Proposition 0.1. *For PINNs combined with DDM, we further assume for $i = 1, 2$ that a_i is the constant coefficient and $b_i \in L^\infty(\Omega_i)$ is a variable coefficient. If the parameters satisfy the following requirements $0 < a_{min} < a_i < a_{max}$, $0 < b_{min} < b_i < b_{max}$, $0 < \lambda_2 < 1$, $\lambda_1 > 0$, $0 < J_{min} < J_i < J_{max}$, $0 < Q_{min} < Q_i < Q_{max}$ and $0 < S_{min} < S_i < S_{max}$, where*

$$\begin{aligned}
J_i &= 2a_i b_i - \frac{1}{2}b_{max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup|a_i \nabla b_i|,\\
Q_i &= b_i^2 - \frac{1}{2}b_{max}^2 - \frac{1}{2}\lambda_1 - a_i \Delta b_i - \sup|a_i \nabla b_i|,\\
S_i &= (3 + 4\lambda_2)a_i^2 + 2a_i b_i + 2b_{max}^2 + 4\lambda_1 + b_i^2 - a_i\Delta b_i + 2\sup|a_i \nabla b_i| + 2\lambda_3.
\end{aligned} \tag{1}$$

then we have for all $u \in H_0^1(\Omega)$,

$$\begin{aligned}
&\min\{\frac{1}{2}(1-\lambda_2)a_{min}^2, J_{min}, Q_{min}\}\|u - u^*\|_{H_0^2(\Omega)}^2 = \sum_{i=1}^{2}\{\min\{\frac{1}{2}(1-\lambda_2)a_{min}^2, J_{min}, Q_{min}\}\|u - u^*\|_{H_0^2(\Omega_i)}^2\}\\
&\leq \mathbf{E}^{PINNs}(u) - \mathbf{E}^{PINNs}(u^*) \leq \sum_{i=1}^{2} S_{max}\|u - u^*\|_{H_0^2(\Omega_i)}^2\} \leq S_{max}\|u - u^*\|_{H_0^2(\Omega)}^2.
\end{aligned} \tag{2}$$

Proof. We first expand the objective function,

$$\begin{aligned}
&\mathbf{E}^{PINNs}(u) - \mathbf{E}^{PINNs}(u^*)\\
&= \sum_{i=1}^{2}\int_{\Omega_i}|\nabla\cdot(a_i\nabla u) - b_i u + f - \nabla\cdot(a_i\nabla u^*) + b_i u^* - f|^2 dx + \underbrace{\lambda_1\int_\Gamma |u_2 - u_1 - \varphi - u_2^* + u_1^* + \varphi|^2 ds}_{(6)}\\
&+ \underbrace{\lambda_2\int_\Gamma |a_2\nabla u_2\cdot\mathbf{n} - a_1\nabla u_1\cdot\mathbf{n} - \psi - a_2\nabla u_2^*\cdot\mathbf{n} + a_1\nabla u_1^*\cdot\mathbf{n} + \psi|^2 ds}_{(7)} + \lambda_3\int_{\partial\Omega}|u - g - u^* + g|^2 ds.\\
&= \sum_{i=1}^{2}\left\{\int_{\Omega_i} a_i^2(\Delta\tilde{u})^2 dx + \int_{\Omega_i} b_i^2(\tilde{u})^2 dx - 2\int_{\Omega_i}(b_i\tilde{u})(a_i\Delta\tilde{u})dx\right\} + (6) + (7) + \lambda_3\int_{\partial\Omega}\tilde{u}^2 ds\\
&\overset{Green}{=} \sum_{i=1}^{2}\left\{\int_{\Omega_i} a_i^2(\Delta\tilde{u})^2 dx + \int_{\Omega_i} b_i^2(\tilde{u})^2 dx + 2\int_{\Omega_i}\nabla\cdot(b_i\tilde{u})(a_i\nabla\tilde{u})dx - 2\int_\Gamma(b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n}ds\right\} + (6) + (7) + \lambda_3\int_{\partial\Omega}\tilde{u}^2 ds\\
&= \sum_{i=1}^{2}\left\{\int_{\Omega_i} a_i^2(\Delta\tilde{u})^2 dx + \int_{\Omega_i} b_i^2(\tilde{u})^2 dx + 2\int_{\Omega_i} b_i a_i(\nabla\tilde{u})^2 dx + 2\int_{\Omega_i}(\nabla b_i)\tilde{u}a_i\nabla\tilde{u}dx - 2\int_\Gamma(b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n})ds\right\}\\
&+ (6) + (7) + \lambda_3\int_{\partial\Omega}\tilde{u}^2 ds\\
&= \sum_{i=1}^{2}\left\{\int_{\Omega_i} a_i^2(\Delta\tilde{u})^2 dx + \int_{\Omega_i} b_i^2(\tilde{u})^2 dx + 2\int_{\Omega_i} b_i a_i(\nabla\tilde{u})^2 dx + \int_{\Omega_i}(\nabla b_i)a_i\nabla(\tilde{u}^2)dx - 2\int_\Gamma(b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n})ds\right\}\\
&+ (6) + (7) + \lambda_3\int_{\partial\Omega}\tilde{u}^2 ds\\
&\overset{Green}{=} \sum_{i=1}^{2}\left\{\int_{\Omega_i} a_i^2(\Delta\tilde{u})^2 dx + \int_{\Omega_i} 2a_i b_i(\nabla\tilde{u})^2 dx + \int_{\Omega_i}\left(b_i^2 - a_i\Delta b_i\right)\tilde{u}^2 dx + \underbrace{\int_\Gamma(\nabla b_i)a_i\tilde{u}^2\cdot\mathbf{n}ds}_{(4)} - 2\int_\Gamma(b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n})ds\right\}\\
&+ (6) + (7) + \lambda_3\int_{\partial\Omega}\tilde{u}^2 ds.
\end{aligned} \tag{3}$$

