

**Proposition 0.1.** For PINNs combined with DDM, we further assume for  $i = 1, 2$  that  $a_i$  is the constant coefficient and  $b_i \in L^\infty(\Omega_i)$  is a variable coefficient. If the parameters satisfy the following requirements  $0 < a_{\min} < a_i < a_{\max}$ ,  $0 < b_{\min} < b_i < b_{\max}$ ,  $0 < \lambda_2 < 1$ ,  $\lambda_1 > 0$ ,  $0 < J_{\min} < J_i < J_{\max}$ ,  $0 < Q_{\min} < Q_i < Q_{\max}$  and  $0 < S_{\min} < S_i < S_{\max}$ , where

$$\begin{aligned} J_i &= 2a_i b_i - \frac{1}{2}b_{\max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup |a_i \nabla b_i|, \\ Q_i &= b_i^2 - \frac{1}{2}b_{\max}^2 - \frac{1}{2}\lambda_1 - a_i \Delta b_i - \sup |a_i \nabla b_i|, \\ S_i &= (3 + 4\lambda_2)a_i^2 + 2a_i b_i + 2b_{\max}^2 + 4\lambda_1 + b_i^2 - a_i \Delta b_i + 2 \sup |a_i \nabla b_i| + 2\lambda_3. \end{aligned} \quad (1)$$

then we have for all  $u \in H_0^1(\Omega)$ ,

$$\begin{aligned} \min\left\{\frac{1}{2}(1 - \lambda_2)a_{\min}^2, J_{\min}, Q_{\min}\right\}\|u - u^*\|_{H_0^2(\Omega)}^2 &= \sum_{i=1}^2 \left\{\min\left\{\frac{1}{2}(1 - \lambda_2)a_{\min}^2, J_{\min}, Q_{\min}\right\}\|u - u^*\|_{H_0^2(\Omega_i)}^2\right\} \\ &\leq \mathbf{E}^{PINNs}(u) - \mathbf{E}^{PINNs}(u^*) \leq \sum_{i=1}^2 S_{\max}\|u - u^*\|_{H_0^2(\Omega_i)}^2 \leq S_{\max}\|u - u^*\|_{H_0^2(\Omega)}^2. \end{aligned} \quad (2)$$

*Proof.* We first expand the objective function,

$$\begin{aligned} &\mathbf{E}^{PINNs}(u) - \mathbf{E}^{PINNs}(u^*) \\ &= \sum_{i=1}^2 \int_{\Omega_i} |\nabla \cdot (a_i \nabla u) - b_i u + f - \nabla \cdot (a_i \nabla u^*) + b_i u^* - f|^2 dx + \underbrace{\lambda_1 \int_{\Gamma} |u_2 - u_1 - \varphi - u_2^* + u_1^* + \varphi|^2 ds}_{(6)} \\ &\quad + \underbrace{\lambda_2 \int_{\Gamma} |a_2 \nabla u_2 \cdot \mathbf{n} - a_1 \nabla u_1 \cdot \mathbf{n} - \psi - a_2 \nabla u_2^* \cdot \mathbf{n} + a_1 \nabla u_1^* \cdot \mathbf{n} + \psi|^2 ds}_{(7)} + \lambda_3 \int_{\partial\Omega} |u - g - u^* + g|^2 ds. \\ &= \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} b_i^2 (\tilde{u})^2 dx - 2 \int_{\Omega_i} (b_i \tilde{u}) (a_i \Delta \tilde{u}) dx \right\} + (6) + (7) + \lambda_3 \int_{\partial\Omega} \tilde{u}^2 ds \\ &\stackrel{Green}{=} \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} b_i^2 (\tilde{u})^2 dx + 2 \int_{\Omega_i} \nabla \cdot (b_i \tilde{u}) (a_i \nabla \tilde{u}) dx - 2 \int_{\Gamma} (b_i \tilde{u}) (a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right\} + (6) + (7) + \lambda_3 \int_{\partial\Omega} \tilde{u}^2 ds \\ &= \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} b_i^2 (\tilde{u})^2 dx + 2 \int_{\Omega_i} b_i a_i (\nabla \tilde{u})^2 dx + 2 \int_{\Omega_i} (\nabla b_i) \tilde{u} a_i \nabla \tilde{u} dx - 2 \int_{\Gamma} (b_i \tilde{u}) (a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right\} \\ &\quad + (6) + (7) + \lambda_3 \int_{\partial\Omega} \tilde{u}^2 ds \\ &= \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} b_i^2 (\tilde{u})^2 dx + 2 \int_{\Omega_i} b_i a_i (\nabla \tilde{u})^2 dx + \int_{\Omega_i} (\nabla b_i) a_i \nabla (\tilde{u}^2) dx - 2 \int_{\Gamma} (b_i \tilde{u}) (a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right\} \\ &\quad + (6) + (7) + \lambda_3 \int_{\partial\Omega} \tilde{u}^2 ds \\ &\stackrel{Green}{=} \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} 2a_i b_i (\nabla \tilde{u})^2 dx + \int_{\Omega_i} (b_i^2 - a_i \Delta b_i) \tilde{u}^2 dx + \underbrace{\int_{\Gamma} (\nabla b_i) a_i \tilde{u}^2 \cdot \mathbf{n} ds}_{(4)} - 2 \int_{\Gamma} (b_i \tilde{u}) (a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right\} \\ &\quad + (6) + (7) + \lambda_3 \int_{\partial\Omega} \tilde{u}^2 ds. \end{aligned} \quad (3)$$

