Physics 15c: Wave Phenomena

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Abstract

These are notes[1] for Harvard's *Physics 15c*, an introductory class on waves, as taught by Professor Melissa Franklin[2] in Fall 2021. We will cover most of the textbook *Waves and Oscillations: A prelude to quantum mechanics* by Walter Fox Smith. *The Physics of Waves* by Professor Howard Georgi is optional. Note that not all components of the class can be fully reproduced in this set of notes, including in-class demonstrations and discussions. All omissions and errors are those of the transcribers.

Course description: Forced oscillation and resonance; coupled oscillators and normal modes; Fourier series; Electromagnetic waves, radiation, longitudinal oscillations, sound; traveling waves; signals, wave packets and group velocity; two- and three-dimensional waves; polarization; geometrical and physical optics; interference and diffraction. Optional topics: Water waves, holography, x-ray crystallography, solitons, music, quantum mechanics, and waves in the early universe.

Contents

[1]With thanks to Eric K. Zhang for the template.
[2]With preceptor Dr. Anna Klales and teaching fellows Jerry Ling and Brendon Bullard, as well as lab staff Professor Markus Greiner, lab preceptor James Mitchell and lab teaching fellow Paloma Ocola

1 September 1st, 2021

Prof. Melissa Franklin is elite! We also get to meet Prof. Markus Greiner and TF Paloma Ocola, who will run the PSI labs. Times are Thu 3-6, 6:45-9:45, Fri 3-6.

1.1 Lab logistics

Labs — at least in the beginning — will be quite mechanical and primitive, i.e. building an interferometer. This will continue through the first half of the course.

In the second half of the course, we will be working on group final projects, where students will have freedom of choice for exploration, e.g. laser trapping, quantum erasers, holographic optical elements, etc. This will culminate in a poster presentation to the class.

1.2 Syllabus

See `https://canvas.harvard.edu/courses/90652/assignments/syllabus`.

Melissa Franklin is an experimental particle physicist. Think Physically!

Textbooks: Waves and Oscillations: A prelude to quantum mechanics by Walter Fox Smith,
The Physics of Waves by Howard Georgi

1.2.1 Grading

- Lab attendance: 25%
- Problem sets: 15%
- Midterm: 20%
- Final exam: 30%
- Participation: 10%

1.3 Math we need and concepts we will learn

Math: Trig identities, Linear algebra (for coupled oscillator equations), complex notation for waves, dimensional analysis, Maxwell's equations, dimensional analysis.

Waves: how to generate them, impose boundary conditions, interference, self-interference, resonance, high frequency waves in boxes (quantum mechanics), wave scattering

1.4 Demonstrations

- Resonance of tuning forks
- Speaker polarity and destructive interference
- Music box resonance amplification
- Pendulum oscillations
- Decomposition of gas light

1.5 A mass on a spring

https://canvas.harvard.edu/courses/90652/files/12961642?module_item_id=990221.

We will derive a function $x(t)$ that predicts the motion of the mass as a function of time. We have

$$\boldsymbol{F} = -k\boldsymbol{x} = m\ddot{\boldsymbol{x}} \implies \boxed{\ddot{\boldsymbol{x}} + \frac{k}{m}\boldsymbol{x} = 0.} \tag{1}$$

Observe that in this equation of motion, we have

- Linear differential equation (only 1st powers of $\boldsymbol{x}$)
- Second order differential equation
- Thus, linear combinations of solutions are also solutions
- We also derive, from boundary conditions, that $\boxed{\omega = \sqrt{k/m}}$

2 September 8th, 2021

Reminder: do Canvas quiz, one-minute paper before next lecture.

Reading: Smith Chapters 1-2

2.1 Simple harmonic oscillation

Recall the **Hooke's law force**

$$\boldsymbol{F}_{\text{net}} = m\boldsymbol{a} \tag{2}$$

which has equation of motion

$$\ddot{\boldsymbol{x}} + \omega^2 \boldsymbol{x} = 0 \tag{3}$$

The most general solution for this differential equation is

$$x(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t) \tag{4}$$

where A_1, A_2 are completely determined by initial conditions and ω is fixed by the system.

Completeness: It is a second order differential equation, and because there are two initial conditions (position and velocity).

Uniqueness: kind of hand-waved it, but essentially any other equivalent solution can be rewritten in this form. For example, this solution can also be written as

$$x(t) = C\cos(\omega t + \phi) \tag{5}$$

where the ϕ phase shift encapsulates the sine component. See problem set 0 problem 4.

2.2 Oscillation anatomy

We define the following quantities for oscillation

- **Period,** T — time for mass to complete one oscillation cycle

$$\boxed{T = 2\pi/\omega} \tag{6}$$

- **Amplitude,** A — the max value of the position measured from equilibrium

$$x(t) = C\cos(\omega t + \phi) \tag{7}$$

  Observe that amplitude is determined by initial conditions

$$C = \sqrt{A_1^2 + A_2^2} \tag{8}$$

- **Angular frequency,** ω **and frequency** f — rate of change of phase, with units of radians/second. Recall that 1 cycle $= 2\pi$ radians. We also have

$$\boxed{f = \frac{\omega}{2\pi}, \qquad T = \frac{1}{f}} \tag{9}$$

- **Phase,** ϕ —

$$x(t) = C\cos(\omega t + \phi) \tag{10}$$

  ϕ is the initial phase and $\omega t + \phi$ is the total phase.

  Note. If the sinusoid is shifted to the right, the phase is negative. Similarly for the opposite shift.

We also note that, via differentiation,

$$v_{\max} = \omega A, \qquad a_{\max} = \omega^2 A \tag{11}$$

2.3 Complex solution

Recall **Euler's formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{12}$$

This form suggests an ansatz of the form

$$x(t) = Ce^{\alpha t} \tag{13}$$

which we can check in the SHO formula has a solution of $\alpha = \pm i\omega$. Then, the general solution is

$$x(t) = C_1 e^{+i\omega t} + C_2 e^{-i\omega t} \tag{14}$$

where C_1, C_2 can be complex.

Note. We can interpret this solution as one wave moving forwards and one wave moving backwards.

2.3.1 Complex numbers

We review complex numbers in this section. Complex numbers are vectors in the "complex plane," and can be written in either Cartesian ($z = a + ib$) or Polar ($z = Re^{i\theta}$) form.

We define the following quantities of a complex number:

$$\mathrm{Re}(z) = a, \qquad \mathrm{Im}(z) = b \tag{15}$$

Recall the **complex conjugate** of a complex number $z = a + ib = Re^{+i\theta}$ is $z^* = a - ib = Re^{-i\theta}$

The **norm**, or magnitude', of a complex number is

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z^* z} \tag{16}$$

and its **argument**, or phase, is

$$\arg(z) = \begin{cases} \arctan(b/a), & a \geq 0 \\ \arctan(b/a) + \pi, & a < 0 \end{cases} . \tag{17}$$

To model physical systems, we must impose conditions such that $x(t)$ is always real. Thus, from the general solution

$$x(t) = C_1 e^{+i\omega t} + C_2 e^{-i\omega t} \tag{18}$$

we can impose conditions on C_1, C_2. For a real system, we require $C_1 = C_2^*$, then

$$x(t) = Ce^{i\omega t} + C^* e^{-i\omega t} = C_0 \left[e^{i(\omega t + \phi)} + e^{-i(\omega t + \phi)} \right] = 2C_0 \cos(\omega t + \phi) \tag{19}$$

where $C \equiv C_0 e^{i\phi}$ and $C_0 \equiv |C_1|$.

The complex representation is a mathematical trick — the math in the complex plan is very convenient, and the (real, physical part of the) solution from before remains the same.

Consider the system after time evolution. In our original expression, we have

$$\cos[\omega(t + \Delta t)] = \cos\omega\Delta t \cos\omega t - \sin\omega\Delta t \sin\omega t. \tag{20}$$

For the complex expression, we simply have

$$e^{-i\omega(t+\Delta t)} = \underbrace{e^{-i\omega\Delta t}}_{\text{time evolution}} \underbrace{e^{-i\omega t}}_{\text{original}}, \tag{21}$$

which is an **irreducible solution**. Note that in this case, we can express time evolution as a **rotation in the complex plane**, where the physical oscillation is simply the projection of the rotation onto the real line.

2.4 Energy of the simple harmonic oscillator

$$\boldsymbol{F} = -\boldsymbol{\nabla} U \tag{22}$$

The total mechanical energy of a mass on a spring is **conserved**, i.e. E = K + U.

Since we have

$$x(t) = A\cos(\omega t), \qquad v(t) = -\omega A \sin(\omega t) \tag{23}$$

we can substitute, simplify, and find

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \tag{24}$$

which is constant and time-independent.

Next time: Hooke's Law and the pendulum

3 September 13th, 2021

Section is **not** mandatory.

3.1 Energy of the simple harmonic oscillator

Consider the Taylor expansion of an arbitrary potential around its equilibrium given by

$$V(x) = V(x_0) + V'(x_{\text{eq}})(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \cdots \tag{25}$$

Note that at equilibrium, the second term (first derivative) is 0. The physical interpretation of the second derivative in the Taylor expansion is the spring constant $k = V''(x_0)$.

Example 3.1 (The pendulum)**.** Note that the potential energy is

$$U(\theta) = mgL(1 - \cos\theta) \approx \frac{1}{2}(mgL)\theta^2 \tag{26}$$

for small θ. In the tangential direction, we have

$$-mg\sin\theta = m(l\ddot{\theta}) \tag{27}$$

from which we can derive using the SHO equation that

$$\boxed{\omega_{\text{pendulum}} = \sqrt{\frac{g}{L}}.} \tag{28}$$

For larger θ our approximation overestimates the force, so the real potential is flatter, so the true period is longer, and our approximation *underestimates* the period of oscillation (since it overestimates the force).

For a general potential $U(x)$, we can obtain the frequency of oscillation by

$$\omega = \sqrt{\frac{U''(x_0)}{m}}. \tag{29}$$

3.2 Damped simple harmonic oscillators

All real systems have damping. Drag is velocity-dependent, which motivates a velocity dependent term to our SHO:

$$\boldsymbol{F}_d = -b\boldsymbol{v} \implies m\ddot{\boldsymbol{x}} + b\dot{\boldsymbol{x}} + k\boldsymbol{x} = 0. \tag{30}$$

Let the damping constant be $\gamma \equiv b/m$,, and we keep $\omega_0^2 \equiv k/m$ as usual. We can write

$$\ddot{\boldsymbol{x}} + \gamma\dot{\boldsymbol{x}} + \omega_0^2\boldsymbol{x} = 0. \tag{31}$$

We use an ansatz to solve this equation, motivated by physical demos, containing a sine wave oscillation and an exponential decay.

$$x = X_0 e^{Rt} \tag{32}$$

Substitute this expression and its derivatives into the equation of motion, we find

$$R^2 + \gamma R + \omega_0 = 0 \implies R = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} \tag{33}$$

Note that we require R to be imaginary for oscillation, as in $x = X_0 e^{Rt}$.

Thus, we write the solutions for $x(t)$ as

$$x(t) = x(0)e^{-\gamma t/2} e^{\pm\sqrt{(\gamma/2)^2 - \omega_0^2}t} \tag{34}$$

Since the angular frequency changes, we define the **damped frequency**

$$\omega = \sqrt{\omega_0^2 - \gamma^2/4}. \tag{35}$$

The most general solution for $x(t)$ is given by a linear combination of the two solutions:

$$x(t) = Ce^{-\gamma t/2} e^{\pm i\omega t} \implies \boxed{x(t) = C_1 e^{-\gamma t/2} e^{i\omega t} + C_2 e^{-\gamma t/2} e^{-i\omega t}} \tag{36}$$

where γ describes the damping term, and ω describes the new damped frequency.

For real solutions, we require $C_1 = C_2^*$ where $C_1 = Ce^{i\phi}, C_2 = Ce^{-i\phi}$. Then we have

$$\boxed{x(t) = Ce^{-\gamma t/2} 2\cos(\omega t + \phi)} \tag{37}$$

after taking the real component.

