Rational Functions

I. Polynomial Functions

A polynomial function of degree n, for some nonnegative integer n, is a function of the form:

$$\boldsymbol{P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers (called coefficients of the polynomial), a_n is called the leading coefficient.

Examples:

- $P(x) = x^2 + 5$
  It has degree 2 with $a_2 = 1$, $a_1 = 0$, and $a_0 = 5$.

- $Q(x) = 3x^3 - x^2 - 2x + 2$
  It has degree 3 with $a_3 = 3, a_2 = -1$, $a_1 = -2$, and $a_0 = 2$.

- $R(x) = 10$
  It has degree 0 with $a_0 = 10$. Any constant function is a polynomial function.

II. Rational Functions

A rational function is a ratio of two (2) polynomial functions and can be written in the form:

$$\boldsymbol{f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0}}$$

Examples:

- $f(x) = \dfrac{x^2 + 5}{x + 2}$; where $P(x) = x^2 + 5$ and $Q(x) = x + 2$.
- $g(x) = \dfrac{1}{x}$; where $P(x) = 1$ and $Q(x) = x$.
- $h(x) = 3x^3 + 4$; where $P(x) = 3x^3 + 4$ and $Q(x) = 1$.

It can be seen from the last example that any polynomial function is a rational function.

Recall that if the denominator of a fraction is zero, then the fraction is undefined. Similarly, if the denominator of a rational function is zero, then it is undefined. The **domain** of a rational function is any value of x where the rational function is defined. Hence, the domain is the set of all real numbers except those that cause the denominator to have a value of 0.

Example

Find the domain of the functions below

a. $f(x) = \frac{7x-1}{8x-2}$
b. $g(x) = \frac{3x}{2x+1}$
c. $h(x) = \frac{x+4}{x-4}$

Solution:

The functions are defined for all real numbers except values of x such that the denominators are equal to zero. Finding those values of x, we equate each denominator to zero and solve for x.

a. The denominator of f is $8x-2$. Equating to 0 and solving for x,

$$8x-2=0$$
$$8x=2$$
$$x=\frac{1}{4}.$$

Therefore, the domain of f is all real numbers except $\frac{1}{4}$. In symbol, $\left\{x \in \mathbb{R} \middle| x \neq \frac{1}{4}\right\}$.

b. The denominator of g is $2x+1$. Equating to 0 and solving for x,

$$2x+1=0$$
$$2x=-1$$
$$x=-\frac{1}{2}.$$

Therefore, the domain is $\left\{x \in \mathbb{R} \middle| x \neq -\frac{1}{2}\right\}$.

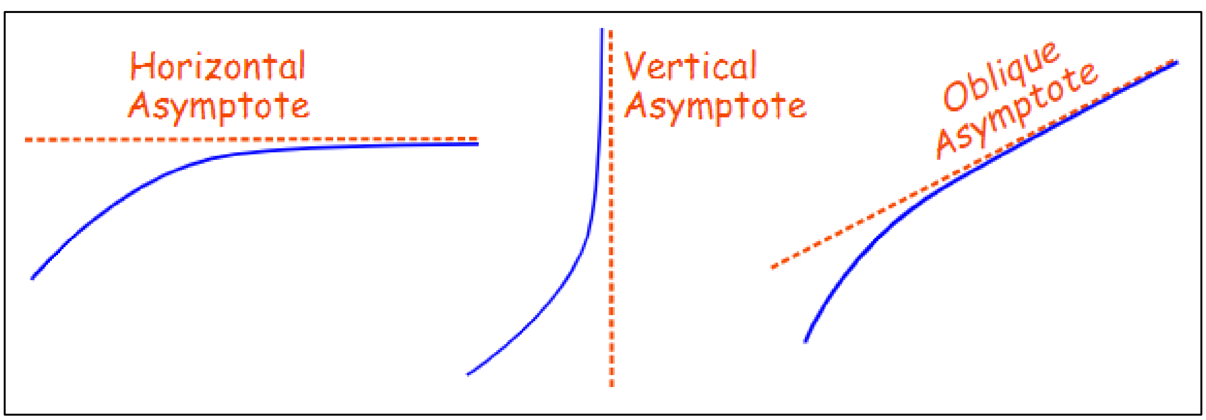
c. The denominator of h is $x-4$. Equating to 0 and solving for x,

$$x-4=0$$
$$x=4.$$

Therefore, the domain is $\{x \in \mathbb{R} | x \neq 4\}$.

III. Graphing a Rational Function

An **asymptote** is a line that a curve approaches as it heads toward infinity. A rational expression can have:



Source: https://www.mathsisfun.com/algebra/asymptote.html

An $\boldsymbol{x}$ **– intercept** is a point where the graph crosses the x – axis. It is also known as the **zero of the function**. The $\boldsymbol{y}$ **– intercept** is the point where the graph crosses the y – axis.

IV. **Rational Equations**

A **rational equation** is an equation that contains at least one (1) rational expression.

Examples:

a. $\frac{1}{x} = \frac{6}{x^2}$

b. $\frac{x-5}{x^2-1} = \frac{6}{x^2-1}$

The solution to the equation can be obtained algebraically.

Examples:

Solve the rational equations above.

Solution:

a. Multiplying both sides by x^2 and simplifying,

$$x^2 \cdot \frac{1}{x} = x^2 \cdot \frac{6}{x^2}$$

$$\frac{x^2}{x} = \frac{6x^2}{x^2}$$

$$x = 6.$$

b. Multiplying both sides by $x^2 - 1$,

$$(x^2 - 1) \cdot \frac{x-5}{x^2-1} = (x^2 - 1) \cdot \frac{6}{x^2-1}$$

$$x - 5 = 6.$$

Adding 5 to both sides,

$$x - 5 + 5 = 6 + 5$$

$$x = 11.$$

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