We analyze the upper and lower bounds of the objective function separately. We first estimate the upper bound

$$(4) = \int_{\Gamma} a_i(\tilde{u})^2(\nabla b_i) \cdot \mathbf{n} ds = - \int_{\Gamma} -a_i(\tilde{u})^2(\nabla b_i) \cdot \mathbf{n} ds \geq - \left| \int_{\Gamma} -a_i(\tilde{u})^2(\nabla b_i) \cdot \mathbf{n} ds \right| = - \left| \int_{\Gamma} a_i(\tilde{u})^2(\nabla b_i) \cdot \mathbf{n} ds \right|$$

$$\geq - \sup |a_i \nabla b_i| \cdot \left| \int_{\Gamma} \tilde{u}^2 ds \right| \stackrel{Trace}{\geq} - \sup |a_i \nabla b_i| \cdot \left\{ \int_{\Omega_i} \tilde{u}^2 ds + \int_{\Omega_i} (\nabla \tilde{u})^2 ds \right\}.$$

Then the upper bound of equation (3) is estimated as

$$(3) \geq \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} [2a_i b_i - \sup |a_i \nabla b_i|] (\nabla \tilde{u})^2 dx + \int_{\Omega_i} [b_i^2 - a_i \Delta b_i - \sup |a_i \nabla b_i|] (\tilde{u})^2 dx \right\}$$

$$+ \underbrace{\sum_{i=1}^2 \left\{ -2 \int_{\Gamma} (b_i \tilde{u})(a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right\}}_{(5)} + (6) + (7) + \lambda_3 \int_{\partial \Omega} \tilde{u}^2 ds$$

$$\geq \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} [2a_i b_i - \sup |a_i \nabla b_i|] (\nabla \tilde{u})^2 dx + \int_{\Omega_i} [b_i^2 - a_i \Delta b_i - \sup |a_i \nabla b_i|] (\tilde{u})^2 dx \right\}$$

$$+ \underbrace{\sum_{i=1}^2 \left\{ -2 \int_{\Gamma} (b_i \tilde{u})(a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right\}}_{(5)} + (6) + (7).$$

Then estimating the upper bound of the objective function we only have to analyze the values of (5) + (6) + (7) to end the analysis.