4 September 15th, 2021

4.1 Damped simple harmonic oscillators cont.

Continuing from last time, we consider three distinct cases for the radical R:

$$\omega_0^2 > \gamma^2/4, \qquad \text{underdamped (complex)} \tag{38}$$
$$\omega_0^2 < \gamma^2/4, \qquad \text{overdamped (real, no oscillations)} \tag{39}$$
$$\omega_0^2 = \gamma^2/4, \qquad \text{critically damped} \tag{40}$$

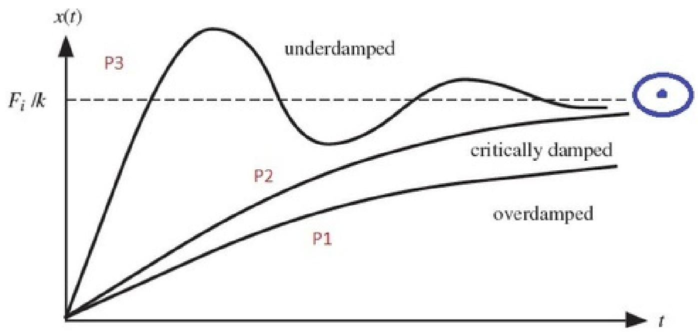

Figure 1: Damped harmonic oscillator behavior

Note the characteristic oscillating and damping times. From observation, a system that is underdamped will oscillate and decay.

In the **1st underdamped case**,

$$x(t) = e^{-\gamma t/2}\cos(\omega t + \phi) \tag{41}$$

if $\gamma \ll \omega_0$, we have

$$\omega = \left[\omega_0 - \left(\frac{\gamma}{2}\right)^2\right]^{\frac{1}{2}} \implies \omega = \omega_0\left[-\left(\frac{\gamma}{2}\right)^2\right]^{\frac{1}{2}} \tag{42}$$
$$\tag{43}$$

In the **2nd strongly damped case**: $\omega_0^2 < \gamma^2/4$ and $\omega_0 \ll \gamma$. Both solutions are real and there is no oscillation (ω_0/γ is small).

$$x(t) = x_0 e^{-\gamma t/2} e^{\pm\sqrt{\gamma^2/4-\omega_0^2}t} \tag{44}$$
$$= x_0 e^{-\gamma t/2} e^{\pm(\gamma/2-\omega_0^2/\gamma)t}. \tag{45}$$

The general solution is given by

$$x(t) = C_1 e^{-\omega_0^2 t/2} + C_2 e^{-\gamma t}. \tag{46}$$

In the **3rd critically damped case**: $\omega_0 = \gamma/2$. No oscillations, only *one* solution. We have $R = -\gamma/2, \qquad x(t) = x(0)e^{Rt}$. Then we have

$$x(t) = x_0 e^{-\gamma t/2} \implies x(t) = C_1 e^{-\gamma t/2} + C_2 t e^{-\gamma t/2} \tag{47}$$

the second part of the general solution comes from taking the limit of an underdamped system: a critical point.

Example 4.1 (LC Circuit)**.** This is analogous to an LC circuit, where it charges up the capacitor and then the inductor. In an LRC circuit, the $Q_{\max}$ is an underdamped harmonic system, as the resistor dissipates the current.

We see demos of glycerin damping and eddy current damping.

4.2 Damped driven harmonic oscillator

We will consider a wheel connected to block via a spring. The rotating wheel will drive the spring with some force $\boldsymbol{F}$. We will increase the damping on the system (using glycerin) to only consider the effect of the driving force on the system.

The driver drives the spring with a force

$$F = kr\cos(\omega t)$$

For **low frequencies**, ($\omega < \omega_0$), the drive and the response are **in phase** with each other.

For **high frequencies**, ($\omega > \omega_0$), the drive and the response are **out of phase** with each other.

At the **resonant frequency**, ($\omega = \omega_0$), the drive leads the response by $\pi/\mathbf{2}$.

4.2.1 Solving the driven simple harmonic oscillator

We have $x(t) = x(0)e^{i\omega t}$. The equation of motion is given by

$$m\ddot{\boldsymbol{x}} + b\dot{\boldsymbol{x}} + k\boldsymbol{x} = \underbrace{kr\cos\omega t}_{\boldsymbol{F}_{\text{driving}}} \tag{48}$$

When there is no damping, $b = 0$. In this case, we can solve

$$-mx_0\omega^2 e^{i\omega t} + kxe^{i\omega t} = kre^{i\omega t} \implies x(t) = \frac{kr}{-m\omega^2 + k} = \frac{\omega_0^2 r}{\omega_0^2 - \omega^2} \implies \omega_0^2 = \frac{k}{m} \tag{49}$$

At **perfect resonance**, we get infinities. So, let's add damping $b \neq 0$:

$$m\ddot{\boldsymbol{x}} + b\dot{\boldsymbol{x}} + k\boldsymbol{x} = kr\cos\omega t \tag{50}$$

Let the solution be $x(t) = x_0 e^{i\omega t}$. Then

$$(-m\omega^2 + ib\omega + k)x_0 e^{i\omega t} = +kre^{i\omega t}. \tag{51}$$

Let $r = b/m$. Then, we find the amplitude is

$$\boxed{x_0 = \frac{r\omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma\omega}} \tag{52}$$

where $\omega_0 \equiv \sqrt{k/m}, \Gamma \equiv b/m$.

Example 4.2 (Motors)**.** We will consider three cases: a motor runs at the natural frequency of the oscillator ω_0, the motor runs faster, and the motor runs slower.

1. Motor runs at natural frequency and resonance occurs. If $\omega_d = \omega_0$ then

$$x(t) = \frac{r\omega_0}{i\Gamma}(\cos\omega_0 t + i\sin\omega_0 t) \tag{53}$$

when we take the real part, we have the solution is

$$+\frac{r\omega_0}{\Gamma}\sin\omega_0 t \tag{54}$$

Here, the amplitude is $\boxed{r\omega_0/\Gamma}$. Since Γ is small, we have an amplitude greater than the original amplitude.

Example 4.3 (Z boson resonance)**.** The Z boson takes exactly the classical resonance form.

$$\sigma = \sigma_0 \frac{\Gamma^2/4}{(E-E_R)^2+\Gamma^2/4} \tag{55}$$

with resonance width $1/\tau$.

5 September 20th, 2021

5.1 Damped driven harmonic oscillators

As before, we consider three cases of the relationship between drive frequency and natural frequency.

1. Motor runs at **natural frequency** and resonance occurs. The phase of the mass is 90 degrees behind the driver.

   See last week's notes for details of derivation.

2. **Driver frequency is above natural frequency** where $\omega \gg \omega_0 \implies \omega^2 > \omega_0^2 \implies \omega^2 > i\Gamma\omega$.

$$x(t) = \frac{r\omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma\omega} e^{i\omega t} \implies x(t) = -r\frac{\omega_0^2}{\omega^2}\cos\omega t \tag{56}$$

3. **Driving frequency is much less than natural frequency**: $\omega \ll \omega_0$.

$$x(t) = \frac{r\omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma\omega} e^{i\omega t} \approx r\cos\omega t \tag{57}$$

   and it is **in phase**.

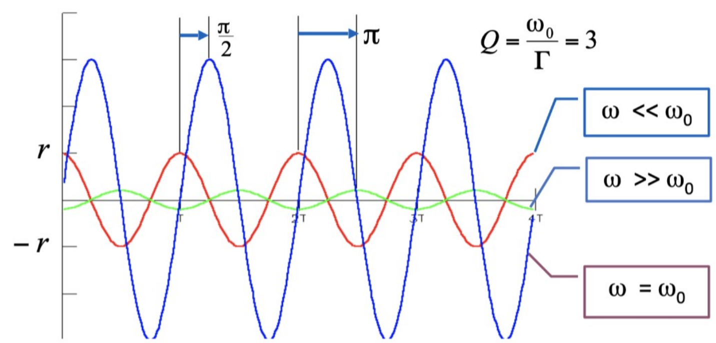

Figure 2: Damped driven harmonic oscillator relative phases

Let's consider the phase of the oscillation. Recall

$$x(t) = \frac{r\omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma\omega} e^{i\omega t}, \qquad \text{Amplitude} = \sqrt{xx^*}, \qquad \phi = \arg\left(\frac{r\omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma\omega}\right) = \cot^{-1}\left(\frac{r\omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma\omega}\right) \tag{58}$$

Resonance in quantum mechanics and particle physics is completely analogous. Here, the damping factor is the *inverse lifetime* of a particle.

5.2 Beats and coupled oscillators

Demo. Two tuning forks produce beats!! Resonant transfer from one tuning fork to another. Detuned forks produce beats.

- It seems like increased detuning increases the frequency of the beats

5.2.1 Beats

When two notes are played, the pressure variation int he eardrum is the **sum** of the two oscillations. When they are *close in frequency*, “beating" is heard at a frequency that does not match the two frequencies.

Consider two oscillations with same amplitude

$$x_1(t) = A\cos\omega_1 t, \qquad x_2(t) = A\cos\omega_2 t, \qquad x(t) = x_1(t) + x_2(t) = A(\cos\omega_1 t + \cos\omega_2 t \tag{59}$$

We can use the sum of cosines trigonometric identity $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$ to derive

$$x(t) = 2\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right). \tag{60}$$

The slow oscillation is modulating the amplitude. Then, we can define the period of the beat as the wide peak-peak distance.

Frequency of rapid oscillations: $\boxed{\omega_{\text{fast}} = \frac{1}{2}(\omega_1+\omega_2)}$ Beat frequency: $\boxed{\omega_{\text{slow}} = \frac{1}{2}(\omega_1-\omega_2)}$ (although we often define beat frequency without the factor of $\frac{1}{2}$ because we only hear the peak-peak frequency).

The ω_{fast} corresponds to the small oscillations that make up the larger oscillation and ω_{slow} is responsible for the long-scale oscillations.

5.3 Coupled oscillators

Demo. coupled pendulums – two identical pendulums with same mass and length connected by a spring:

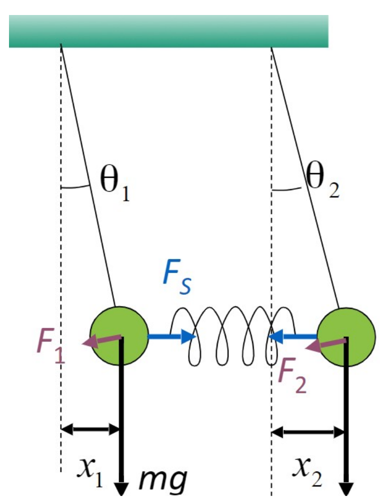

Figure 3: Coupled pendulums

In the small oscillation limit for the coupled oscillator system:

$$x_1 \approx L\theta_1, x_2 \approx L\theta_2, \qquad |\boldsymbol{F}_{\text{spring}}| \approx |k(x_1 - x_2)|, \qquad \boldsymbol{F}_g = mg\sin\theta \approx \frac{mg}{L}x. \tag{61}$$

The equation of motion for mass 1 is given by

$$m\ddot{\boldsymbol{x}}_1 = -\frac{mg}{L}\boldsymbol{x}_1 - k(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{62}$$

The analogous equation for mass 2 is simply

$$m\ddot{\boldsymbol{x}}_2 = -\frac{mg}{L}\boldsymbol{x}_2 - k(\boldsymbol{x}_2 - \boldsymbol{x}_1). \tag{63}$$

Note. This looks like our equation of motion for a single harmonic oscillator with an extra term. This is a *coupled differential equation*. We propose that the system can be solved by adding and subtracting the two equations.

6 September 22nd, 2021

6.0.1 Matrix math

We review matrix math: addition and subtraction, multiplication, determinants, cofactor expansion, inversion, and characteristic equations for eigenvalues and eigenvectors.

6.1 Coupled oscillators

We can solve the system of coupled oscillators from the previous lecture with the **brute force method** where we use an ansantz $x = e^{Xt}$ and some silly algebra.

