# Math 114, Problem Set 7 (due Monday, November 4)

November 1, 2013

(1) Let $V_0, V_1, V_2, V_3, \ldots$ be real vector spaces with norms

$$|| \bullet ||_n : V_n \to \mathbb{R}_{\geq 0}\,.$$

Given an element $\vec{v} = (v_n)_{n \geq 0} \in \prod_{n \geq 0} V_n$, let

$$||\vec{v}|| = \sum_{n \geq 0} ||v_n||_n \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Let $V \subseteq \prod_{n \geq 0} V_n$ be the subset consisting of those elements $\vec{v}$ such that $||\vec{v}|| < \infty$. Show that $V$ is real vector space and that $\vec{v} \mapsto ||\vec{v}||$ is a norm on $V$. If each $V_n$ is a Banach space, show that $V$ is a Banach space. We will refer to $V$ as the $\ell^1$-*sum* of the Banach spaces $\{V_n\}_{n \geq 0}$.

(2) Suppose we are given a sequence $E_0, E_1, E_2, \ldots \subseteq \mathbb{R}^m$ of pairwise disjoint measurable subsets of $\mathbb{R}^m$. Let $E = \bigcup E_n$. Show that $L^1(E)$ is isomorphic to the $\ell^1$-sum of the Banach spaces $L^1(E_n)$.

(3) Let $E \subseteq \mathbb{R}^n$ be a measurable set, let $p$ and $q$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and suppose that $f \in L^p(E)$, $g \in L^q(E)$ are functions satisfying

$$\int_E fg = ||f||_{L^p} ||g||_{L^q}.$$

Prove that either $f = 0$, or there exists a nonnegative real number $\lambda$ such that $|g| = \lambda |f|^{p/q}$ almost everywhere.

(4) Let $p, q, r > 1$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $E \subseteq \mathbb{R}^n$ be measurable, and let $f \in L^p(E)$ and $g \in L^q(E)$. Show that the product function $fg$ belongs to $L^r(E)$, and that

$$||fg||_{L^r} \leq ||f||_{L^p} ||g||_{L^q}.$$

(1) Let $V_0, V_1, V_2, V_3, \ldots$ be real vector spaces with norms

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Let $\vec{v}, \vec{w} \in V$ and $\lambda \in \mathbb{R}$

Define. $\vec{v} + \vec{w} = (v_n + w_n)_{n\geq 0}$. $\lambda\vec{v} = (\lambda \vec{v_n})_{n\geq 0}$

$\Rightarrow$ $V$ : vector space.

Next we show that $\|\cdot\|$ is a norm.

① $\|v\| \geq 0$.

$\|v\| = 0 \Leftrightarrow \|v_n\|_n = 0$ for all $n$. $\Leftrightarrow \vec{v} = 0$.

② $\lambda \|\vec{v}\| = \sum_{n\geq 0} \lambda \|v_n\|_n = \sum_{n\geq 0} \|\lambda v_n\|_n = \|\lambda \vec{v}\|$

③ $\|\vec{v_1}\| + \|\vec{v_2}\| = \sum_{n\geq 0}\left(\|v_{1.n}\|_n + \|v_{2.n}\|_n\right) \geq \sum_{n\geq 0} \|v_{1.n} + v_{2.n}\|_n$

$= \|\vec{v_1} + \vec{v_2}\|$ $\boxed{//}$

If each $V_n$ : Banach space. (complete normed linear space)

(Claim) $V$ : Banach space

Let $\{v_i\}$ be a Cauchy sequence in $V$.

For all $\varepsilon > 0$. $\exists N$ s.t. $j, k > N \Rightarrow \|v_j - v_k\| < \varepsilon$.

$\Rightarrow$ $v_i = (v_{n.i})_{n\geq 0}$. each sequence $v_{n,i}$ : Cauchy

Since each $V_n$ : Banach space.

$\lim_{i\to\infty} v_{n.i} = v_n \in V_n$. $v = (v_n)_{n\geq 0} = \lim_{i\to\infty} v_i$

Now we must show that $v \in V$.

(Claim) $\|v\| < \infty$

for $i \geq N$. $\|v_{n,i} - v_n\|_n < \varepsilon_n$ & $\sum_n \varepsilon_n < \varepsilon$

$\therefore \|v\| - \|v_i\| \leq \|v - v_i\| < \varepsilon \Rightarrow \|v\| < \|v_i\| + \varepsilon < \infty$

$(v_i \in V \Rightarrow \|v_i\| < \infty)$

$\boxed{\phantom{x}}$

(2) Suppose we are given a sequence $E_0, E_1, E_2, \ldots \subseteq \mathbb{R}^m$ of pairwise disjoint measurable subsets of $\mathbb{R}^m$. Let $E = \bigcup E_n$. Show that $L^1(E)$ is isomorphic to the $\ell^1$-sum of the Banach spaces $L^1(E_n)$.

$\Lambda :=$ the $\ell^1$-sum of the $L^1(E_n)$

Define map $\varphi : L^1(E) \to \Lambda$ by $\varphi(f) = (f|_{E_n})_{n\geq 0}$

Then

$$\|\varphi(f)\|_{\ell^1} = \sum_{n\geq 0} \|(f|_{E_n})\|_{L^1(E_n)} = \sum_{n\geq 0} \int_{E_n} |(f|_{E_n})|$$

$$= \int_E |f| = \|f\|_{L^1(E)} < \infty$$

$\Rightarrow$ map : well-defined & preserves norms on $L^1(E)$ and $\Lambda$.

