

ME8135 - State Estimation - Assignment 1

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1 Question 1

\mathbf{x} is a random variable of length K :

$$\mathbf{x} = \mathcal{N}(\mathbf{0}, \mathbf{1}) \quad (1)$$

a) What type of random variable is the following random variable?

$$\mathbf{y} = \mathbf{x}^\top \mathbf{x} \quad (2)$$

\mathbf{y} is a continuous random variable.

b) Calculate the mean and variance of \mathbf{y} .

First a few notes as helpers:

$$\mathbf{y} = \sum_i^K x_i^2 = (x_1^2 + \cdots + x_K^2)$$

and

$$E[x_i x_i] = 1, E[x_i x_j] = 0 \text{ when } i \neq j$$

by the trivial case of (2.41), $E[\mathbf{x}\mathbf{x}^\top] = \mathbf{\Sigma}$.

The mean is:

$$\begin{aligned} \mu &= E[\mathbf{y}] = E\left[\sum_i^K x_i^2\right] \\ &= E[x_1 x_1] + \cdots + E[x_K x_K] \\ &= K \end{aligned}$$

The variance is:

$$\begin{aligned}
\text{Var}(\mathbf{y}) &= E[(\mathbf{y} - \mu)^2] = E[(\mathbf{y} - K)^2] \\
&= E[\mathbf{y}^2 - 2\mathbf{y}K + K^2] \\
&= E[\mathbf{y}^2] - 2KE[\mathbf{y}] + K^2 \\
&= E[\mathbf{y}^2] - K^2 \\
&= E\left[\left(\sum_i^K x_i^2\right)^2\right] - K^2 \\
&= E\left[\sum_i^K x_i^4 + \sum_{i=1}^K \sum_{j \neq i}^K x_i^2 x_j^2\right] - K^2 \\
&= \sum_i^K E[x_i x_i x_i x_i] + \sum_{i=1}^K \sum_{j \neq i}^K E[x_i x_i x_j x_j] - K^2 \\
&= \sum_i^K 3 + \sum_{i=1}^K \sum_{j \neq i}^K 1 - K^2 \\
&= 3K + K(K-1) - K^2 \\
&= 3K - K + K^2 - K^2 \\
&= 2K
\end{aligned}$$

where the expectations are found using (2.40) and the earlier helper note to obtain:

$$E[x_i x_i x_i x_i] = 3E[x_i x_i]E[x_i x_i] = 3$$

and

$$\begin{aligned}
E[x_i x_i x_j x_j] &= E[x_i x_i]E[x_j x_j] + E[x_i x_j]E[x_i x_j] + E[x_i x_j]E[x_i x_j] \\
&= 1 + 0 + 0 = 1
\end{aligned}$$

c) Using Python, plot the PDF of \mathbf{y} for $K = 1, 2, 3, 10, 100$.

