**let:**

$\mathbf{t}$ = Time index in **months**, $t = 0,1,2,\ldots,\mathbf{T}$
$\mathbf{T}$ = Total number of months in the simulation horizon
$\mathbf{FV}_t$ = Future value of retirement account at time $t$
$\mathbf{W}$ = Initial retirement account value ("nest egg")
$\mathbf{r}_a$ = Annual interest rate
$\mathbf{r}$ = Monthly interest rate (i.e., $\mathbf{r}_a$ divided by 12)

$\mathbf{fa}_i$ = Annual inflation rate for cost component $\mathbf{C}_i$
$\mathbf{f}_i$ = Monthly inflation rate (i.e. $\mathbf{f}_a/12$ ) for cost component $\mathbf{C}$
$\mathbf{C}_i$ = Initial (base) cost for component $i$ , where $i = 1,2,...,\mathbf{K}$
$\mathbf{C}_{i,t}$ = Future value of the cost component $i$ at time $t$

**month:** $t = 1$

$$\mathbf{C}_{1,1} = (1+\mathbf{f}_1)\mathbf{C}_1$$ cost component $1$ at month (i.e. t): $1$ inflated by: $\mathbf{f}_1$

$$\mathbf{C}_{2,1} = (1+\mathbf{f}_2)\mathbf{C}_2$$ cost component $2$ at month (i.e. t): $1$ inflated by: $\mathbf{f}_2$

$$\mathbf{C}_{3,1} = (1+\mathbf{f}_3)\mathbf{C}_3$$ cost component $3$ at month (i.e. t): $1$ inflated by: $\mathbf{f}_3$

$$\mathbf{FV}_1 = \mathbf{W}(1+\mathbf{r}) - (\mathbf{C}_{1,1}+\mathbf{C}_{2,1}+\mathbf{C}_{3,1})$$ (Future Value after 1 period)

**since:** $\mathbf{K} = 3$

$$\mathbf{FV}_1 = \mathbf{W}(1+\mathbf{r}) - \sum_{k=1}^{\mathbf{K}} \mathbf{C}_{k,1}$$ (Future Value after 1 period)

$$\mathbf{FV}_1 = \mathbf{W}(1+\mathbf{r}) - \sum_{k=1}^{\mathbf{K}} (1+\mathbf{f}_k)\mathbf{C}_k$$ substituting for $\mathbf{C}_k$

**month:** $t = 2$

$$\mathbf{C}_{1,2} = (1+\mathbf{f}_1)\mathbf{C}_{1,1}$$ cost component $1$ at month (i.e. t): $2$ inflated by: $\mathbf{f}_1$

$$= (1+\mathbf{f}_1)(1+\mathbf{f}_1)\mathbf{C}_1$$ substituting for $\mathbf{C}_{1,1}$

$$= (1+\mathbf{f}_1)^2\mathbf{C}_1$$ combining terms

$$\mathbf{C}_{2,2} = (1+\mathbf{f}_2)\mathbf{C}_{2,1}$$ cost component $2$ at month (i.e. t): $2$ inflated by: $\mathbf{f}_2$

$$= (1+\mathbf{f}_2)(1+\mathbf{f}_2)\mathbf{C}_2$$ substituting for $\mathbf{C}_{2,1}$

$$= (1+\mathbf{f}_2)^2\mathbf{C}_2$$ combining terms

$$\mathbf{C}_{3,2} = (1+\mathbf{f}_3)\mathbf{C}_{3,1}$$ cost component $3$ at month (i.e. t): $2$ inflated by: $\mathbf{f}_3$

$$= (1+\mathbf{f}_3)(1+\mathbf{f}_3)\mathbf{C}_3$$ substituting for $\mathbf{C}_{1,1}$