We analyze the upper and lower bounds of the objective function separately. We first estimate the upper bound

$$
\begin{aligned}
(4) &= \int_\Gamma a_i(\tilde{u})^2(\nabla b_i)\cdot \mathbf{n} ds = -\int_\Gamma -a_i(\tilde{u})^2(\nabla b_i)\cdot \mathbf{n} ds \geq -\left|\int_\Gamma -a_i(\tilde{u})^2(\nabla b_i)\cdot \mathbf{n} ds\right| = -\left|\int_\Gamma a_i(\tilde{u})^2(\nabla b_i)\cdot \mathbf{n} ds\right| \\
&\geq -\sup|a_i\nabla b_i|\cdot\left|\int_\Gamma \tilde{u}^2 ds\right| \overset{Trace}{\geq} -\sup|a_i\nabla b_i|\cdot\left\{\int_{\Omega_i}\tilde{u}^2 ds + \int_{\Omega_i}(\nabla\tilde{u})^2 ds\right\}.
\end{aligned} \tag{4}
$$

Then the upper bound of equation (3) is estimated as

$$
\begin{aligned}
(3) \geq& \sum_{i=1}^{2}\left\{\int_{\Omega_i} a_i^2(\Delta\tilde{u})^2 dx + \int_{\Omega_i}\left[2a_ib_i - \sup|a_i\nabla b_i|\right](\nabla\tilde{u})^2 dx + \int_{\Omega_i}\left[b_i^2 - a_i\Delta b_i - \sup|a_i\nabla b_i|\right](\tilde{u})^2 dx\right\} \\
&+ \underbrace{\sum_{i=1}^{2}\left\{-2\int_\Gamma (b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n})ds\right\}}_{(5)} + (6) + (7) + \lambda_3\int_{\partial\Omega}\tilde{u}^2 ds \\
\geq& \sum_{i=1}^{2}\left\{\int_{\Omega_i} a_i^2(\Delta\tilde{u})^2 dx + \int_{\Omega_i}\left[2a_ib_i - \sup|a_i\nabla b_i|\right](\nabla\tilde{u})^2 dx + \int_{\Omega_i}\left[b_i^2 - a_i\Delta b_i - \sup|a_i\nabla b_i|\right](\tilde{u})^2 dx\right\} \\
&+ \underbrace{\sum_{i=1}^{2}\left\{-2\int_\Gamma (b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n})ds\right\}}_{(5)} + (6) + (7).
\end{aligned} \tag{5}
$$

