

let:

t = Time index in **months**, $t = 0, 1, 2, \dots, T$

T = Total number of months in the simulation horizon

FV_t = Future value of retirement account at time t

W = Initial retirement account value ("nest egg")

r_a = Annual interest rate

r = Monthly interest rate (i.e., r_a divided by 12)

$f_{a,i}$ = Annual inflation rate for cost component C_i

f_i = Monthly inflation rate (i.e. $f_{a,i}/12$) for cost component C_i

C_i = Initial (base) cost for component i , where $i = 1, 2, \dots, K$

$C_{i,t}$ = Future value of the cost component i at time t

month: $t = 1$

$$C_{1,1} = (1 + f_1)C_1$$

$$C_{2,1} = (1 + f_2)C_2$$

$$C_{3,1} = (1 + f_3)C_3$$

cost component 1 at month (i.e. t): 1 inflated by: f_1

cost component 2 at month (i.e. t): 1 inflated by: f_2

cost component 3 at month (i.e. t): 1 inflated by: f_3

$$FV_1 = W(1 + r) - (C_{1,1} + C_{2,1} + C_{3,1})$$

(Future Value after 1 period)

since: $K = 3$

$$FV_1 = W(1 + r) - \sum_{k=1}^K C_{k,1}$$

(Future Value after 1 period)

$$FV_1 = W(1 + r) - \sum_{k=1}^K (1 + f_k)C_k$$

substituting for C_k

month: $t = 2$

$$C_{1,2} = (1 + f_1)C_{1,1}$$

$$= (1 + f_1)(1 + f_1)C_1$$

$$= (1 + f_1)^2 C_1$$

cost component 1 at month (i.e. t): 2 inflated by: f_1

substituting for $C_{1,1}$

combining terms

$$C_{2,2} = (1 + f_2)C_{2,1}$$

$$= (1 + f_2)(1 + f_2)C_2$$

$$= (1 + f_2)^2 C_2$$

cost component 2 at month (i.e. t): 2 inflated by: f_2

substituting for $C_{2,1}$

combining terms

$$C_{3,2} = (1 + f_3)C_{3,1}$$

$$= (1 + f_3)(1 + f_3)C_3$$

$$= (1 + f_3)^2 C_3$$

cost component 3 at month (i.e. t): 2 inflated by: f_3

substituting for $C_{3,1}$

combining terms

in general: $C_{k,t} = (1 + f_k)^t C_k$

by induction, $(1 + f_k)$ applied every time period t

$$FV_2 = FV_1(1 + r) - (C_{1,2} + C_{2,2} + C_{3,2})$$

(Future Value after 2 period)

$$= FV_1(1 + r) - \sum_{k=1}^K C_{k,2}$$

where $t = 2$

$$= FV_1(1 + r) - \sum_{k=1}^K (1 + f_k)^2 C_k$$

since $C_{k,t} = (1 + f_k)^t C_k \rightarrow C_{k,2} = (1 + f_k)^2 C_k$

$$FV_2 = [W(1 + r) - \sum_{k=1}^K (1 + f_k)C_k](1 + r) - \sum_{k=1}^K (1 + f_k)^2 C_k$$

substitute for FV_1

$$= W(1 + r)(1 + r) - (1 + r) \sum_{k=1}^K (1 + f_k)C_k - \sum_{k=1}^K (1 + f_k)^2 C_k$$

distribute $(1 + r)$

$$FV_2 = W(1 + r)^2 - (1 + r) \sum_{k=1}^K (1 + f_k)C_k - \sum_{k=1}^K (1 + f_k)^2 C_k$$