$$= (1+\mathbf{f}_3)^2\mathbf{C}_3$$ combining terms

**in general:** $\mathbf{C}_{k,t} = (1+\mathbf{f}_k)^t\mathbf{C}_k$ by induction, $(1+\mathbf{f}_k)$ applied every time period $\mathbf{t}$

$$\mathbf{FV}_2 = \mathbf{FV}_1(1+\mathbf{r}) - (\mathbf{C}_{1,2}+\mathbf{C}_{2,2}+\mathbf{C}_{3,2})$$ (Future Value after 2 period)

$$= \mathbf{FV}_1(1+\mathbf{r}) - \sum_{k=1}^{\mathbf{K}} \mathbf{C}_{k,2}$$ where $t = 2$

$$= \mathbf{FV}_1(1+\mathbf{r}) - \sum_{k=1}^{\mathbf{K}} (1+\mathbf{f}_k)^2\mathbf{C}_k$$ since $\mathbf{C}_{k,t} = (1+\mathbf{f}_k)^t\mathbf{C}_k \rightarrow \mathbf{C}_{k,2} = (1+\mathbf{f}_k)^2\mathbf{C}_k$

$$\mathbf{FV}_2 = [\mathbf{W}(1+\mathbf{r}) - \sum_{k=1}^{\mathbf{K}} (1+\mathbf{f}_k)\mathbf{C}_k](1+\mathbf{r}) - \sum_{k=1}^{\mathbf{K}} (1+\mathbf{f}_k)^2\mathbf{C}_k$$ substitute for $\mathbf{FV}_1$

$$= \mathbf{W}(1+\mathbf{r})(1+\mathbf{r}) - (1+\mathbf{r})\sum_{k=1}^{\mathbf{K}} (1+\mathbf{f}_k)\mathbf{C}_k - \sum_{k=1}^{\mathbf{K}} (1+\mathbf{f}_k)^2\mathbf{C}_k$$ distribute $(1+\mathbf{r})$

$$\mathbf{FV}_2 = \mathbf{W}(1+\mathbf{r})^2 - (1+\mathbf{r})\sum_{k=1}^{\mathbf{K}} (1+\mathbf{f}_k)\mathbf{C}_k - \sum_{k=1}^{\mathbf{K}} (1+\mathbf{f}_k)^2\mathbf{C}_k$$ combine $(1+\mathbf{r})$

**month:** $t = 3$

$$\mathbf{FV}_3 = \mathbf{FV}_2(1+\mathbf{r}) - \sum_{k=1}^{\mathbf{K}} \mathbf{C}_{k,3} \qquad \text{where } t = 3$$

$$= \mathbf{FV}_2(1+\mathbf{r}) - \sum_{k=1}^{\mathbf{K}} (1+\mathbf{f}_k)^3\mathbf{C}_k \qquad \text{since } \mathbf{C}_{k,t} = (1+\mathbf{f}_k)^t\mathbf{C}_k \rightarrow \mathbf{C}_{k,3} = (1+\mathbf{f}_k)^3\mathbf{C}_k$$

$$= [\mathbf{W}(1+\mathbf{r})^2 - (1+\mathbf{r})\sum_{k=1}^{\mathbf{K}}(1+\mathbf{f}_k)\mathbf{C}_k - \sum_{k=1}^{\mathbf{K}}(1+\mathbf{f}_k)^2\mathbf{C}_k](1+\mathbf{r}) - \sum_{k=1}^{\mathbf{K}}(1+\mathbf{f}_k)^3\mathbf{C}_k \qquad \text{substitute } \mathbf{FV}_2 \text{ from above}$$

$$= \mathbf{W}(1+\mathbf{r})^2(1+\mathbf{r}) - (1+\mathbf{r})(1+\mathbf{r})\sum_{k=1}^{\mathbf{K}}(1+\mathbf{f}_k)\mathbf{C}_k - (1+\mathbf{r})\sum_{k=1}^{\mathbf{K}}(1+\mathbf{f}_k)^2\mathbf{C}_k - \sum_{k=1}^{\mathbf{K}}(1+\mathbf{f}_k)^3\mathbf{C}_k \qquad \text{distribute } (1+\mathbf{r})$$