Then estimating the upper bound of the objective function we only have to analyze the values of $(5)+(6)+(7)$to end the analysis.

$$
\begin{aligned}
&(5)+(6)+(7) \\
&= \sum_{i=1}^{2} -2\int_\Gamma (b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n})ds + \lambda_1\int_\Gamma(\tilde{u}_1-\tilde{u}_2)^2 ds + \lambda_2\int_\Gamma\left|a_2\nabla\tilde{u}\cdot\mathbf{n} - a_1\nabla\tilde{u}\cdot\mathbf{n}\right|^2 ds \\
&\overset{Cauchy-Schwarz}{\geq} \sum_{i=1}^{2} -2\left(\int_\Gamma |b_i\tilde{u}|^2 ds\right)^{\frac{1}{2}}\left(\int_\Gamma |a_i\nabla\tilde{u}\cdot\mathbf{n}|^2 ds\right)^{\frac{1}{2}} + \lambda_1\int_\Gamma|\tilde{u}_2-\tilde{u}_1|^2 ds + \lambda_2\int_\Gamma\left|a_2\nabla\tilde{u}\cdot\mathbf{n} - a_1\nabla\tilde{u}\cdot\mathbf{n}\right|^2 ds \\
&\geq \sum_{i=1}^{2} -\int_\Gamma |b_i\tilde{u}|^2 ds - \int_\Gamma |a_i\nabla\tilde{u}\cdot\mathbf{n}|^2 ds + \lambda_1\int_\Gamma|\tilde{u}_2-\tilde{u}_1|^2 ds + \lambda_2\int_\Gamma\left|a_2\nabla\tilde{u}\cdot\mathbf{n} - a_1\nabla\tilde{u}\cdot\mathbf{n}\right|^2 ds \\
&= -\int_\Gamma (b_1^2\tilde{u}_1^2 + b_2^2\tilde{u}_2^2)ds - \int_\Gamma (a_1\nabla\tilde{u}_1\cdot\mathbf{n})^2 + (a_2\nabla\tilde{u}_2\cdot\mathbf{n})^2 ds + \lambda_1\int_\Gamma|\tilde{u}_2-\tilde{u}_1|^2 ds + \lambda_2\int_\Gamma\left|a_2\nabla\tilde{u}\cdot\mathbf{n} - a_1\nabla\tilde{u}\cdot\mathbf{n}\right|^2 ds \\
&= \int_\Gamma\left(\lambda_1(\tilde{u}_2-\tilde{u}_1)^2 - b_1^2\tilde{u}_1^2 - b_2\tilde{u}_2^2\right)ds + \int_\Gamma\left\{\lambda_2\left(a_2\nabla\tilde{u}\cdot\mathbf{n} - a_1\nabla\tilde{u}\cdot\mathbf{n}\right)^2 - (a_1\nabla\tilde{u}\cdot\mathbf{n})^2 - (a_2\nabla\tilde{u}\cdot\mathbf{n})^2\right\}ds \\
&\geq \int_\Gamma \lambda_1\tilde{u}_2^2 + \lambda_1\tilde{u}_1^2 - 2\lambda_1\tilde{u}_1\tilde{u}_2 - b_{max}^2\tilde{u}_1^2 - b_{max}^2\tilde{u}_2^2 ds \\
&+ \underbrace{\int_\Gamma \lambda_2\left(a_2\nabla\tilde{u}_2\cdot\mathbf{n}\right)^2 + \lambda_2\left(a_1\nabla\tilde{u}_1\cdot\mathbf{n}\right)^2 - 2\lambda_2\left(a_1\nabla\tilde{u}_1\cdot\mathbf{n}\right)\left(a_2\nabla\tilde{u}_2\cdot\mathbf{n}\right) - \left(a_1\nabla\tilde{u}_1\cdot\mathbf{n}\right)^2 - \left(a_2\nabla\tilde{u}_2\cdot\mathbf{n}\right)^2 ds}_{(8)} \\
&= \int_\Gamma\left\{\left(\lambda_1 - b_{max}^2\right)\tilde{u}_1^2 + \left(\lambda_1 - b_{max}^2\right)\tilde{u}_2^2 - 2\lambda_1\tilde{u}_1\tilde{u}_2\right\}ds + (8) \\
&\geq \int_\Gamma\left\{\left(\lambda_1 - b_{max}^2\right)|\tilde{u}_1|^2 + \left(\lambda_1 - b_{max}^2\right)|\tilde{u}_2|^2 - 2\lambda_1|\tilde{u}_1||\tilde{u}_2|\right\}ds + (8) \\
&\geq \int_\Gamma (\lambda_1 - b_{max}^2)|\tilde{u}_1||\tilde{u}_2| - 2\lambda_1|\tilde{u}_1||\tilde{u}_2| ds + (8).
\end{aligned} \tag{6}
$$

