Math 114, Problem Set 5 (due Monday, October 21)

October 14, 2013

(1) Let $E \subseteq \mathbb{R}^n$ be a measurable set, and let $f_0 \leq f_1 \leq f_2 \leq \cdots$ be an increasing sequence of integrable functions on E for which the the sequence of integrals $\{\int_E f_i\}_{i \geq 0}$ is bounded. Show that the sequence $\{f_i\}$ converges almost everywhere to an integrable function f, and that $\int_E f$ is a limit of the sequence $\{\int_E f_i\}_{i \geq 0}$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Show that $\int_{\mathbb{R}} f$ is a limit of the sequence of real numbers $\{\int_{-n}^{n} f\}_{n \geq 0}$. Here $\int_{-n}^{n} f$ denotes the integral $\int_{[-n,n]} f|_{[-n,n]}$.

(3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function, and suppose that $\int_B f|_B = 0$ for every open box $B \subseteq \mathbb{R}^n$. Prove that f vanishes almost everywhere.

(4) Let E be the subset of $[0, 1]$ consisting of those real numbers whose decimal expansion contains infinitely many occurrences of the digit 7. Show that E is a measurable set, and compute its measure.

(1) Let $E \subseteq \mathbb{R}^n$ be a measurable set, and let $f_0 \le f_1 \le f_2 \le \cdots$ be an increasing sequence of integrable functions on E for which the the sequence of integrals $\{\int_E f_i\}_{i\ge0}$ is bounded. Show that the sequence $\{f_i\}$ converges almost everywhere to an integrable function f, and that $\int_E f$ is a limit of the sequence $\{\int_E f_i\}_{i\ge0}$.

$\{\int_E f_i\}_{i\ge0}$: non-decreasing, bounded

$\Rightarrow$ converges to some real number α.

For each $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} \|f_i - f_{i-1}\|_{L^1(E)} = \sum_{i=1}^{k} \int_E |f_i - f_{i-1}| = \sum_{i=1}^{k} \int_E (f_i - f_{i-1})$$

$$= \int_E f_k - \int_E f_0 .$$

$$\therefore \sum_{i=1}^{k} \|f_i - f_{i-1}\|_{L^1(E)} \longrightarrow \lim_{k\to\infty} \int_E f_k - \int_E f_0 = \alpha - \int_E f_0 < \infty$$

$\Rightarrow$ quick convergence of $\{\int_E f_i\}_{i\ge0}$.

$\{f_i\}$ converges pointwise a.e. to a function f.

$$\int_E |f| = \int_E \lim_{k\to\infty} |f_k| \le \int_E \lim_{k\to\infty} \sum_{i=1}^{k} |f_i - f_{i-1}| + |f_0|$$

$$= \lim_{k\to\infty} \int_E \sum_{i=1}^{k} |f_i - f_{i-1}| + |f_0| = \lim_{k\to\infty} \int_E (f_k - f_0) + |f_0|$$

$$= \lim_{k\to\infty} \int_E f_k - \int_E f_0 + \int_E |f_0| < \infty$$

Let's show $\int_E f$ is a limit of $\{\int_E f_i\}$.

The Lebesgue Dominated Convergence Theorem *Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \le g$ on E for all n.*

If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

$\{f_i\} \to f$ pointwise a.e. on E.
& f: integrable over E.
$\Rightarrow$ by Dominated Convergence Theorem,

$$\int_E f = \lim_{i\to\infty} \int_E f_i$$

$\blacksquare$

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Show that $\int_{\mathbb{R}} f$ is a limit of the sequence of real numbers $\{\int_{-n}^{n} f\}_{n\geq 0}$. Here $\int_{-n}^{n} f$ denotes the integral $\int_{[-n,n]} f|_{[-n,n]}$.

Define the function $f_n = \begin{cases} f(x), & x \in [-n, n] \\ 0, & \text{else} \end{cases}$

$\Rightarrow \int_{-n}^{n} f = \int_{\mathbb{R}} f_n$

$|f_n|$: bounded above by $|f|$ ($\because$ integrable)

$\{f_n\}$ converges pointwise to f.
by Dominated Convergence Theorem,

$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n = \lim_{n\to\infty} \int_{-n}^{n} f.$$

$\blacksquare$

(3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function, and suppose that $\int_B f|_B = 0$ for every open box $B \subseteq \mathbb{R}^n$. Prove that f vanishes almost everywhere.

$f(x) = f_+(x) - f_-(x)$

$f_+(x) := f|_{E^+}$. $E^+ = \{x \in \mathbb{R}^n : f(x) > 0\} \subset \mathbb{R}^n$

$f_-(x) := f|_{E^-}$ $E^- = \{x \in \mathbb{R}^n : f(x) \leq 0\} \subset \mathbb{R}^n$

By the measurability of f, both E^+. E^- : measurable.

If $\int_{E^+} f^+ = 0$, then $f_+ = 0$ a.e.

(E.T.S.) $\int_{E^+} f^+ = 0$ (same for E^-)

Measurable sets can be decomposed into the union of an F_δ-set and a set of measure zero.

(F_δ : countable union of closed set)

$E^+ = F \cup S$. (F : F_δ. $m(S) = 0$)

$$\int_{E^+} f_+ = \int_{E_+} f = \int_F f + \int_S f = \int_F f = 0$$

$\because$

$$\int_{\text{closed}} f = \int_{\mathbb{R}^n} f - \int_{\text{open}} f = 0$$

open sets in $\mathbb{R}^n$ can be expressed as the countable union of disjoint open boxes, together with their boundaries which measure is 0.

▨

(4) Let E be the subset of $[0,1]$ consisting of those real numbers whose decimal expansion contains infinitely many occurrences of the digit 7. Show that E is a measurable set, and compute its measure.

$\{E_k\}_{k=1}^{\infty}$: a family of measurable subsets of $\mathbb{R}^d$

$E_k := \{x \in [0.1] :$ the decimal expansion of x contains a 7 at the k-th place $\}$

& $m(E_k) = \frac{1}{10}$

Let $E := \{x \in \mathbb{R}^d : x \in E_k$ for infinitely many $k\}$

$= \limsup_{k\to\infty} E_k$

$\Rightarrow E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$: measurable ($\because E_k$: measurable)

(Claim) $m(E) = 1$

$E^c \cap [0.1] = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k^c \cap [0.1]$

$\Rightarrow m(E^c \cap [0.1]) \leq \sum_{n=1}^{\infty} m\left(\bigcap_{k\geq n} E_k^c \cap [0.1]\right)$

$= \sum_{n=1}^{\infty} \inf_{k\geq n} \left(\frac{9}{10}\right)^{k-n} = 0$

$\Rightarrow m(E) = m([0.1]) - m(E^c \cap [0.1])$

$= 1 - 0 = 1$

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