$$(5) + (6) + (7)$$

$$= \sum_{i=1}^2 -2 \int_{\Gamma} (b_i \tilde{u})(a_i \nabla \tilde{u} \cdot \mathbf{n}) ds + \lambda_1 \int_{\Gamma} (\tilde{u}_1 - \tilde{u}_2)^2 ds + \lambda_2 \int_{\Gamma} |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 ds$$

$$\stackrel{Cauchy-Schwarz}{\geq} \sum_{i=1}^2 -2 \left( \int_{\Gamma} |b_i \tilde{u}|^2 ds \right)^{\frac{1}{2}} \left( \int_{\Gamma} |a_i \nabla \tilde{u} \cdot \mathbf{n}|^2 ds \right)^{\frac{1}{2}} + \lambda_1 \int_{\Gamma} |\tilde{u}_2 - \tilde{u}_1|^2 ds + \lambda_2 \int_{\Gamma} |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 ds$$

$$\geq \sum_{i=1}^2 - \int_{\Gamma} |b_i \tilde{u}|^2 ds - \int_{\Gamma} |a_i \nabla \tilde{u} \cdot \mathbf{n}|^2 ds + \lambda_1 \int_{\Gamma} |\tilde{u}_2 - \tilde{u}_1|^2 ds + \lambda_2 \int_{\Gamma} |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 ds$$

$$= - \int_{\Gamma} (b_1^2 \tilde{u}_1^2 + b_2^2 \tilde{u}_2^2) ds - \int_{\Gamma} (a_1 \nabla \tilde{u}_1 \cdot \mathbf{n})^2 + (a_2 \nabla \tilde{u}_2 \cdot \mathbf{n})^2 ds + \lambda_1 \int_{\Gamma} |\tilde{u}_2 - \tilde{u}_1|^2 ds + \lambda_2 \int_{\Gamma} |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 ds$$

$$= \int_{\Gamma} (\lambda_1 (\tilde{u}_2 - \tilde{u}_1)^2 - b_1^2 \tilde{u}_1^2 - b_2^2 \tilde{u}_2^2) ds + \int_{\Gamma} \left\{ \lambda_2 (a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n})^2 - (a_1 \nabla \tilde{u} \cdot \mathbf{n})^2 - (a_2 \nabla \tilde{u} \cdot \mathbf{n})^2 \right\} ds$$

$$\geq \int_{\Gamma} \lambda_1 \tilde{u}_2^2 + \lambda_1 \tilde{u}_1^2 - 2\lambda_1 \tilde{u}_1 \tilde{u}_2 - b_{max}^2 \tilde{u}_1^2 - b_{max}^2 \tilde{u}_2^2 ds$$

$$+ \underbrace{\int_{\Gamma} \lambda_2 (a_2 \nabla \tilde{u}_2 \cdot \mathbf{n})^2 + \lambda_2 (a_1 \nabla \tilde{u}_1 \cdot \mathbf{n})^2 - 2\lambda_2 (a_1 \nabla \tilde{u}_1 \cdot \mathbf{n})(a_2 \nabla \tilde{u}_2 \cdot \mathbf{n}) - (a_1 \nabla \tilde{u}_1 \cdot \mathbf{n})^2 - (a_2 \nabla \tilde{u}_2 \cdot \mathbf{n})^2 ds}_{(8)}$$

$$= \int_{\Gamma} \{ (\lambda_1 - b_{max}^2) \tilde{u}_1^2 + (\lambda_1 - b_{max}^2) \tilde{u}_2^2 - 2\lambda_1 \tilde{u}_1 \tilde{u}_2 \} ds + (8)$$

$$\geq \int_{\Gamma} \{ (\lambda_1 - b_{max}^2) |\tilde{u}_1|^2 + (\lambda_1 - b_{max}^2) |\tilde{u}_2|^2 - 2\lambda_1 |\tilde{u}_1| |\tilde{u}_2| \} ds + (8)$$

$$\geq \int_{\Gamma} (\lambda_1 - b_{max}^2) |\tilde{u}_1| |\tilde{u}_2| - 2\lambda_1 |\tilde{u}_1| |\tilde{u}_2| ds + (8).$$