We further simplify the above equation

$$
\begin{aligned}
&(5)+(6)+(7)\\
&=(\lambda_1+b_{max}^2)\left[-\int_\Gamma |\tilde{u}_1||\tilde{u}_2|ds\right]+(8)\\
&\geq\left(-\frac{\lambda_1}{2}-\frac{1}{2}b_{max}^2\right)\int_\Gamma|\tilde{u}_1|^2ds+\left(\frac{-\lambda_1}{2}-\frac{1}{2}b_{max}^2\right)\int_\Gamma|\tilde{u}_2|^2ds+(8)\\
&=\sum_{i=1}^{2}\left[\left(\frac{-\lambda_1}{2}-\frac{1}{2}b_{max}^2\right)\int_\Gamma|\tilde{u}_i|^2ds\right]+(8)\\
&\overset{Trace}{\geq}\sum_{i=1}^{2}\left[\left(-\frac{\lambda_1}{2}-\frac{1}{2}b_{max}^2\right)\int_{\Omega_i}|\tilde{u}_i^2|dx+\left(-\frac{\lambda_1}{2}-\frac{1}{2}b_{max}^2\right)\int_{\Omega_i}|\nabla\tilde{u}_i^2|dx\right]+(8)
\end{aligned}
\tag{7}
$$

Thus the objective function is

$$
\begin{aligned}
(3)&=\mathbf{E}^{PINNs}(u)-\mathbf{E}^{PINNs}(u^*)\\
&\geq\sum_{i=1}^{2}\left\{\int_{\Omega_i}a_i^2(\Delta\tilde{u})^2dx+\int_{\Omega_i}\left[2a_ib_i-\sup|a_i\nabla b_i|\right](\nabla\tilde{u})^2dx+\int_{\Omega_i}\left[b_i^2-a_i\Delta b_i-\sup|a_i\nabla b_i|\right](\tilde{u})^2dx\right\}\\
&+\sum_{i=1}^{2}\left[(-\frac{\lambda_1}{2}-\frac{1}{2}b_{max}^2)\int_{\Omega_i}|\tilde{u}_i|^2dx+(-\frac{\lambda_1}{2}-\frac{1}{2}b_{max}^2)\int_{\Omega_i}|\nabla\tilde{u}_i|^2dx+(8)\right]\\
&\geq\sum_{i=1}^{2}\left\{\int_{\Omega_i}a_i^2(\Delta\tilde{u})^2dx+\int_{\Omega_i}\left[2a_ib_i-\frac{1}{2}b_{max}^2-\frac{\lambda_1}{2}-\sup|a_i\nabla b_i|\right](\nabla\tilde{u})^2dx\right.\\
&\left.+\int_{\Omega_i}\left[b_i^2-\frac{1}{2}b_{max}^2-\frac{\lambda_1}{2}-a_i\Delta b_i-\sup|a_i\nabla b_i|\right]|\tilde{u}^2|dx\right\}+(8).
\end{aligned}
\tag{8}
$$