$$\mathbf{FV}_3 = \mathbf{W}(1+\mathbf{r})^3 - (1+\mathbf{r})^2\sum_{k=1}^{\mathbf{K}}(1+\mathbf{f}_k)\mathbf{C}_k - (1+\mathbf{r})\sum_{k=1}^{\mathbf{K}}(1+\mathbf{f}_k)^2\mathbf{C}_k - \sum_{k=1}^{\mathbf{K}}(1+\mathbf{f}_k)^3\mathbf{C}_k \qquad \text{combine } (1+\mathbf{r})$$

**collect:** $\mathbf{C}_i$ **from each** $\sum$

collect terms (in columns): $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$

$$\begin{aligned}
\mathbf{FV}_3 = {} & \mathbf{W}(1+\mathbf{r})^3 \\
& -(1+\mathbf{r})^2(1+\mathbf{f}_1)\ \mathbf{C}_1 - (1+\mathbf{r})^2(1+\mathbf{f}_2)\ \mathbf{C}_2 - (1+\mathbf{r})^2(1+\mathbf{f}_3)\ \mathbf{C}_3 \\
& -(1+\mathbf{r})\ (1+\mathbf{f}_1)^2\mathbf{C}_1 - (1+\mathbf{r})\ (1+\mathbf{f}_2)^2\mathbf{C}_2 - (1+\mathbf{r})\ (1+\mathbf{f}_3)^2\mathbf{C}_3 \\
& - \qquad (1+\mathbf{f}_1)^3\mathbf{C}_1 - \qquad (1+\mathbf{f}_2)^3\mathbf{C}_2 - \qquad (1+\mathbf{f}_3)^3\mathbf{C}_3
\end{aligned}$$

**by inference:**
**all (1+r) & (1+f) exponents add to: t**
**(1+f) exponent range: 1 to t**
**(1+r) exponent range: 0 to** $(\mathbf{t}-\mathbf{1})$

$$\begin{aligned}
\mathbf{FV}_t = {} & \mathbf{W}(1+\mathbf{r})^t \\
& -(1+\mathbf{r})^{t-1}(1+\mathbf{f}_1)\quad \mathbf{C}_1 - (1+\mathbf{r})^{t-1}(1+\mathbf{f}_2)\quad \mathbf{C}_2 - (1+\mathbf{r})^{t-1}(1+\mathbf{f}_3)\quad \mathbf{C}_3 \\
& -(1+\mathbf{r})^{t-2}(1+\mathbf{f}_1)^2\quad \mathbf{C}_1 - (1+\mathbf{r})^{t-2}(1+\mathbf{f}_2)^2\ \mathbf{C}_2 - (1+\mathbf{r})^{t-2}(1+\mathbf{f}_3)^2\quad \mathbf{C}_3 \\
& -\ldots \\
& -(1+\mathbf{r})\quad (1+\mathbf{f}_1)^{t-1}\mathbf{C}_1 - (1+\mathbf{r})\quad (1+\mathbf{f}_2)^{t-1}\mathbf{C}_2 - (1+\mathbf{r})\quad (1+\mathbf{f}_3)^{t-1}\mathbf{C}_3 \\
& - \qquad (1+\mathbf{f}_1)^t\quad \mathbf{C}_1 - \qquad (1+\mathbf{f}_2)^t\quad \mathbf{C}_2 - \qquad (1+\mathbf{f}_3)^t\quad \mathbf{C}_3
\end{aligned}$$

