ME8135 - State Estimation - Assignment 1

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May 2023

1 Question 1

$\mathbf{x}$ is a random variable of length K:

$$\mathbf{x} = \mathcal{N}(\mathbf{0}, \mathbf{1}) \tag{1}$$

a) What type of random variable is the following random variable?

$$\mathbf{y} = \mathbf{x}^\top \mathbf{x} \tag{2}$$

$\mathbf{y}$ is a continuous random variable.

b) Calculate the mean and variance of $\mathbf{y}$.

First a few notes as helpers:

$$\mathbf{y} = \mathbf{x}^\top \mathbf{x} = \sum_i^K x_i^2$$

and

$$E[x_i x_i] = 1, E[x_i x_j] = 0 \text{ when } i \neq j$$

by the trivial case of (2.41), $E[\mathbf{x}\mathbf{x}^\top] = \mathbf{\Sigma}$.

The mean is:

$$\begin{aligned}
\mu = E[\mathbf{y}] &= E[\sum_i^K x_i^2] \\
&= \sum_i^K E[x_i^2] \\
&= \sum_i^K 1 \\
&= K
\end{aligned}$$

The variance is:

$$\begin{aligned}
\mathrm{Var}(\mathbf{y}) &= E[(\mathbf{y}-\mu)^2] = E[(\mathbf{y}-K)^2] \\
&= E[\mathbf{y}^2 - 2\mathbf{y}K + K^2] \\
&= E[\mathbf{y}^2] - 2KE[\mathbf{y}] + K^2 \\
&= E[\mathbf{y}^2] - K^2 \\
&= E[(\sum_i^K x_i^2)^2] - K^2 \\
&= E[\sum_i^K x_i^4 + \sum_{i=1}^K \sum_{j\neq i}^K x_i^2 x_j^2] - K^2 \\
&= \sum_i^K E[x_i x_i x_i x_i] + \sum_{i=1}^K \sum_{j\neq i}^K E[x_i x_i x_j x_j] - K^2 \\
&= \sum_i^K 3 + \sum_{i=1}^K \sum_{j\neq i}^K 1 - K^2 \\
&= 3K + K(K-1) - K^2 \\
&= 3K - K + K^2 - K^2 \\
&= 2K
\end{aligned}$$

where the expectations are found using (2.40) and the earlier helper note to obtain:

$$E[x_i x_i x_i x_i] = 3E[x_i x_i]E[x_i x_i] = 3$$

and

$$\begin{aligned}
E[x_i x_i x_j x_j] &= E[x_i x_i]E[x_j x_j] + E[x_i x_j]E[x_i x_j] + E[x_i x_j]E[x_i x_j] \\
&= 1 + 0 + 0 = 1
\end{aligned}$$

c) Using Python, plot the PDF of $\mathbf{y}$ for $K = 1, 2, 3, 10, 100$.