Now analyze equation (8)

$$
\begin{aligned}
(8)&=\int_\Gamma(\lambda_2-1)|a_2\nabla\tilde{u}_2\cdot\mathbf{n}|^2+(\lambda_2-1)|a_1\nabla\tilde{u}_1\cdot\mathbf{n}|^2-2\lambda_2(a_1\nabla\tilde{u}_1\cdot\mathbf{n})(a_2\nabla\tilde{u}_2\cdot\mathbf{n})ds\\
&\geq\int_\Gamma(\lambda_2-1)|a_2\nabla\tilde{u}_2\cdot\mathbf{n}||a_1\nabla\tilde{u}_1\cdot\mathbf{n}|-2\lambda_2|a_1\nabla\tilde{u}_1\cdot\mathbf{n}||a_2\nabla\tilde{u}_2\cdot\mathbf{n}|ds\\
&=\int_\Gamma(-\lambda_2-1)|a_1\nabla\tilde{u}_1\cdot\mathbf{n}||a_2\nabla\tilde{u}_2\cdot\mathbf{n}|ds\\
&=\left(\frac{1}{2}\lambda_2+\frac{1}{2}\right)\int_\Gamma-2|a_1\nabla\tilde{u}_1\cdot\mathbf{n}||a_2\nabla\tilde{u}_2\cdot\mathbf{n}|ds.
\end{aligned}
\tag{9}
$$

We further simplify the above equation

$$
\begin{aligned}
(8)&\geq\left(-\frac{1}{2}\lambda_2-\frac{1}{2}\right)\int_\Gamma(a_1\nabla\tilde{u}_1\cdot\mathbf{n})^2ds+\left(-\frac{1}{2}\lambda_2-\frac{1}{2}\right)\int_\Gamma(a_2\nabla\tilde{u}_2\cdot\mathbf{n})^2ds\\
&\geq\left(-\frac{1}{2}-\frac{1}{2}\lambda_2\right)a_1^2\int_\Gamma|\nabla\tilde{u}_1\cdot\mathbf{n}|^2ds+\left(-\frac{1}{2}-\frac{1}{2}\lambda_2\right)a_2^2\int_\Gamma|\nabla\tilde{u}_2\cdot\mathbf{n}|^2ds\\
&=\sum_{i=1}^{2}\left(-\frac{1}{2}-\frac{1}{2}\lambda_2\right)a_i^2\int_\Gamma|\nabla\tilde{u}_i\cdot\mathbf{n}|^2ds\\
&\overset{Trace}{\geq}\sum_{i=1}^{2}\left[\left(-\frac{1}{2}-\frac{1}{2}\lambda_2\right)a_i^2\int_{\Omega_i}|\nabla\tilde{u}_i|^2dx+\left(-\frac{1}{2}-\frac{1}{2}\lambda_2\right)a_i^2\int_{\Omega_i}\Delta\tilde{u}_i{}^2dx\right].
\end{aligned}
\tag{10}
$$

We bring (8) into the objective function

$$
\begin{aligned}
(3) = \sum_{i=1}^{2} \Bigg\{ & \int_{\Omega_i} \left(a_i^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_2 a_i^2 \right) (\Delta\tilde{u})^2 dx + \int_{\Omega_i} \left(2a_i b_i - \frac{1}{2}b_{max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup|a_i \nabla b_i| \right) (\nabla\tilde{u})^2 dx \\
& + \int_{\Omega_i} \left(b_i^2 - \frac{1}{2}b_{max}^2 - \frac{1}{2}\lambda_1 - a_i\Delta b_i - \sup|a_i\nabla b_i| \right) (\tilde{u})^2 dx \Bigg\} \\
\geq \sum_{i=1}^{2} \Bigg\{ & \int_{\Omega_i} \frac{1}{2}(1-\lambda_2) a_i^2 (\Delta\tilde{u})^2 dx + \int_{\Omega_i} \left[2a_i b_i - \frac{1}{2}b_{max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup|a_i\nabla b_i| \right] (\nabla\tilde{u})^2 dx \\
& + \int_{\Omega_i} \left[b_i^2 - \frac{1}{2}b_{max}^2 - \frac{1}{2}\lambda_1 - a_i\Delta b_i - \sup|a_i\nabla b_i| \right] (\tilde{u})^2 dx \Bigg\}.
\end{aligned}
\tag{11}
$$