Define map $\rho : \Lambda \to L^1(E)$ by

$$\rho((f_n)_{n\geq 0}) = f_n(x) \text{ for } x \in E_n$$

$$= \sum_{n\geq 0} f_n(x) \cdot I_{E_n}(x)$$

$\Rightarrow$ well-defined &

$$\|\rho((f_n)_{n\geq 0})\|_{L^1(E)} = \int_E |\rho((f_n)_{n\geq 0})| = \sum_{n\geq 0} \int_{E_n} |f_n|$$

$$= \sum_{n\geq 0} \|f_n\|_{L^1(E_n)} = \|(f_n)_{n\geq 0}\|_{\ell_1} < \infty$$

$\therefore$ $\rho$ is an inverse to $\varphi$.

$\Rightarrow$ $\varphi$ : isomorphisms of Banach spaces.

$\Rightarrow$ $L^1(E)$ : isomorphic to $\Lambda$.

$\blacksquare$

(3) Let $E \subseteq \mathbb{R}^n$ be a measurable set, let $p$ and $q$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and suppose that $f \in L^p(E)$, $g \in L^q(E)$ are functions satisfying

$$\int_E fg = ||f||_{L^p}||g||_{L^q}.$$

Prove that either $f = 0$, or there exists a nonnegative real number $\lambda$ such that $|g| = \lambda|f|^{p/q}$ almost everywhere.

Assume $\|f\|_p$ and $\|g\|_q$ are in $(0.\infty)$.

(Claim) $xy \le \frac{x^p}{p} + \frac{y^p}{q}$ when $x.y \ge 0$

$\left( \because x = |f|, y = |g| : |fg| \le \frac{|f|^p}{p} + \frac{|g|^q}{q} \quad \int |fg| d\mu \le \frac{1}{p} + \frac{1}{q} = 1 \right)$

Fix $y \ge 0$, $\varphi(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy$. for $x \ge 0$

$\varphi'(x) = x^{p-1} - y$. $\varphi''(x) = (p-1)x^{p-2}$

$\Rightarrow \varphi(x) \ge \varphi\left(y^{\frac{1}{p-1}}\right) = y^q\left(\frac{1}{p} + \frac{1}{q}\right) - y^q = 0$

$\therefore \frac{x^p}{p} + \frac{y^q}{q} \ge xy$

$\blacksquare$

It holds when $x = y^{\frac{1}{p-1}} \Rightarrow x^p = y^q$

Let $x = \frac{f}{\|f\|_p}$. $y = \frac{g}{\|g\|_q}$

$\Rightarrow \frac{fg}{\|f\|_p \|g\|_q} \le \frac{1}{p} \cdot \frac{f^p}{\|f\|_p^p} + \frac{1}{q} \cdot \frac{g^q}{\|g\|_q^q}$

$$\Rightarrow \int_E fg \le \left(\frac{1}{p}+\frac{1}{q}\right)\|f\|_p \cdot \|g\|_q = \|f\|_p \cdot \|g\|_q$$

Since it holds, $\dfrac{|f|^p}{\|f\|_p^p} = \dfrac{|g|^q}{\|g\|_q^q}$ a.e.

$$\Rightarrow |g|^q = \frac{\|g\|_q^q}{\|f\|_p^p} \cdot |f|^p \quad \text{a.e.}$$

$$\Rightarrow |g| = \lambda |f|^{p/q} \quad \text{a.e.}$$

$\boxed{/\!/}$

(4) Let $p, q, r > 1$ be real numbers satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Let $E \subseteq \mathbb{R}^n$ be measurable, and let $f \in L^p(E)$ and $g \in L^q(E)$. Show that the product function $fg$ belongs to $L^r(E)$, and that

$$||fg||_{L^r} \le ||f||_{L^p}||g||_{L^q}.$$

$$f \in L^p(E) \Rightarrow \int |f|^p < \infty \Rightarrow \int |f|^{r\cdot\frac{p}{r}} < \infty \Rightarrow f^r \in L^{\frac{p}{r}}(E)$$

$$g \in L^q(E) \Rightarrow \int |g|^q < \infty \Rightarrow \int |g|^{r\cdot\frac{q}{r}} < \infty \Rightarrow g^r \in L^{\frac{q}{r}}(E)$$

$$= L^{\frac{p}{p-r}}(E)$$

Since $\frac{r}{p} + \frac{p-r}{p} = 1$. by Holder's Inequality,

$$\int_E f^r \cdot g^r \le \|f^r\|_{\frac{p}{r}} \cdot \|g^r\|_{\frac{p}{p-r}}$$

$$= \left(\int (f^r)^{p/r}\right)^{\frac{r}{p}} \cdot \left(\int (g^r)^{\frac{q}{r}}\right)^{\frac{r}{q}}$$

$$= \left(\int |f|^p\right)^{\frac{r}{p}} \cdot \left(\int |g|^q\right)^{\frac{r}{q}} \Rightarrow \|fg\|_r \le \|f\|_p \|g\|_q$$

$\therefore fg \in L^r(E)$

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