**let: FC be the Future Cost then**
$\mathbf{FC}_{i,t}$ **cost: i at time: t**

$$\mathbf{FC}_{i,t} = -(1+\mathbf{r})^{t-1}(1+\mathbf{f}_i)\mathbf{C}_i - (1+\mathbf{r})^{t-2}(1+\mathbf{f}_i)^2\mathbf{C}_i - \ldots - (1+\mathbf{r})(1+\mathbf{f}_i)^{t-1}\mathbf{C}_i - (1+\mathbf{r})^0(1+\mathbf{f}_i)^t\mathbf{C}_i \qquad \text{parameterize } \mathbf{C}_i \text{ and } \mathbf{t}$$

**then:**

$$\mathbf{FC}_{i,t}\frac{(1+\mathbf{f}_i)}{(1+\mathbf{r})} = -(1+\mathbf{r})^{t-2}(1+\mathbf{f}_i)^2\mathbf{C}_i - (1+\mathbf{r})^{t-3}(1+\mathbf{f}_i)^3\mathbf{C}_i - \ldots - (1+\mathbf{r})^0(1+\mathbf{f}_i)^t\mathbf{C}_i - (1+\mathbf{r})^{-1}(1+\mathbf{f}_i)^{t+1}\mathbf{C}_i \qquad \text{multiply by } \frac{(1+\mathbf{f}_i)}{(1+\mathbf{r})}$$

**and:**

$$\mathbf{FC}_{i,t} - \mathbf{FC}_{i,t}\frac{(1+\mathbf{f}_i)}{(1+\mathbf{r})} = -(1+\mathbf{r})^{t-1}(1+\mathbf{f}_i)\mathbf{C}_i - (1+\mathbf{r})^{t-2}(1+\mathbf{f}_i)^2\mathbf{C}_i - \ldots\ - (1+\mathbf{r})(1+\mathbf{f}_i)^{t-1}\mathbf{C}_i - (1+\mathbf{r})^0(1+\mathbf{f}_i)^t\mathbf{C}_i \qquad \text{from } \mathbf{FC}_{i,t}$$

$$+(1+\mathbf{r})^{t-2}(1+\mathbf{f}_i)^2\mathbf{C}_i + (1+\mathbf{r})^{t-3}(1+\mathbf{f}_i)^3\mathbf{C}_i + \ldots + (1+\mathbf{r})^0(1+\mathbf{f}_i)^t\mathbf{C}_i + (1+\mathbf{r})^{-1}(1+\mathbf{f}_i)^{t+1}\mathbf{C}_i \qquad \text{subtract } \mathbf{FC}_{i,t}\frac{(1+\mathbf{f}_i)}{(1+\mathbf{r})}$$

$$\mathbf{FC}_{i,t} - \mathbf{FC}_{i,t}\frac{(1+\mathbf{f}_i)}{(1+\mathbf{r})} = -(1+\mathbf{r})^{t-1}(1+\mathbf{f}_i)\mathbf{C}_i \qquad + (1+\mathbf{r})^{-1}(1+\mathbf{f}_i)^{t+1}\mathbf{C}_i$$

$$\mathbf{FC}_{i,t}[1 - \frac{(1+\mathbf{f}_i)}{(1+\mathbf{r})}] = -(1+\mathbf{r})^{t-1}(1+\mathbf{f}_i)\mathbf{C}_i + (1+\mathbf{r})^{-1}(1+\mathbf{f}_i)^{t+1}\mathbf{C}_i \qquad \text{factor out } \mathbf{FC}_{i,t}$$

$$\mathbf{FC}_{i,t}[\frac{(1+\mathbf{r})}{(1+\mathbf{r})} - \frac{(1+\mathbf{f}_i)}{(1+\mathbf{r})}] = -(1+\mathbf{r})^{t-1}(1+\mathbf{f}_i)\mathbf{C}_i + (1+\mathbf{r})^{-1}(1+\mathbf{f}_i)^{t+1}\mathbf{C}_i \qquad \text{substitute } \frac{(1+\mathbf{r})}{(1+\mathbf{r})} \text{ for 1}$$

