n_{j1}}\omega \\
&= \sum_{i=1}^{n}{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}v_{oi} - \sum_{i=1}^{n-1}S\left({}^{n_{10}}_{n_{(i+1)0}}\mathbf{R}\,{}^{n_{(i+1)0}}o_{i+1}\right) \\
&\qquad \times \sum_{j=1}^{i}{}^{n_{10}}_{n_{j0}}\mathbf{R}\,{}^{n_{j0}}_{n_{j1}}\omega \\
&= \sum_{i=1}^{n}{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}v_{oi} \\
&\qquad -\sum_{j=1}^{n-1}\left\{\left[\sum_{i=j}^{n-1}S\left({}^{n_{10}}_{n_{(i+1)0}}\mathbf{R}\,{}^{n_{(i+1)0}}o_{i+1}\right)\right]{}^{n_{10}}_{n_{j0}}\mathbf{R}\,{}^{n_{j0}}_{n_{j1}}\omega\right\} \\
&= \sum_{i=1}^{n}{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}v_{oi} \\
&\qquad -\sum_{i=1}^{n-1}\left\{\left[\sum_{j=i}^{n-1}S\left({}^{n_{10}}_{n_{(j+1)0}}\mathbf{R}\,{}^{n_{(j+1)0}}o_{j+1}\right)\right]{}^{n_{10}}_{n_{i0}}\mathbf{R}\,{}^{n_{i0}}_{n_{i1}}\omega\right\} \\
&\qquad (n \geq 2)
\end{aligned} \tag{24}
$$

By combining (19) and (24), the velocity of terminal can be expressed as:

$$
\begin{aligned}
&\begin{bmatrix} {}^{n_{10}}v_{on} \\ {}^{n_{10}}_{n_{n1}}\omega \end{bmatrix} = \sum_{i=1}^{n}\mathbf{J}_{Ri}\begin{bmatrix} {}^{n_{i0}}v_{oi} \\ {}^{n_{i0}}_{n_{i1}}\omega \end{bmatrix} (k \geq 2) \\
&\mathbf{J}_{Ri} = \begin{bmatrix} {}^{n_{10}}_{n_{i0}}\mathbf{R} & -\left[\sum_{j=i}^{n-1}S\left({}^{n_{10}}_{n_{(j+1)0}}\mathbf{R}\,{}^{n_{(j+1)0}}o_{j+1}\right)\right]{}^{n_{10}}_{n_{i0}}\mathbf{R} \\ 0_{3\times 3} & {}^{n_{10}}_{n_{i0}}\mathbf{R} \end{bmatrix} (i < n) \\
&\mathbf{J}_{Rn} = \begin{bmatrix} {}^{n_{10}}_{n_{i0}}\mathbf{R} & 0_{3\times 3} \\ 0_{3\times 3} & {}^{n_{10}}_{n_{i0}}\mathbf{R} \end{bmatrix} (i = n)
\end{aligned} \tag{25}
$$

Based on the forces situations of the 4SPS+SP PM [14], it is known that there are four active force along SPS legs and two constrained forces in SP leg. The two constrained forces are exerted on r_{i0} at S joint and parallel with x_i and y_i axes, respectively.

Let ${}^{n_{10}}F$ and ${}^{n_{10}}T$ be the external force and torque applied on terminal platform relative to base $\{n_{10}\}$. Let F_{rij} $(i=1,2,\ldots,n;\ j=1,2,3,4)$ be the active forces and F_{pij} $(i=1,2,\ldots,n;\ j=1,2)$ be the constrained forces of

PM i for n(4-SPS+SP) S-PM. From (25) and the principle of virtual work, we obtain,

$$F_{s1}^{\mathrm{T}} v_{r1} + F_{s2}^{\mathrm{T}} v_{r2} + \cdots + F_{sn}^{\mathrm{T}} v_{rn} = -\begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} {}^{n_{10}}v_{on} \\ {}^{n_{10}}_{n_{n1}}\omega \end{bmatrix}$$

$$= -\begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix}^{\mathrm{T}} \sum_{i=1}^{k} \mathbf{J}_{Ri} \begin{bmatrix} {}^{n_{i0}}v_{oi} \\ {}^{n_{i0}}_{n_{i1}}\omega \end{bmatrix} = -\begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix}^{\mathrm{T}} \sum_{i=1}^{k} \mathbf{J}_{Ri} \mathbf{J}_i^{-1} v_{ri}$$

$$F_{si} = \begin{bmatrix} F_{ri1} & F_{ri2} & F_{ri3} & F_{ri4} & F_{pi1} & F_{pi2} \end{bmatrix}^{\mathrm{T}} \tag{26}$$

where F_{si} $(i=1,2,\ldots,n)$ is the six-dimensional vector formed by active and constrained forces/torques of PM i.