We note that there are two **normal modes**, where both masses oscillate at the same frequency. There is the *pendulum mode*, where the pendulums are oscillating in the same direction, and the *breathing mode*, where the pendulums are oscillating in opposite directions. We note that in the breathing mode, the frequency is *greater than* the frequency of the pendulum mode because of the restoring force.

We can also use the **symmetry/physical intuition method**. This method allows us to derive equations of motion that are in fact *harmonic* in new coordinates. In essence, the substitution rotates into the frame of $x_1 + x_2$ and $x_1 - x_2$, which are precisely the frequencies of the beats.

Finally, we can apply the very insightful **matrix method**. We first cast the e.o.m. into matrix form

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} \frac{mg}{L} + k & -k \\ -k & \frac{mg}{L} + k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies M\ddot{X} = -KX.$$

We then promote it to the complex plane $X = \text{Re}[Z]$, and guess a normal mode solution (same frequency for all masses) given by

$$Z = e^{i\omega t} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{64}$$

where the $A_1, A_2 \in \mathbb{C}$ contains all of our phase information. Plugging our guess into the matrix equation, we can write the eigenvalue equation, as well as the characteristic equation given by

$$\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = M^{-1}K \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \qquad (M^{-1}K - I\omega^2) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0. \tag{65}$$

We can solve this characteristic equation for our eigenvalues, which correspond to ω and eigenvectors, which correspond to *normal modes*.

For this system, we have

$$\omega_p^2 = \frac{g}{L}, \qquad \omega_b^2 = \frac{g}{L} + \frac{2k}{m} \implies |e_p\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |e_b\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{66}$$

and we can write

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = c_p e^{i\omega_p t}|e_p\rangle + c_b e^{-\omega_b t}|e_b\rangle, \qquad c_p = A_p e^{i\phi_p}, \;\; c_b = A_b e^{i\phi_b} \tag{67}$$

and

$$\boxed{x(t) = \text{Re}(z) = A_p \cos{(\omega_p t + \phi_p)}|e_p\rangle + A_b \cos{(\omega_b t + \phi_b)}|e_b\rangle,} \tag{68}$$

which shows that our solution can be written as a linear combination of the normal modes!

7 September 27th, 2021

7.1 Resonance and normal modes

Note. Writing the equations is the hardest part of solving coupled systems!

Larger coupling produces greater splitting of normal mode oscillation frequency.

Definition 7.1 (Normal mode)**.** A way in which a system can move in a steady state in which all parts of system move with same **frequency** and **phases**. Note that parts may have different amplitudes.

In a 1D system, the number of normal modes is equal to the number of masses.

Given the complex set of two equations, the initial conditions determine the coefficients. We can consider the real part and use trig identities to derive the normal modes, with frequency average $\frac{\omega_p+\omega_s}{2}$ and difference $\frac{\omega_p-\omega_s}{2}$, which are the beats. We see two oscillations: an envelope oscillation (beat) and a more frequent oscillation.

We can write coupled equations of x_1, x_2 in matrix form

$$\mathcal{M}\ddot{\boldsymbol{X}} = -\boldsymbol{K}\boldsymbol{X}, \qquad \boldsymbol{X} = Re(\boldsymbol{Z}), \qquad \boldsymbol{Z} = e^{i\omega t}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{69}$$

Assume one frequency of both oscillations to solve for normal modes. The eigenvalue equation gives the frequencies ω^2, and the eigenvectors give the normal modes.

We can write down the general solutions, which I missed :(

Note. If someone asks us to prove that this is a general solution, direct them to a mathematics course. -Melissa Franklin

7.2 Driven coupled oscillations

Demo. Wine glass resonance.

We discuss the power delivered by the motor/absorbed by the system. The solution is in the notes.

We consider

$$\langle P_{drive}\rangle = \frac{F_0 A\omega_d \sin\delta}{2} = \tag{70}$$

Note. If we want to deliver a lot of power, we should drive the system at the resonant frequency.

At resonance, the **full-width at half-max** (FWHM) is $\boxed{FWHM = \frac{\omega_0}{Q} = \gamma}$. This width is defined by ω_-, ω_+.

As we've done many times before, we can write and solve this matrix equation.

In the undrivne case, we considier the homogeneous equation with $F = 0$.

8 September 29th, 2021

Demo (continued)**.** Plotted normal modes of wine glass.

8.1 Driven coupled harmonic oscillators

Recall the equations of motion from last time where we solved for the homogeneous solution.

8.1.1 The non-homogeneous solution

Recall our ansatz $\boldsymbol{X} = \mathrm{Re}(\boldsymbol{Z}), \boldsymbol{Z} = \boldsymbol{C}e^{i\omega_d t}$ where $\boldsymbol{C} = (C_1, C_2)$. We will substitute this solution into the matrix equation with a driver. The equation becomes

$$(\mathcal{M}^{-1}\boldsymbol{K} - \omega_d^2 I)\boldsymbol{C}e^{i\omega_d t} = \mathcal{M}^{-1}Fe^{i\omega_d t} \tag{71}$$

Note that we have written our driving force as an exponential in place of our $\cos\theta$.

Note. We have **two** simultaneous equations. We have K, ω_d fixed, and $\boldsymbol{C}$ is **not** fixed.

We find that

$$\boxed{C_1 = \frac{F_0}{m}\left(\frac{k/m + g/l - \omega_d^2}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)}\right), \qquad C_2 = \frac{F_0}{m}\frac{k/m}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)}.} \tag{72}$$

We note that if $\omega_d = \omega_1 \implies \frac{C_1}{C_2} = 1$ and if $\omega_d = \omega_2 \implies \frac{C_1}{C_2} = -1$.

In the case that $C_1 = C_2$, we recover our normal mode associated with $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and in the case that $C_1 = -C_2$, we recover the $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ mode.

Thus we have solutions where the driver is the **same** as one of the normal modes of the *undriven* system.

The solution to the driven problem is sum of the homogeneous and non-homogeneous solutions

$$x_1 = \alpha\cos(\omega_1 t + \phi_1) + \beta\cos(\omega_2 t + \phi_2) + C_1\cos\omega_d t \tag{73}$$
$$x_2 = \alpha\cos(\omega_1 t + \phi_1) - \beta\cos(\omega_2 t + \phi_2) + C_2\cos\omega_d t. \tag{74}$$

Demo. Driven coupled harminic oscillators.

8.2 Waves

Consider N coupled pendulums with long strings. We will model a wave as simply the limit as $N \to \infty$. We will then derive the wave equation (using normal modes) and solve it!

Consider a mass-spring transmission line

We have introduced a coordinate system ξ_i defined above.

Note. We have complete symmetry along the $\hat{\boldsymbol{x}}$ axis.

The EOM of the n-th mass is given by

$$m\ddot{\boldsymbol{\xi}}_n = -k(\xi_n - \xi_{n-1}) - k(\xi_n - \xi_{n+1}). \tag{75}$$

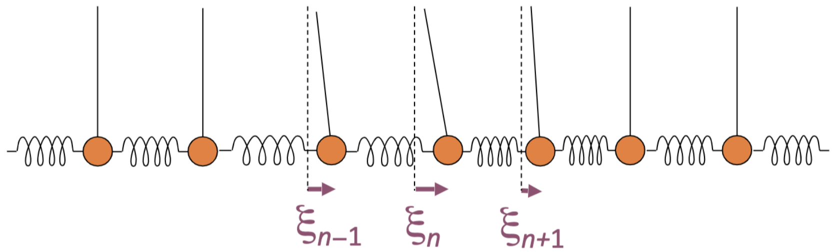

Figure 4: N coupled pendulums

Note we have neglected gravity due to large L.

We will use the position of equilibrium x to specify the mass instead of indexing by n:

$$\xi_n \to \xi(x), \qquad \xi_{n\pm1} \to \xi(x \pm \Delta x). \tag{76}$$

We note that ξ is only defined at discreet points of interval Δx. We seek to take $\Delta x \to 0$ to make $\xi(x)$ a *continuous* function.

We can "take the limit" by taking the Taylor expansion of $\xi(x \pm \Delta x)$ at small x given by

$$\xi(x + \Delta x) = \xi(x) + \xi'(x)\Delta x + \frac{1}{2}\xi''(x)(\Delta x)^2 + \cdots \tag{77}$$

$$\xi(x - \Delta x) = \xi(x) - \xi'(x)\Delta x + \frac{1}{2}\xi''(x)(\Delta x)^2 - \cdots \tag{78}$$

Substituting these expansions into our original equation and dividing through by Δx, we recover

$$\boxed{\frac{m}{\Delta x}\frac{d^2}{dt^2}\xi(x) = k\Delta x\frac{d^2}{dx^2}\xi(x).} \tag{79}$$

Where $\rho_l \equiv \frac{m}{\Delta x}$ is the **(linear) mass density** and $K = k\Delta x$ is the **elastic modulus**.

Note. Using the elastic modulus K, we can rewrite Hooke's law as

$$\boldsymbol{F} = -k\Delta L = -K\frac{\Delta L}{L}. \tag{80}$$

We have found the **wave equation**, given by

$$\boxed{\rho_l\frac{\partial^2}{\partial t^2}\xi(x,t) = K\frac{\partial^2}{\partial x^2}\xi(x,t).} \tag{81}$$

Recall we derived this equation from $N \to \infty$ coupled oscillators $\implies \exists$ an *infinite* number of normal modes.

We can write our coupled oscillators equation

$$m\frac{d^2}{dt^2}\boldsymbol{\xi} = \boldsymbol{K}\boldsymbol{\xi}, \qquad \boldsymbol{\xi} \equiv \begin{pmatrix} \vdots \\ \xi_{n-1} \\ \xi_n \\ \xi_{n+1} \\ \vdots \end{pmatrix}, \qquad \boldsymbol{K} \equiv \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & -2k & k & & \\ & k & -2k & k & \\ & & k & -2k & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{82}$$

As we have done many times before, we guess an ansatz of the form $\boldsymbol{\xi}(x,t) = \boldsymbol{a}(x)e^{i\omega t}$. Plugging this into our wave equation above, we have

$$-\rho_l \omega^2 \boldsymbol{a}(x) = K\frac{d^2}{dx^2}\boldsymbol{a}(x) \implies \boldsymbol{a}(x) = Ae^{\pm ikx}, \quad k = \omega\sqrt{\frac{\rho_l}{K}} \implies \boxed{\xi(x,t) = Ae^{i\omega t}e^{\pm ikx}.} \tag{83}$$

Here, k is the *wavenumber*. Ignoring A, the real part is $\cos kx \pm \omega t$.

Recall that the wavelength $\lambda \equiv \frac{2\pi}{k}$.

9 October 4th, 2021

We recall the real component of the wave equation is given by

$$\xi(x,t) = \cos(kx \pm \omega t) \tag{84}$$

where $\pm$ corresponds to backwards and forwards moving waves, respectively.

9.0.1 Propagation velocity

The wave velocity is given by the condition

$$kx \pm \omega t = const \implies x = \frac{const \mp \omega t}{k} \implies \frac{dx}{dt} = \mp \frac{\omega}{k} \tag{85}$$

and

$$k \equiv \omega \sqrt{\frac{\rho_l}{K}} \implies \boxed{\frac{dx}{dt} = \mp c_w \equiv \mp \sqrt{\frac{K}{\rho_l}}} \tag{86}$$

where c_w the speed of the wave is determined completely by physical properties.