$$\mathbf{FC}_{i,t}[\frac{1+\mathbf{r}-1-\mathbf{f}_i}{(1+\mathbf{r})}] = -(1+\mathbf{r})^{t-1}(1+\mathbf{f}_i)\mathbf{C}_i + (1+\mathbf{r})^{-1}(1+\mathbf{f}_i)^{t+1}\mathbf{C}_i$$

$$\mathbf{FC}_{i,t}[\frac{\mathbf{r}-\mathbf{f}_i}{(1+\mathbf{r})}] = -(1+\mathbf{r})^{t-1}(1+\mathbf{f}_i)\mathbf{C}_i + (1+\mathbf{r})^{-1}(1+\mathbf{f}_i)^{t+1}\mathbf{C}_i$$

$$\mathbf{FC}_{i,t}(\mathbf{r}-\mathbf{f}_i) = -(1+\mathbf{r})^t(1+\mathbf{f}_i)\mathbf{C}_i + (1+\mathbf{r})(1+\mathbf{r})^{-1}(1+\mathbf{f}_1)^{t+1}\mathbf{C}_i \qquad \text{multiply by } (1+\mathbf{r})$$

$$\mathbf{FC}_{i,t}(\mathbf{r}-\mathbf{f}_i) = -(1+\mathbf{r})^t(1+\mathbf{f}_i)\mathbf{C}_i + (1+\mathbf{f}_1)^{t+1}\mathbf{C}_i \qquad (1+\mathbf{r})(1+\mathbf{r})^{-1} == 1$$

$$\mathbf{FC}_{i,t}(\mathbf{r}-\mathbf{f}_i) = \mathbf{C}_i[(1+\mathbf{f}_i)^{t+1} - (1+\mathbf{r})^t(1+\mathbf{f}_i)] \qquad \text{factor out } \mathbf{C}_i$$

$$\mathbf{FC}_{i,t} = \frac{\mathbf{C}_i[(1+\mathbf{f}_i)^{t+1} - (1+\mathbf{r})^t(1+\mathbf{f}_i)]}{(\mathbf{r}-\mathbf{f}_i)} \qquad \text{divide by } (\mathbf{r}-\mathbf{f}_i)$$

range of k: $\mathbf{1} .. \mathbf{K}$

$$\mathbf{FV}_t = \mathbf{W}(1+\mathbf{r})^t + \mathbf{FC}_{1,t} + \mathbf{FC}_{2,t} + \ldots + \mathbf{FC}_{K-1,t} + \mathbf{FC}_{K,t}$$

where: $\mathbf{FC}_{k,t} = -(1+\mathbf{r})^{t-1}(1+\mathbf{f}_k)\mathbf{C}_k - \ldots - (1+\mathbf{r})^0(1+\mathbf{f}_k)^t\mathbf{C}_k$

$$\mathbf{FV}_t = \mathbf{W}(1+\mathbf{r})^t + \sum_{k=1}^{\mathbf{K}} \mathbf{FC}_{k,t}$$

where: $\mathbf{FC}_{k,t} = \dfrac{\mathbf{C}_k[(1+\mathbf{f}_k)^{t+1} - (1+\mathbf{r})^t(1+\mathbf{f}_k)]}{(\mathbf{r}-\mathbf{f}_k)}$

let

$$\mathbf{FC}_t = \sum_{k=1}^{\mathbf{K}} \mathbf{FC}_{k,t}$$

where $\mathbf{FC}_{k,t} = \dfrac{\mathbf{C}_k\left[(1+\mathbf{f}_k)^{t+1} - (1+\mathbf{r})^t(1+\mathbf{f}_k)\right]}{\mathbf{r}-\mathbf{f}_k}$

then

$$\mathbf{FV}_t = \mathbf{W}(1+\mathbf{r})^t + \mathbf{FC}_t$$

let

$$\mathbf{A}_{k,t} = (1+\mathbf{f}_k)^{t+1} - (1+\mathbf{r})^t(1+\mathbf{f}_k)$$

then

$$\mathbf{FC}_{k,t} = \mathbf{C}_k \cdot A_{k,t}$$

and

$$\mathbf{FC}_t = \sum_{k=1}^{\mathbf{K}} \mathbf{FC}_{k,t} = \sum_{k=1}^{\mathbf{K}} \mathbf{C}_k \cdot A_{k,t}$$