So we obtain

$$
\begin{aligned}
(3) \geq \min & \left\{ \frac{1}{2}(1-\lambda_2)a_{min}^2, \min\left[2a_i b_i - \frac{1}{2}b_{max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup|a_i\nabla b_i| \right] \right. \\
& \left. , \min\left[b_i^2 - \frac{1}{2}b_{max}^2 - \frac{1}{2}\lambda_1 - a_i\Delta b_i - \sup|a_i\nabla b_i| \right] \right\} \|u-u^*\|_{H_0^2(\Omega)}^2 \\
& \geq \min\left\{ \frac{1}{2}(1-\lambda_2)a_{min}^2, J_{min}, Q_{min} \right\} \|u-u^*\|_{H_0^2(\Omega)}^2
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
J_{min} &= \min J_i = \min\left\{ 2a_i b_i - \frac{1}{2}b_{max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup|a_i\nabla b_i| \right\}, i = 1,2, \\
Q_{min} &= \min Q_i = \min\left\{ b_i^2 - \frac{1}{2}b_{max}^2 - \frac{1}{2}\lambda_1 - a_i\Delta b_i - \sup|a_i\nabla b_i| \right\}, i = 1,2.
\end{aligned}
\tag{13}
$$

Now we analyze the upper bound of equation (3)

$$
\begin{aligned}
(3) = \sum_{i=1}^{2} & \left\{ \int_{\Omega_i} a_i^2(\Delta\tilde{u})^2 dx + \int_{\Omega_i} 2a_i b_i (\nabla\tilde{u})^2 dx + \int_{\Omega_i} \left(b_i^2 - a_i\Delta b_i \right) \tilde{u}^2 dx + \underbrace{\int_{\Gamma} (\nabla b_i) a_i \tilde{u}^2 \cdot \mathbf{n} ds}_{(4)} - 2\int_{\Gamma} (b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n}) ds \right\} \\
& + (6) + (7) + \lambda_3 \int_{\partial\Omega} \tilde{u}^2 ds.
\end{aligned}
\tag{14}
$$

Where (4) we analyze separately

$$
(4) = \int_{\Gamma} a_i(\tilde{u})^2 \nabla b_i \cdot \mathbf{n} ds \leq \int_{\Gamma} \left| a_i(\tilde{u})^2 \nabla b_i \cdot \mathbf{n} \right| ds \leq \sup|a_i\nabla b_i| \left| \int_{\Gamma} \tilde{u}^2 ds \right| \overset{Trace}{\leq} \sup|a_i\nabla b_i| \left\{ \int_{\Omega_i} \tilde{u}^2 ds + \int_{\Omega_i} (\nabla\tilde{u})^2 ds \right\}.
\tag{15}
$$

The simplification result of (4) is brought to the objective function

$$
\begin{aligned}
(3) \leq & \sum_{i=1}^{2} \left\{ \int_{\Gamma} a_i^2(\Delta\tilde{u})^2 dx + \int_{\Omega_i} \left[2a_i b_i + \sup|a_i\nabla b_i| \right] (\nabla\tilde{u})^2 dx + \int_{\Omega_i} \left[b_i^2 - a_i\Delta b_i + \sup|a_i\nabla a_i| \right] \tilde{u}^2 dx \right\} \\
& + \underbrace{\sum_{i=1}^{2} \left\{ -2\int_{\Gamma} (b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n}) ds \right\}}_{(5)} + (6) + (7) + \lambda_3 \int_{\partial\Omega} \tilde{u}^2 ds.
\end{aligned}
\tag{16}
$$