Equation (26) leads to:

$$\begin{bmatrix} F_{s1}^{\mathrm{T}} & \cdots & F_{sn}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} v_{r1} \\ \vdots \\ v_{rn} \end{bmatrix} = -\begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix}^{\mathrm{T}}$$

$$\times \begin{bmatrix} \mathbf{J}_{R1}\mathbf{J}_1^{-1} & \cdots & \mathbf{J}_{Rn}\mathbf{J}_n^{-1} \end{bmatrix} \begin{bmatrix} v_{r1} \\ \vdots \\ v_{rn} \end{bmatrix} \tag{27a}$$

Equation (27a) leads to:

$$F_{si} = -(\mathbf{J}_{Ri}\mathbf{J}_i^{-1})^{\mathrm{T}} \begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix} \tag{27b}$$

From (27b), F_{rij} and F_{pij} $(i=1,2,\ldots,n;\ j=1,2,3,4)$ can be solved when ${}^{n_{10}}F$ and ${}^{n_{10}}T$ are given.

For SPS legs, the active force F_{rij} $(i=1,2,\ldots,n;$ $j=1,2,3,4)$ produce tension deformation along r_i. Let δd_{rij} be the tension deformation along r_{ij} due to the active forces F_{rij}, then,

$$F_{rij} = k_{rij}\delta d_{rij}, \quad k_{rij} = \frac{ES_{ij}}{r_{ij}} \tag{28}$$

where E is the modular of elasticity and S_{ij} is the j-th SPS leg's cross section of PM i.

For SP legs, the constrained forces F_{pij} $(i=1,2,\ldots,n;$ $j=1,2)$ at S joint in r_{i0} can be equivalent to one force F_{tij} at o_i which is parallel with F_{pij} and one torque T_{tij} which is perpendicular with F_{pij} and r_{i0}. It leads to:

$$F_{tij} = F_{pij} \tag{29a}$$

$$\begin{aligned} T_{ti1} &= r_{i0} \times F_{pi1} = F_{pi1} r_{i0}(z_i \times x_i) = F_{pi1} r_{i0} y_i, \\ T_{ti1} &= F_{pi1} r_{i0} \\ T_{s2} &= r_{i0} \times F_{pi2} = F_{pi2} r_{i0}(z_i \times y_i) = -F_{pi2} r_{i0} x_i, \\ T_{s2} &= -F_{pi2} r_{i0} \end{aligned} \tag{29b}$$

F_{tij} produces flexibility deformation and T_{tij} produces bending deformation in r_{i0}. Let δd_{tij} be the flexibility deformation of r_{i0} due to F_{tij}. Let $\delta\theta_{tij}$ be the bending deformation of r_{i0} due to the constrained torques T_{pij}.

F_{tij} produces flexibility deformation δd_{tij}, thus we obtain:

$$F_{tij} = k_{tij}\delta d_{tij}, \quad k_{tij} = \frac{3EI}{r_{ij}^3} \tag{30a}$$

where I is the moment inertia.

T_{tij} produces bending deformation $\delta\theta_{tij}$, thus we obtain:

$$T_{tij} = s_{tij}\delta\theta_{tij}, \quad s_{tij} = \frac{EI}{r_{ij}} \tag{30b}$$

Equations (29a) and (29b) lead to:

$$F_{mi} = \mathbf{G}_i F_{si}, \quad F_{mi} = \begin{bmatrix} F_{ri1} \\ F_{ri2} \\ F_{ri3} \\ F_{ri4} \\ F_{ti1} \\ F_{ti2} \\ T_{ti1} \\ T_{ti2} \end{bmatrix},$$

$$\mathbf{G}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & r_{i0} & 0 \\ 0 & 0 & 0 & 0 & 0 & -r_{i0} \end{bmatrix} \tag{31}$$

