Math 114, Problem Set 1 (due Monday, September 16)

September 9, 2013

(1) Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function satisfying $f(0) = f(\pi) = 0$, and define real numbers $a_1, a_2, \ldots$ by the formula

$$a_n = \frac{2}{\pi} \int_0^\pi \sin(nx) f(x) dx.$$

Show that the sum $\sum_{n>0} a_n^2$ converges (hint: compare the sum with the integral $\int_0^\pi f(x)^2 dx$).

(2) Let $f : [0, \pi] \to \mathbb{R}$ be the discontinuous function given by the formula

$$f(x) = \begin{cases} 1 & \text{if } \frac{\pi}{4} < x < \frac{3\pi}{4} \\ 0 & \text{otherwise.} \end{cases}$$

Determine the real numbers $a_n = \frac{2}{\pi} \int_0^\pi \sin(nx) f(x) dx$. Using the identity

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \tan^{-1}(1) = \frac{\pi}{4},$$

compute the value of the infinite sums

$$g(x) = \sum_{n>0} a_n \sin(nx)$$

when $x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$.

(3) Let $V \subseteq \mathbb{R}^n$ be a linear subspace of dimension $< n$. Show that the outer measure $\mu^*(V)$ is equal to zero.

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$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$. (Fourier series)

$\underbrace{\int_0^\pi \left(f(x) - \sum_{n=1}^{N} a_n \sin nx\right)^2 dx}_{(*)} \geq 0$ (sharp when N is large.)

Since f is continuous, f achieves some absolute maximum M on a compact domain. $\Rightarrow |f(x)| \leq M$ on $x \in [0.\pi]$

$$\left(f(x) - \sum_{n=1}^{N} a_n \sin nx\right)^2 \leq \left(f^2 + \sum_{n=1}^{N} a_n^2 \sin^2 nx\right)(1 + \cdots + 1)$$
$$\leq \left(f^2 + \sum_{n=1}^{N} a_n^2\right)(N+1) \leq (N+1)\left(M^2 + \sum_{n=1}^{N} a_n^2\right)$$

$\therefore (*)$: finite-valued.

$$(*) \iff \int_0^\pi f(x)^2 dx - 2\int_0^\pi f(x)\sum_{n=1}^{N} a_n \sin nx\, dx + \int_0^\pi \left(\sum_{n=1}^{N} a_n \sin nx\right)^2 dx \geq 0$$
$$\iff 2\sum_{n=1}^{N} a_n \int_0^\pi f(x)\sin nx\, dx - \sum_{m,n=1}^{N} a_m a_n \int_0^\pi \sin mx \sin nx\, dx$$
$$\leq \int_0^\pi f(x)^2 dx$$

(Riemann integration commutes with finite summation of integrable functions.)

$$2\sum_{n=1}^{N} a_n \int_0^\pi f(x)\sin nx\, dx = 2\sum_{n=1}^{N} a_n \cdot \frac{\pi}{2} a_n = \pi \sum_{n=1}^{N} a_n^2$$

$$\int_0^\pi \sin mx \sin nx\, dx = \begin{cases} 0 & .\ m \neq n \\ \frac{\pi}{2} & .\ m = n \end{cases}$$

$$\therefore (*) \iff \pi \sum_{n=1}^{N} a_n^2 - \frac{\pi}{2}\sum_{n=1}^{N} a_n^2 \leq \int_0^\pi f(x)^2 dx$$
$$\iff \frac{\pi}{2}\sum_{n=1}^{N} a_n^2 \leq \int_0^\pi f(x)^2 dx \quad \text{for any } N.$$

$\therefore \sum_{n=1}^{\infty} a_n^2 \leq \frac{2}{\pi}\int_0^\pi f(x)^2 dx \leq 2M^2$. by MCT, $\sum_{n=1}^{\infty} a_n^2$ converges.

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Determine the real numbers $a_n = \frac{2}{\pi}\int_0^\pi \sin(nx)f(x)dx$. Using the identity

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when $x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$.

$$a_n = \frac{2}{\pi}\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \sin nx \, dx = \frac{2}{\pi n}\left(\cos\frac{\pi n}{4} - \cos\frac{3\pi n}{4}\right) = \frac{4}{\pi n}\sin\frac{\pi n}{2}\cdot\sin\frac{\pi n}{4}$$

$$\Rightarrow g(x) = \sum_{n>0}\frac{4}{\pi n}\sin\frac{\pi n}{2}\sin\frac{\pi n}{4}\sin nx$$

