Constraints for metric estimation

$$\mathbf{d} \in \mathbb{R}^N, \quad \mathbf{d}[i] = \|\mathbf{x}_i - \mathbf{u}\|^2_{\mathbf{M}}$$

- Assuming $\mathbf{M}$ and $\mathbf{d}$ are known, we have a *closed form expression for* $\mathbf{u}$:

$$\mathbf{u} = \frac{1}{2}\mathbf{M}^\dagger\mathbf{R}^\dagger(\mathbf{a}_\mathbf{M} - \mathbf{Q}_\Gamma\mathbf{d})$$

$\mathbf{R}$: "Feature matrix" that depends on item feature vectors **(known)**

$\mathbf{a}_\mathbf{M}$: "Item-metric vector" constructed from ground truth metric and item features

$\mathbf{Q}_\Gamma$: "Selection matrix" selects rows based on items used in comparisons **(known)**

- This leads to a condition that our metric and distances must satisfy:

$$\boxed{\mathbf{0} = (\mathbf{I} - \mathbf{R}\mathbf{R}^\dagger)(\mathbf{a}_\mathbf{M} - \mathbf{Q}_\Gamma\mathbf{d})}$$

A joint estimator for ideal point and metric

- Estimate the metric **and** distances via constrained SDP

$$(\widehat{\mathbf{M}}, \widehat{\mathbf{d}}, \widehat{\zeta}) = \arg\min_{\mathbf{M},\mathbf{d},\zeta} \ell(\mathbf{d}, \mathbf{y}) + \gamma_1 \|\zeta\|_1 + \gamma_2 \|\mathbf{M}\|_F^2 + \gamma_3 \|\mathbf{d}\|_2^2$$

$$\text{s.t.} \qquad -\zeta \leq (\mathbf{I} - \mathbf{R}\mathbf{R}^\dagger)(\mathbf{a}_\mathbf{M} - \mathbf{Q}_\Gamma \mathbf{d}) \leq \zeta$$

$$\zeta \geq \mathbf{0}, \quad \mathbf{M} \succeq \mathbf{0}$$

- Use estimate for metric and distances to solve directly for ideal point

$$\hat{\mathbf{u}} = \frac{1}{2}(\widehat{\mathbf{M}}\mathbf{R}^\top\mathbf{R}\widehat{\mathbf{M}} + \alpha\mathbf{I})^{-1}\widehat{\mathbf{M}}\mathbf{R}^\top(\mathbf{a}_{\widehat{\mathbf{M}}} - \mathbf{Q}_\Gamma\hat{\mathbf{d}})$$