9.0.2 Fourier series

Any periodic function $f(t)$ with period T can be expressed as a Fourier series

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T} \right) \tag{87}$$

where

$$a_0 \equiv \frac{1}{T} \int_0^T dt\ f(t), \qquad a_n \equiv \frac{2}{T} \int_0^T dt\ f(t) \cos \frac{2\pi nt}{T}, \qquad b_n \equiv \frac{2}{T} \int_0^T dt\ f(t) \sin \frac{2\pi nt}{T}. \tag{88}$$

9.1 Infinite string normal modes

The find normal modes, we assume that the displacement of the string is separable and takes the form

$$\psi(x,t) = A(x)B(t). \tag{89}$$

We assume that the time evolution of the string at all points is the same. We can substitute this solution into our wave equation

$$\frac{\partial^2}{\partial t^2}\psi = v_p^2 \frac{\partial^2}{\partial x^2}\psi \implies A(x)\frac{\partial^2}{\partial t^2}B(t) = v_p^2 B(t)\frac{\partial^2}{\partial x^2}A(x). \tag{90}$$

Recall that v_p is the propagation speed of the wave. Using the method of separation of variables, we have

$$\frac{1}{v_p^2 B}\frac{\partial^2}{\partial t^2}B(t) = \frac{1}{A(x)}\frac{\partial^2}{\partial x^2}A(x) = -k_n^2 \tag{91}$$

and find that

$$A(x) = C_n \sin(k_n x + \alpha_n), \qquad B(t) = B_n \sin(k_n v_p t + \beta_n) \tag{92}$$

where $\omega_n = v_p k_n$ is the *dispersion relation.* Then the normal modes of an infinite string are

$$\boxed{\psi_n(x,t) = A_n \sin(\omega_n t + \beta_n) \sin(k_n x + \alpha_n), \qquad \omega_n = v_p k_n}$$

and k_n can be anything. We have absorbed all the constants into our constant A_n. Our normal modes are standing waves.

9.2 Finite case of continuous string

Suppose we fix a string of finite length to a wall and attach the other end to a massless ring constrained to move along a rod. The general solution for the normal modes are given above. The boundary conditions tell us

$$\psi(x,t)\big|_{x=0} = 0 \implies \alpha_n = 0, \qquad \psi(x,t)\big|_{x=L} = 0 \implies k_n L = \frac{\pi}{2} + n\pi, n \in \mathbb{N}. \tag{93}$$

Then we have the first three modes

$$n = 1: \qquad k_1 = \frac{\pi}{2L}, \qquad \lambda_1 = \frac{2\pi}{k} = 4L, \tag{94}$$

$$n = 2: \qquad k_2 = \frac{3\pi}{2L}, \qquad \lambda_2 = \frac{4L}{3} \tag{95}$$

$$n = 3: \qquad k_3 = \frac{3\pi}{2L}, \qquad \lambda_3 = \frac{4L}{5} \tag{96}$$

The general solution for a fixed string at one end is a linear combination of the normal modes

$$\boxed{\psi(x,t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t + \beta_n) \sin(k_n x), \qquad k_n = \frac{(2n-1)\pi}{2L}.} \tag{97}$$

9.3 Finite case of continuous string (fixed at both ends)

We can do the same process. The result is identical to the above with

$$k_n = \frac{n\pi}{L}. \tag{98}$$

Note. We can determine the coefficients A_n using Fourier's trick. Recall that

$$\int_0^L dx \ \sin k_n x \sin k_m x = \frac{L}{2}\delta_{nm} \tag{99}$$

because we have a complete orthonormal set. Then the coefficients are simply given by

$$\boxed{A_n \equiv \frac{2}{L}\int_0^L dx \ \psi(x,0) \sin k_n x.} \tag{100}$$

Example 9.1 (Plucking a string)**.** What is the evolution of the string if we release it from rest with following shape with $w = 0.75$.

$$\psi(x) = \begin{cases} x, & x \leq w \\ \frac{w(1-x)}{1-w}, & x > w \end{cases}. \tag{101}$$

The visualization of this is in the notes/provided notebook.

10 October 6th, 2021

Announcements

- Psets due Thurs, but allowed 5 free 1-day extensions. (No need to ask)
- Exam is in 2 weeks from today, cover through pset 5 (released tomorrow, lecture material through lecture 10)
- No pset day after midterm
- Friday practice exam ?
- Go through all HW + subskills, will be given extra subskill type problems. Subskill-type problems **will be on the midterm**
- Exam in class, group portion 20 minutes

10.1 Normal modes on a string (review)

10.1.1 Infinite string

Recall we guess the wavefunction as separable in position and time.

$$\psi(x,t) = A(x)B(t) \tag{102}$$

We then plug in this ansatz into the wave equation (diffusion equation).

$$\frac{\partial^2}{\partial t^2}\psi(x,t) = \nu_p^2 \frac{\partial^2}{\partial x^2}\psi(x,t) \tag{103}$$

where ν_p is the phase velocity $\frac{\omega}{k}$????

We find separable normal mode solutions of the form

$$\psi_n(x,t) = A_n \sin(\omega_n t + \beta_n)\sin(k_n x + a_n) \tag{104}$$

Any behavior of the string can be written as a sum of normal modes, so we can decompose any wave into a sum of such normal modes over the entire basis.

10.1.2 Finite string

By restricting the length of the string, only certain wavenumbers satisfy the boundary conditions. That is, we write the overall wavefunction as

$$\psi(x,t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t + \beta_n)\sin(k_n x) \tag{105}$$

If the string starts from rest, sine = 1 at $t = 0$, so in fact

$$\psi(x,t) = \sum_{n=1}^{\infty} A_n \sin(k_n x), \tag{106}$$

with

$$k_n = \frac{(2n-1)\pi}{2L} \text{ node at one end,} \qquad k_n = \frac{n\pi}{L} \text{ node at both ends} \tag{107}$$

Demo. Sand on an oscillating disc! We'll see patterns of sand on the discs according to normal modes. By touching the edge of the disc, we enforce a boundary condition, and achieve a node.

Note that normal modes are **orthogonal**! So we can find the coefficients A_n by taking the inner product. this is a discrete-ish analog of the Fourier series.

$$A_m = \frac{2}{L}\int_0^L \sin(k_m x)\psi \; dx = \frac{2}{L}\sum_{n=1}^{\infty}\int_0^L A_n \sin(k_m x)\cdot\sin(k_n x) \; dx. \tag{108}$$

This is quite nice, both for compression and for easy time evolution.

Theorem 10.1 (Fourier's theorem)**.** *Fourier's theorem says that you can describe any* ***periodic function*** *as a discrete sum of sines and cosines:*

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right]. \tag{109}$$

Moreover, we can use a Fourier series to describe a function over a finite domain by extending its repetition: the series will still be correct over the support.

11 October 13th, 2021

The contents of this class are incorporated into the previous and next lecture's notes.

12 October 18th, 2021

Announcements

- Study for test! Will help you learn. Learn subskills.
- Not a test of speed, should be simpler, no Mathematica/calculator needed.

12.1 Travelling waves at a boundary

We learned that $f(x - vt)$ is a solution to the wave equation with a pulse going in the positive x direction; $f(x + vt)$ goes in the negative direction. We recall the wave equation:

$$\frac{\partial^2 f}{\partial t^2} = v_p^2 \frac{\partial^2 f}{\partial x^2}, \qquad v_p = \sqrt{\frac{K}{\rho_\ell}} = \sqrt{\frac{T_{\text{string}}}{\rho_\ell}} \approx \frac{\text{elastic}}{\text{inertia}} \tag{110}$$

is the phase velocity.

We can describe our pulse in terms of position x or wave number k: either the wavefunction, or the wavenumber decomposition (via Fourier?). In QM, we have

$$p = \frac{\hbar}{\lambda}, \qquad k = \frac{2\pi}{\lambda} \tag{111}$$

12.1.1 A string

Last time, we considered mechanical waves on a string with tension T and linear mass density ρ. Then, it has $v = \frac{\partial \xi}{\partial t}$.

Definition 12.1 (Impedance)**.** We define the *impedance* as $Z = \frac{T}{\nu_p} = \sqrt{\rho T}$.

We have found solutions for a normal mode on an infinite string and added initial conditions and boundary conditions.

$$\psi_n(x,t) = A_n \sin(\omega_n t + \beta_n) \sin(k_n x + a_n) \tag{112}$$

where the initial conditions fix A_n, β_n and the boundary conditions fix k_n, α_n.

Now, we consider a transverse mechanical wave on a string. We can derive (force as a component of tension)

$$\boldsymbol{F} = -T \sin\theta = -T\frac{d\xi}{dx} = -T\xi'(x \pm v_p t), \qquad v_{\text{piece of string}} = \frac{d\xi}{dt} = v_p \xi'(x \pm v_p t) \tag{113}$$

We can think of $\frac{\boldsymbol{F}}{\boldsymbol{v}} = -\frac{T}{v_p} = \sqrt{T\rho}$.

12.1.2 Disturbance

By linearity, we will construct a disturbance by a linear combination of two traveling wave solutions.

$$g(x,t) = f(x - vt) + f(x + vt) \tag{114}$$

Consider a rope tied down at $x = L$. We have a boundary condition $\xi(x = L) = 0$. Then, we will have an incident, reflected, and transmitted wave. We do not observe a transmitted wave. Moreover, the boundary conditions tells us that at the wall

$$\xi_{\text{inc}}(L - vt) + \xi_{\text{ref}}(L + vt) = 0 \implies \xi_{\text{ref}} = -\xi_{\text{inc}} \tag{115}$$

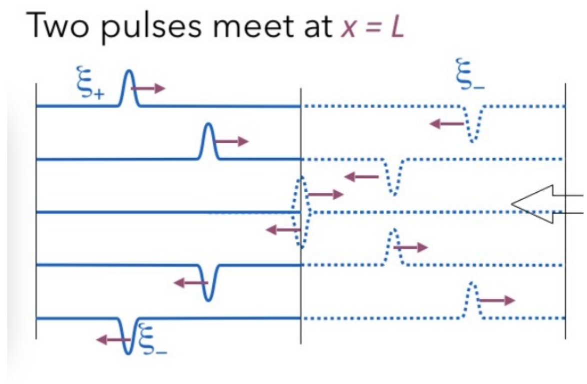

Figure 5: Mirror waves

We visualize this as ξ_+, ξ_- hit each other and cancel each other at the boundary so the constraint ($\xi = 0$) at $x = L$ is satisfied. Then, we essentially see the ξ_+ on the way in and ξ_- on the way out. These are called **mirror waves**, so the dotted waves inside the wall don't exist.

Similarly, in the example of the square pulse, we can find the Fourier components via Fourier decomposition, but we can also decompose it simply by symmetry and boundary conditions.

Now, imagine the end is not fixed but has an attached ring that can move up and down in y. Then, our boundary condition is

$$\frac{\partial \xi_{\text{inc}}(x - v_p t)}{\partial x} + \frac{\partial \xi_{\text{ref}}(x + v_p t)}{\partial x} = 0 \tag{116}$$

At $x = L$, we have $\xi_{\text{inc}}(L - v_p t) = \xi_{\text{ref}}(L - v_p t)$.

We consider a pulse on a complex string and see the simulation with boundary conditions

$$\xi_{\text{inc}}(0, t) + \xi_{\text{ref}}(0, t) = \xi_{\text{trans}}(0, t). \tag{117}$$

Moreover, the slope has to be the same

$$\frac{\partial}{\partial x} \xi_L(x, t)\bigg|_{x=0} = \frac{\partial}{\partial x} \xi_R(x, t)\bigg|_{x=0} \tag{118}$$

and

$$-\frac{1}{v_1} \xi'_{\text{inc}} + \frac{1}{v_1} \xi'_{\text{ref}} = -\frac{1}{v_2} \xi'_{\text{trans}} \tag{119}$$

We can rewrite and solve this equation and obtain

$$\boxed{\xi_{\text{trans}} = \frac{2v_2}{v_2 + v_1} \xi_{\text{inc}}, \qquad \xi_{\text{ref}} = \frac{v_2 - v_1}{v_2 + v_1} \xi_{\text{inc}}.} \tag{120}$$

We consider the following conditions:

1. If $\rho_2 = \infty$, then $v_2 = \sqrt{\frac{T}{\rho}} = 0$. Thus, $\xi_{\text{trans}} = 0, \xi_{\text{ref}} = -\xi_{\text{inc}}$.
2. If $\rho_1 < \rho_2 < \infty$, then $v_2 < v_1$, so $-1 < R < 0$, and $T < 1$
3. If $\rho_1 = \rho_2, v_2 = v_1$, so $R = 0, T = 1$.
4. If $\rho_2 \ll \rho_1, v_2 \gg v_1$, then ξ_{trans}

A whip is tapered, so that as you crack it the mass density decreases so the tip breaks the sound barrier!