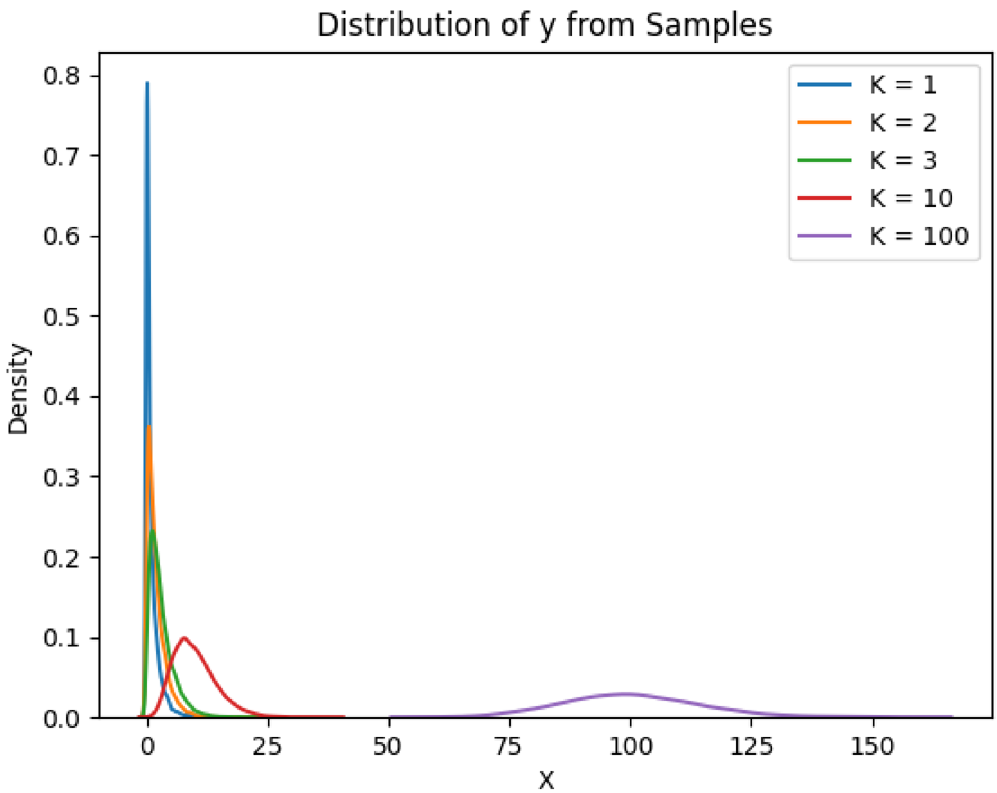


Figure 1: Sampled plot of the PDF of $\mathbf{y}$ for K = 1, 2, 3, 10, 100

2 Question 2

$\mathbf{x}$ is a random variable of length N:

$$\mathbf{x} = \mathcal{N}(\mu, \mathbf{\Sigma}) \tag{3}$$

a) Assume $\mathbf{x}$ is transformed linearly, i.e. $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $N \times N$ matrix. Calculate the mean and covariance of $\mathbf{y}$. Show the derivations.

   In this case of matrix $\mathbf{A}$ we can apply the expectation operator directly as is done in (2.58).

$$\mu_y = E[\mathbf{y}] = E[\mathbf{A}\mathbf{x}] = \mathbf{A}E[\mathbf{x}] = \mathbf{A}\mu$$
$$\mathbf{\Sigma}_{yy} = E[(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)^\top] = \mathbf{A}E[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^\top]\mathbf{A}^\top = \mathbf{A}\mathbf{\Sigma}\mathbf{A}^\top$$

b) Repeat a), when $\mathbf{y} = (\mathbf{A_1} + \mathbf{A_2})$

$$\mathbf{A_1}\mathbf{x} + \mathbf{A_2}\mathbf{x} = (\mathbf{A_1} + \mathbf{A_2})\mathbf{x} = \mathbf{A}\mathbf{x}$$

   Now substitute into a).

$$\mu_y = \mathbf{A}\mu = (\mathbf{A_1} + \mathbf{A_2})\mu$$
$$\mathbf{\Sigma}_{yy} = \mathbf{A}\mathbf{\Sigma}\mathbf{A}^\top = (\mathbf{A_1} + \mathbf{A_2})\mathbf{\Sigma}(\mathbf{A_1} + \mathbf{A_2})^\top$$

c) If $\mathbf{x}$ is transformed by a nonlinear differentiable function, i.e. $\mathbf{y} = f(\mathbf{x})$, compute the covariance matrix of $\mathbf{y}$. Show the derivation.

The Jacobian, $\mathbf{J}$, can be used in place of a linear transform in order to use a nonlinear differentiable function. It can be substituted into the covariance derivation from a).

$$\mathbf{\Sigma}_{yy} = E[(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)^\top] = \mathbf{J}E[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^\top]\mathbf{J}^\top = \mathbf{J\Sigma J}^\top$$

d) Apply c) when

$$\mathbf{x} = \begin{bmatrix} \rho \\ \theta \end{bmatrix}, \mathbf{\Sigma} = \begin{bmatrix} \sigma^2_{\rho\rho} & \sigma^2_{\rho\theta} \\ \sigma^2_{\rho\theta} & \sigma^2_{\theta\theta} \end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix} \tag{4}$$

Compute the covariance of y analytically. This models how range-bearing measurements in the polar coordinate frame are converted to a Cartesian coordinate frame.

