Division C Astronomy

Harvard Undergraduate
Science Olympiad

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Abstract

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Problem 1 (Tidal Forces). Suppose we are living in a two-dimensional world where there exists a $1/r$ gravitational force instead of our $1/r^2$ force in three dimensions. Let this gravitational force be given by

$$\vec{F}_g = -G_2 m_1 m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^2}.$$

The **tidal force** $\vec{F}_t$ on a 2particle mass m from a 2star mass M is given by the difference between this gravitational force $\vec{F}_g$ and the **fictitious translational force** $\vec{F}_s$

$$\vec{F}_t = \vec{F}_g - \vec{F}_s = G_2 Mm \frac{\vec{p} - \vec{r}}{|\vec{p} - \vec{r}|^2} - G_2 Mm \frac{\vec{p}^2}{|\vec{p}|^2}$$

where $\vec{p}$ is the vector from the 2particle to the 2star and $\vec{r}$ is the vector that describes the radius of our 2particle.

Problem 1.a. Find the tidal force on a 2particle of mass m on the surface of a circular 2planet with radius r orbiting a 2star with mass M at a distance $a \gg r$ as a function of the angle θ shown below. Take the center of the 2planet to be the origin of your coordinate system and assume that the 2star is at $(a, 0)$. Then θ is the angle from the x-axis.

Hint. Manipulate the given equation and Taylor expand around a reasonable variable in calculations.

Solution. In this problem, $\vec{r} = (r\cos\theta, r\sin\theta)$ and $\vec{a} = (a, 0)$. Then we have the tidal force given by

$$\vec{F}_t = G_2 Mm \frac{\vec{a} - \vec{r}}{|\vec{a} - \vec{r}|^2} - G_2 Mm \frac{\vec{a}^2}{|\vec{a}|^2} \tag{1}$$

We can simplify this equation by Taylor expanding the first term in powers of $\vec{r}$ and using the fact that $\vec{a} \gg \vec{r}$. Then the above equation becomes:

$$\begin{aligned}
\vec{F}_t &= G_2 Mm \frac{\vec{a} - \vec{r}}{(\vec{a} - \vec{r}) \cdot (\vec{a} - \vec{r})} - G_2 Mm \frac{\vec{a}^2}{|\vec{a}|^2} && (2)\\
&\approx G_2 Mm \vec{a}(|\vec{a}|^2 - 2\vec{a} \cdot \vec{r})^{-1} - G_2 Mm \frac{\vec{a}}{|\vec{a}|^2} - G_2 Mm \frac{\vec{r}}{|\vec{a}|^2} && (3)\\
&\approx G_2 Mm \frac{\vec{a}}{|\vec{a}|^2}(1 - 2\vec{a} \cdot \vec{r}/|\vec{a}|^2)^{-1} - G_2 Mm \frac{\vec{a}}{|\vec{a}|^2} - G_2 Mm \frac{\vec{r}}{|\vec{a}|^2} && (4)\\
&\approx G_2 Mm \frac{\vec{a}}{|\vec{a}|^2}(1 + 2\vec{a} \cdot \vec{r}/|\vec{a}|^2 + \cdots)^{-1} - G_2 Mm \frac{\vec{a}}{|\vec{a}|^2} - G_2 Mm \frac{\vec{r}}{|\vec{a}|^2} && (5)\\
&\approx G_2 Mm \left(2 \cdot \frac{\vec{a}(\vec{a} \cdot \vec{r})}{|\vec{a}|^4} - \frac{\vec{r}}{|\vec{a}|^2}\right) = \boxed{\frac{G_2 Mm}{a^2}\left(2\hat{a}(\hat{a} \cdot \vec{r}) - \vec{r}\right)} && (6)
\end{aligned}$$

In this derivation, we first expand the first term. Observe that the $\vec{r} \cdot \vec{r}$ in the denominator vanishes because $\vec{a} \gg \vec{r}$ and the numerator becomes $\vec{a}$, ie. $\vec{a} - \vec{r} \approx \vec{a}$. We can then Taylor expand the first term around $\vec{r} = 0$ and take the significant terms. Combining like terms yields our final answer as desired.