13 October 25th, 2021

Announcements

- Mean for midterm was 92, std was 7.
- Error on the mean of a Gaussian is $\sigma_\mu = \frac{\sigma}{\sqrt{N}}$, where N is the number of measurements.
- How to not hook up things in an experimental lab without breaking things

Recall last time, we introduced impedance on a string, which is a measure of resistance to motion (cause over effect). Consider a string with two pieces with different densities ρ_1, ρ_2 and wave velocities v_1, v_2 respectively. We define

$$Z \equiv \frac{\text{Force}}{\text{Velocity}} = \sqrt{T\rho} \tag{121}$$

where T the tension.

Moreover, recall the reflection and transmission coefficients given respectively by

$$R \equiv \frac{v_2 - v_1}{v_1 + v_2} = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \qquad T \equiv \frac{2v_2}{v_1 + v_2} = \frac{2Z_1}{Z_1 + Z_2} \tag{122}$$

where we have equivalently expressed these coefficients in terms of impedance.

13.1 Energy of traveling waves and impedance matching

We watch a video showing transferring of energy between two waves, discussing impedance and related concepts.

Kangaroo hide has stronger tensile strength than cow hide. Tapered whips minimizes reflections and maximizes transmission along the length of the string. We watch a demo of Daniel cracking a whip.

We can perform impedance matching at various discontinuities in a string with changing densities.

Note. The wave equation is the same as the equation for an LC circuit, for some $V = V_0 e^{i(kx-\omega t)}, I = I_0 e^{i(kx-\omega t)}$:

$$\frac{\partial^2 V}{\partial t^2} = \frac{LC}{(\Delta x)^2}\frac{\partial^2 V}{\partial x^2}, \qquad \boxed{v_p = \frac{1}{\Delta x}\sqrt{LC}} \tag{123}$$

Then the impedance is the ratio of the voltage and the current, given by

$$Z = \frac{V}{I} = \sqrt{\frac{L}{C}} \sim \frac{\boldsymbol{F}}{\boldsymbol{v}}. \tag{124}$$

The analog circuit is some voltage and current and some resistance where Z is the effective impedance. We can simply set $R = \sqrt{L/C}$ to *match impedances.*

Note. Matching impedance **minimizes signal reflection** and **maximizes power output**.

14 October 27th, 2021

Announcements

- We continue our discussion of impedance and energy of strings.

Energy on a string. Consider the pulse generator with the scope. We see a reflection in impedance of the scope. Also, we discussed two strings with different densities ρ_1, ρ_2 that were connected. Similarly, we can consider a transmission line with different resistances, and a voltage pulse that is sent through the circuit. We should all take Physics 113, a new course which is easier than Physics 123.

14.1 Energy on a string

Recall the string breakdown. We calculate the kinetic and potential energy, and energy/momentum transfer down the string.

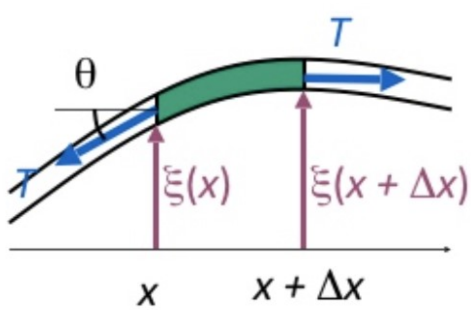

Figure 6: String

We derive the **kinetic energy**

$$E_k = \frac{1}{2}\rho_e\ dx\ \omega^2 \xi_0^2 e^{2i(kx-\omega t)} \tag{125}$$

where ω is the frequency of the wave and ξ_0 is the amplitude. Then, the total energy of the string is found by integrating the above over the entire string.

Moreover, we can find the **potential energy** as

$$\frac{E_s}{\Delta x} = T \cdot ds = \frac{T}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 dx \tag{126}$$

where this is written as an energy density for potential energy.

We can see that the total kinetic energy density and total potential energy density is the same:

$$\frac{E_k}{dx} = \frac{E_s}{dx}, \qquad T\frac{\partial \xi}{\partial x} = \rho\frac{\partial \xi}{\partial t} \tag{127}$$

The total energy density on the string is then

$$\frac{dE}{dx} = 2\frac{dE_k}{dx} \tag{128}$$

We can also derive the **energy passing a point at time** t as

$$\frac{dE}{dt} = v_{\text{wave}}\frac{dE}{dx} \implies \left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2} v\rho\omega^2 \xi_0^2. \tag{129}$$

The wave has the most kinetic energy where the string has the most spring potential energy, which occurs when it is stretched the most, since here, $T = U_{\text{spring}}$.

14.2 Longitudinal waves and sound

Now, we ask if there is momentum along the horizontal direction of a transverse wave. Since the string is more stretched at equilibrium points, the density is lower... will we see density waves?

We can consider **longitudinal waves** now. Note that we can map longitudinal waves' density pressure to transverse waves displacement, in which case many of our equations still work. The density oscillates, and we can derive similar equations based on $\rho_l + \Delta\rho_l$.

We watch a simulation of this phenomenon.

As molecules move back and forth, the displacement $A(x, t)$ is again

$$A = A_0 \cos(kx - \omega t + \phi) \tag{130}$$

Note that density is proportional to velocity, so

$$\rho \propto \rho_0 + \frac{dA}{dt} \tag{131}$$

where the second term is essentially velocity. Thus,

$$\rho = \rho_0 + \Delta\rho \sin(kx - \omega t + \phi). \tag{132}$$

From the string wave equation, we derive:

$$\mu\frac{\partial^2 A}{\partial t^2} = T\frac{\partial^2 A}{\partial x^2}, \qquad \rho\frac{\partial^2 A}{\partial t^2} = P\frac{\partial^2 A}{\partial x^2} \tag{133}$$

$$\tag{134}$$

Assuming constant temperature in the gas, and the ideal gas law $PV = nRT$, we have constant entropy, so the sound wave does not increase entropy:

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} = \sqrt{\frac{\gamma P}{\rho}} \tag{135}$$

where $\gamma = \frac{C_P}{C_V} = \frac{f+2}{f}$ with f the degrees of freedom for a given gas. For example, $\gamma = \frac{5}{3}$ for monatomic gases, and $\gamma = \frac{7}{5}$ for diatomic gases (3 translational, 1 rotational, 1 vibrational mode).

For gas at temperature T, we have

$$v_{\text{RMS}} = \sqrt{\frac{3RT}{m}}, \qquad v_{\text{sound}} = \sqrt{\frac{\gamma}{3}} v_{\text{RMS}} \tag{136}$$

which make sense physically.

14.2.1 Open tubes (wind instruments)

Consider the three cases for a tube: open-open, closed-closed, and open-closed. Note open-open and closed-closed have the same wavenumbers (but not the same shape). Recall

$$v = f\lambda. \tag{137}$$

Note. An open end of pipe acts as a free end for sound. Wind instruments are pipes with at least one open end.

15 November 1st, 2021

Announcements

- Today we move on from sound in air and cover solid and liquid

15.1 Sound

Recall the speed of any wave (and sound in particular) is:

$$v = \sqrt{\frac{K}{\rho_l}} = \sqrt{\gamma \frac{RT}{m}} \tag{138}$$

which is about 342 m/s at room temp for air.

The energy transfer rate is

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2} v \rho_l \omega^2 \xi_0^2 \tag{139}$$

where v is speed, ρ_l is linear mass density, ω is frequency, and ξ_0 is equilibrium amplitude. Generally, we can write the amplitude as $\xi(x,t) = \xi_0 \cos(kx - \omega t + \phi)$

15.1.1 Sound in solids and liquids

Consider a steel rod, we have

$$F = K\frac{\Delta l}{l}, \;\; Y \equiv \frac{K}{A} \implies \frac{F}{A} = Y\frac{\Delta l}{l} \tag{140}$$

where Y is the Young's modulus, with K depending only on material properties and thickness.

We derive

$$v_s = \sqrt{\frac{K}{\rho_l}} = \sqrt{\frac{YA}{\rho_l}} = \sqrt{\frac{Y}{\rho_v}}, \qquad \rho_v = \rho_l / A \tag{141}$$

where ρ_v is the volume density. Note that v_{steel} is 5700 m/s, which is much faster than in air.

In **liquids**, we derive

$$P = \frac{F}{A} = -M_B \frac{\Delta l}{L} \tag{142}$$

where M_B is the bulk modulus, so

$$v = \sqrt{\frac{K}{\rho_l}} = \sqrt{\frac{AM_B}{\rho_l l A}} = \sqrt{\frac{M_B}{\rho_v}} \tag{143}$$

This is 1450 for water.

15.2 Sound Intensity

Returning to air.

$$I \equiv \frac{P}{A} \propto |\psi^2| \tag{144}$$

Note. Human hearing is logarithmic, so we use the decibel system:

$$dB = 10 \log_{10} \frac{I}{I_0} \tag{145}$$

We treat most sound sources as point sources. Thus, we consider surfaces of spheres:

$$I = \frac{P}{A} = \frac{P}{4\pi r^2} \tag{146}$$

15.3 Doppler effect

We consider when the source or observer of sound is moving.

15.3.1 Moving source

Let f_0 be the frequency of the source in the source frame, and c be the speed of sound in air, and v_s be the speed of the source. Then, we derive that the frequency heard by the observer is

$$f' = \frac{1}{T'} = \frac{1}{T}\left(\frac{c}{c-v}\right) = f_0\left(\frac{c}{c-v}\right). \tag{147}$$

15.3.2 Moving observer

Let f_0 be the frequency of the source in the source frame, and c be the speed of sound in air, and v_0 be the speed of the observer. Now, we derive

$$f' = \left(\frac{v_0 + c}{c}\right) f_0. \tag{148}$$

15.3.3 Observer and source moving

Now, we find

$$f' = \left(\frac{c + v_0}{c - v_s}\right) f_0. \tag{149}$$

15.3.4 Sound barrier

If the source is moving faster than the speed of sound, then the interference of waves produces interference to create very high pressure regions

15.4 3D waves

We extend the 1D wave equation to the 3D wave equation:

$$\frac{\partial^2 \psi(x,y,z,t)}{\partial t^2} = v^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(x,y,z,t) = v^2 \boldsymbol{\nabla}^2 \psi(x,y,z,t) \tag{150}$$

There are many solutions to the 3D wave equation. We discuss the **plane wave**:

$$\psi(x,y,z,t) = \psi_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \phi), \qquad \boldsymbol{k} = (k_x, k_y, k_z), \qquad \boldsymbol{r} = (x,y,z) \tag{151}$$

This is a solution as long as $\omega = v|\boldsymbol{k}|$.

16 November 3rd, 2021

Announcements

- Wow so sparse today

16.1 Interference

Recall the 3D wave equation (assuming an isotropic medium, so same velocity in each direction).

$$\frac{\partial^2 \psi(x,y,z,t)}{\partial t^2} = v^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi(x,y,z,t) = v^2 \boldsymbol{\nabla}^2 \psi(x,y,z,t) \tag{152}$$

One special solution is the **plane wave**:

$$\psi(x,y,z,t) = \psi_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \phi), \qquad \omega = v|\boldsymbol{k}|, gh\boldsymbol{k} = (k_x, k_y, k_z), \qquad \boldsymbol{r} = (x,y,z) \tag{153}$$

As a complex exponential, the above is

$$\psi(\boldsymbol{r}, t) = Re\left[\psi_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \phi)}\right] \tag{154}$$

We work on some problems which visualize interference of two plane waves. Totally constructive interference occurs at peak-peak, trough-trough, in spherical waves. We experience a demo of interference from two speakers in the room.