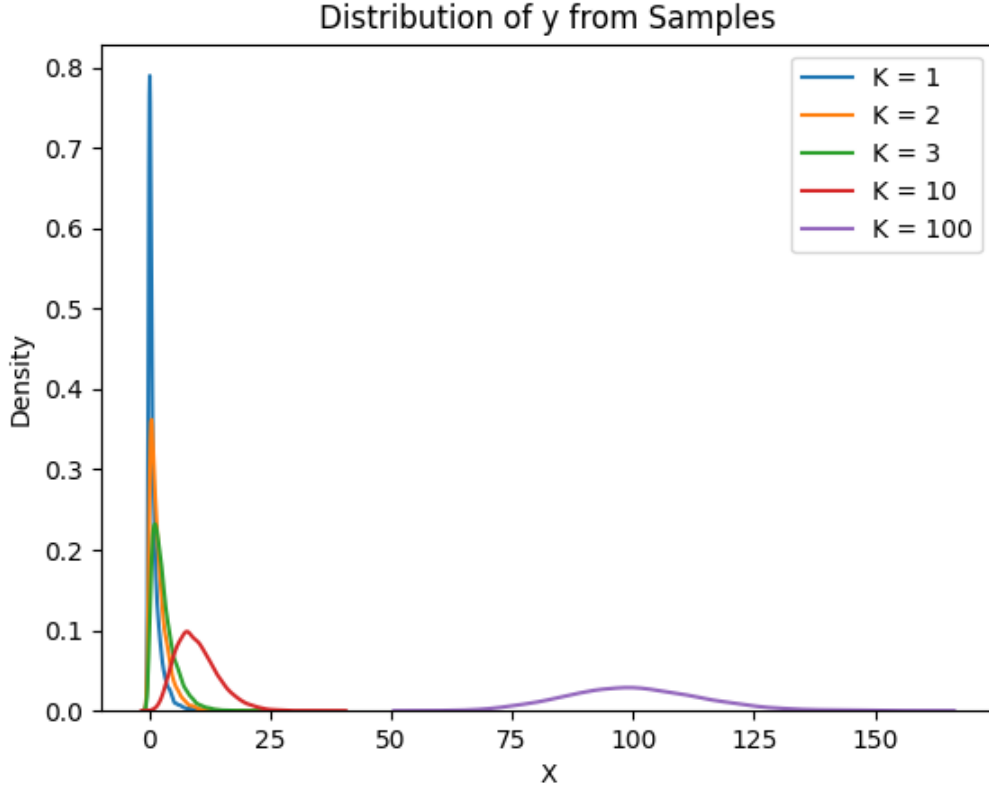


Figure 1: Sampled plot of the PDF of \mathbf{y} for $K = 1, 2, 3, 10, 100$

2 Question 2

\mathbf{x} is a random variable of length N :

$$\mathbf{x} = \mathcal{N}(\mu, \Sigma) \quad (3)$$

- a) Assume \mathbf{x} is transformed linearly, i.e. $\mathbf{y} = \mathbf{A}\mathbf{x}$, where \mathbf{A} is an $N \times N$ matrix. Calculate the mean and covariance of \mathbf{y} . Show the derivations.

I take the question to mean the covariance matrix. In this case of matrix \mathbf{A} we can apply the expectation operator directly as is done in (2.58).

$$\begin{aligned} \mu_y &= E[\mathbf{y}] = E[\mathbf{A}\mathbf{x}] = \mathbf{A}E[\mathbf{x}] = \mathbf{A}\mu \\ \Sigma_{yy} &= E[(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)^\top] = \mathbf{A}E[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^\top]\mathbf{A}^\top = \mathbf{A}\Sigma\mathbf{A}^\top \end{aligned}$$

- b) Repeat a), when $\mathbf{y} = (\mathbf{A}_1 + \mathbf{A}_2)\mathbf{x}$

$$\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{x} = (\mathbf{A}_1 + \mathbf{A}_2)\mathbf{x} = \mathbf{A}\mathbf{x}$$

Now substitute into a).

$$\mu_y = \mathbf{A}\mu = (\mathbf{A}_1 + \mathbf{A}_2)\mu$$

$$\Sigma_{yy} = \mathbf{A}\Sigma\mathbf{A}^\top = (\mathbf{A}_1 + \mathbf{A}_2)\Sigma(\mathbf{A}_1 + \mathbf{A}_2)^\top$$

- c) If \mathbf{x} is transformed by a nonlinear differentiable function, i.e. $\mathbf{y} = f(\mathbf{x})$, compute the covariance matrix of \mathbf{y} . Show the derivation.

The Jacobian, \mathbf{J} , can be used in place of a linear transform in order to use a nonlinear differentiable function. It can be substituted into the covariance derivation from a).

$$\Sigma_{yy} = E[(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)^\top] = \mathbf{J}E[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^\top]\mathbf{J}^\top = \mathbf{J}\Sigma\mathbf{J}^\top$$

- d) Apply c) when

$$\mathbf{x} = \begin{bmatrix} \rho \\ \theta \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_{\rho\rho}^2 & \sigma_{\rho\theta}^2 \\ \sigma_{\rho\theta}^2 & \sigma_{\theta\theta}^2 \end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix} \rho \cos \theta \\ \rho \sin \theta \end{bmatrix} \quad (4)$$

Compute the covariance of \mathbf{y} analytically. This models how range-bearing measurements in the polar coordinate frame are converted to a Cartesian coordinate frame.

