Math 114, Problem Set 3 (due Monday, September 30)

September 24, 2013

(1) Let $E \subseteq \mathbb{R}^n$ be a measurable set with $\mu(E) < \infty$. Show that for each $\epsilon > 0$, there exists a set $E' \subseteq \mathbb{R}^n$ which is a finite disjoint union of open boxes satisfying

$$\mu(E - E'), \mu(E' - E) < \epsilon.$$

(2) Let $f_1, f_2, \ldots : \mathbb{R}^n \to \mathbb{R}$ be a sequence of measurable functions and suppose that for each $\vec{x} \in \mathbb{R}^n$, the sequence $\{f_i(\vec{x})\}$ is bounded. Show that the function $f(\vec{x}) = \limsup\{f_i(\vec{x})\}$ is measurable.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that f is *Borel measurable* if, for every real number t, the set $\{x \in \mathbb{R} : f(x) \leq t\}$ is Borel measurable.

(3) Prove that if $f, g : \mathbb{R} \to \mathbb{R}$ are Borel measurable functions, then the composition $g \circ f$ is Borel measurable.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Show that there exists a Borel measurable function g which is equal to f almost everywhere.

(1) Let $E \subseteq \mathbb{R}^n$ be a measurable set with $\mu(E) < \infty$. Show that for each $\epsilon > 0$, there exists a set $E' \subseteq \mathbb{R}^n$ which is a finite disjoint union of open boxes satisfying

$$\mu(E - E'), \mu(E' - E) < \epsilon.$$

From Problem Set II. we know that for $\varepsilon > 0$.

$\exists$ compact set $K \subseteq E$ s.t. $\mu(E) - \varepsilon \leq \mu(K) \leq \mu(E)$

$\Rightarrow$ let $\{B_i\}$: covering of K with countably many open boxes

s.t. $\mu(K) \leq \mu(\bigcup_{i=0}^{\infty} B_i) \leq \mu(K) + \varepsilon$

Since K is compact.

$\exists$ finite subcover of K $\Rightarrow E' = \bigcup_{k=0}^{n} B_k \left(= \bigcup_{i=0}^{\infty} B_i \right)$

We can assume that $\{B_k\}$ are disjoint.

($\because$ closure contains K, boundaries of the boxes have measure zero.)

$\therefore \mu(E) - \varepsilon \leq \mu(K) \leq \mu(E') \leq \mu(K) + \varepsilon$

$\Rightarrow \mu(E - E') = \mu(E) - \mu(E \cap E') \leq \mu(E) - \mu(K) \leq \varepsilon$

$\mu(E' - E) = \mu(E') - \mu(E \cap E') \leq \mu(E') - \mu(K) \leq \varepsilon$

($\because E \supset K$. $E' \supset K \Rightarrow E \cap E' \supset K \Rightarrow \mu(E \cap E') \geq \mu(K)$)

$\blacksquare$

(2) Let $f_1, f_2, \ldots : \mathbb{R}^n \to \mathbb{R}$ be a sequence of measurable functions and suppose that for each $\vec{x} \in \mathbb{R}^n$, the sequence $\{f_i(\vec{x})\}$ is bounded. Show that the function $f(\vec{x}) = \limsup\{f_i(\vec{x})\}$ is measurable.

$\{f_i(\vec{x})\}$: bounded $\Rightarrow \limsup\{f_i(\vec{x})\}$ exists.

For each $\vec{x} \in \mathbb{R}^n$, $f(\vec{x}) \le \alpha$

$\Leftrightarrow$ non-increasing sequence $\sup_{m \ge M} f_m(\vec{x})$ converges to some number $\le \alpha$ as $M \to \infty$.

$\Leftrightarrow$ for each $n \in \mathbb{Z}_+$,

$\sup_{m \ge M} f_m(\vec{x}) \le \alpha + \frac{1}{n}$ for sufficiently large M.

$\Leftrightarrow$ for each $n \in \mathbb{Z}_+$,

$\exists$ sufficiently large M s.t. $f_m(\vec{x}) \le \alpha + \frac{1}{n}$ for all $m \ge M$

$\therefore \{x \in \mathbb{R}^n : f(\vec{x}) \le \alpha\}$

$= \bigcap_{n=1}^{\infty} \bigcup_{M=1}^{\infty} \bigcap_{m \ge M} \{\vec{x} \in \mathbb{R}^n : f_m(\vec{x}) \le \alpha + \frac{1}{n}\}$

: measurable ($\because$ ① f_m : measurable. ② countable intersection or union of measurable sets is also measurable)

$\therefore f$: measurable $\blacksquare$

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that f is *Borel measurable* if, for every real number t, the set $\{x \in \mathbb{R} : f(x) \leq t\}$ is Borel measurable.