We further simplify the above equation

$$\begin{aligned}
& (5) + (6) + (7) \\
& = (-\lambda_1 + b_{max}^2) \left[ -\int_{\Gamma} |\tilde{u}_1| |\tilde{u}_2| ds \right] + (8) \\
& \geq \left( \frac{\lambda_1}{2} - \frac{1}{2} b_{max}^2 \right) \int_{\Gamma} |\tilde{u}_1|^2 ds + \left( \frac{-\lambda_1}{2} - \frac{1}{2} b_{max}^2 \right) \int_{\Gamma} |\tilde{u}_2|^2 ds + (8) \\
& = \sum_{i=1}^2 \left[ \left( \frac{-\lambda_1}{2} - \frac{1}{2} b_{max}^2 \right) \int_{\Gamma} |\tilde{u}_i|^2 ds \right] + (8) \tag{7} \\
& \stackrel{Trace}{\geq} \sum_{i=1}^2 \left[ \left( \frac{-\lambda_1}{2} - \frac{1}{2} b_{max}^2 \right) \int_{\Omega_i} |\tilde{u}_i^2| dx + \left( \frac{-\lambda_1}{2} - \frac{1}{2} b_{max}^2 \right) \int_{\Omega_i} |\nabla \tilde{u}_i^2| dx \right] + (8) \\
& = \int_{\Gamma} (-\lambda_1 - b_{max}^2) |\tilde{u}_1| |\tilde{u}_2| ds + (8)
\end{aligned}$$

Thus the objective function is

$$\begin{aligned}
(3) & = \mathbf{E}^{PINNs}(u) - \mathbf{E}^{PINNs}(u^*) \\
& \geq \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} [2a_i b_i - \sup |a_i \nabla b_i|] (\nabla \tilde{u})^2 dx + \int_{\Omega_i} [b_i^2 - a_i \Delta b_i - \sup |a_i \nabla b_i|] (\tilde{u})^2 dx \right\} \\
& + \sum_{i=1}^2 \left[ \left( \frac{-\lambda_1}{2} - \frac{1}{2} b_{max}^2 \right) \int_{\Omega_i} |\tilde{u}_i|^2 dx + \left( \frac{-\lambda_1}{2} - \frac{1}{2} b_{max}^2 \right) \int_{\Omega_i} |\nabla \tilde{u}_i|^2 dx + (8) \right] \tag{8} \\
& \geq \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} \left[ 2a_i b_i - \frac{1}{2} b_{max}^2 - \frac{\lambda_1}{2} - \sup |a_i \nabla b_i| \right] (\nabla \tilde{u})^2 dx \right. \\
& \left. + \int_{\Omega_i} \left[ b_i^2 - \frac{1}{2} b_{max}^2 - \frac{\lambda_1}{2} - a_i \Delta b_i - \sup |a_i \nabla b_i| \right] |\tilde{u}|^2 dx \right\} + (8).
\end{aligned}$$

Now analyze equation (8)

$$\begin{aligned}
(8) & = \int_{\Gamma} -2(a_1 \nabla \tilde{u}_1 \cdot \mathbf{n})(a_2 \nabla \tilde{u}_2 \cdot \mathbf{n}) ds = \int_{\Gamma} -2a_1 a_2 \nabla \tilde{u}_1 \cdot \nabla \tilde{u}_2 \cdot (\mathbf{n})^2 ds \\
& = \int_{\Gamma} -2a_1 a_2 \nabla \tilde{u}_1 \cdot \nabla \tilde{u}_2 ds = -2a_1 a_2 \int_{\Gamma} \nabla \tilde{u}_1 \nabla \tilde{u}_2 ds \\
& \stackrel{Cauchy-Schwarz}{\geq} -2a_1 a_2 \left[ \left( \int_{\Gamma} (\nabla \tilde{u}_1)^2 ds \right)^{\frac{1}{2}} \left( \int_{\Gamma} (\nabla \tilde{u}_2)^2 ds \right)^{\frac{1}{2}} \right] \\
& \geq -a_1 a_2 \left[ \int_{\Gamma} (\nabla \tilde{u}_1)^2 ds + \int_{\Gamma} (\nabla \tilde{u}_2)^2 ds \right] \tag{9} \\
& = \int_{\Gamma} (\lambda_2 - 1) |a_2 \nabla \tilde{u}_2 \cdot \mathbf{n}|^2 + (\lambda_2 - 1) |a_1 \nabla \tilde{u}_1 \cdot \mathbf{n}|^2 - 2\lambda_2 (a_1 \nabla \tilde{u}_1 \cdot \mathbf{n})(a_2 \nabla \tilde{u}_2 \cdot \mathbf{n}) ds \\
& \geq \int_{\Gamma} (\lambda_2 - 1) |a_2 \nabla \tilde{u}_2 \cdot \mathbf{n}| |a_1 \nabla \tilde{u}_1 \cdot \mathbf{n}| - 2\lambda_2 |a_1 \nabla \tilde{u}_1 \cdot \mathbf{n}| |a_2 \nabla \tilde{u}_2 \cdot \mathbf{n}| ds \\
& = \int_{\Gamma} (-\lambda_2 - 1) |a_1 \nabla \tilde{u}_1 \cdot \mathbf{n}| |a_2 \nabla \tilde{u}_2 \cdot \mathbf{n}| ds \\
& = \left( \frac{1}{2} \lambda_2 + \frac{1}{2} \right) \int_{\Gamma} -2 |a_1 \nabla \tilde{u}_1 \cdot \mathbf{n}| |a_2 \nabla \tilde{u}_2 \cdot \mathbf{n}| ds.
\end{aligned}$$