Plugging in $\vec{r} = (r\cos\theta, r\sin\theta)$ and $\vec{a} = (a, 0), \hat{a} = (1, 0)$, our tidal force becomes:

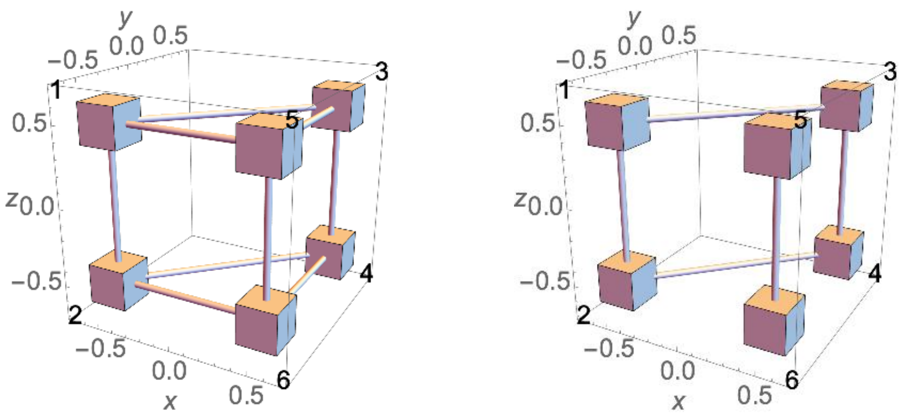
$$\boxed{\frac{G_2 Mm}{a^2}(r\cos\theta, -r\sin\theta)} \tag{7}$$

□

Problem 2 (Disintegrating Rods). Consider the following six-mass system connected by massless rods. The mass' positions are given by:

$$\vec{r}_1 = \left(-\tfrac{l}{2}, -\tfrac{l}{2}, \tfrac{l}{2}\right) \qquad \vec{r}_2 = \left(-\tfrac{l}{2}, -\tfrac{l}{2}, -\tfrac{l}{2}\right) \qquad \vec{r}_3 = \left(\tfrac{l}{2}, \tfrac{l}{2}, \tfrac{l}{2}\right)$$
$$\vec{r}_4 = \left(\tfrac{l}{2}, \tfrac{l}{2}, -\tfrac{l}{2}\right) \qquad \vec{r}_5 = \left(\tfrac{l}{2}, -\tfrac{l}{2}, \tfrac{l}{2}\right) \qquad \vec{r}_6 = \left(\tfrac{l}{2}, -\tfrac{l}{2}, -\tfrac{l}{2}\right)$$

where $l = 1$ and each block has mass m with moment of inertia I with respect to all **principal axes**. A principal axis is an axis that arises from symmetry (ie. any axes perpendicular to a plane of symmetry, any axis parallel to a rotational axis of symmetry). Objects rotating about a principal axis require no additional torques to keep it rotating.



(a) System before disintegration (b) System after disintegration

Consider the system at (a) rotating with angular velocity

$$\vec{\omega} = \omega_0(\hat{x} + \hat{y})$$

and the system (b) at time $t = 0$ immediately after some of the rods spontaneously disintegrate.

Problem 2.a. Find the forces and torques of each of the masses immediately after the rods disintegrate.

Hint. A small change in momentum $d\vec{p}$ of any one of the blocks can be described with the ***cross product*** *$d\theta \times \vec{p}$*

$$d\vec{p} = d\theta \times \vec{p}.$$

This is because the magnitude of the momentum never changes, only its direction — an infinitesimal change in the momentum is simply an infinitesimal change in the angle of the initial momentum vector. Use this fact, combined with the ***no-slip condition***

$$\vec{v} = \vec{\omega} \times \vec{r}$$

to determine the forces.

Solution. Using the relationships given above, we can derive the following relationship for force:

$$\vec{F} = \vec{\omega} \times m\vec{v} = \vec{\omega} \times m\left(\vec{\omega} \times \vec{r}\right) \tag{8}$$

where $\vec{r}$ for the force is defined by the vector from the center of rotation/rotational axis to the blocks and $\vec{\omega}$ is the angular velocity vector given in the problem. Moreover, observe that immediately after disintegration (at time $t = 0$), the angular velocity of each resulting component remains unchanged.