16.1.1 Constructive interference

Let ψ_1, ψ_2 be two waves given by

$$\psi_1 = \psi_0 e^{i\overbrace{(kr_1 - \omega t + \phi_1)}^{\Phi_1}}, \qquad \psi_2 = \psi_0 e^{i\overbrace{(kr_2 - \omega t + \phi_2)}^{\Phi_2}}. \tag{155}$$

We can qualitatively determine *constructive interference* when the phase difference of the waves is an integer multiple of 2π:

$$\Phi_2 - \Phi_1 = k(r_2 - r_1) + (\phi_2 - \phi_1) = 2\pi n. \tag{156}$$

Here, $\phi_2 - \phi_1$ is the initial phase of the sources. If they are in phase, this is 0.

$$\Delta r = r_2 - r_1 = \frac{2\pi n}{k} = \lambda n. \tag{157}$$

In general, we can always convert between *phase* and *path length*, since one cycle of the wave is 2π radians.

$$\boxed{\Delta\phi = \frac{2\pi}{\lambda}\Delta s.} \tag{158}$$

We can derive that the scale factor has units of rad/m, so it is $k = \frac{2\pi}{\lambda}$.

16.2 Electromagnetic waves

We recall Maxwell's equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\frac{\rho}{\epsilon_0},\qquad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\qquad \boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{dt},\qquad \boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}. \tag{159}$$

In order, this is Gauss's Law, Gauss's Law for magnetic monopoles, and Maxwell-Faraday's law, and Ampere's Law. These equations govern $\boldsymbol{E}$ and $\boldsymbol{B}$ waves. The *divergence* $\boldsymbol{\nabla}\cdot\boldsymbol{A}$ measures how much the vector field spreads out. The *curl* $\boldsymbol{\nabla}\times\boldsymbol{A}$ measures how much the field rotates around a point.

In a *vacuum*, we have $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$, and $\boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$. We derive the wave equation from Maxwell's equations in a vacuum:

$$\mu_0\epsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2}=\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{B}}{\partial t}=-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E})=-[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E})-(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{E}]=(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{E}\implies\boxed{\mu_0\epsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2}=\boldsymbol{\nabla}^2\boldsymbol{E}.} \tag{160}$$

Thus, the speed of light is $c=\frac{1}{\sqrt{\mu_0\epsilon_0}}$.

We consider the *plane wave soluion*:

$$\boldsymbol{E}(\boldsymbol{r},t)=Re\left[\boldsymbol{E_0}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right],\qquad \omega=c|\boldsymbol{k}|. \tag{161}$$

Using Maxwell's and dropping the real dependence,

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0\implies\boldsymbol{\nabla}\cdot\left(\boldsymbol{E}_0e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right)=0. \tag{162}$$

We show that

$$\boldsymbol{k}\cdot\boldsymbol{B}_0=0 \tag{163}$$

Demo. Light-bulbs being lit up when oriented in the correct field direction. This shows how the standing $\boldsymbol{E}$ and standing $\boldsymbol{B}$ fields are perpendicular and alternate. $\boldsymbol{E}$ creates $\boldsymbol{B}$ and vice versa.

17 November 8th, 2021

Recall

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \boldsymbol{E}, \qquad \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \boldsymbol{B} \tag{164}$$

with wave velocity $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ with ϵ_0 the electric permittivity and μ_0 the magnetic permeability. Note that there is **no dispersion**, all EM frequencies travel at the same speed.

The solution to the wave equation are plane waves

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}, \qquad \boldsymbol{B} = \boldsymbol{B}_0 e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} \tag{165}$$

where $B_0 = \frac{E_0}{c}$. For a travelling wave, $\boldsymbol{E}$ is **in phase** with $\boldsymbol{B}$.

17.1 EM standing waves

We can create a standing wave by adding

$$\boldsymbol{E}_1 = E_0 \cos(kz - \omega t)\hat{\boldsymbol{x}}, \qquad \boldsymbol{E}_2 = E_0 \cos(kz + \omega t)\hat{\boldsymbol{x}} \implies \boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = 2E_0 \cos kz \cos \omega t \ \hat{\boldsymbol{x}} \tag{166}$$

To compute $\boldsymbol{B}$, we can use Maxwell's equations to find

$$-\frac{\partial \boldsymbol{B}}{\partial t} = -2E_0 k \sin kz \cos \omega t \hat{\boldsymbol{y}} \implies \boldsymbol{B} = E_0 \frac{k}{\omega} \sin kz \sin \omega t \hat{\boldsymbol{y}} \tag{167}$$

Thus, for a standing wave, $\boldsymbol{E}$ and $\boldsymbol{B}$ are **out of phase by** $\frac{\pi}{2}$. Since $|\boldsymbol{E}|$ is maximum when $|\boldsymbol{B}| = 0$, this affects energy flow.

17.2 Boundary conditions

We will now consider boundary conditions. Consider a conducting wall, so $\boldsymbol{E} = 0$ at the wall, and there must be a reflected wave with flipped sign.

Thus, we calculate the total $\boldsymbol{E}$ on LHS is

$$E_{\text{total}}^{\text{LHS}} = \frac{E_0}{2}[\cos(kz - \omega t) - \cos(-kz - \omega t)] = E_0 \sin(\omega t) \sin(kz)\hat{\boldsymbol{x}} \tag{168}$$

which is just a standing wave! We can show

$$\boldsymbol{B}_{\text{total}} = \frac{E_0}{c} \cos(\omega t) \cos(kz)\hat{\boldsymbol{y}} \tag{169}$$

Then, $\boldsymbol{B}$ and $\boldsymbol{E}$ are **not** in phase.

17.2.1 Directional energy flux

The energy transfer per unit area per unit time has units Watts/m^2. The energy density in waves is

$$\epsilon \equiv \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \tag{170}$$

We also introduce the Poynting vector $\boldsymbol{S}$

$$\boldsymbol{S} \equiv \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B}) \tag{171}$$

in the direction $\hat{\boldsymbol{k}}$, which is parallel to the velocity of the wave.

For a travelling wave, $\boldsymbol{E}$ and $\boldsymbol{B}$ are in phase, so

$$\boldsymbol{S} = \frac{1}{\mu_0}\frac{E^2}{c}\hat{\boldsymbol{k}} = c\epsilon_0 E^2\hat{\boldsymbol{k}} = c\epsilon \tag{172}$$

where ϵ is the energy density. Thus, energy flow is the product of velocity and energy density.

Example 17.1 (Plane wave)**.** If $\boldsymbol{E} = E_0\cos(kz - \omega t)$, then

$$\langle\epsilon\rangle = \frac{1}{2}\epsilon_0 E_0^2, \qquad \langle\boldsymbol{S}\rangle = \frac{1}{2}c\epsilon_0 E_0^2, \qquad \langle\boldsymbol{S}\rangle = v\langle\epsilon\rangle$$

Example 17.2 (Standing wave)**.**

$$\boldsymbol{E} = A\cos kz\cos\omega t\ \hat{\boldsymbol{x}}, \qquad \boldsymbol{B} = \frac{A}{c}\sin kz\sin\omega t\ \hat{\boldsymbol{y}}, \qquad \boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B}) = \frac{1}{\mu_0}A^2\cos kz\sin kz\cos\omega t\sin\omega t\hat{\boldsymbol{z}} \tag{173}$$

Averaging over one period of oscillation,

$$\cos\omega t\sin\omega t = \frac{1}{2}\sin 2\omega t \implies \langle\sin 2\omega t\rangle_T = 0 \tag{174}$$

so there is no net energy flow, which makes sense because the energy oscillates back and forth between the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

Note (Momentum)**.** We can calculate the force on the atom, and find that the momentum comes from $q\boldsymbol{v}\times\boldsymbol{B}$ which is always in the $\hat{\boldsymbol{z}}$ direction. This turns out to be a small force for the Earth. Radiation pressure for photons is $p = \frac{E}{c}/$

17.3 Polarization

Here, $\boldsymbol{E}$ defines the **polarization direction**. In the simplest case we have $\boldsymbol{E} = E_0\cos(kz - \omega t)\hat{\boldsymbol{x}}$, while it can be a superposition of two solutions like $\boldsymbol{E} = E_0\cos(kz - \omega t + \phi_1)\hat{\boldsymbol{x}} + E_0\cos(kz - \omega t + \phi_2)\hat{\boldsymbol{y}}$. More generally,

$$\boldsymbol{E}(z,t) = Re[\boldsymbol{\psi}_0 e^{i(kz-\omega t)}], \qquad \boldsymbol{\psi}_0 = \psi_1\hat{\boldsymbol{x}} + \psi_2\hat{\boldsymbol{y}}, \qquad \psi_1 = A_1 e^{i\phi_1}, \qquad \psi_2 = A_2 e^{i\phi_2} \tag{175}$$

and

$$\boldsymbol{E}(z,t) = Re\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}e^{i(kz-\omega t)}. \tag{176}$$

Example 17.3 (Linear Polarization)**.** Let $\phi_1 = \phi_2$, $A_1 = A_2$. Then,

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = E_0\cos(kz - \omega t)\hat{\boldsymbol{x}} + E_0\cos(kz - \omega t)\hat{\boldsymbol{y}} \tag{177}$$

The field forms a 45° angle with the x-axis, and oscillates back and forth along this line. We say $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = E_0\begin{pmatrix}1\\ 1\end{pmatrix}$, but linear polarization can be for any arbitrary angle θ

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = E_0\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

Example 17.4 (Circular Polarization)**.** Now we consider $\phi_1 \neq \phi_2$, $\Delta\phi = \pi/2$

$$\boldsymbol{E}_1 = E_0 \cos(kz - \omega t)\hat{\boldsymbol{x}}, \qquad \boldsymbol{E}_2 = E_0 \sin(kz - \omega t)\hat{\boldsymbol{y}} = E_0 \cos(kz - \omega t - \pi/2)\hat{\boldsymbol{y}} \tag{178}$$

and

$$\boldsymbol{E}_{\text{total}} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = \text{Re}\left[E_0\hat{\boldsymbol{x}} + E_0 \underbrace{e^{-i\pi/2}}_{-i}\hat{\boldsymbol{y}}\right] e^{i(kz-\omega t)} = \text{Re}\left[E_0 \begin{pmatrix} 1 \\ -i \end{pmatrix}\right] e^{i(kz-\omega t)}, \qquad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E_0 \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{179}$$

We consider what this looks like, and find that it goes in a circle in the xy plane over time, so this is circular polarization.

Example 17.5 (Elliptical Polarization)**.** For completeness, consider

$$\boldsymbol{E}_1 = E_0 \cos(kx - \omega t)\hat{\boldsymbol{x}}, \qquad \boldsymbol{E}_2 = 2E_0 \sin(kx - \omega t)\hat{\boldsymbol{y}} \implies \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -i \end{pmatrix}. \tag{180}$$

More generally, we have

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = e^{i\phi} \begin{pmatrix} A\cos\theta - iB\sin\theta \\ A\sin\theta + iB\cos\theta \end{pmatrix}. \tag{181}$$

We can also consider arbitrary polarization in the $\hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ directions

$$\boldsymbol{E}_{\text{initial}} = E_x e^{i(kx-\omega t)}\hat{\boldsymbol{x}} + E_y e^{i(kz-\omega t)} \tag{182}$$

The polarizer absorbs the $\hat{\boldsymbol{y}}$ component so

$$\boldsymbol{E}_{\text{final}} = E_x e^{i(kx-\omega t)}\hat{\boldsymbol{x}} \tag{183}$$

Malus' Law states that intensity is proportional to E^2 of light, so

$$I_{\text{final}} = I_{\text{initial}} \cos^2\theta \tag{184}$$

We can make circularly polarized light using interesting materials. If the velocity of light in material is $v = \frac{c}{n}$, then when $n_x \neq n_y$ waves must have the same frequency but different λ in different materials. Using these boundary conditions, we can solve $n_1\lambda_1 = n_2\lambda_2$ and make $\frac{1}{4}$ wave plates.

18 November 10th, 2021

Announcements

- There is a lot to get through.