**let $\mathbf{K} = \mathbf{1}$**
**for the special case of only one cost component**

$$\mathbf{FV}_t = \mathbf{W}(1+\mathbf{r})^t + \sum_{k=1}^{\mathbf{1}} \mathbf{FC}_{k,t}$$

$$\mathbf{FV}_t = \mathbf{W}(1+\mathbf{r})^t + \mathbf{FC}_{1,t}$$

$$\mathbf{FV}_t = \mathbf{W}(1+\mathbf{r})^t + \frac{\mathbf{C}_1[(1+\mathbf{f}_1)^{t+1} - (1+\mathbf{r})^t(1+\mathbf{f}_1)]}{(\mathbf{r}-\mathbf{f}_1)}$$

where: $\mathbf{FC}_{k,t} = \dfrac{\mathbf{C}_k[(1+\mathbf{f}_k)^{t+1} - (1+\mathbf{r})^t(1+\mathbf{f}_k)]}{(\mathbf{r}-\mathbf{f}_k)}$

$$\mathbf{FV}_t = \mathbf{W}(1+\mathbf{r})^t + \frac{\mathbf{C}_1(1+\mathbf{f}_1)^{t+1} - \mathbf{C}_1(1+\mathbf{r})^t(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}$$

distribute $\mathbf{C}_1$

$$\mathbf{FV}_t = \mathbf{W}(1+\mathbf{r})^t + \frac{\mathbf{C}_1(1+\mathbf{f}_1)^{t+1}}{(\mathbf{r}-\mathbf{f}_1)} - \frac{\mathbf{C}_1(1+\mathbf{r})^t(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}$$

separate numerator

$$\mathbf{FV}_t = \mathbf{W}(1+\mathbf{r})^t - \frac{\mathbf{C}_1(1+\mathbf{r})^t(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)} + \frac{\mathbf{C}_1(1+\mathbf{f}_1)^{t+1}}{(\mathbf{r}-\mathbf{f}_1)}$$

reorganize terms

$$\mathbf{FV}_t = (1+\mathbf{r})^t[\mathbf{W} - \frac{\mathbf{C}_1(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}] + \frac{\mathbf{C}_1(1+\mathbf{f}_1)^{t+1}}{(\mathbf{r}-\mathbf{f}_1)}$$

factor out: $(1+\mathbf{r})^t$

$$\mathbf{FV}_t = (1+\mathbf{r})^t[\mathbf{W} - \frac{\mathbf{C}_1(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}] + (1+\mathbf{f}_1)^t[\frac{\mathbf{C}_1(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}]$$

factor out: $(1+\mathbf{f}_1)^t$

**Claude:**

- If $r > f_1$, the investment grows faster than the withdrawals

- If r = $f_1$, a different formula would be needed (since you'd have division by zero)
- If r < $f_1$, the withdrawals grow faster than the investment, guaranteeing ruin