combine $(1 + r)$

month: $t = 3$

$$\begin{aligned}
 \mathbf{FV}_3 &= \mathbf{FV}_2(1+r) - \sum_{k=1}^K \mathbf{C}_{k,3} && \text{where } t = 3 \\
 &= \mathbf{FV}_2(1+r) - \sum_{k=1}^K (1+f_k)^3 \mathbf{C}_k && \text{since } \mathbf{C}_{k,t} = (1+f_k)^t \mathbf{C}_k \rightarrow \mathbf{C}_{k,3} = (1+f_k)^3 \mathbf{C}_k \\
 &= [\mathbf{W}(1+r)^2 - (1+r) \sum_{k=1}^K (1+f_k) \mathbf{C}_k - \sum_{k=1}^K (1+f_k)^2 \mathbf{C}_k](1+r) - \sum_{k=1}^K (1+f_k)^3 \mathbf{C}_k && \text{substitute } \mathbf{FV}_2 \text{ from above} \\
 &= \mathbf{W}(1+r)^2(1+r) - (1+r)(1+r) \sum_{k=1}^K (1+f_k) \mathbf{C}_k - (1+r) \sum_{k=1}^K (1+f_k)^2 \mathbf{C}_k - \sum_{k=1}^K (1+f_k)^3 \mathbf{C}_k && \text{distribute } (1+r) \\
 \mathbf{FV}_3 &= \mathbf{W}(1+r)^3 - (1+r)^2 \sum_{k=1}^K (1+f_k) \mathbf{C}_k - (1+r) \sum_{k=1}^K (1+f_k)^2 \mathbf{C}_k - \sum_{k=1}^K (1+f_k)^3 \mathbf{C}_k && \text{combine } (1+r)
 \end{aligned}$$

collect: \mathbf{C}_i from each \sum

$$\begin{aligned}
 \mathbf{FV}_3 &= \mathbf{W}(1+r)^3 && \text{collect terms (in columns):} \\
 &\quad - (1+r)^2(1+f_1) \mathbf{C}_1 - (1+r)^2(1+f_2) \mathbf{C}_2 - (1+r)^2(1+f_3) \mathbf{C}_3 && \mathbf{C}_1 \\
 &\quad - (1+r)(1+f_1)^2 \mathbf{C}_1 - (1+r)(1+f_2)^2 \mathbf{C}_2 - (1+r)(1+f_3)^2 \mathbf{C}_3 && \mathbf{C}_2 \\
 &\quad - (1+f_1)^3 \mathbf{C}_1 - (1+f_2)^3 \mathbf{C}_2 - (1+f_3)^3 \mathbf{C}_3 && \mathbf{C}_3
 \end{aligned}$$

by inference:

all $(1+r)$ & $(1+f)$ exponents add to: t
 $(1+f)$ exponent range: 1 to t
 $(1+r)$ exponent range: 0 to $(t-1)$

$$\begin{aligned}
 \mathbf{FV}_t &= \mathbf{W}(1+r)^t \\
 &\quad - (1+r)^{t-1}(1+f_1) \mathbf{C}_1 - (1+r)^{t-1}(1+f_2) \mathbf{C}_2 - (1+r)^{t-1}(1+f_3) \mathbf{C}_3 \\
 &\quad - (1+r)^{t-2}(1+f_1)^2 \mathbf{C}_1 - (1+r)^{t-2}(1+f_2)^2 \mathbf{C}_2 - (1+r)^{t-2}(1+f_3)^2 \mathbf{C}_3 \\
 &\quad \dots \\
 &\quad - (1+r)(1+f_1)^{t-1} \mathbf{C}_1 - (1+r)(1+f_2)^{t-1} \mathbf{C}_2 - (1+r)(1+f_3)^{t-1} \mathbf{C}_3 \\
 &\quad - (1+f_1)^t \mathbf{C}_1 - (1+f_2)^t \mathbf{C}_2 - (1+f_3)^t \mathbf{C}_3
 \end{aligned}$$

let: \mathbf{FC} be the Future Cost then
 $\mathbf{FC}_{i,t}$ cost: i at time: t

$$\mathbf{FC}_{i,t} = -(1+r)^{t-1}(1+f_i) \mathbf{C}_i - (1+r)^{t-2}(1+f_i)^2 \mathbf{C}_i - \dots - (1+r)(1+f_i)^{t-1} \mathbf{C}_i - (1+r)^0(1+f_i)^t \mathbf{C}_i$$

parameterize \mathbf{C}_i and t

then:

$$\mathbf{FC}_{i,t} \frac{(1+f_i)}{(1+r)} = -(1+r)^{t-2}(1+f_i)^2 \mathbf{C}_i - (1+r)^{t-3}(1+f_i)^3 \mathbf{C}_i - \dots - (1+r)^0(1+f_i)^t \mathbf{C}_i - (1+r)^{-1}(1+f_i)^{t+1} \mathbf{C}_i$$