We further simplify the above equation

$$\begin{aligned}
(8) &\geq \left(-\frac{1}{2}\lambda_2 - \frac{1}{2}\right) \int_{\Gamma} (a_1 \nabla \tilde{u}_1 \cdot \mathbf{n})^2 ds + \left(-\frac{1}{2}\lambda_2 - \frac{1}{2}\right) \int_{\Gamma} (a_2 \nabla \tilde{u}_2 \cdot \mathbf{n})^2 ds \\
&\geq \left(-\frac{1}{2} - \frac{1}{2}\lambda_2\right) a_1^2 \int_{\Gamma} (\nabla \tilde{u}_1)^2 ds + \left(-\frac{1}{2} - \frac{1}{2}\lambda_2\right) a_2^2 \int_{\Gamma} (\nabla \tilde{u}_2)^2 ds \\
&= \sum_{i=1}^2 \left(-\frac{1}{2} - \frac{1}{2}\lambda_2\right) a_i^2 \int_{\Gamma} (\nabla \tilde{u}_i)^2 ds \\
&\stackrel{Trace}{\geq} \sum_{i=1}^2 \left[ \left(-\frac{1}{2} - \frac{1}{2}\lambda_2\right) a_i^2 \int_{\Omega_i} (\nabla \tilde{u}_i)^2 dx + \left(-\frac{1}{2} - \frac{1}{2}\lambda_2\right) a_i^2 \int_{\Omega_i} (\Delta \tilde{u}_i)^2 dx \right].
\end{aligned} \tag{10}$$

We bring (8) into the objective function

$$\begin{aligned}
(3) &= \sum_{i=1}^2 \left\{ \int_{\Omega_i} \left( a_i^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_2 a_i^2 \right) (\Delta \tilde{u})^2 dx + \int_{\Omega_i} \left( 2a_i b_i - \frac{1}{2}b_{max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup |a_i \nabla b_i| \right) (\nabla \tilde{u})^2 dx \right. \\
&\quad \left. + \int_{\Omega_i} \left( b_i^2 - \frac{1}{2}b_{max}^2 - \frac{1}{2}\lambda_1 - a_i \Delta b_i - \sup |a_i \nabla b_i| \right) (\tilde{u})^2 dx \right\} \\
&\geq \sum_{i=1}^2 \left\{ \int_{\Omega_i} \frac{1}{2}(1 - \lambda_2) a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} \left[ 2a_i b_i - \frac{1}{2}b_{max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup |a_i \nabla b_i| \right] (\nabla \tilde{u})^2 dx \right. \\
&\quad \left. + \int_{\Omega_i} \left[ b_i^2 - \frac{1}{2}b_{max}^2 - \frac{1}{2}\lambda_1 - a_i \Delta b_i - \sup |a_i \nabla b_i| \right] (\tilde{u})^2 dx \right\}.
\end{aligned} \tag{11}$$