**let $\mathbf{FV}_t = 0$, all funds expended, depletion, solve for t**

| | |
|---|---|
| $0 = (1+\mathbf{r})^t[\mathbf{W} - \frac{\mathbf{C}_1(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}] + (1+\mathbf{f}_1)^t[\frac{\mathbf{C}_1(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}]$ | $\mathbf{FV}_t = 0$ |
| $-(1+\mathbf{f}_1)^t[\frac{\mathbf{C}_1(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}] = (1+\mathbf{r})^t[\mathbf{W} - \frac{\mathbf{C}_1(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}]$ | **rearrange** |
| $-(1+\mathbf{f}_1)^t = (1+\mathbf{r})^t[\mathbf{W} - \frac{\mathbf{C}_1(1+\mathbf{f}_1)}{(\mathbf{r}-\mathbf{f}_1)}][\frac{(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)}]$ | **multiply by** $\frac{(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)}$ |
| $-(1+\mathbf{f}_1)^t = (1+\mathbf{r})^t[\mathbf{W}\frac{(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)} - 1]$ | **multiply by** $\frac{(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)}$ |
| $-\frac{(1+\mathbf{f}_1)^t}{(1+\mathbf{r})^t} = \frac{\mathbf{W}(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)} - 1$ | **divide by** $(1+\mathbf{r})^t$ |
| $\frac{(1+\mathbf{f}_1)^t}{(1+\mathbf{r})^t} = 1 - \frac{\mathbf{W}(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)}$ | **multiply by** $-\mathbf{1}$ |
| $\ln(1+\mathbf{f}_1)^t - \ln(1+\mathbf{r})^t = \ln[1 - \frac{\mathbf{W}(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)}]$ | **take the** $\ln$ |
| $t\ln(1+\mathbf{f}_1) - t\ln(1+\mathbf{r}) = \ln[1 - \frac{\mathbf{W}(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)}]$ | $\ln(x^y) = y\ln(x)$ |
| $t[\ln(1+\mathbf{f}_1) - \ln(1+\mathbf{r})] = \ln[1 - \frac{\mathbf{W}(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)}]$ | **factor out: t** |
| $t = \frac{\ln[1 - \frac{\mathbf{W}(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)}]}{\ln(1+\mathbf{f}_1) - \ln(1+\mathbf{r})}$ | **divide by:** $\ln(1+\mathbf{f}_1) - \ln(1+\mathbf{r})$ |
| $t = \frac{\ln[1 - \frac{\mathbf{W}(\mathbf{r}-\mathbf{f}_1)}{\mathbf{C}_1(1+\mathbf{f}_1)}]}{\ln[\frac{(1+\mathbf{f}_1)}{(1+\mathbf{r})}]}$ | $\ln(x-y) = \ln(x)/\ln(y)$ |

**let $\mathbf{f_1} = 0$ , costs don't increase over: $\mathbf{t}$**

$$t = \frac{\ln[1 - \frac{\mathbf{W}(\mathbf{r} - \mathbf{f}_1)}{\mathbf{C}_1(1 + \mathbf{f}_1)}]}{\ln[\frac{(1 + \mathbf{f}_1)}{(1 + \mathbf{r})}]}$$

**definition of $\mathbf{t}$ with single cost component: $\mathbf{C_1}$**

$$t = \frac{\ln[1 - \frac{\mathbf{Wr}}{\mathbf{C}_1}]}{\ln[\frac{1}{(1 + \mathbf{r})}]}$$

$\mathbf{f_1 = 0}$

$$t = \frac{\ln[1 - \frac{\mathbf{Wr}}{\mathbf{C}_1}]}{\ln(1) - \ln(1 + \mathbf{r}))}$$

$\ln(x/y) = \ln(x) - \ln(y)$

$$t = \frac{\ln[1 - \frac{\mathbf{Wr}}{\mathbf{C}_1}]}{-\ln(1 + \mathbf{r})}$$

$\ln(1) = 0$

$$t = \frac{\ln[1 - \frac{\mathbf{Wr}}{\mathbf{C}_1}]}{-\mathbf{r}}$$

**$\mathbf{r}$ is small, then $\ln(1 + \mathbf{r}) \approx \mathbf{r}$**

$$t = -\frac{1}{\mathbf{r}}\ln[1 - \frac{\mathbf{Wr}}{\mathbf{C}_1}]$$

**fibonacci equation (almost)**

$$t = \frac{1}{\mathbf{r}}[-\ln(1 - \frac{\mathbf{Wr}}{\mathbf{C}_1})]$$

**move the minus sign**

$$t = \frac{1}{\mathbf{r}}[-\ln(\frac{\mathbf{C}_1 - \mathbf{Wr}}{\mathbf{C}_1})]$$