Equations (29a) and (29b) lead to:

$$\begin{aligned} F_{mi} &= \mathbf{K}_{mi}\delta d_{mi}, \quad \delta d_{mi} = \mathbf{C}_{mi} F_{mi}, \quad \mathbf{C}_{mi} = \mathbf{K}_{mi}^{-1} \\ \delta d_{mi} &= \begin{bmatrix} d_{ri1} & d_{ri2} & d_{ri3} & d_{ri4} & d_{ti1} & d_{ti2} & \theta_{ti1} & \theta_{ti2} \end{bmatrix} \\ \mathbf{K}_{mi} &= \mathrm{diag}\begin{bmatrix} k_{ri1} & k_{ri2} & k_{ri3} & k_{ri4} & k_{ti1} & k_{ti2} & s_{ti1} & s_{ti2} \end{bmatrix} \end{aligned} \tag{32}$$

Equations (27b) and (31) lead to:

$$F_{mi}^{\mathrm{T}} = F_{si}^{\mathrm{T}}\mathbf{G}_i^{\mathrm{T}} = \begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix}^{\mathrm{T}} (\mathbf{J}_{Ri}\mathbf{J}_i^{-1})\mathbf{G}_i^{\mathrm{T}} \tag{33}$$

Based on the principle of virtual work and (33),

$$F_{m1}^{\mathrm{T}}\delta d_{m1} + F_{m2}^{\mathrm{T}}\delta d_{m2} + \cdots + F_{mn}^{\mathrm{T}}\delta d_{mn} = -\begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix}^{\mathrm{T}}$$

$$\delta\rho = -\begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix}^{\mathrm{T}} \sum_{i=1}^{n} (\mathbf{J}_{Ri}\mathbf{J}_i^{-1})\mathbf{G}_i^{\mathrm{T}}\delta d_{mi} \tag{34}$$

where $\delta\rho$ is the deformation of terminal platform. Equation (34) leads to:

$$\delta\rho = \sum_{i=1}^{n} (\mathbf{J}_{Ri}\mathbf{J}_i^{-1})\mathbf{G}_i^{\mathrm{T}}\delta d_{mi} \tag{35}$$

Equations (31), (32) and (35) lead to:

$$\delta d_{mi} = \mathbf{C}_{mi}\mathbf{G}_i F_{si} = -\mathbf{C}_{mi}\mathbf{G}_i(\mathbf{J}_{Ri}\mathbf{J}_i^{-1})^{\mathrm{T}} \begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix}$$

$$\delta\rho = -\sum_{i=1}^{n} (\mathbf{J}_{Ri}\mathbf{J}_i^{-1})\mathbf{G}_i^{\mathrm{T}}\mathbf{C}_{mi}\mathbf{G}_i(\mathbf{J}_{Ri}\mathbf{J}_i^{-1})^{\mathrm{T}} \begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix} \tag{36}$$

Equations (35) and (36) lead to:

$$\begin{bmatrix} {}^{n_{10}}F \\ {}^{n_{10}}T \end{bmatrix} = \mathbf{K}\delta\rho,$$

$$\mathbf{K} = -\left[\sum_{i=1}^{n} (\mathbf{J}_{Ri}\mathbf{J}_i^{-1})\mathbf{G}_i^{\mathrm{T}}\mathbf{C}_{mi}\mathbf{G}_i(\mathbf{J}_{Ri}\mathbf{J}_i^{-1})^{\mathrm{T}}\right]^{-1} \tag{37}$$

Here $\mathbf{K}$ is a 6×6 stiffness matrix of n(4-SPS+SP) S-PM. The elastic deformation can be solved from (36) when external force and torque applied on terminal platform are given. The stiffness matrix $\mathbf{K}$ of the n(4-SPS+SP) S-PM can be solved from (37).

For this n(4-SPS+SP) S-PM, the active forces produce $4n$ longitudinal deformations along SPS active legs, while the constrained force produces $2n$ flexibility deformation and $2n$ bending deformation in SP legs. The $8n$ deformations that are present in SPS and SP legs affect the deformation and stiffness.

4. Analytically Solved Example

In this part, the 3(4-SPS+SP) S-PM is analysed to illustrate the kinematics, statics and stiffness model.

Figure 3 is a 3(4-SPS+SP) S-PM formed by three 4-SPS+SP PMs connected in series. Set $E = 2.11\times 10^{11}$ Pa, $EI = 26{,}502\,\mathrm{N\,m^2}$, $S_i = 0.0013\,\mathrm{m^2}$, $G = 80\times 10^9$ Pa, $I_p = 2.5120\times 10^{-7}\,\mathrm{m^4}$. The dimension parameters of the serial-PM and the length, velocity and acceleration of r_{ij} ($i = 1,2,3$; $j = 1,2,3,4$) are given in Table 1.