We analyze the last three terms of the objective function

$$
\begin{aligned}
(5)+(6)+(7) &= \sum_{i=1}^{2}\left\{-2\int_{\Gamma}(b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n})ds\right\}+\lambda_1\int_{\Gamma}|\tilde{u}_2-\tilde{u}_1|^2ds+\lambda_2\int_{\Gamma}|a_2\nabla\tilde{u}\cdot\mathbf{n}-a_1\nabla\tilde{u}\cdot\mathbf{n}|^2ds\\
&\le \sum_{i=1}^{2}\left\{\left|-2\int_{\Gamma}(b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n})ds\right|\right\}+\lambda_1\int_{\Gamma}|\tilde{u}_2-\tilde{u}_1|^2ds+\lambda_2\int_{\Gamma}|a_2\nabla\tilde{u}\cdot\mathbf{n}-a_1\nabla\tilde{u}\cdot\mathbf{n}|^2ds\\
&= \sum_{i=1}^{2}\left\{\left|2\int_{\Gamma}(b_i\tilde{u})(a_i\nabla\tilde{u}\cdot\mathbf{n})ds\right|\right\}+\lambda_1\int_{\Gamma}|\tilde{u}_2-\tilde{u}_1|^2ds+\lambda_2\int_{\Gamma}|a_2\nabla\tilde{u}\cdot\mathbf{n}-a_1\nabla\tilde{u}\cdot\mathbf{n}|^2ds\\
&\overset{Cauchy-Schwarz}{\le} \sum_{i=1}^{2}\left\{2\left(\int_{\Gamma}|b_i\tilde{u}|^2ds\right)^{\frac{1}{2}}\left(\int_{\Gamma}|a_i\nabla\tilde{u}\cdot\mathbf{n}\right)^{\frac{1}{2}}\right\}+\lambda_1\int_{\Gamma}|\tilde{u}_2-\tilde{u}_1|^2ds\\
&+\lambda_2\int_{\Gamma}|a_2\nabla\tilde{u}\cdot\mathbf{n}-a_1\nabla\tilde{u}\cdot\mathbf{n}|^2ds\\
&\le \sum_{i=1}^{2}\left\{\int_{\Gamma}|b_i\tilde{u}|^2ds+\int_{\Gamma}|a_i\nabla\tilde{u}\cdot\mathbf{n}|^2ds\right\}+\lambda_1\int_{\Gamma}|\tilde{u}_2-\tilde{u}_1|^2ds+\lambda_2\int_{\Gamma}|a_2\nabla\tilde{u}\cdot\mathbf{n}-a_1\nabla\tilde{u}\cdot\mathbf{n}|^2ds\\
&= \int_{\Gamma}\left(b_1^2\tilde{u}_1^2+b_2^2\tilde{u}_2^2+\lambda_1|\tilde{u}_2-\tilde{u}_1|^2\right)ds+\int_{\Gamma}\left\{(a_1\nabla\tilde{u}\cdot\mathbf{n})^2+(a_2\nabla\tilde{u}\cdot\mathbf{n})^2+\lambda_2|a_2\nabla\tilde{u}\cdot\mathbf{n}-a_1\nabla\tilde{u}\cdot\mathbf{n}\right\}ds\\
&= \int_{\Gamma}\left\{(\lambda_1+b_{max}^2)\tilde{u}_1^2+(\lambda_1+b_{max}^2\tilde{u}_2^2-2\lambda_1\tilde{u}_1\tilde{u}_2\right\}ds\\
&+\int_{\Gamma}\left\{(1+\lambda_2)(a_1\nabla\tilde{u}\cdot\mathbf{n})^2+(1+\lambda_2)(a_2\nabla\tilde{u}\cdot\mathbf{n})^2-2\lambda_2(a_1\nabla\tilde{u}\cdot\mathbf{n})(a_2\nabla\tilde{u}\cdot\mathbf{n})\right\}ds\\
&\le \int_{\Gamma}\left\{(\lambda_1+b_{max}^2)\tilde{u}_1^2+(\lambda_1+b_{max}^2)\tilde{u}_2^2+\lambda_1\tilde{u}_1^2+\lambda_1\tilde{u}_2^2\right\}ds\\
&+\int_{\Gamma}\left\{(1+\lambda_2)(a_1\nabla\tilde{u}\cdot\mathbf{n})^2+(1+\lambda_2)(a_2\nabla\tilde{u}\cdot\mathbf{n}^2+\lambda_2(a_1\nabla\tilde{u}\cdot\mathbf{n}^2+\lambda_2(a_2\nabla\tilde{u}\cdot\mathbf{n})^{\mathbf{2}}\right\}ds\\
&= \int_{\Gamma}(2\lambda_1+b_{max}^2)\tilde{u}_1^2+(2\lambda_1+b_{max}^2)\tilde{u}_2^2ds+\int_{\Gamma}(1+2\lambda_2)(a_1\nabla\tilde{u}\cdot\mathbf{n}^2+(1+2\lambda_2)(a_2\nabla\tilde{u}\cdot\mathbf{n}^2ds\\
&\le \sum_{i=1}^{2}\left\{(2\lambda_1+b_{max}^2)\int_{\Gamma}\tilde{u}_i^2ds+(1+2\lambda_2)\int_{\Gamma}(a_i\nabla\tilde{u}\cdot\mathbf{n})^2ds\right\}\\
&\le \sum_{i=1}^{2}\left\{(2\lambda_1+b_{max}^2)\int_{\Gamma}\tilde{u}_i^2ds+(1+2\lambda_2)a_i^2\int_{\Gamma}(\nabla\tilde{u}_i)^2ds\right\}\\
&\overset{Trace}{\le} \sum_{i=1}^{2}\left\{(2\lambda_1+b_{max}^2)\int_{\Omega_i}\tilde{u}_i^2dx\right.\\
&\left.+(2\lambda_1+b_{max}^2)\int_{\Omega_i}(\nabla\tilde{u}_i)^2dx+(1+2\lambda_2)a_i^2\int_{\Omega_i}(\nabla\tilde{u}_i)^2dx+(1+2\lambda_2)a_i^2\int_{\Omega_i}(\Delta\tilde{u}_i)^2dx\right\}.
\end{aligned}
\tag{17}
$$

