Homework 3

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The Uniform-Pareto

$$
\begin{aligned}
p(x \mid \theta) &= \frac{1}{\theta - 0} I_{0,\theta}(x) = \frac{1}{\theta} I_{0,\theta}(x) \\
p(\theta \mid x) &\propto \left[\frac{1}{\theta} I_{0,\theta}(x)\right] \left[\frac{\alpha\beta^\alpha}{\theta^{\alpha+1}} I_{\beta,\inf}(\theta)\right] \\
p(\theta \mid x) &\propto \frac{1}{\theta^{\alpha+2}} I_{\beta,\inf}(\theta) I_{0,\theta}(x) \\
p(\theta \mid x) &\sim \text{Pareto}(\alpha + 1, \max(\beta, x))
\end{aligned}
$$

The Bayes Estimator or Bayes Procedure

a.

$$
\begin{aligned}
\rho(\delta(x), x) &= E[L(\theta, \delta(x)) \mid x] \\
\rho(\delta(x), x) &= E[c(\theta - \delta(x))^2 \mid x] \\
\rho(\delta(x), x) &= cE[(\theta - \delta(x))^2 \mid x] \\
\rho(\delta(x), x) &= c\left\{E[\theta^2 \mid x] + E[-2\theta\delta(x) \mid x] + E[\delta(x)^2 \mid x]\right\} \\
\rho(\delta(x), x) &= c\left\{E[\theta^2 \mid x] - 2\delta(x)E[\theta \mid x] + \delta(x)^2\right\} \\
\rho(\delta(x), x) &= cE[\theta^2 \mid x] - 2c\delta(x)E[\theta \mid x] + c\delta(x)^2 \\
\frac{\partial\rho(\delta(x), x)}{\partial\delta(x)} &= \frac{\partial\left\{cE[\theta^2 \mid x] - 2c\delta(x)E[\theta \mid x] + c\delta(x)^2\right\}}{\partial\delta(x)} = 0 \\
\frac{\partial\left\{cE[\theta^2 \mid x] - 2c\delta(x)E[\theta \mid x] + c\delta(x)^2\right\}}{\partial\delta(x)} &= -2cE[\theta \mid x] + 2c\delta(x) = 0 \\
2c\delta(x) &= 2cE[\theta \mid x] \\
\delta(x) &= E[\theta \mid x]
\end{aligned}
$$

b.

$$
\begin{aligned}
\rho(\delta(x), x) &= E[L(\theta, \delta(x)) \mid x] \\
\rho(\delta(x), x) &= E[w(\theta)(g(\theta) - \delta(x))^2 \mid x] \\
\rho(\delta(x), x) &= E[w(\theta)g(\theta)^2 \mid x] + E[-2w(\theta)g(\theta)\delta(x) \mid x] + E[w(\theta)\delta(x)^2 \mid x] \\
\rho(\delta(x), x) &= E[w(\theta)g(\theta)^2 \mid x] - 2\delta(x)E[w(\theta)g(\theta) \mid x] + \delta(x)^2 E[w(\theta) \mid x] \\
\frac{\partial \rho(\delta(x), x)}{\partial \delta(x)} &= \frac{\partial \left\{ E[w(\theta)g(\theta)^2 \mid x] - 2\delta(x)E[w(\theta)g(\theta) \mid x] + \delta(x)^2 E[w(\theta) \mid x] \right\}}{\partial \delta(x)} = 0 \\
\frac{\partial \rho(\delta(x), x)}{\partial \delta(x)} &= -2E[w(\theta)g(\theta) \mid x] + 2\delta(x)E[w(\theta) \mid x] = 0 \\
2\delta(x)E[w(\theta) \mid x] &= 2E[w(\theta)g(\theta) \mid x] \\
\delta(x) &= \frac{E[w(\theta)g(\theta) \mid x]}{E[w(\theta) \mid x]}
\end{aligned}
$$

Basic Decision Theory

For an action rule a, the posterior risk can be found using the expression:

$$
r(a) = L(\theta_1, a)\pi(\theta_1) + L(\theta_2, a)\pi(\theta_2)
$$

We can then determine the posterior risk for A:

$$
\begin{aligned}
r(a_1) &= 0 \cdot \frac{4}{5} + 4 \cdot \frac{1}{5} = \frac{4}{5} \\
r(a_2) &= 3 \cdot \frac{4}{5} + 6 \cdot \frac{1}{5} = \frac{18}{5} \\
r(a_3) &= 1 \cdot \frac{4}{5} + 0 \cdot \frac{1}{5} = \frac{4}{5} \\
r(a_4) &= 3 \cdot \frac{4}{5} + 0 \cdot \frac{1}{5} = \frac{12}{5} \\
r(a_5) &= 4 \cdot \frac{4}{5} + 1 \cdot \frac{1}{5} = \frac{17}{5}
\end{aligned}
$$

As we can see above, a_1 and a_3 are Bayes actions because they both minimize posterior risk.