18.1 Circular Polarization

Last time, we talked about electromagnetic fields with boundary conditions, and electromagnetic wave polarization. Recall our derivation of linear polarization in the easy case. We also review circularly polarized light, and touch on elliptically polarized light.

18.1.1 Malus' law

The intensity of electromagnetic wave is proportional to E^2, so

$$E_{\text{final}} = E_{\text{initial}} \cos^2 \theta \tag{185}$$

We perform some calculations with this law.

18.2 Index of refraction

Reference: Feynman Lectures Volume 1, Ch. 31 Index of Refraction.

Velocity of light is

$$v = \frac{c}{n}. \tag{186}$$

The two materials are side by side with n_1, n_2: the frequency is the same but wavelength changes.

Consider a material with different $n_x \neq n_y$. Then, field propagates differently in $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$.

$$\Delta\phi = \frac{2\pi\ell}{\lambda_x} - \frac{2\pi\ell}{\lambda_y}. \tag{187}$$

We can choose ℓ for a particular ω, such that $\Delta\phi = \frac{\pi}{2}$, which gives us circular polarization as a $\frac{1}{4}$ wave plate. We can also create a $\frac{1}{2}$ wave plate with $\Delta\phi = \pi$.

18.3 Snell's law

We consider two derivations

18.3.1 Principle of least time

The path is an extremal path. It is a minimum, as nearby pathlengths are similar and add constructively; further away, they add destructively. We see a demo for how much the path length changes when further away from the minimum.

Snell's Law says that $\theta_{\text{incidence}} = \theta_{\text{reflection}}$. Fermat added for refraction to use the principle of least time:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{188}$$

18.3.2 Boundary conditions on a physical membrane

We have ρ, T with ψ_L for $x < 0$ and ρ', T' with ψ_R for $X > 0$. The 2D wave solution is $\psi = A\sin(k_x x)\sin(k_y y)\cdot$ $\sin(\omega t + \phi$ **WHAT IS THIS**

In the most general case, $\psi_L = \psi_{\text{incoming}} + \psi_{\text{reflected}}, \qquad \psi_R = \psi_{\text{transmitted}}$

$$\psi_L = Ae^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \sum_\alpha R_\alpha e^{i(\boldsymbol{k}_\alpha\cdot\boldsymbol{r}-\omega t)}, \qquad \psi_R = \sum_\beta T_\beta e^{i(\boldsymbol{k}_\beta\cdot\boldsymbol{r}-\omega t)} \tag{189}$$

Since the frequency is the same, we have

$$|k_\alpha|^2 = \frac{\omega^2\rho}{T} = \frac{\omega^2}{v^2}, \qquad |k_\beta|^2 = \frac{\omega^2\rho'}{T'} = \frac{\omega^2}{v'^2} \tag{190}$$

At the membrane boundary $x = 0$, $\psi_L = \psi_R$. This is only true if $k_{\alpha,y} = k_{\beta,y} = k_y$ just the y components of k since the boundary condition is at $x = 0$. Then,

$$k_{\alpha,x} = -\sqrt{\frac{\omega^2}{v^2} - k_y^2}, \qquad k_{\beta,x} = -\sqrt{\frac{\omega^2}{v'^2} - k_y^2} \implies k_{\alpha,x} = k_x \tag{191}$$

Now, considering the $\boldsymbol{k}$ vectors, we see that

$$|\boldsymbol{k}|\sin\theta = |\boldsymbol{k}'|\sin\theta' \implies n\sin\theta = n'\sin\theta' \tag{192}$$

which is Snell's Law.

18.4 Dispersion

If the medium is a dispersive medium, then different wavelengths travels at different speeds. Then a pulse will get wider and wider.

We see the demo of the laser sound: hitting a longitudinal wave.

19 November 15th, 2021

Ear takes 10 ms from pressure to neuron, eye takes 40ms from light on retina to neuron. 30 ft away = both are same.

19.1 Dispersion

Consider a string with a wave traveling down it, if we have $v(\lambda)$ with different velocities for different wavelengths, then the shape of the wave will change as it travels.

Consider a pulse created by adding two waves

$$y_1(x,t) = A\cos(k_1x - \omega_1 t), \qquad y_2(x,t) = A\cos(k_2x - \omega_2 t). \tag{193}$$

Recall the envelope. We define

$$\omega_+ = \frac{\omega_1 + \omega_2}{2}, \qquad \omega_- = \frac{\omega_1 - \omega_2}{2}, \qquad k_+ = \frac{k_1 + k_2}{2}, \qquad k_- = \frac{k_1 - k_2}{2}. \tag{194}$$

We can rewrite $y_1 + y_2$ in terms of these new definitions. Recall $v_{\text{phase}} = \frac{\omega}{k}$. We have $v_{\text{group}} = \frac{\partial\omega}{\partial k}$.

The dispersion relation is, for instance,

$$k = \frac{\sqrt{\omega^2 - \omega_0^2}}{v}. \tag{195}$$

Then, v_{phase} depends on λ. On a real piano string, $v = \frac{\omega}{k} = \sqrt{\frac{T}{\rho_l}}$. We find $\frac{\omega}{k} = v\sqrt{1 + \alpha k^2}$

Example 19.1 (Waveguide)**.**

$$v_{\text{group}} = \frac{\partial\omega}{\partial k} < c = c\sqrt{1 - \frac{\omega_0^2}{\omega^2}} \tag{196}$$

Example 19.2 (Water waves)**.** If $\lambda >>$ depth of water, we get a **tsunami**.

$$\omega = \sqrt{gH}k \tag{197}$$

where H is the depth. Here, $v_p = v_g = \sqrt{gH}$.

In summary,

$$\underbrace{v_{\text{phase}}}_{\text{no information}} = \frac{\omega}{k}, \qquad \underbrace{v_{\text{group}}}_{\text{information}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{198}$$

There are three cases for comparing v_g, v_p. But $v_p > c$ is possible.

19.2 Electromagnetic waves revisited

Recall that an electromagnetic wave carries energy via the Poynting vector

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} \tag{199}$$

We consider the power on shell. Now, note that when we accelerate the charge q the $\boldsymbol{E}$ field struggles to catch up, generating an interesting field.

UNCLEAR WHAT IS GOING ON HERE, REVIEW LATER

19.3 Fourier transform and delta function

We require some mathematical machinery for generating and describing a complicated pulse on a string.

We can describe a travelling pulse

$$f(x,t) = \int_{-\infty}^{\infty} d\omega\ F(\omega) e^{-i\omega t} e^{ikx} \tag{200}$$

19.3.1 Delta function

The delta function $\delta(x)$ has properties

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} = \delta(\omega-\omega'), \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega')\delta(\omega-\omega')d\omega' = F(\omega) \tag{201}$$

For any wave, we can describe it in either wavevector or frequency space, $f(t), F(\omega)$. An infinite plane wave we know the exact frequency, while a delta function has no frequency.

From this, we can roughly estimate the classical uncertainty relation

$$\Delta f \Delta t \geq \frac{1}{4\pi} \tag{202}$$

In quantum mechanics, the Heisenberg uncertainty relation is

$$\Delta E \Delta t \geq \frac{h}{4\pi}, \qquad \Delta p \Delta x \geq \frac{h}{4\pi} \tag{203}$$

19.3.2 Transmitting a signal through a dispersive medium

Say we wish to transfer

$$f_{\text{signal}} = f_s \cos\omega_s t \tag{204}$$

So, we use a carrier wave $\cos\omega_0 t$,

$$f(t) = f_s \cos\omega_s t \cos\omega_0 t = \frac{1}{2} f_s [\cos(\omega_0 + \omega s)t + \cos(\omega_0 - \omega_s)t] \tag{205}$$

We can consider the dispersion curve. Still frequency dependent attenuation.

20 November 17th, 2021

Announcements

- We have one problem set drop
- Homework 9 due on Tues, homework 10 out Tuesday due during reading pd
- Final will have *problem* problems

20.1 Huygen's principle

Let each point on a wavefront be replaced by a point source. Then the outgoing wave is the same (one point source vs. replacing each point with a point source).

20.1.1 Deriving Snell's law with Huygen's principle

Replace all points in the wavefront of an incoming wavefront with point sources. Using some tools of geometry, we can find the difference in pathlengths of incoming and outgoing ray. Comparing angles, we obtain Snell's law.

INCLUDE PICTURE OF DERIVATION LATER

20.1.2 Huygen for double slit interference

To quantitatively determine whether a point will be constructive or destructive interference, we can sum up both waves, or we can just check the relative phase of the waves at that point.

$$\Delta\Phi = \begin{cases} 0, & \text{constructive} \\ \pi, & \text{destructive} \end{cases}. \tag{206}$$

Note. Other things might contribute to phase. We will discuss this later.

Demo. We can do this with light! Recall the lab, where we project an image from the computer onto a screen. We have a mask, lens, a Fourier plane with an iris, and then another lens before the screen. We compare different double slit widths and spacings. For fixed narrow widths, increasing the spacing between them decreases the constructive interference and creates shorter bright spots.

We can draw the diagrams and perform the math for two sources, comparing the distance (and then amplitude) of the point sources at some point p. Let the two rays have source distances r_1, r_2 and the middle parallel ray have distance r From this, we derive

$$\boldsymbol{E}_1 = E_0 e^{i(kr_1 - \omega t)}, \qquad \boldsymbol{E}_2 = E_0 e^{i(kr_2 - \omega t)} \tag{207}$$

and

$$\begin{aligned} \boldsymbol{E}_{\text{total}} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = E_0 e^{-i\omega t}\left[e^{ikr_1} + e^{ikr_2}\right] &= E_0 e^{-i\omega t}\left[e^{ik(r-\Delta r)} + e^{ik(r+\Delta r}\right] \\ &= E_0 e^{-i\omega t} e^{ikr}\left[e^{-ik\Delta r} + e^{ik\Delta r}\right] \\ &\implies \boldsymbol{E}_p = 2E_0 e^{ikr - i\omega t}\cos(k\Delta r) \end{aligned} \tag{208}$$

This is the field at point p, where $\Delta r = \frac{d\sin\theta}{2}$. The field is wildly oscillating. Note that we observe the intensity is

$$I \propto \text{Amplitude}^2 \tag{209}$$

and our amplitude is $2E_0\cos(k\Delta r)$, so

$$\frac{I(\theta)}{I(0)} = \left[\frac{E_p(\theta)}{E_p(0)}\right]^2 \propto \cos^2\left[\frac{kd\sin\theta}{2}\right]. \tag{210}$$

this is the **far field limit** which is good for all θ. The bright bands occur when $k = \frac{2\pi}{\lambda}$:

$$\frac{kd\sin\theta}{2} = \pm n\pi, \qquad n = 0, 1, 2, \ldots, \qquad d\sin\theta = \pm n\lambda \tag{211}$$

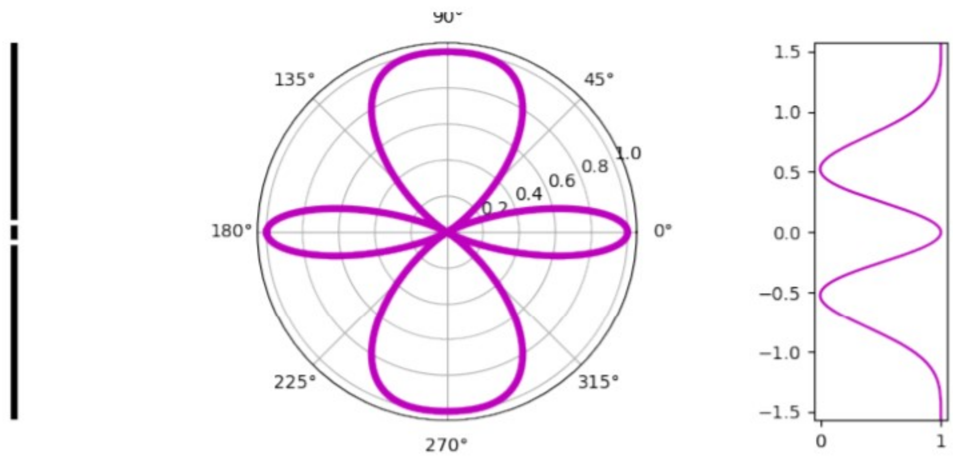

Figure 7: Interference pattern on polar plot

We consider the interference patterns not only on a 1D screen, but also on a 2D polar plot. We can derive the separation distance by observing that there is constructive interference along the axes of the sources, so the sources are separated by an integer multiple of λ. Moreover, since there is no other line of constructive interference between $0°, 90°$, then this pathlength must be exactly λ.