Finally,

$$
(5)+(6)+(7)=\sum_{i=1}^{2}\left\{(2\lambda_1+b_{max}^2)\int_{\Omega_i}\tilde{u}_i^2dx+\left[2\lambda_1+b_{max}^2+(1+2\lambda_2)a_i^2\right]\int_{\Omega_i}(\nabla\tilde{u}_i)^2dx+(1+2\lambda_2)\int_{\Omega_i}(\Delta\tilde{u}_i)^2dx\right\}
\tag{18}
$$

In summary

$$
\begin{aligned}
(3) \leq \sum_{i=1}^{2} \Bigg\{ & \int_\Gamma a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} [2a_i b_i + \sup |a_i \nabla b_i|] (\nabla \tilde{u})^2 dx + \int_{\Omega_i} \left[b_i^2 - a_i \Delta b_i + \sup |a_i \nabla a_i|\right] \tilde{u}^2 dx] \\
& + (2\lambda_1 + b_{max}^2) \int_{\Omega_i} \tilde{u}_i^2 dx + \left[2\lambda_1 + b_{max}^2 + (1+2\lambda_2) a_i^2\right] \int_{\Omega_i} (\nabla \tilde{u}_i)^2 dx + (1+2\lambda_2) \int_{\Omega_i} (\Delta \tilde{u}_i)^2 dx + \lambda_3 \int_{\Omega_2} \tilde{u}^2 dx + \lambda_3 \int_{\Omega_2} (\nabla \tilde{u})^2 dx \Bigg\} \\
= \sum_{i=1}^{2} \Bigg\{ & \int_{\Omega_i} \left[(2+2\lambda_2) a_i^2\right] (\Delta \tilde{u})^2 dx + \int_{\Omega_i} \left[2a_i b_i + b_{max}^2 + (1+2\lambda_2) a_i^2 + 2\lambda_1 + \lambda_3 + \sup |a_i \nabla b_i|\right] (\nabla \tilde{u})^2 dx \\
& + \int_{\Omega_i} \left[b_i^2 + 2\lambda_1 + b_{max}^2 - a_i \Delta b_i + \lambda_3 + \sup |a_i \nabla b_i|\right] \tilde{u}^2 dx \Bigg\}
\end{aligned} \tag{19}
$$

So the upper bound is

$$
(3) \leq \max \left\{ (3+4\lambda_2) a_i^2 + 2a_i b_i + 2b_{max}^2 + 4\lambda_1 + b_i^2 - a_i \Delta b_i + 2\lambda_3 + 2 \sup |a_i \nabla b_i| \right\} \|u - u^*\|_{H_0^2(\Omega)}^2 = S_{max} \|u - u^*\|_{H_0^2(\Omega)}^2 \tag{20}
$$

□