For brevity we will replace each term with a variable then compute the covariance. The terms can be later substituted back.

$$\begin{aligned} \Sigma_{yy} &= \mathbf{J}\Sigma\mathbf{J}^\top \\ &= \begin{bmatrix} \frac{\partial f_1}{\partial \rho} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial \rho} & \frac{\partial f_2}{\partial \theta} \end{bmatrix} \begin{bmatrix} \sigma_{\rho\rho}^2 & \sigma_{\rho\theta}^2 \\ \sigma_{\rho\theta}^2 & \sigma_{\theta\theta}^2 \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial \rho} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial \rho} & \frac{\partial f_2}{\partial \theta} \end{bmatrix}^\top \\ &= \begin{bmatrix} \cos \theta & -\rho \sin \theta \\ \sin \theta & \rho \cos \theta \end{bmatrix} \begin{bmatrix} \sigma_{\rho\rho}^2 & \sigma_{\rho\theta}^2 \\ \sigma_{\rho\theta}^2 & \sigma_{\theta\theta}^2 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\rho \sin \theta & \rho \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \\ &= \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \\ &= \begin{bmatrix} (ae + bg)a + (af + bh)b & (ae + bg)c + (af + bh)d \\ (ce + dg)a + (cf + dh)b & (ce + dg)c + (cf + dh)d \end{bmatrix} \end{aligned}$$

Where the covariance terms are as follows and are left unsimplified for clarity:

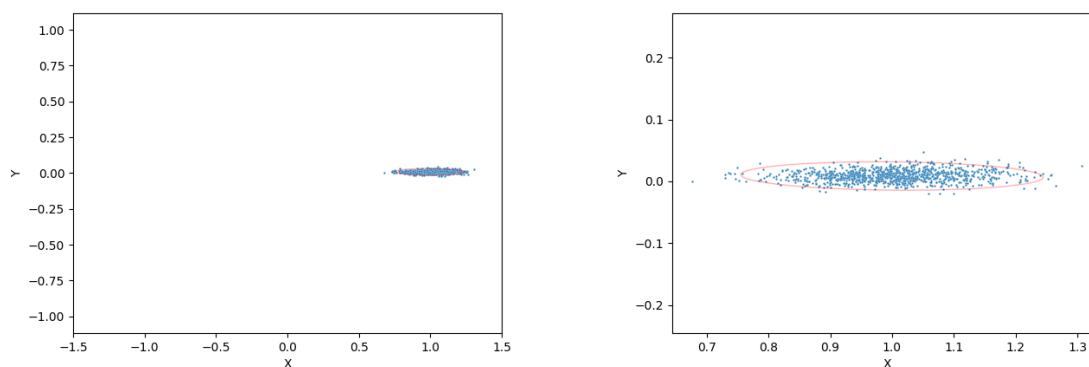
$$\begin{aligned} (ae + bg)a + (af + bh)b &= (\sigma_{\rho\rho}^2 \cos \theta + \sigma_{\rho\theta}^2 \sin \theta) \cos \theta + (\sigma_{\rho\theta}^2 \cos \theta - \sigma_{\theta\theta}^2 \sin \theta)(-\rho \sin \theta) \\ (ae + bg)c + (af + bh)d &= (\sigma_{\rho\rho}^2 \cos \theta + \sigma_{\rho\theta}^2 \sin \theta) \sin \theta + (\sigma_{\rho\theta}^2 \cos \theta - \sigma_{\theta\theta}^2 \sin \theta)(\rho \cos \theta) \\ (ce + dg)a + (cf + dh)b &= (\sigma_{\rho\rho}^2 \sin \theta + \sigma_{\rho\theta}^2 \cos \theta) \cos \theta + (\sigma_{\rho\theta}^2 \sin \theta + \sigma_{\theta\theta}^2 \cos \theta)(-\rho \sin \theta) \\ (ce + dg)c + (cf + dh)d &= (\sigma_{\rho\rho}^2 \sin \theta + \sigma_{\rho\theta}^2 \cos \theta) \sin \theta + (\sigma_{\rho\theta}^2 \sin \theta + \sigma_{\theta\theta}^2 \cos \theta)(\rho \cos \theta) \end{aligned}$$

- e) Simulate d) using the Monte Carlo simulation, i.e. assume

$$\mathbf{x} = \begin{bmatrix} 1m \\ 0.5^\circ \end{bmatrix}, \Sigma = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix} \quad (5)$$

Sample 1000 points from this distribution and plot the transformed results on x-y coordinates. Plot the uncertainty ellipse, calculated from part d). Overlay the ellipse on the point samples.

I assume the given \mathbf{x} here as the mean, μ , and apply it to equation 3. I also assume the given variance of θ is in degrees. The result of the latter is a clearer contrast between the inputs. The uncertainty ellipse represents a 95% confidence. The scale of the axes are equal in their respective plots.



(a) Origin centered.

(b) Mean centered.

Figure 2: Plots of the transformed distribution.