Math 114, Problem Set 4 (due Monday, October 7)

September 29, 2013

(1) Let $E \subseteq \mathbb{R}^n$ be a measurable set and let $f : E \to \mathbb{R}$ be a nonnegative measurable function. Show that the set $\{x \in E : f(x) \neq 0\}$ has measure zero if and only if $\int_E f = 0$.

(2) Let $E \subseteq \mathbb{R}^n$ be a measurable set, and let $f_1, f_2, \ldots$ be a sequence of measurable functions on E which converges pointwise to another function $f : E \to \mathbb{R}^n$. Show that there exists a sequence of subsets

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E$$

such that $\mu(E - \bigcup E_i) = 0$ and the sequence $\{f_i|_{E_j}\}$ converges uniformly to $f|_{E_j}$ for each j.

(3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Show that for each $\epsilon > 0$, there exists a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that the set $\{x \in \mathbb{R}^n : f(x) \neq g(x)\}$ has measure $< \epsilon$.

(4) Let $E \subseteq \mathbb{R}^m$ and $E' \subseteq \mathbb{R}^n$ be measurable sets. Show that $E \times E'$ is a measurable subset of $\mathbb{R}^{m+n}$, and that $\mu_{\mathbb{R}^{m+n}}(E \times E') = \mu_{\mathbb{R}^m}(E)\mu_{\mathbb{R}^n}(E')$. Here $\mu_{\mathbb{R}^k}$ denotes Lebesgue measure on $\mathbb{R}^k$ (hint: reduce to the case where $\mu_{\mathbb{R}^m}(E) < \infty$ and study the function $S \mapsto \mu_{\mathbb{R}^{m+n}}(E \times S)$).

(1) Let $E \subseteq \mathbb{R}^n$ be a measurable set and let $f : E \to \mathbb{R}$ be a nonnegative measurable function. Show that the set $\{x \in E : f(x) \neq 0\}$ has measure zero if and only if $\int_E f = 0$.

($\Rightarrow$) $f = 0$ a.e. on E.

Let $\forall E' \subset E$. $m(E') < \infty$

$g : E' \to \mathbb{R}$ (bounded. measurable) with $0 \le g \le f$.

$S \subset E'$. $S = \{x \in E' : g(x) = 0\}$

Since $f = 0$ a.e. on $E' \subset E$. $0 \le g \le f$

$\Rightarrow g = 0$ a.e. on E' $\Rightarrow m(E' - S) = 0$.

$$\therefore \int_E f = \sup_g \left\{ \int_{E'} g \right\}$$

$$= \sup_g \left\{ \int_S g + \int_{E'-S} g \right\} = \sup_g \left\{ 0 + \int_{E'-S} g \right\}$$

$$\le \sup_g \left\{ m(E'-S) \cdot \left(\sup_{E'-S} g \right) \right\} = 0$$

$\square$

($\Leftarrow$) $\int_E f = 0$. $(\lambda > 0)$

Let $E_\lambda \subset E$. $E_\lambda = \{x \in E : f(x) \ge \lambda\}$

by Chebyshev's inequality. $m(E_\lambda) \le \frac{1}{\lambda} \int_E f$

$T \subset E$. $T := \{x \in E : f(x) > 0\}$.

$T = \cap_\lambda E_\lambda$. Choose $\lambda = \frac{1}{n}$ $(n \in \mathbb{N})$

$\Rightarrow m(T) \le \frac{1}{n} \int_E f = 0$ for any $n \in \mathbb{N}$.

$\therefore m(T) = 0$

$\square$

(2) Let $E \subseteq \mathbb{R}^n$ be a measurable set, and let $f_1, f_2, \ldots$ be a sequence of measurable functions on E which converges pointwise to another function $f : E \to \mathbb{R}^n$. Show that there exists a sequence of subsets

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E$$

such that $\mu(E - \bigcup E_i) = 0$ and the sequence $\{f_i|_{E_j}\}$ converges uniformly to $f|_{E_j}$ for each j.