So we obtain

$$\begin{aligned}
(3) &\geq \min \left\{ \frac{1}{2}(1 - \lambda_2) a_{min}^2, \min \left[ 2a_i b_i - \frac{1}{2}b_{max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup |a_i \nabla b_i| \right] \right. \\
&\quad \left. , \min \left[ b_i^2 - \frac{1}{2}b_{max}^2 - \frac{1}{2}\lambda_1 - a_i \Delta b_i - \sup |a_i \nabla b_i| \right] \right\} \|u - u^*\|_{H_0^2(\Omega)}^2 \\
&\geq \min \left\{ \frac{1}{2}(1 - \lambda_2) a_{min}^2, J_{min}, Q_{min} \right\} \|u - u^*\|_{H_0^2(\Omega)}^2
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
J_{min} &= \min J_i = \min \left\{ 2a_i b_i - \frac{1}{2}b_{max}^2 - \frac{1}{2}a_i^2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 a_i^2 - \sup |a_i \nabla b_i| \right\}, i = 1, 2, \\
Q_{min} &= \min Q_i = \min \left\{ b_i^2 - \frac{1}{2}b_{max}^2 - \frac{1}{2}\lambda_1 - a_i \Delta b_i - \sup |a_i \nabla b_i| \right\}, i = 1, 2.
\end{aligned} \tag{13}$$

Now we analyze the upper bound of equation (3)

$$\begin{aligned}
(3) &= \sum_{i=1}^2 \left\{ \int_{\Omega_i} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} 2a_i b_i (\nabla \tilde{u})^2 dx + \int_{\Omega_i} (b_i^2 - a_i \Delta b_i) \tilde{u}^2 dx + \underbrace{\int_{\Gamma} (\nabla b_i) a_i \tilde{u}^2 \cdot \mathbf{n} ds}_{(4)} - 2 \int_{\Gamma} (b_i \tilde{u}) (a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right\} \\
&\quad + (6) + (7) + \lambda_3 \int_{\partial\Omega} \tilde{u}^2 ds.
\end{aligned} \tag{14}$$

Where (4) we analyze separately

$$(4) = \int_{\Gamma} a_i (\tilde{u})^2 \nabla b_i \cdot \mathbf{n} ds \leq \int_{\Gamma} |a_i (\tilde{u})^2 \nabla b_i \cdot \mathbf{n}| ds \leq \sup |a_i \nabla b_i| \left| \int_{\Gamma} \tilde{u}^2 ds \right| \stackrel{Trace}{\leq} \sup |a_i \nabla b_i| \left\{ \int_{\Omega_i} \tilde{u}^2 ds + \int_{\Omega_i} (\nabla \tilde{u})^2 ds \right\}. \tag{15}$$

The simplification result of (4) is brought to the objective function

$$\begin{aligned}
 (3) \leq & \sum_{i=1}^2 \left\{ \int_{\Gamma} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} [2a_i b_i + \sup |a_i \nabla b_i|] (\nabla \tilde{u})^2 dx + \int_{\Omega_i} [b_i^2 - a_i \Delta b_i + \sup |a_i \nabla a_i|] \tilde{u}^2 dx \right\} \\
 & + \underbrace{\sum_{i=1}^2 \left\{ -2 \int_{\Gamma} (b_i \tilde{u})(a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right\}}_{(5)} + (6) + (7) + \lambda_3 \int_{\partial \Omega} \tilde{u}^2 ds.
 \end{aligned} \tag{16}$$