Then solving for the four masses in the 1234-frame, we have

$$\vec{F}_1 = ml\omega_0^2\hat{z} \qquad \vec{F}_2 = -ml\omega_0^2\hat{z} \tag{9}$$

$$\vec{F}_3 = ml\omega_0^2\hat{z} \qquad \vec{F}_4 = -ml\omega_0^2\hat{z} \tag{10}$$

And for the 56-dumbbell, we have

$$\vec{F}_5 = ml\omega_0^2\hat{z} \qquad \vec{F}_6 = -ml\omega_0^2\hat{z} \tag{11}$$

The torques on all the masses is simplay zero after disintegration because all components are rotating about their principal axes — they require no torques to continue rotating. Mathematically, we can consider the change in angular momentum. We have

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{\vec{L}_{system}}{I_\perp} \times \vec{L}_{block} \tag{12}$$

where

$$\vec{L} \equiv \vec{\vec{\mathcal{I}}}\vec{\omega} \tag{13}$$

and $\vec{\vec{\mathcal{I}}}$ is the **inertia tensor**. This is simply a result of the conservation of angular momentum. As the blocks rotate, the angular momenta of the blocks does something crazy (since $\vec{v}$ is always changing), while the angular momentum of the components will be conserved. In particular, the angular momenta of the blocks is rotating around the angular momentum of the component (ie. the frame or dumbbell). We can again use the cross product in our calculation. Evaluating this above equation will yield

$$\vec{\tau}_1 = \vec{\tau}_2 = \vec{\tau}_3 = \vec{\tau}_4 = \vec{\tau}_5 = \vec{\tau}_6 = 0. \tag{14}$$

Alternatively, upon inspection, we can observe that $\vec{L}_{system}$ and $\vec{L}_{block}$ point in the same direction, which means the cross product vanishes as desired. □

Problem 2.b. Determine the equations of motion of the "1234-frame" and the "56-dumbbell" as a function of time after disintegration.

Hint. You may want to consider rotation matrices in three-dimensions.

Solution. (Qualitative answer). The 1234-frame will remain unmoved and continue rotation in space. The 56-dumbbell will fly off with some tangential velocity and continue rotation with angular velocity $\vec{\omega}$.

(Quantitative solution.) We can model the system using the rotation matrix in the plane perpendicular to $\vec{\omega}$. In this case, the objects will rotate in the $x = -y$ plane with an angular frequency $\|\vec{\omega}\|$. This rotation matrix R is given by

$$R = \begin{pmatrix} \cos\left(\frac{\omega_0 t}{\sqrt{2}}\right)^2 & \sin\left(\frac{\omega_0 t}{\sqrt{2}}\right)^2 & \frac{1}{\sqrt{2}}\sin\omega_0 t\sqrt{2} \\ \sin\left(\frac{\omega_0 t}{\sqrt{2}}\right)^2 & \cos\left(\frac{\omega_0 t}{\sqrt{2}}\right)^2 & \frac{-1}{\sqrt{2}}\sin\omega_0 t\sqrt{2} \\ \frac{-1}{\sqrt{2}}\sin\omega_0 t\sqrt{2} & \frac{1}{\sqrt{2}}\sin\omega_0 t\sqrt{2} & \cos\omega_0 t\sqrt{2} \end{pmatrix} \tag{15}$$

This matrix describes the motion of any object rotating around the origin with angular velocity $\vec{\omega}$.