For brevity we will replace each term with a variable then compute the covariance. The terms can be later substituted back.

$$\begin{aligned}
\mathbf{\Sigma}_{yy} &= \mathbf{J\Sigma J}^\top \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial\rho} & \frac{\partial f_1}{\partial\theta} \\ \frac{\partial f_2}{\partial\rho} & \frac{\partial f_2}{\partial\theta} \end{bmatrix} \begin{bmatrix} \sigma^2_{\rho\rho} & \sigma^2_{\rho\theta} \\ \sigma^2_{\rho\theta} & \sigma^2_{\theta\theta} \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial\rho} & \frac{\partial f_1}{\partial\theta} \\ \frac{\partial f_2}{\partial\rho} & \frac{\partial f_2}{\partial\theta} \end{bmatrix}^\top \\
&= \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix} \begin{bmatrix} \sigma^2_{\rho\rho} & \sigma^2_{\rho\theta} \\ \sigma^2_{\rho\theta} & \sigma^2_{\theta\theta} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \\
&= \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \\
&= \begin{bmatrix} (ae+bg)a+(af+bh)b & (ae+bg)c+(af+bh)d \\ (ce+dg)a+(cf+dh)b & (ce+dg)c+(cf+dh)d \end{bmatrix}
\end{aligned}$$

Substituting the terms the covariance terms we obtain the following which are left unsimplified for clarity:

$$\begin{aligned}
(ae+bg)a+(af+bh)b &= (\sigma^2_{\rho\rho}\cos\theta + \sigma^2_{\rho\theta}\sin\theta)\cos\theta + (\sigma^2_{\rho\theta}\cos\theta - \sigma^2_{\theta\theta}\sin\theta)(-\rho\sin\theta) \\
(ae+bg)c+(af+bh)d &= (\sigma^2_{\rho\rho}\cos\theta + \sigma^2_{\rho\theta}\sin\theta)\sin\theta + (\sigma^2_{\rho\theta}\cos\theta - \sigma^2_{\theta\theta}\sin\theta)(\rho\cos\theta) \\
(ce+dg)a+(cf+dh)b &= (\sigma^2_{\rho\rho}\sin\theta + \sigma^2_{\rho\theta}\cos\theta)\cos\theta + (\sigma^2_{\rho\theta}\sin\theta + \sigma^2_{\theta\theta}\cos\theta)(-\rho\sin\theta) \\
(ce+dg)c+(cf+dh)d &= (\sigma^2_{\rho\rho}\sin\theta + \sigma^2_{\rho\theta}\cos\theta)\sin\theta + (\sigma^2_{\rho\theta}\sin\theta + \sigma^2_{\theta\theta}\cos\theta)(\rho\cos\theta)
\end{aligned}$$

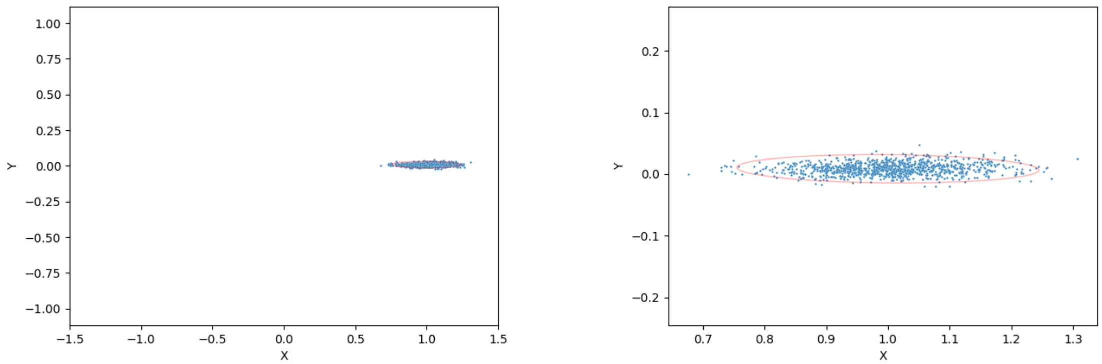
e) Simulate d) using the Monte Carlo simulation, i.e. assume

$$\mathbf{x} = \begin{bmatrix} 1m \\ 0.5^\circ \end{bmatrix}, \mathbf{\Sigma} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix} \tag{5}$$

Sample 1000 points from this distribution and plot the transformed results on x-y coordinates. Plot the uncertainty ellipse, calculated from part d). Overlay the ellipse on the point samples.

I assume the given $\mathbf{x}$ here as the mean, μ, and apply it to equation 3. I also assume the given variance of θ is in degrees. The result of the latter is a clearer contrast between the inputs.

The uncertainty ellipse represents a 95% confidence. The scale of the axes are equal in their respective plots.



(a) Origin centered.

(b) Mean centered.

Figure 2: Plots of the transformed distribution.