We analyze the last three terms of the objective function

$$\begin{aligned}
 (5) + (6) + (7) &= \sum_{i=1}^2 \left\{ -2 \int_{\Gamma} (b_i \tilde{u})(a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right\} + \lambda_1 \int_{\Gamma} |\tilde{u}_2 - \tilde{u}_1|^2 ds + \lambda_2 \int_{\Gamma} |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 ds \\
 &\leq \sum_{i=1}^2 \left\{ \left| -2 \int_{\Gamma} (b_i \tilde{u})(a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right| \right\} + \lambda_1 \int_{\Gamma} |\tilde{u}_2 - \tilde{u}_1|^2 ds + \lambda_2 \int_{\Gamma} |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 ds \\
 &= \sum_{i=1}^2 \left\{ \left| 2 \int_{\Gamma} (b_i \tilde{u})(a_i \nabla \tilde{u} \cdot \mathbf{n}) ds \right| \right\} + \lambda_1 \int_{\Gamma} |\tilde{u}_2 - \tilde{u}_1|^2 ds + \lambda_2 \int_{\Gamma} |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 ds \\
 &\stackrel{Cauchy-Schwarz}{\leq} \sum_{i=1}^2 \left\{ 2 \left( \int_{\Gamma} |b_i \tilde{u}|^2 ds \right)^{\frac{1}{2}} \left( \int_{\Gamma} |a_i \nabla \tilde{u} \cdot \mathbf{n}|^2 ds \right)^{\frac{1}{2}} \right\} + \lambda_1 \int_{\Gamma} |\tilde{u}_2 - \tilde{u}_1|^2 ds \\
 &\quad + \lambda_2 \int_{\Gamma} |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 ds \\
 &\leq \sum_{i=1}^2 \left\{ \int_{\Gamma} |b_i \tilde{u}|^2 ds + \int_{\Gamma} |a_i \nabla \tilde{u} \cdot \mathbf{n}|^2 ds \right\} + \lambda_1 \int_{\Gamma} |\tilde{u}_2 - \tilde{u}_1|^2 ds + \lambda_2 \int_{\Gamma} |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 ds \\
 &= \int_{\Gamma} (b_1^2 \tilde{u}_1^2 + b_2^2 \tilde{u}_2^2 + \lambda_1 |\tilde{u}_2 - \tilde{u}_1|^2) ds + \int_{\Gamma} \{ (a_1 \nabla \tilde{u} \cdot \mathbf{n})^2 + (a_2 \nabla \tilde{u} \cdot \mathbf{n})^2 + \lambda_2 |a_2 \nabla \tilde{u} \cdot \mathbf{n} - a_1 \nabla \tilde{u} \cdot \mathbf{n}|^2 \} ds \\
 &= \int_{\Gamma} \{ (\lambda_1 + b_{max}^2) \tilde{u}_1^2 + (\lambda_1 + b_{max}^2) \tilde{u}_2^2 - 2\lambda_1 \tilde{u}_1 \tilde{u}_2 \} ds \\
 &\quad + \int_{\Gamma} \{ (1 + \lambda_2)(a_1 \nabla \tilde{u} \cdot \mathbf{n})^2 + (1 + \lambda_2)(a_2 \nabla \tilde{u} \cdot \mathbf{n})^2 - 2\lambda_2 (a_1 \nabla \tilde{u} \cdot \mathbf{n})(a_2 \nabla \tilde{u} \cdot \mathbf{n}) \} ds \\
 &\leq \int_{\Gamma} \{ (\lambda_1 + b_{max}^2) \tilde{u}_1^2 + (\lambda_1 + b_{max}^2) \tilde{u}_2^2 + \lambda_1 \tilde{u}_1^2 + \lambda_1 \tilde{u}_2^2 \} ds \\
 &\quad + \int_{\Gamma} \{ (1 + \lambda_2)(a_1 \nabla \tilde{u} \cdot \mathbf{n})^2 + (1 + \lambda_2)(a_2 \nabla \tilde{u} \cdot \mathbf{n})^2 + \lambda_2 (a_1 \nabla \tilde{u} \cdot \mathbf{n})^2 + \lambda_2 (a_2 \nabla \tilde{u} \cdot \mathbf{n})^2 \} ds \\
 &= \int_{\Gamma} (2\lambda_1 + b_{max}^2) \tilde{u}_1^2 + (2\lambda_1 + b_{max}^2) \tilde{u}_2^2 ds + \int_{\Gamma} (1 + 2\lambda_2)(a_1 \nabla \tilde{u} \cdot \mathbf{n})^2 + (1 + 2\lambda_2)(a_2 \nabla \tilde{u} \cdot \mathbf{n})^2 ds \\
 &\leq \sum_{i=1}^2 \left\{ (2\lambda_1 + b_{max}^2) \int_{\Gamma} \tilde{u}_i^2 ds + (1 + 2\lambda_2) \int_{\Gamma} (a_i \nabla \tilde{u} \cdot \mathbf{n})^2 ds \right\} \\
 &\leq \sum_{i=1}^2 \left\{ (2\lambda_1 + b_{max}^2) \int_{\Gamma} \tilde{u}_i^2 ds + (1 + 2\lambda_2) a_i^2 \int_{\Gamma} (\nabla \tilde{u}_i)^2 ds \right\} \\
 &\stackrel{Trace}{\leq} \sum_{i=1}^2 \left\{ (2\lambda_1 + b_{max}^2) \int_{\Omega_i} \tilde{u}_i^2 dx \right. \\
 &\quad \left. + (2\lambda_1 + b_{max}^2) \int_{\Omega_i} (\nabla \tilde{u}_i)^2 dx + (1 + 2\lambda_2) a_i^2 \int_{\Omega_i} (\nabla \tilde{u}_i)^2 dx + (1 + 2\lambda_2) a_i^2 \int_{\Omega_i} (\Delta \tilde{u}_i)^2 dx \right\}.
 \end{aligned} \tag{17}$$