For the stationary 1234-frame after disintegration, we can dot R with the position vectors we used in **Problem 2.a** to determine the equations of motions of the rotation of our vectors around $\vec{\omega}$. We can then add the initial conditions (initial position of each block). Evaluating for each of the blocks, we have

$$\vec{s}_1(t) = \left(-\frac{l}{2} + \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, -\frac{l}{2} - \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, \frac{l}{2} + \frac{l\cos\sqrt{2}\omega_0 t}{2}\right) \tag{16}$$

$$\vec{s}_2(t) = \left(-\frac{l}{2} - \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, -\frac{l}{2} + \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, -\frac{l}{2} - \frac{l\cos\sqrt{2}\omega_0 t}{2}\right) \tag{17}$$

$$\vec{s}_3(t) = \left(\frac{l}{2} + \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, \frac{l}{2} - \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, \frac{l}{2} + \frac{l\cos\sqrt{2}\omega_0 t}{2}\right) \tag{18}$$

$$\vec{s}_4(t) = \left(\frac{l}{2} - \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, \frac{l}{2} + \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, -\frac{l}{2} - \frac{l\cos\sqrt{2}\omega_0 t}{2}\right) \tag{19}$$

For the 56-dumbbell, we can do the same thing as for the blocks in the 1234-frame but we must add in the tangential velocity of the dumbbell. We can again use

$$\vec{v}_{cm} = \vec{\omega} \times \vec{r}_{cm} \tag{20}$$

where $\vec{r}_{cm}$ is the vector from the origin to the center of mass of the dumbbell. Then our equations of motion for blocks 5 and 6 are

$$\vec{s}_5(t) = \left(\frac{l}{2} + \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, -\frac{l}{2} - \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, \frac{l}{2} - l\omega_0 + \frac{l\cos\sqrt{2}\omega_0 t}{2}\right) \tag{21}$$

$$\vec{s}_6(t) = \left(\frac{l}{2} - \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, -\frac{l}{2} + \frac{l\sin\sqrt{2}\omega_0 t}{2\sqrt{2}}, -\frac{l}{2} - l\omega_0 - \frac{l\cos\sqrt{2}\omega_0 t}{2}\right) \tag{22}$$

□

Problem 3 (Special Relativity). Special relativity is the generally accepted physical theory regarding the relationship between space and time — spacetime. According to Einstein's original formulation, it is based on two postulates

- The laws of physics are invariant with respect to all inertial reference frames (non-accelerating frames of reference)
- The speed of light c in a vacuum is the same for all observers, regardless of the motion or light source of the observer

Events with coordinates with respect to two reference frames S and S' are related by **Lorentz transformations**. These transformations relate the positions, times, and lengths from one reference frame S to another S' moving at a speed v relative to S with the **Lorentz factor** γ given by

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

The Lorentz transformations give us the following relationships/phenomena:

1. Time Dilation: a time difference $\Delta t'$ measured by a clock in S' is *longer* than a time difference Δt measured by a clock in S.
$$\Delta t' = \gamma \Delta t$$
2. Length Contraction: a length $\Delta l'$ measured in a moving frame S' is *shorter* than the length Δl measured in the rest frame S.
$$\Delta l' = \Delta l / \gamma$$
3. Velocity addition: if an object is moving at a velocity $\vec{u}\,'$ in the moving frame S', then its velocity $\vec{u}$ in the rest frame S is given by
$$\vec{u} = \frac{v + u'}{1 + vu'/c^2}$$

In relativistic units, we can set $c = 1$ and add in factors of c at the end of calculations appropriately for units.

Problem 3.a. Consider a relativistic rocket length L that is traveling from Earth to Mars at a speed v_1. A relativistic astronaut walks from the rear to the front of the rocket with a speed v_2 with respect to the rocket.

How long does the astronaut's walk take according to a clock at the rear of the ship?

Problem 3.b. How long does the walk take according to the astronaut's clock?

Solution. (a) Because we are in the ship's frame, the time measured by the clock is simply $\boxed{L/v_2}$.

(b) Because the astronaut's clock is *younger* than the clocks measured by the clocks on the train, the astronaut's clock measures a time

$$\boxed{\frac{L\sqrt{1-v_2^2/c^2}}{v_2}}$$

The astronaut always moves with a speed v_2, the clocks in the ship frame measure the ticks of the astronaut's watch slowed down by a factor of $\gamma_2 = \frac{1}{\sqrt{1-v_2^2/c^2}}$.

□