The pose of PM i ($i = 1,2,3,4$) is solved as follows.

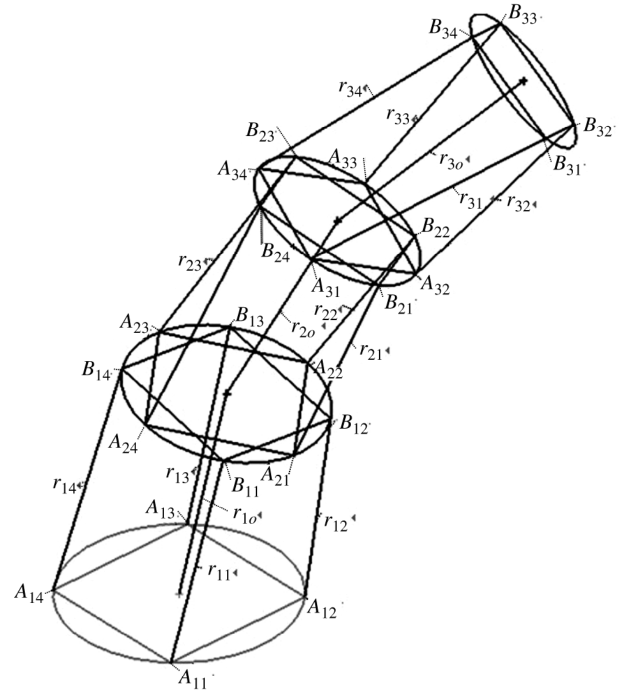

Figure 3. Sketch of 3(4SPS+SP) serial-PM.

The results show that the single 4-SPS+SP PM have 16 solutions (Table 2). From (7), (10a) and (2b), the corresponding pose parameters can be obtained. To determine the acceptable analytic solutions from multisolutions, the simulation mechanism is created [15] using CAD software and the forward kinematics can be solved by the simulation mechanism. By comparing, it is known that the simulation solution coincides with the first solution solved by the analytic method for PM i. Using this solution, the reasonable pose of the upper platform PM i is solved in Table 3.

The position, velocity and acceleration of terminal platform of 3(4-SPS+SP) S-PM are solved in Table 4.

When the workloads applied at o_k is given as ${}^{n_0}F_o = (-20\ -30\ -60)^{\mathrm{T}}N$, ${}^{n_0}T_o = (0\ 0\ 0)^{\mathrm{T}}\,\mathrm{N\,m}$, the active forces and constrained forces in r_{ij} and r_{oi} for PM i can be solved in Table 5.

The tension, flexibility, bending deformations of r_{ij} can be solved in Table 6.

The deformation of the terminal platform of 3(4-SPS+SP) S-PM is derived as:

$$\delta\rho = (0.0031\ -0.0002\ 0.0063\ -0.0101\ -0.0241\ -0.0164)^{\mathrm{T}}$$

The stiffness matrix of 3(4-SPS+SP) S-PM is derived as follows:

$$\mathbf{K} = 10^4 \begin{bmatrix} -0.5420 & -0.1218 & -0.7521 & -0.0512 & -0.1134 & -0.0659 \\ -0.1218 & -0.6972 & -0.4363 & 0.6850 & -0.5036 & 0.3216 \\ -0.7521 & -0.4363 & -2.0434 & -0.5091 & -0.0497 & -0.1609 \\ -0.0512 & 0.6850 & -0.5091 & -3.3027 & 2.0427 & -1.1804 \\ -0.1134 & -0.5036 & -0.0497 & 2.0427 & -1.4824 & 0.8866 \\ -0.0659 & 0.3216 & -0.1609 & -1.1804 & 0.8866 & -0.6532 \end{bmatrix}$$

Table 1
The Dimension Parameters, Length, Velocity and Acceleration of r_{ij}

	L (m)	l_1 (m)	l_2 (m)	r_1 (m)	r_2 (m)	r_3 (m)	r_4 (m)	v_{r1} (m/s)	v_{r2} (m/s)	v_{r3} (m/s)	v_{r4} (m/s)	a_{r1} (m/s^2)	a_{r2} (m/s^2)	a_{r3} (m/s^2)	a_{r4} (m/s^2)
PM 1	0.5	0.45	0.4	0.70	0.66	0.76	0.80	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
PM 2	0.4257	0.4	0.35	0.62	0.58	0.70	0.75	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
PM 3	0.3758	0.35	0.3	0.75	0.61	0.65	0.78	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