20.1.3 N equally spaced slits

For N equally spaced slits, we obtain

$$\frac{I(\alpha)}{I(0)} = \left(\frac{\sin(\frac{N\alpha}{2})}{N\sin(\frac{\alpha}{2})}\right)^2. \tag{212}$$

Demo. Diffraction gratings

20.1.4 Thin film interference

A slight tint for the viewer when looking at camera lenses or glasses.

We can solve the math with reflection and transmission coefficients. Alternatively, we consider the picture where a ray of light passes through a glass in air. There are two reflected rays, one from the air-glass barrier (front of glass) and one from the glass-air barrier (back of glass), and these rays will interfere.

Demo. Shining a light around a spherical ball, we see a bright spot in the center, showing that light behaves as a wave.

21 November 22nd, 2021

The contents of this class are incorporated into the previous and next lecture's notes.

22 November 29th, 2021

Recall the intensity for different interference patterns.

For *double split interference*, we have

$$\frac{I(\theta)}{I(0)} \approx \cos^2 \frac{\pi d \sin\theta}{\lambda}, \qquad d\sin\theta = n\lambda. \tag{213}$$

For *N-slit interference* (diffraction grating), we have

$$\frac{I(\theta)}{I(0)} \approx \left[\frac{\sin N\alpha/2}{N\sin\alpha/2}\right]^2, \qquad d\sin\theta = n\lambda. \tag{214}$$

We note that the first destructive bound is at $n = 1$.

For diffraction from a *single slit*, we have

$$\frac{I(\theta)}{I(0)} \approx \left[\frac{\sin\beta/2}{\beta/2}\right]^2, \qquad \beta \equiv kD\sin\theta, \qquad D\sin\theta = n\lambda \tag{215}$$

For interference and diffraction, we have

$$\frac{I(\theta)}{I(0)} \approx \left[\frac{\sin\beta/2}{\beta/2}\frac{\sin N\alpha/2}{N\sin\alpha/2}\right]^2, \qquad \alpha \equiv kd\sin\theta, \qquad \beta \equiv kD\sin\theta. \tag{216}$$

We note that for interference and diffraction, the diffraction destroys interference in between the central cluster and tails.

22.1 Rayleigh criterion

When light enters the eye, it causes a diffraction pattern on the back of your retina. If the angle θ between two rays of light is small, then the peaks will overlap.

The *Rayleigh criterion* specifies the *minimum distance* for two objects need to be resolvable. The minimum resolvable angular separation is

$$\boxed{\theta = \frac{1.22\lambda}{D}} \tag{217}$$

for circular apertures. The constant comes from Bessel functions!

22.2 Free space propagation

We recall that the far-field diffraction pattern is the Fourier transform (squared).

Consider a transparency T where we let light through if transparent and no light if not transparent. We can think of the transparency as a sum of point sources. Let us define

$$T(x) = \begin{cases} 1, & \text{transparent} \\ 0, & \text{opaque} \end{cases}. \tag{218}$$

Let $E(X)$ be the electric field at position X. Then

$$E(X) \propto \int dx\ T(x)e^{ikr}. \tag{219}$$

We have distance z between the transparency and the plate, and x, X be the point on the transparency and plate, respectively. Then

$$r^2 = z^2 - (X - x)^2 \implies r = z\left[1 + \frac{(X-x)^2}{z^2}\right]^{1/2}. \tag{220}$$

We can consider three areas. The pattern near the field, the "middle-zone" where the image gets some fringes, and far region where the image does **not** look like the transparency.

In the near-field region, the image on the screen looks like a shadow and we make no approximations. r is the same as above.

In the "middle zone", we see *Fresnel diffraction*. In this region, $X - x \ll z$ and we take a Taylor expansion and

$$r \approx z\left[1 + \frac{1}{2}\frac{(X-x)^2}{z^2}\right] \approx z + \frac{1}{2z}\left(X^2 + x^2 - 2Xx\right). \tag{221}$$

Far away, we see *Fraunhofer diffraction* where we neglect the x^2 term because on this scale, x^2 becomes small. Then

$$r \approx z + \frac{1}{2z}\left(X^2 - 2Xx\right). \tag{222}$$

It is important to note that in Franhofer diffraction, we have

$$E(X) \propto \int dx \; T(x) e^{-ik\frac{Xx}{z}} \tag{223}$$

which is a Fourier transform of $T(x)$, where $x \mapsto \frac{kX}{z}$. We note that the z and X^2 terms in r for the Fraunhofer diffraction produce constants that we absorb out of the integral.

22.2.1 Physical intuition

We can think of the wave form near the aperture as a wave in three dimensions. This can be written as a sum of plane waves with different wave vectors $\boldsymbol{k}_x$.

Because the distances between the wave packets are so small and the distance they travel is so large, they arrive at different times in different positions, which extracts the spatial frequencies.

22.3 Lenses

When a plane wave hits a lens, the lens will focus the light at some point f, the *focal length.* We have

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} \tag{224}$$

where d_o, d_i are the distance of the object and image, respectively.

Note. If the incoming wave is at an angle, it will focus in a different position. The job of a lens, then, is to convert incoming plane waves into spherical waves and vice versa. This introduces a phase shift of $e^{-i\pi x^2}\lambda f$. To turn the plan waves into spherical waves, we must speed up the light near the edges!

Moreover, we note that this phase shift induces a Fourier transform in the Fresnel region because $z = f$.

We then discussed the application of lenses in 15c lab.

23 December 1st, 2021

Today, we will think about quantum mechanics with the help of some demos.

Demo. The diffraction pattern of a slit in a block and the inverse are the same. This is because the bump and a corresponding trough have the same frequency Fourier decomposition.

Claim (Melissa Franklin)**.** "All the important things in the diffraction pattern are the same: the gaps, the holes, the bananas, the ducks."

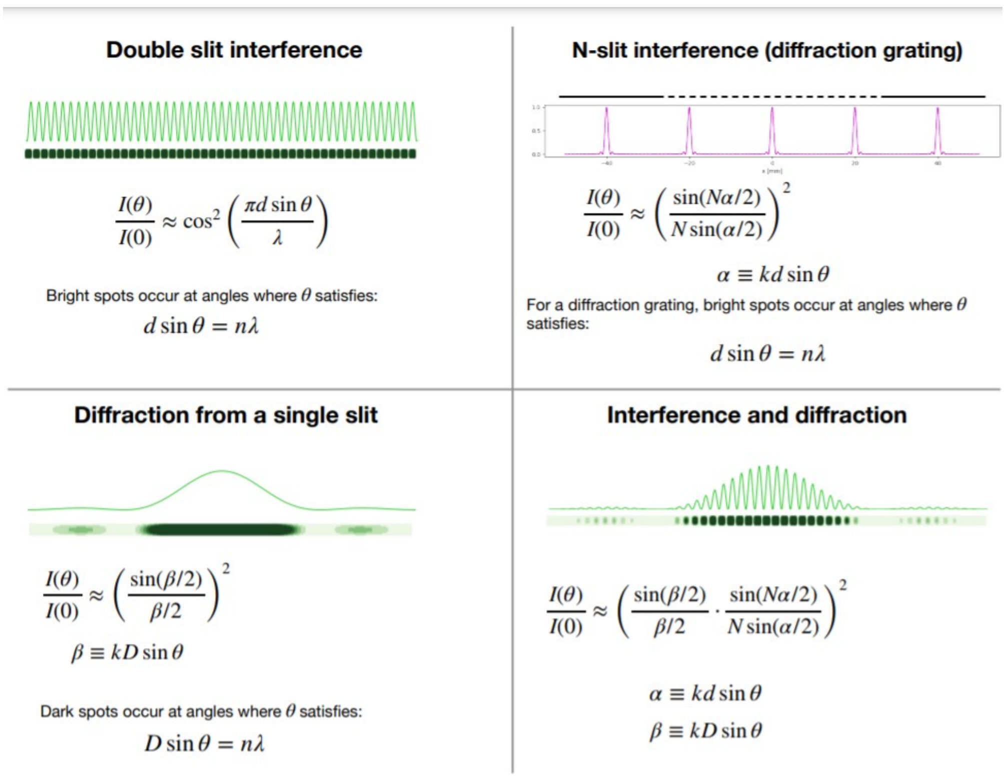

Figure 8: Various interference patterns

23.1 Photoelectric Effect

Hertz found that the number of *liberated* electrons depend on the frequency f of light. Classically, $I \propto E^2$. Quantum mechanically, energy is *quantized.* $E = hf$ where h is Planck's constant.

We can consider an electron that is liberated in a metal (induced by a laser where we can control the frequency and intensity). Then we can also change the voltage across the plate which induces an electric field. We increase this voltage until *no particles* are able to traverse the distance between the plates. We can then obtain the kinetic energy given by

$$T = eV \tag{225}$$

where e is the fundamental charge.

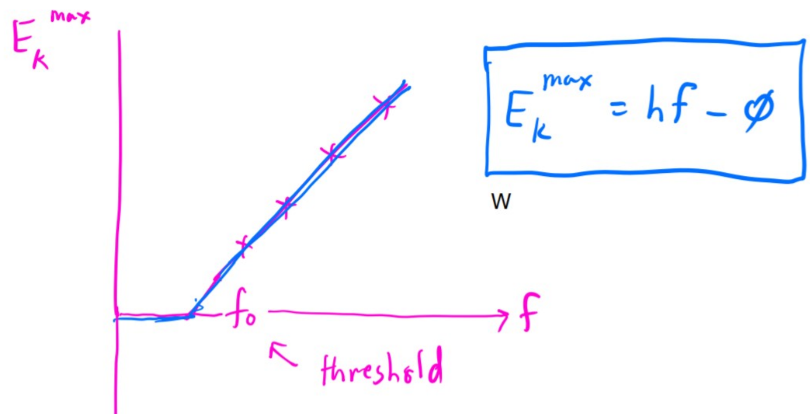

Figure 9: Energy of liberated electrons due to photoelectric effect and work function

After exceeding the work function ϕ, the energy increases linearly. Thus, we have that the electron energy is

$$T_{\max} = hf - \phi \tag{226}$$

where h is the slope, ϕ is the work function (and projected intercept).

Note (De Broglie wavelength)**.** We will find that all objects have a *de Broglie wavelength*. Recall that this is the length scale at which objects begin to behave *both* like a particle and a wave.

Consider the demo of slits with polarizers. We see that the intensity only shows an interference pattern if the slits are polarized 45°. This is because after the last filter, we have reprojected the particles 45° and we obtain the same pattern as a regular double slit experiment. This is related to the *quantum eraser* experiment.

23.1.1 Double slit revisited

This treatment of the double slit is taken from Feynman Volume 3 Chapter 1.

Let's consider double slit experiments with different beams, bullets and water waves. In water, we have $I \propto h^2$, where h is the height of the water wave. We find that the intensity distribution is identical between the water wave and electron double split experiment.

In the interference experiment with the electron, the square of the wave function $|\phi_1 + \phi_2|^2$ is interpreted as the probability amplitude. This is analogous to the intensity of water through two slits $I_{12} = |h_1 + h_2|^2$ where h_i is the height of hole i. Note that probability amplitudes allow for interference.

The conclusion is that we cannot know what slit the electron went through in the double slit experiment.

23.1.2 Electron Diffraction

An electron is also a wave! We can observe its interference in the demo.

The **de Broglie wavelength** is defined as $\lambda = h/p$.