**make $\mathbf{C}_1$ the common denominator**

$$t = \frac{1}{\mathbf{r}}[-[\ln(\mathbf{C}_1 - \mathbf{Wr}) - \ln(\mathbf{C}_1)]]$$

$\ln(\mathbf{x}/\mathbf{y}) = \ln(\mathbf{x}) - \ln(\mathbf{y})$

$$t = \frac{1}{\mathbf{r}}[\ln(\mathbf{C}_1) - \ln(\mathbf{C}_1 - \mathbf{Wr})]$$

**distribute minus sign**

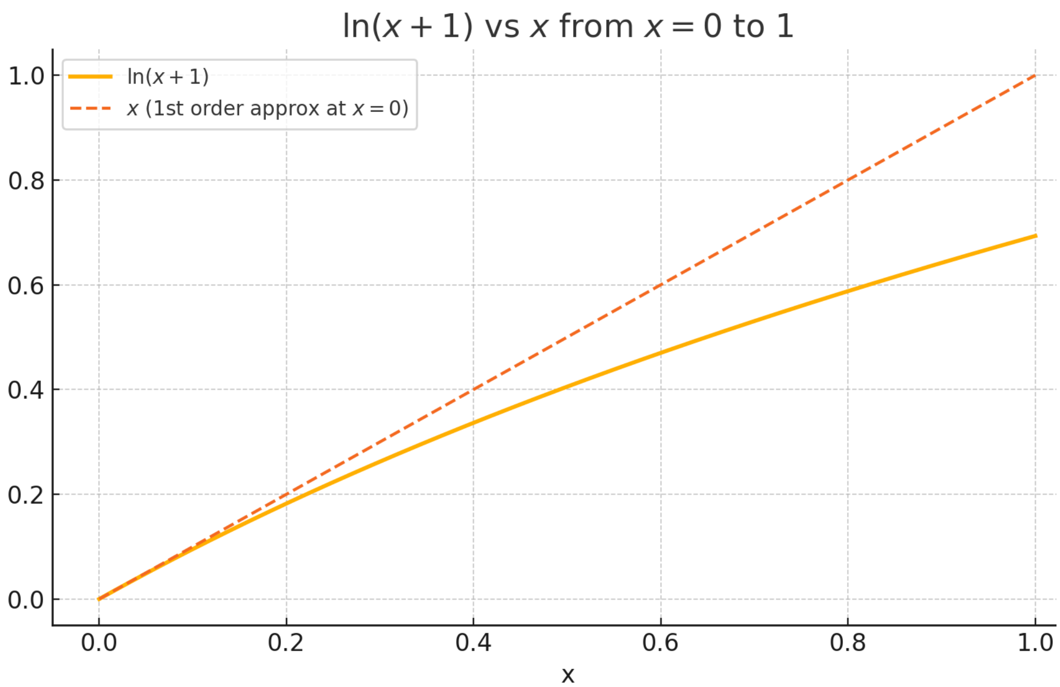
$$t = \frac{1}{\mathbf{r}}[\ln(\frac{\mathbf{C}_1}{\mathbf{C}_1 - \mathbf{Wr}})]$$

$\ln(\mathbf{x}) - \ln(\mathbf{y}) = \ln(\mathbf{x}/\mathbf{y})$

$$t = \frac{1}{\mathbf{r}}\ln(\frac{\mathbf{C}_1}{\mathbf{C}_1 - \mathbf{Wr}})$$

**fibonacci equation**

**<u>fibonacci equation: A Proper Derivation of the 7 Most Important Equations for Your Retirement</u>**




**if *ln(r)* is small, then *ln(1+r)=r***