① $$g\left(\frac{\pi}{4}\right) = \sum_{n>0}\frac{4}{n\pi}\cdot\sin\frac{\pi n}{2}\cdot\sin^2\frac{\pi n}{4}$$

$$= \sum_{k>0}\frac{4}{\pi(2k-1)}(-1)^k\sin^2\frac{\pi(2k-1)}{4}$$

$$= \sum_{k>0}\frac{4}{\pi(2k-1)}\cdot\frac{(-1)^{k+1}}{2} = \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2}$$

② $$g\left(\frac{3}{4}\pi\right) = \sum_{n>0}\frac{4}{\pi n}\sin\frac{\pi n}{2}\cdot\sin\frac{\pi n}{4}\cdot\sin\frac{3\pi n}{4}$$

$$= \sum_{k>0}\frac{4}{\pi(2k-1)}(-1)^{k+1}\frac{1}{2}$$

$$= \sum_{k>0}\frac{2}{\pi(2k-1)}(-1)^{k+1} = \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2}$$

$\blacksquare$

(3) Let $V \subseteq \mathbb{R}^n$ be a linear subspace of dimension $< n$. Show that the outer measure $\mu^*(V)$ is equal to zero.

Let $\{v_1, \cdots, v_k\}$: orthonormal basis for the k-dimensional linear subspace $V \subset \mathbb{R}^n$.

P : k-dimensional parallelepiped spanned by $\{v_1, \cdots, v_k\}$

- E.T.S. outer measure $\mu^*(P) = 0$.

($\because$ ① V: countable union of translates of P
② outer measure is translation-invariant
③ countable subadditivity)

Following collections of boxes all cover P.

- Collection 0

$C_0 := \{ B^{(0)}_{i_1 \cdots i_k} : 0 \le i_1, i_2, \cdots, i_k \le 1, \in \mathbb{Z} \}$

where $B^{(0)}_{i_1 \cdots i_k}$ is an open n-box of sidelength 2 centred at the vertex of the parallelepiped given by $\sum_{\alpha=1}^{k} i_\alpha v_\alpha$.

- This is a cover of P, since

$\| \frac{1}{2} \sum_{\alpha=1}^{k} v_\alpha \| \le \frac{1}{2^k} \sum_{\alpha=1}^{k} \| v_\alpha \| \le \frac{k}{2^k} < 1$

- The total volume of $C_0 = 2^n \cdot 2^k = 2^{n+k}$.

- Collection 1

$C_1 := \{ B^{(1)}_{i_1 \cdots i_k} : 0 \le i_1, i_2, \cdots, i_k \le 2, \in \mathbb{Z} \}$

where $B^{(1)}_{i_1 \cdots i_k}$ is an open n-box of sidelength 1 centred at the vertex of the parallelepiped given by $\sum_{\alpha=1}^{k} \frac{1}{2} i_\alpha v_\alpha$.

- This is a cover of P, since

$$\left\|\frac{1}{2}\sum_{\alpha=1}^{k} V_\alpha\right\| \le \frac{1}{2^k}\sum_{\alpha=1}^{k}\|V_\alpha\| \le \frac{k}{2^k} < 1$$

- The total volume of $C_1 = 1^n \cdot 3^k = 3^k$

- Collection 2

$C_2 := \{ B^{(2)}_{i_1 \cdots i_k} : 0 \le i_1, i_2, \cdots, i_k \le 3, \in \mathbb{Z} \}$

where $B^{(2)}_{i_1\cdots i_k}$ is an open n-box of sidelength $\frac{2}{3}$ centred at the vertex of the parallelepiped given by $\sum_{\alpha=1}^{k} \frac{1}{3} i_\alpha V_\alpha$.

- This is a cover of P.
- The total volume of $C_2 = \left(\frac{2}{3}\right)^n \cdot 4^k = \frac{2^{n+2k}}{3^n}$

$\vdots$

- Collection m

$C_m := \{ B^{(m)}_{i_1 \cdots i_k} : 0 \le i_1, i_2, \cdots, i_k \le m+1, \in \mathbb{Z} \}$

where $B^{(m)}_{i_1\cdots i_k}$ is an open n-box of sidelength $\frac{2}{m+1}$ centred at the vertex of the parallelepiped given by $\sum_{\alpha=1}^{k} \frac{1}{m+1} i_\alpha V_\alpha$.

- This is a cover of P.
- The total volume of $C_m = \left(\frac{2}{m+1}\right)^n \cdot (m+2)^k = 2^n \cdot \frac{(m+2)^k}{(m+1)^n}$

$\therefore C_m \to 0$ as $m \to \infty$. $\Rightarrow \mu^*(P) = 0$ $\quad \therefore \mu^*(V) = 0$

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