Table 2
The 16 Solutions of t_{11}, t_{21} and t_{31}

t_{11}	0.10928	−0.10928	0.2222	−0.2222
	6.95	−6.95	7.1614	−7.1614
	$0.079-0.218i$	$-0.079+0.218i$	$-0.73266i$	$0.733i$
	$-1.236i$	$1.236i$	$0.079+0.218i$	$-0.079-0.218i$
t_{21}	0.16182	−0.16182	0.2718	−0.272
	4.8737	−4.8737	4.9752	−4.975
	$0.064-0.253i$	$0.064+0.253i$	$-0.754i$	$0.754i$
	$-1.217i$	$1.2168i$	$0.064+0.253i$	$-0.064-0.253i$
T_{31}	0.19711	−0.19711	0.25784	−0.258
	4.3722	−4.3722	4.5613	−4.561
	$0.273-0.069i$	$-0.273+0.069i$	$-0.849i$	$0.849i$
	$-1.1236i$	$1.1236i$	$0.273+0.069i$	$-0.273-0.069i$

Table 3
The Pose of Single 4-SPS+SP PM

	α (°)	β (°)	λ (°)	X_o (m)	Y_o (m)	Z_o (m)
PM 1	21.8014	12.4736	−22.8961	0.1460	0.0584	0.7108
PM 2	19.9651	18.3838	−35.4613	0.1948	0.0708	0.6236
PM 3	75.6454	22.3011	−40.2939	0.0641	0.2503	0.6300

Table 4
The Position, Velocity and Acceleration of Terminal Platform of 3(4-SPS+SP) S-PM

${}^{n0}o_k$ (m)	${}^{n0}v_k$ (m/s)	${}^{n0}\ \omega_k$ (rad/s)	${}^{n0}a_k$ (m/s^2)	${}^{n0}\varepsilon_k$ (rad/s^2)
0.6696	0.1060	0.0296	0.1086	0.0313
0.7198	0.0988	0.0438	0.0968	0.0439
1.7168	0.2460	0.0225	0.2457	0.0231

Table 5
Active Forces and Constrained Forces in r_{ij} and r_{oi}

	F_{a1}	F_{a2}	F_{a3}	F_{r4}	F_{p1}	F_{p2}
PM 1	122.7	−77.592	94.002	−73.932	7.8868	21.363
PM 2	128.99	−62.19	95.754	−93.604	3.4628	8.8176
PM 3	−298.21	330.93	−226.3	257.91	0.90207	−30.954

Table 6
Deformations in r_{ij} and r_{oi}

	δr_1 (10^{-7} m)	δr_2 (10^{-7} m)	δr_3 (10^{-7} m)	δr_4 (10^{-7} m)	δd_1 (10^{-5} m)	δd_2 (10^{-3} m)	$\delta\theta_1$ (10^{-4} rad)	$\delta\theta_2$ (10^{-3} rad)
PM 1	12.96	−7.73	10.78	−8.93	61.25	16.59	25.2	6.84
PM 2	12.07	−5.44	10.12	−10.60	19.78	50.36	9.03	2.30
PM 3	−3.38	3.05	−22.20	−40.36	−5.73	1.97	2.52	8.67

