Math 114, Problem Set 6 (due Monday, October 28)

October 21, 2013

(1) Let X be a metric space, and let $\{x_n\}_{n\geq 0}$ be a sequence of points in X which satisfies the following conditions:

   (a) For every subsequence $\{x_{i_0}, x_{i_1}, \ldots\}$ of $\{x_n\}_{n\geq 0}$, there exists a further subsequence $\{x_{i_{j_0}}, x_{i_{j_1}}, \ldots\}$ which converges.

   (b) For any pair of convergent subsequences $\{x_{i_0}, x_{i_1}, x_{i_2}, \ldots\}$, $\{x_{j_0}, x_{j_1}, x_{j_2}, \ldots\}$ of $\{x_n\}_{n\geq 0}$, the limits $\lim\{x_{i_n}\}$ and $\lim\{x_{j_n}\}$ are the same.

Show that the sequence $\{x_n\}_{n>0}$ converges.

(2) Let E be a measurable subset of $\mathbb{R}^m \times \mathbb{R}^n$. For each $x \in \mathbb{R}^m$, let $E_x = \{y \in \mathbb{R}^n : (x, y) \in E\}$. Show that E has measure zero if and only if the sets E_x have measure zero for almost every x.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} 1 & \text{if } (\exists n \in \mathbf{Z}_{\geq 0})[n \leq x, y < n+1] \\ -1 & \text{if } (\exists n \in \mathbf{Z}_{\geq 0})[n \leq x < n+1 \leq y < n+2] \\ 0 & \text{otherwise.} \end{cases}$$

For each $x \in \mathbb{R}$, let f_x denote the function given by $f_x(y) = f(x, y)$. For each $y \in \mathbb{R}$, let f_y denote the function given by $f_y(x) = f(x, y)$. Show that the functions

$$x \mapsto \int_{\mathbb{R}} f_x \qquad y \mapsto \int_{\mathbb{R}} f_y$$

are integrable, and compute their integrals (in other words, compute the double integrals $\int(\int f(x,y)dx)dy$ and $\int(\int f(x,y)dy)dx$.) Why does the result not contradict Fubini's theorem?

(4) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the inequality

$$\phi(\frac{x+y}{2}) \leq \frac{\phi(x) + \phi(y)}{2}$$

for all $x, y \in \mathbb{R}$. Show that ϕ is convex: that is, for each real number $\lambda \in [0, 1]$, we have

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

for all $x, y \in \mathbb{R}$.

(1) Let X be a metric space, and let $\{x_n\}_{n\geq 0}$ be a sequence of points in X which satisfies the following conditions:

(a) For every subsequence $\{x_{i_0}, x_{i_1}, \ldots\}$ of $\{x_n\}_{n\geq 0}$, there exists a further subsequence $\{x_{i_{j_0}}, x_{i_{j_1}}, \ldots\}$ which converges.

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Show that the sequence $\{x_n\}_{n>0}$ converges.

By (b), $\exists x$ s.t. if $\lim\{x_{i_n}\}$ exists. then $\lim x_{i_n} = x$

Suppose $\{x_n\}$ does not converges to x.

For $\varepsilon > 0$, $\forall k$. $\exists n_k$ s.t. $n_k > k$ & $|x_{n_k} - x| > \varepsilon$

$(n_1 < n_2 < n_3 < \cdots)$

$\therefore \forall k \geq 0$, $|x_{n_k} - x| > \varepsilon$

By (a). there exists a further subsequence of $\{x_{n_k}\}_{k\geq 0}$ which converges to x.

$\Rightarrow \{x_{n_{k_0}}. x_{n_{k_1}}. \cdots\} \subseteq \{x_{n_k}\}$. $\lim\limits_{j\to+\infty} x_{n_{k_j}} = x$

However, for every j. $|x_{n_{k_j}} - x| > \varepsilon$

$\Rightarrow \lim\limits_{j\to\infty} x_{n_{k_j}} \neq x$

$\therefore \{x_n\}$ converges x.