(3) Prove that if $f, g : \mathbb{R} \to \mathbb{R}$ are Borel measurable functions, then the composition $g \circ f$ is Borel measurable.

(E.T.S.) if

$\forall t \in \mathbb{R}.\ f^{-1}((-\infty . t]) = \{x \in \mathbb{R} : f(x) \leq t\}$ is Borel,

then, for all Borel sets $E \subset \mathbb{R}$. $f^{-1}(E)$ is Borel.

$$\left(\because \text{For } \alpha \in \mathbb{R}.\ \{x : (f \circ g)(x) > \alpha\} = (f \circ g)^{-1}[(\alpha . \infty)] = g^{-1}\big[\underbrace{f^{-1}[(\alpha . \infty)]}_{\text{Borel set}}\big] \right)$$

Borel Set $E \subset \mathbb{R}$ can be constructed by taking countable unions. intersections, and complements of the sets $\{(-\infty . t_j] : j \in \mathbb{N}\}$.

$\Rightarrow$ $f^{-1}(E)$ expressible by taking countable unions. intersections, and complements of $\{f^{-1}((-\infty . t_j]) : j \in \mathbb{N}\}$

$\Rightarrow$ Since $\{f^{-1}((-\infty . t_j]) : j \in \mathbb{N}\}$ contains only Borel sets, $f^{-1}(E)$ is Borel.

$\blacksquare$

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Show that there exists a Borel measurable function g which is equal to f almost everywhere.

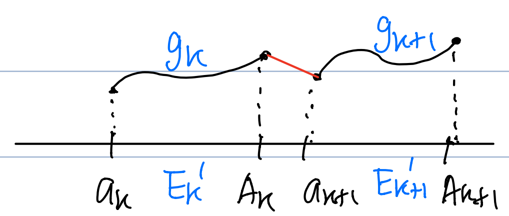
① Prove Lusin's Theorem on all of $\mathbb{R}$.

Fix some small $\eta > 0$.
Apply the finite-measure version of Lusin's Thm to each of the intervals $I_k = \left[k + \frac{\eta}{2^{|k|}},\ k+1 - \frac{\eta}{2^{|k|}}\right]$ for $k \in \mathbb{Z}$.

$\Rightarrow \exists$ continuous functions $g_k : \mathbb{R} \to \mathbb{R}$ s.t. on some $E_k' \subset I_k$ with $m(I_k - E_k') < \varepsilon_k$ s.t $f|_{E_k'} = g(k)$

Let $E' = \bigcup_{k=-\infty}^{\infty} E_k'$ $\Rightarrow$ Note that $m(\mathbb{R} - E') < 9 \cdot \eta$

Each E_k' can be chosen compact $\Rightarrow$ max : A_k. min : a_k



$\Rightarrow$ make it continuous by a straight line segment between $(A_k.\ g_k(A_k))$. $(a_{k+1}.\ g_{k+1}(a_{k+1}))$

$\therefore$ for any $\eta > 0$, $\exists$ continuous ftn $G : \mathbb{R} \to \mathbb{R}$
s.t. $f|_{E'} = G$. $m(\mathbb{R} - E') < 9 \cdot \eta \to 0$

$\boxed{//}$

② (Claim) The pointwise limits of sequences of continuous functions are Borel measurable.

(pf) Almost same as Royden ch3. Prop 9 (p60-61)

③ By Lusin on all of $\mathbb{R}$, we can get a sequence of continuous functions $G^{(1)}, G^{(2)}, \cdots$ such that for each $j \in \mathbb{N}$,
$f = G^{(j)}$ except on a set of measure at most $\frac{1}{j}$.
Take $G: \mathbb{R} \to \mathbb{R}$ to be the Borel function defined by
$G(x) := \lim_{j \to \infty} G^{(j)}(x)$

Since $m(\mathbb{R} \setminus (E')^{(j)}) = \frac{1}{j}$ for all $j \in \mathbb{N}$ $\to$ measure : 0.
$\therefore f(x) = G(x)$ a.e. $\boxed{}$