Finally,

$$(5) + (6) + (7) = \sum_{i=1}^2 \left\{ (2\lambda_1 + b_{max}^2) \int_{\Omega_i} \tilde{u}_i^2 dx + [2\lambda_1 + b_{max}^2 + (1 + 2\lambda_2)a_i^2] \int_{\Omega_i} (\nabla \tilde{u}_i)^2 dx + (1 + 2\lambda_2) \int_{\Omega_i} (\Delta \tilde{u}_i)^2 dx \right\} \quad (18)$$

In summary

$$\begin{aligned} (3) &\leq \sum_{i=1}^2 \left\{ \int_{\Gamma} a_i^2 (\Delta \tilde{u})^2 dx + \int_{\Omega_i} [2a_i b_i + \sup |a_i \nabla b_i|] (\nabla \tilde{u})^2 dx + \int_{\Omega_i} [b_i^2 - a_i \Delta b_i + \sup |a_i \nabla a_i|] \tilde{u}^2 dx \right\} \\ &+ (2\lambda_1 + b_{max}^2) \int_{\Omega_i} \tilde{u}_i^2 dx + [2\lambda_1 + b_{max}^2 + (1 + 2\lambda_2)a_i^2] \int_{\Omega_i} (\nabla \tilde{u}_i)^2 dx + (1 + 2\lambda_2) \int_{\Omega_i} (\Delta \tilde{u}_i)^2 dx + \lambda_3 \int_{\Omega_2} \tilde{u}^2 dx + \lambda_3 \int_{\Omega_2} (\nabla \tilde{u})^2 dx \Big\} \\ &= \sum_{i=1}^2 \left\{ \int_{\Omega_i} [(2 + 2\lambda_2)a_i^2] (\Delta \tilde{u})^2 dx + \int_{\Omega_i} [2a_i b_i + b_{max}^2 + (1 + 2\lambda_2)a_i^2 + 2\lambda_1 + \lambda_3 + \sup |a_i \nabla b_i|] (\nabla \tilde{u})^2 dx \right. \\ &\quad \left. + \int_{\Omega_i} [b_i^2 + 2\lambda_1 + b_{max}^2 - a_i \Delta b_i + \lambda_3 + \sup |a_i \nabla b_i|] \tilde{u}^2 dx \right\} \end{aligned} \quad (19)$$

So the upper bound is

$$(3) \leq \max \left\{ (3 + 4\lambda_2)a_i^2 + 2a_i b_i + 2b_{max}^2 + 4\lambda_1 + b_i^2 - a_i \Delta b_i + 2\lambda_3 + 2 \sup |a_i \nabla b_i| \right\} \|u - u^*\|_{H_0^2(\Omega)}^2 = S_{max} \|u - u^*\|_{H_0^2(\Omega)}^2 \quad (20)$$

□