$\boxtimes$

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} 1 & \text{if } (\exists n \in \mathbf{Z}_{\geq 0})[n \leq x, y < n+1] \\ -1 & \text{if } (\exists n \in \mathbf{Z}_{\geq 0})[n \leq x < n+1 \leq y < n+2] \\ 0 & \text{otherwise.} \end{cases}$$

For each $x \in \mathbb{R}$, let f_x denote the function given by $f_x(y) = f(x,y)$. For each $y \in \mathbb{R}$, let f_y denote the function given by $f_y(x) = f(x,y)$. Show that the functions

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are integrable, and compute their integrals (in other words, compute the double integrals $\int(\int f(x,y)dx)dy$ and $\int(\int f(x,y)dy)dx$.) Why does the result not contradict Fubini's theorem?

$$f_x(y) = \begin{cases} 1. & x \geq 0 \text{ and } [x] \leq y < [x]+1 \\ -1. & x \geq 0 \text{ and } [x]+1 \leq y < [x]+2 \\ 0. & \text{otherwise} \end{cases}$$

$$f_y(x) = \begin{cases} 1. & y \geq 0 \text{ and } [y] \leq x < [y]+1 \\ -1. & y \geq 1 \text{ and } [y]-1 \leq x < [y] \\ 0. & \text{otherwise} \end{cases}$$

$$\Rightarrow \int_{\mathbb{R}} f_x = \int f_x(y)dy = 0.$$

$$\int_{\mathbb{R}} f_y = \int f_y(x)dx = \begin{cases} 1. & y \in [0.1) \\ 0. & \text{else} \end{cases}$$

$$\Rightarrow \int\left(\int f(x.y)dx\right)dy = 1. \quad \int\left(\int f(x.y)dy\right)dx = 0$$

Since $f(x.y)$ is not integrable, it does not contradict Fubini's Theorem.

(pf)

$|f| \geq \chi_{\mathcal{S}}$ where $\mathcal{S} = \{(x.y) : n \leq x.y < n+1.\ n \in \mathbb{Z}_{\geq 0}\}$

$$\mathcal{F} = \bigcup_{n=0}^{\infty} \underbrace{\{(x,y) : n \le x, y < n+1\}}_{\text{measure } = 1} \Rightarrow \mu(\mathcal{F}) = \infty$$

$$\Rightarrow \int_{\mathbb{R}^2} |f| \ge \int_{\mathbb{R}^2} \chi_{\mathcal{F}} = \mu(\mathcal{F}) = \infty$$

$\boxed{//}$

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for all $x, y \in \mathbb{R}$. Show that ϕ is convex: that is, for each real number $\lambda \in [0,1]$, we have

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for all $x, y \in \mathbb{R}$.

① Prove when $\lambda = \alpha/2^i \in [0,1]$ by induction.

Let $\alpha = 2\beta + 1$ (odd number)

$$\phi\left(\frac{\alpha}{2^{i+1}}x + \left(1 - \frac{\alpha}{2^{i+1}}\right)y\right)$$

$$\le \frac{1}{2}\left(\phi\left(\left(\frac{\beta}{2^i}\right)x + \left(1 - \frac{\beta}{2^i}\right)y\right) + \phi\left(\left(\frac{\beta+1}{2^i}\right)x + \left(1 - \frac{\beta+1}{2^i}\right)y\right)\right)$$

$$\le \frac{1}{2}\left(\left(\frac{\beta}{2^i} + \frac{\beta+1}{2^i}\right)\phi(x) + \left(2 - \frac{\beta}{2^i} - \frac{\beta+1}{2^i}\right)\phi(y)\right)$$

$$= \left(\frac{\alpha}{2^{i+1}}\right)\phi(x) + \left(1 - \frac{\alpha}{2^{i+1}}\right)\phi(y)$$

② Generally $\lambda \in [0,1]$

select a convergent sequence $\{\alpha_i/2^i\} \to \lambda$.

Fix $x, y \in \mathbb{R}$ and let $\varepsilon > 0$.

$f(z) := \phi(zx + (1-z)y)$

By continuity and convergence.

$\exists n > 0$ s.t. $\left|\frac{\alpha_n}{2^n} - \lambda\right| < \frac{\varepsilon}{2\max(|\phi(x)|, |\phi(y)|)}$

$\Rightarrow \left|f\left(\frac{\alpha_n}{2^n}\right) - f(\lambda)\right| \leq \frac{\varepsilon}{2}$

$\therefore f(\lambda) \leq f\left(\frac{\alpha_n}{2^n}\right) + \frac{\varepsilon}{2} \leq \frac{\alpha_n}{2^n}\phi(x) + \left(1 - \frac{\alpha_n}{2^n}\right)\phi(y) + \frac{\varepsilon}{2}$

$\leq \lambda\phi(x) + (1-\lambda)\phi(y) + \frac{3}{2}\varepsilon$

Since ε is arbitrarily chosen

$f(\lambda) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$

$\blacksquare$