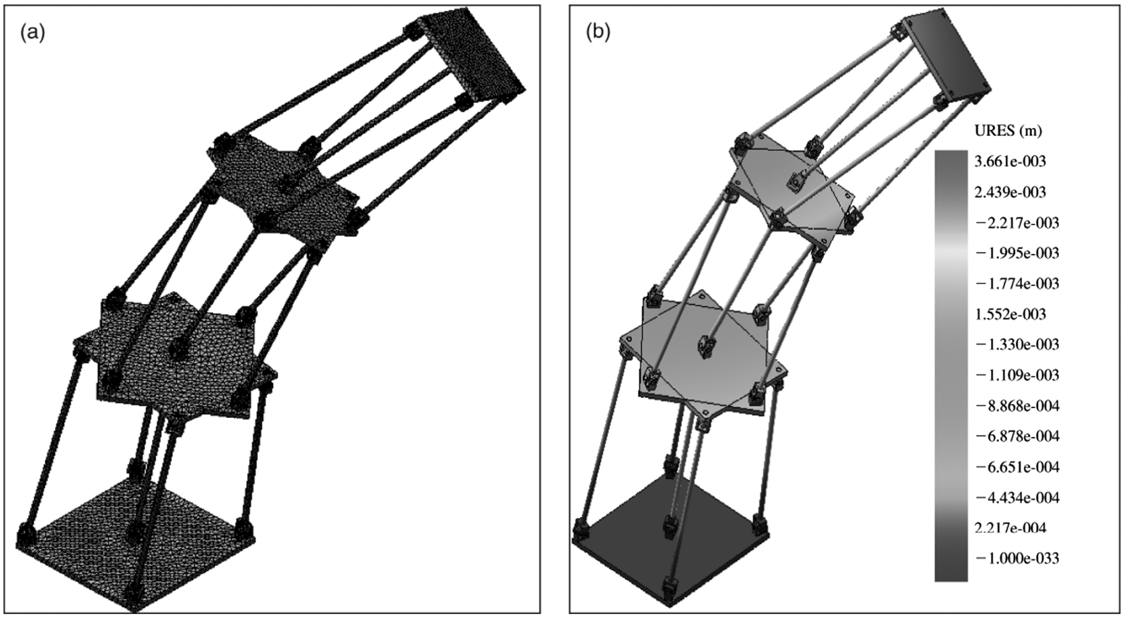

Figure 4. Solved results of elastic deformations of FE model of 3(4-SPS+SP) S-PM: (a) the created FE model of 3(4-SPS+SP) S-PM and (b) the simulated results of 3(4-SPS+SP) S-PM.

The results show that the constrained forces produce large deformations compared with the active forces. The constrained forces have a great influence on the elastic deformation and stiffness of the 3(4-SPS+SP) S-PM. When establishing the stiffness matrix and solving the elastic deformation of this S-PM, the constrained forces must be considered based on its geometric constraints. The stiffness matrix is a symmetrical 6×6 form matrix.

In this section, a 3D assembly mechanism and a finite-element (FE) model of 3(4-SPS+SP) S-PM is established in FE software (see Fig. 4(a)) according to dimension and material parameters described above.

In the FE model, the spherical joint is replaced by three revolute joints. The linear active leg with prismatic joint is formed using the elastic linear rod. The simulated results based on FE model for the deformation of the terminal platform are solved as shown in Fig. 4(b) and Table 7.

It is well known that the FE analysis is a numerical technique for finding approximate solutions. The solved results of FE model built in FE software depend on some key factors such as material parameter, FE dimension and type, reasonable boundary constraints and connection constraints. The results in Table 7 shows that the elastic deformation of FE model of 3(4-SPS+SP) S-PM is basically coincident with that of analytical ones in this section.

Table 7
The Simulated Result Based on FE Model and the Calculated Result Based on Stiffness Model for the Deformation of Terminal Platform

Elastic Deformation of o (mm)		
	FE Model	Analytics
δx	−5.63	−3.1
δy	−2.225	−2
δz	−7.0141	−6.3

5. Conclusion

The kinematics, statics and stiffness of a novel n(4-SPS+SP) S-PM is analysed in this paper. This S-PM has n 4-SPS+SP PM. The forward displacement of 4-SPS+SP PM is derived in close form, and the result shows that the 4-SPS+SP PM has 16 solutions forward displacements. The kinematics of n(4-SPS+SP) serial-PM is solved based on the kinematical result of 4-SPS+SP PM.

The formula for solving the statics of n(4-SPS+SP) S-PM is derived. From this formula, the active and constrained forces in SPS active and SP constrained legs are solved, respectively. The deformations in SPS and SP legs are analysed in detail with both active and constrained forces being considered. Based on the principle of virtual work, the deformation of the terminal platform of n(4-SPS+SP) and the 6×6 stiffness matrix are derived.

The kinematics, statics, elastic deformations and stiffness of 3(4-SPS+SP) are computed. The analytical results are verified by CAD software. The simulation solution of the elastic deformations of the 3(4-SPS+SP) manipulator from the FE model is basically coincident with its analytical solutions. In future work, the dynamics model of this S-PM will be established and the method proposed in this paper will be used for other interesting S-PMs.

Acknowledgements

The authors would like to acknowledge Project (51175447) supported by the National Natural Sciences Foundation of China (NSFC), the Key planned project of Hebei application foundation (11962127D), the research and development project of science and technology of Qinhuangdao city (201101A140).

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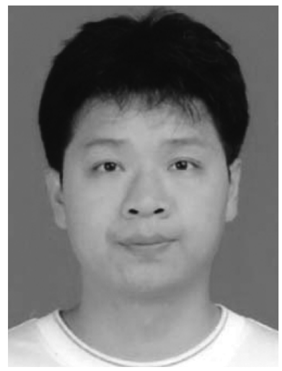
Biographies



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