multiply by $\frac{(1+f_i)}{(1+r)}$

and:

$$\begin{aligned}
 \mathbf{FC}_{i,t} - \mathbf{FC}_{i,t} \frac{(1+f_i)}{(1+r)} &= -(1+r)^{t-1}(1+f_i) \mathbf{C}_i - (1+r)^{t-2}(1+f_i)^2 \mathbf{C}_i - \dots - (1+r)(1+f_i)^{t-1} \mathbf{C}_i - (1+r)^0(1+f_i)^t \mathbf{C}_i && \text{from } \mathbf{FC}_{i,t} \\
 &\quad + (1+r)^{t-2}(1+f_i)^2 \mathbf{C}_i + (1+r)^{t-3}(1+f_i)^3 \mathbf{C}_i + \dots + (1+r)^0(1+f_i)^t \mathbf{C}_i + (1+r)^{-1}(1+f_i)^{t+1} \mathbf{C}_i && \text{subtract } \mathbf{FC}_{i,t} \frac{(1+f_i)}{(1+r)}
 \end{aligned}$$

$$\mathbf{FC}_{i,t} - \mathbf{FC}_{i,t} \left[\frac{(1+f_i)}{(1+r)} \right] = -(1+r)^{t-1}(1+f_i) \mathbf{C}_i + (1+r)^{-1}(1+f_i)^{t+1} \mathbf{C}_i$$

$$\mathbf{FC}_{i,t} \left[1 - \frac{(1+f_i)}{(1+r)} \right] = -(1+r)^{t-1}(1+f_i) \mathbf{C}_i + (1+r)^{-1}(1+f_i)^{t+1} \mathbf{C}_i$$

factor out $\mathbf{FC}_{i,t}$

$$\mathbf{FC}_{i,t} \left[\frac{(1+r)}{(1+r)} - \frac{(1+f_i)}{(1+r)} \right] = -(1+r)^{t-1}(1+f_i) \mathbf{C}_i + (1+r)^{-1}(1+f_i)^{t+1} \mathbf{C}_i$$

substitute $\frac{(1+r)}{(1+r)}$ for 1

$$\mathbf{FC}_{i,t} \left[\frac{1+r-1-f_i}{(1+r)} \right] = -(1+r)^{t-1}(1+f_i) \mathbf{C}_i + (1+r)^{-1}(1+f_i)^{t+1} \mathbf{C}_i$$

$$\mathbf{FC}_{i,t} \left[\frac{r-f_i}{(1+r)} \right] = -(1+r)^{t-1}(1+f_i) \mathbf{C}_i + (1+r)^{-1}(1+f_i)^{t+1} \mathbf{C}_i$$

$$\mathbf{FC}_{i,t}(r-f_i) = -(1+r)^t(1+f_i) \mathbf{C}_i + (1+r)(1+r)^{-1}(1+f_i)^{t+1} \mathbf{C}_i$$

multiply by $(1+r)$

$$\mathbf{FC}_{i,t}(r-f_i) = -(1+r)^t(1+f_i) \mathbf{C}_i + (1+f_i)^{t+1} \mathbf{C}_i$$

$(1+r)(1+r)^{-1} = 1$

$$\mathbf{FC}_{i,t}(r-f_i) = \mathbf{C}_i[(1+f_i)^{t+1} - (1+r)^t(1+f_i)]$$

factor out \mathbf{C}_i

$$\mathbf{FC}_{i,t} = \frac{\mathbf{C}_i[(1+f_i)^{t+1} - (1+r)^t(1+f_i)]}{(r-f_i)}$$

divide by $(r-f_i)$

range of k: 1 .. K

$$\mathbf{FV}_t = \mathbf{W}(1 + \mathbf{r})^t + \mathbf{FC}_{1,t} + \mathbf{FC}_{2,t} + \dots + \mathbf{FC}_{K-1,t} + \mathbf{FC}_{K,t} \quad \text{where: } \mathbf{FC}_{k,t} = - (1 + \mathbf{r})^{t-1}(1 + \mathbf{f}_k)\mathbf{C}_k - \dots - (1 + \mathbf{r})^0(1 + \mathbf{f}_k)^t\mathbf{C}_k$$