Egoroff's Theorem *Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\epsilon > 0$, there is a closed set F contained in E for which*

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \sim F) < \epsilon.$$

Fix $\varepsilon > 0$. Applying Egoroff to $(f_j)_{j\in\mathbb{N}}$ restricted to the domain $B_m(0) \cap E$. $\exists A_m \subset B_m(0) \cap E$

s.t. $m((B_m(0) \cap E) - A_m) < \frac{\varepsilon}{2^m}$ & $f_j|_{A_m} \to f|_{A_m}$ uniformly as $j \to \infty$.

$\Rightarrow f_j \to f$ uniformly when restricted to the domain $E_M = \bigcup_{m=1}^{M} A_m$

$\Rightarrow E_0 \subset E_1 \subset \cdots \subset E$ with unif. conv on each E_M.

(Claim) $\mu(E - \bigcup_{M=1}^{\infty} E_M) = 0$

$$E - \bigcup_{M=1}^{\infty} E_M = E \cap \left(\bigcup_{M=1}^{\infty} E_M\right)^c = E \cap \left(\bigcup_{m=1}^{\infty} A_m\right)^c$$

$$= E \cap \left(\bigcap_{m=1}^{\infty} A_m^c\right)$$

$$= E \cap \bigcap_{m=1}^{\infty} \left\{(B_m(0))^c \cup ((B_m(0) \cap E) - A_m)\right\}$$

$$= \lim_{k\to\infty} \bigcap_{m=k}^{\infty} ((B_m(0) \cap E) - A_m)$$

$$0 \le \mu(E - \bigcup_{m=1}^{\infty} E_m) \le \lim_{m\to\infty} m((B_m(0) \cap E) - A_m) \to 0$$

$$\therefore \mu(E - \bigcup_{m=1}^{\infty} E_m) = 0$$

▨

(3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Show that for each $\epsilon > 0$, there exists a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that the set $\{x \in \mathbb{R}^n : f(x) \neq g(x)\}$ has measure $< \epsilon$.

This is a problem extending the case of Lusin's Theorem for $\mu(E) < \infty$ to the case where $E = \mathbb{R}^n$

Let $\varepsilon > 0$.

& partition $\mathbb{R}^n$ into countably many disjoint unit boxes E_i.

Since measure of the closure of E_i : finite

$\Rightarrow$ by Lusin's Theorem,

$\exists$ closed set $K_i \subseteq \overline{E_i}$. conti ftn $g_i : \mathbb{R}^n \to \mathbb{R}$

s.t. $f = g_i$ on K_i & $\mu(\overline{E_i} - K_i) < \frac{\varepsilon}{2^{i+1}}$

Take $F_i \subseteq \overline{E_i}$ where $\mu(\overline{E_i} - F_i) < \frac{\varepsilon}{2^{i+1}}$

define a continuous bump function $\beta_i : \overline{E_i} \to \mathbb{R}$.

s.t. precisely one on $\overline{E_i} - F_i$ and goes to 0 on the boundaries of $\overline{E_i}$.

$\Rightarrow$ $g_i \beta_i : \overline{E_i} \to \mathbb{R}$ conti

($\because$ product of two conti ftns.)

Let $S_i := \{x \in \overline{E_i} : f(x) \neq g_i(x)\beta_i(x)\}$

$$\mu(S_i) < \mu(\overline{E_i} - K_i) + \mu(\overline{E_i} - F_i) = \frac{\varepsilon}{2^i}$$

Since each $g_i\beta_i$ is zero at the boundaries of the box,

$g : \mathbb{R}^n \to \mathbb{R}$. $g(x) = g_i(x)\beta_i(x)$ for $x \in \overline{E_i}$.

g: continuous function

$f(x) \neq g(x)$ when $x \in \bigcup_{i \geq 0} S_i$

$\Rightarrow \mu\left(\bigcup_{i \geq 0} S_i\right) \leq \sum_{i \geq 0} \mu(S_i) < \sum_{i \geq 0} \frac{\varepsilon}{2^i} = \varepsilon$

g: our desired function. $\blacksquare$