$$\mathbf{FV}_t = \mathbf{W}(1 + \mathbf{r})^t + \sum_{k=1}^K \mathbf{FC}_{k,t} \quad \text{where: } \mathbf{FC}_{k,t} = \frac{\mathbf{C}_k[(1 + \mathbf{f}_k)^{t+1} - (1 + \mathbf{r})^t(1 + \mathbf{f}_k)]}{(\mathbf{r} - \mathbf{f}_k)}$$

let

$$\mathbf{FC}_t = \sum_{k=1}^K \mathbf{FC}_{k,t}$$

$$\text{where } \mathbf{FC}_{k,t} = \frac{\mathbf{C}_k [(1 + \mathbf{f}_k)^{t+1} - (1 + \mathbf{r})^t(1 + \mathbf{f}_k)]}{\mathbf{r} - \mathbf{f}_k}$$

then

$$\mathbf{FV}_t = \mathbf{W}(1 + \mathbf{r})^t + \mathbf{FC}_t$$

let

$$\mathbf{A}_{k,t} = (1 + \mathbf{f}_k)^{t+1} - (1 + \mathbf{r})^t(1 + \mathbf{f}_k)$$

then

$$\mathbf{FC}_{k,t} = \mathbf{C}_k \cdot \mathbf{A}_{k,t}$$

and

$$\mathbf{FC}_t = \sum_{k=1}^K \mathbf{FC}_{k,t} = \sum_{k=1}^K \mathbf{C}_k \cdot \mathbf{A}_{k,t}$$

let $\mathbf{K} = 1$

for the special case of only one cost component

$$\mathbf{FV}_t = \mathbf{W}(1 + \mathbf{r})^t + \sum_{k=1}^1 \mathbf{FC}_{k,t}$$

$$\mathbf{FV}_t = \mathbf{W}(1 + \mathbf{r})^t + \mathbf{FC}_{1,t}$$

$$\mathbf{FV}_t = \mathbf{W}(1 + \mathbf{r})^t + \frac{\mathbf{C}_1[(1 + \mathbf{f}_1)^{t+1} - (1 + \mathbf{r})^t(1 + \mathbf{f}_1)]}{(\mathbf{r} - \mathbf{f}_1)}$$

$$\text{where: } \mathbf{FC}_{k,t} = \frac{\mathbf{C}_k[(1 + \mathbf{f}_k)^{t+1} - (1 + \mathbf{r})^t(1 + \mathbf{f}_k)]}{(\mathbf{r} - \mathbf{f}_k)}$$

$$\mathbf{FV}_t = \mathbf{W}(1 + \mathbf{r})^t + \frac{\mathbf{C}_1(1 + \mathbf{f}_1)^{t+1} - \mathbf{C}_1(1 + \mathbf{r})^t(1 + \mathbf{f}_1)}{(\mathbf{r} - \mathbf{f}_1)}$$

distribute \mathbf{C}_1

$$\mathbf{FV}_t = \mathbf{W}(1 + \mathbf{r})^t + \frac{\mathbf{C}_1(1 + \mathbf{f}_1)^{t+1}}{(\mathbf{r} - \mathbf{f}_1)} - \frac{\mathbf{C}_1(1 + \mathbf{r})^t(1 + \mathbf{f}_1)}{(\mathbf{r} - \mathbf{f}_1)}$$

separate numerator

$$\mathbf{FV}_t = \mathbf{W}(1 + \mathbf{r})^t - \frac{\mathbf{C}_1(1 + \mathbf{r})^t(1 + \mathbf{f}_1)}{(\mathbf{r} - \mathbf{f}_1)} + \frac{\mathbf{C}_1(1 + \mathbf{f}_1)^{t+1}}{(\mathbf{r} - \mathbf{f}_1)}$$

reorganize terms

$$\mathbf{FV}_t = (1 + \mathbf{r})^t \left[\mathbf{W} - \frac{\mathbf{C}_1(1 + \mathbf{f}_1)}{(\mathbf{r} - \mathbf{f}_1)} \right] + \frac{\mathbf{C}_1(1 + \mathbf{f}_1)^{t+1}}{(\mathbf{r} - \mathbf{f}_1)}$$

factor out: $(1 + \mathbf{r})^t$

$$\mathbf{FV}_t = (1 + \mathbf{r})^t \left[\mathbf{W} - \frac{\mathbf{C}_1(1 + \mathbf{f}_1)}{(\mathbf{r} - \mathbf{f}_1)} \right] + (1 + \mathbf{f}_1)^t \left[\frac{\mathbf{C}_1(1 + \mathbf{f}_1)}{(\mathbf{r} - \mathbf{f}_1)} \right]$$

factor out: $(1 + \mathbf{f}_1)^t$

Claude:

- If $\mathbf{r} > \mathbf{f}_1$, the investment grows faster than the withdrawals

- If $r = f_1$, a different formula would be needed (since you'd have division by zero)
- If $r < f_1$, the withdrawals grow faster than the investment, guaranteeing ruin

let $FV_t = 0$, all funds expended, depletion,
solve for t

$$\begin{aligned}
 0 &= (1+r)^t \left[W - \frac{C_1(1+f_1)}{(r-f_1)} \right] + (1+f_1)^t \left[\frac{C_1(1+f_1)}{(r-f_1)} \right] && FV_t = 0 \\
 -(1+f_1)^t \left[\frac{C_1(1+f_1)}{(r-f_1)} \right] &= (1+r)^t \left[W - \frac{C_1(1+f_1)}{(r-f_1)} \right] && \text{rearrange} \\
 -(1+f_1)^t &= (1+r)^t \left[W - \frac{C_1(1+f_1)}{(r-f_1)} \right] \left[\frac{(r-f_1)}{C_1(1+f_1)} \right] && \text{multiply by } \frac{(r-f_1)}{C_1(1+f_1)} \\
 -(1+f_1)^t &= (1+r)^t \left[W \left[\frac{(r-f_1)}{C_1(1+f_1)} \right] - 1 \right] && \text{multiply by } \frac{(r-f_1)}{C_1(1+f_1)} \\
 -\frac{(1+f_1)^t}{(1+r)^t} &= \frac{W(r-f_1)}{C_1(1+f_1)} - 1 && \text{divide by } (1+r)^t \\
 \frac{(1+f_1)^t}{(1+r)^t} &= 1 - \frac{W(r-f_1)}{C_1(1+f_1)} && \text{multiply by } -1 \\
 \ln(1+f_1)^t - \ln(1+r)^t &= \ln \left(1 - \frac{W(r-f_1)}{C_1(1+f_1)} \right) && \text{take the } \ln \\
 t \ln(1+f_1) - t \ln(1+r) &= \ln \left(1 - \frac{W(r-f_1)}{C_1(1+f_1)} \right) && \ln(x^y) = y \ln(x) \\
 t[\ln(1+f_1) - \ln(1+r)] &= \ln \left(1 - \frac{W(r-f_1)}{C_1(1+f_1)} \right) && \text{factor out: } t \\
 t &= \frac{\ln \left(1 - \frac{W(r-f_1)}{C_1(1+f_1)} \right)}{\ln(1+f_1) - \ln(1+r)} && \text{divide by: } \ln(1+f_1) - \ln(1+r) \\
 t &= \frac{\ln \left(1 - \frac{W(r-f_1)}{C_1(1+f_1)} \right)}{\ln \left(\frac{(1+f_1)}{(1+r)} \right)} && \ln(x) - \ln(y) = \ln(x/y) \\
 t &= \frac{\ln \left(1 - \frac{W(r-f_1)}{C_1(1+f_1)} \right)}{\ln(1+f_1) - \ln(1+r)} && \text{from a previous step} \\
 t &= \frac{\ln \left(1 - \frac{W(r-f_1)}{C_1(1+f_1)} \right)}{f_1 - r} && \text{x is small, then } \ln(1+x) \approx x
 \end{aligned}$$

let $f_1 = 0$, costs don't increase over: t

$$t = \frac{\ln \left(1 - \frac{Wr - f_1}{C_1(1 + f_1)} \right)}{\ln \left(\frac{(1 + f_1)}{(1 + r)} \right)}$$

definition of t with single cost component: C_1

$$t = \frac{\ln \left(1 - \frac{Wr}{C_1} \right)}{\ln \left(\frac{1}{(1 + r)} \right)}$$

$$f_1 = 0$$

$$t = \frac{\ln \left(1 - \frac{Wr}{C_1} \right)}{\ln(1) - \ln(1 + r)}$$

$$\ln(x/y) = \ln(x) - \ln(y)$$

$$t = \frac{\ln \left(1 - \frac{Wr}{C_1} \right)}{-\ln(1 + r)}$$

$$\ln(1) = 0$$

$$t = \frac{\ln \left(1 - \frac{Wr}{C_1} \right)}{-r}$$

r is small, then $\ln(1 + r) \approx r$

$$t = -\frac{1}{r} \ln \left(1 - \frac{Wr}{C_1} \right)$$

fibonacci equation (almost)

$$t = \frac{1}{r} \left[-\ln \left(1 - \frac{Wr}{C_1} \right) \right]$$

move the minus sign

$$t = \frac{1}{r} \left[-\ln \left(\frac{C_1 - Wr}{C_1} \right) \right]$$

make C_1 the common denominator

$$t = \frac{1}{r} [- [\ln(C_1 - Wr) - \ln(C_1)]]$$

$$\ln(x/y) = \ln(x) - \ln(y)$$

$$t = \frac{1}{r} [\ln(C_1) - \ln(C_1 - Wr)]$$

distribute minus sign

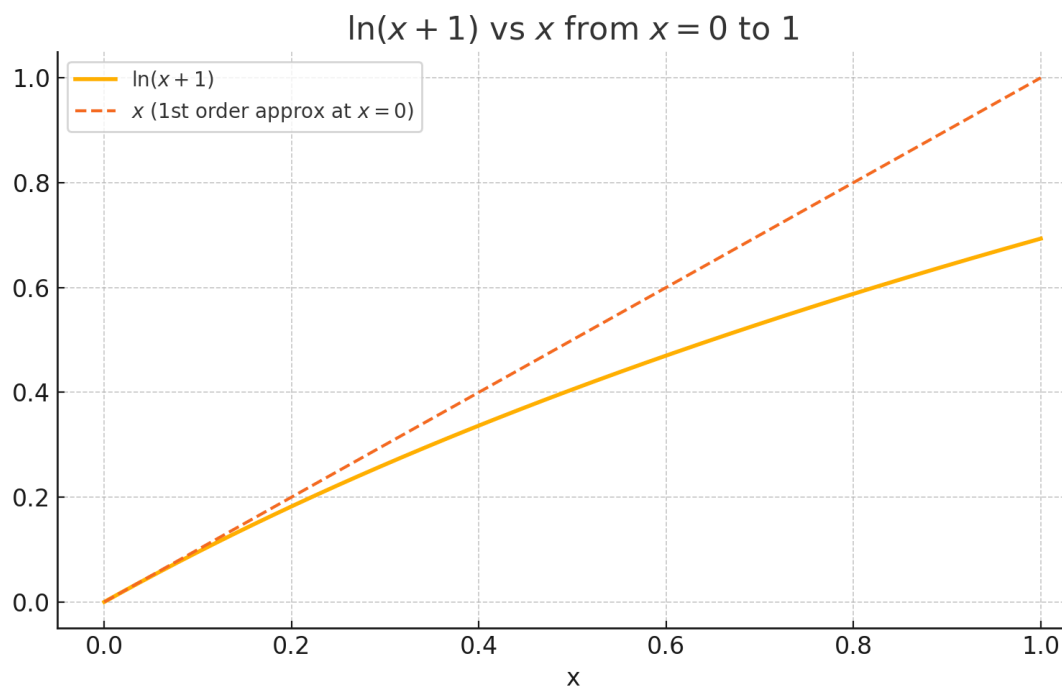
$$t = \frac{1}{r} \left[\ln \left(\frac{C_1}{C_1 - Wr} \right) \right]$$

$$\ln(x) - \ln(y) = \ln(x/y)$$

$$t = \frac{1}{r} \ln \left(\frac{C_1}{C_1 - Wr} \right)$$

fibonacci equation

fibonacci equation: A Proper Derivation of the 7 Most Important Equations for Your Retirement



if $\ln(r)$ is small, then $\ln(1+r)=r$