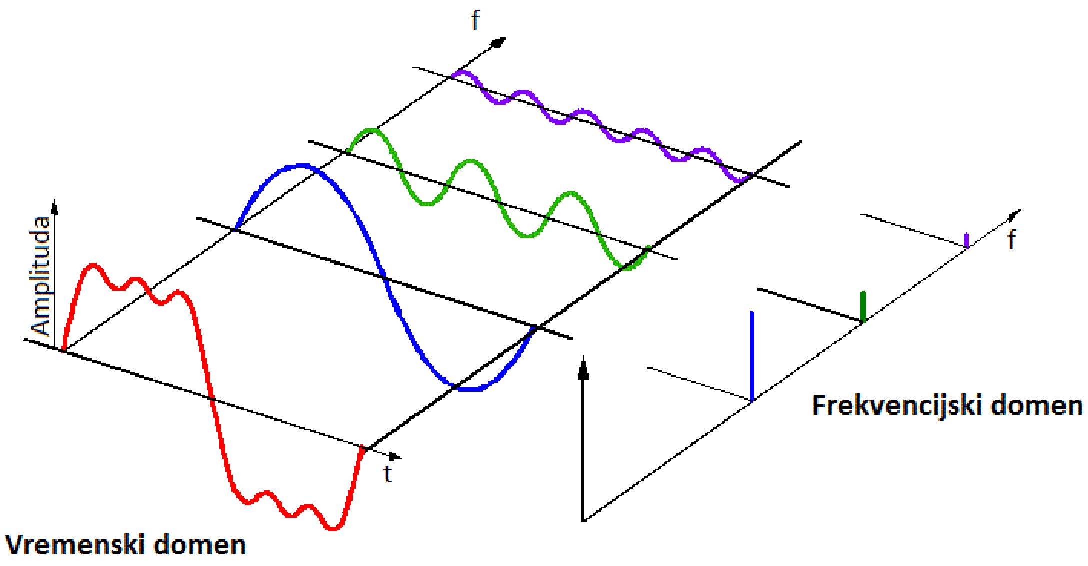
Furijeova transformacija

Furijeova transformacija transformiše signal iz vremenskog u frekvencijski domen, a koristi se kod aperiodičnih signala.



Furijeova transformacija signala $f(t)$ se računa na sledeći način:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Inverzna Furijeova transformacija je:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

$F(j\omega)$ je kompleksna veličina. Njen moduo se naziva spektralna gustina amplituda, a argument spektralna gustina faza.

Energija signala $f(t)$ je:

$$W = \int_{-\infty}^{\infty} f^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

Korelacija signala $f_1(t)$ i $f_2(t)$ je:

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_2(t+\tau) dt$$

Konvolucija signala $f_1(t)$ i $f_2(t)$ je:

$$\rho_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_2(\tau - t) dt$$

Zadaci

1. Dokazati sledeće osobine Furijeove transformacije:

a) spektralna gustina amplituda je parna funkcija, a spektralna gustina faza neparna,

b) Furijeova transformacija signala $f(t)$ je realna funkcija učestanosti ako je $f(t)$ parna, a imaginarna funkcija ako je $f(t)$ neparna i

c) $F\{f(t\text{-}\tau)\} = e^{-j\omega\tau}F\{f(t)\}$

Rešenje:

a) $F(j\omega) = \int\limits_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$

$$F(-j\omega) = \int\limits_{-\infty}^{\infty} f(t)e^{j\omega t}dt = \int\limits_{-\infty}^{\infty}\left[f(t)e^{-j\omega t}\right]^{*}dt = \left[\int\limits_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right]^{*} = F^{*}(j\omega)$$

Pošto su $F(j\omega)$ i $F(-j\omega)$ konjugovano kompleksni sledi da je:

$$\left|F(-j\omega)\right| = \left|F(j\omega)\right| \text{ i}$$
$$\arg\{F(-j\omega)\} = -\arg\{F(j\omega)\}$$

b) $F(j\omega) = \int\limits_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int\limits_{-\infty}^{\infty} f(t)\cos(\omega t)dt - j\int\limits_{-\infty}^{\infty} f(t)\sin(\omega t)dt$

Uzimajući u obzir da je kosinusna funkcija parna, a sinusna neparna, dobijamo da je:

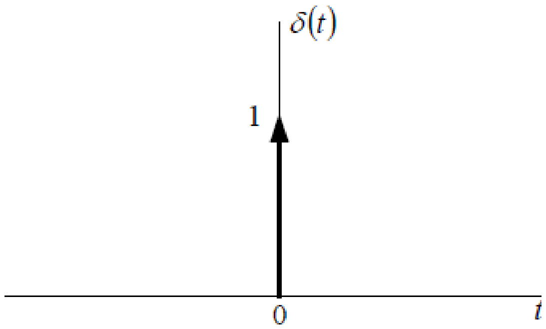
$$(\forall t \in \Re)\ f(t) = f(-t) \Rightarrow F(j\omega) = \int\limits_{-\infty}^{\infty} f(t)\cos(\omega t)dt = 2\int\limits_{0}^{\infty} f(t)\cos(\omega t)dt \in \Re$$

$$(\forall t \in \Re)\ f(-t) = -f(t) \Rightarrow F(j\omega) = -j\int\limits_{-\infty}^{\infty} f(t)\sin(\omega t)dt = -2j\int\limits_{0}^{\infty} f(t)\sin(\omega t)dt \in \Im$$

c) $F\{f(t\text{-}\tau)\} = \int\limits_{-\infty}^{\infty} f(t-\tau)e^{-j\omega t}dt = \int\limits_{-\infty}^{\infty} f(\mu)e^{-j\omega(\mu+\tau)}d\mu = e^{-j\omega\tau}F\{f(t)\}$

2. Odrediti Furijeovu transformaciju Dirakovog delta impulsa.

Jediničnu impulsnu funkciju (Dirakov impuls) grafički predstavljamo na način koji je prikazan slikom. Oznaka "1" na slici ne predstavlja vrednost signala u tački t=0, već je to vrednost površine koja se nalazi ispod ove funkcije (odnosno integral impulsne funkcije po celom prostoru nezavisne promenljive t).



Slika 1 - ***Grafički prikaz Dirakovog impulsa (jedinične impulsne funkcije)***

$$\Delta(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1 \cdot e^{-j\omega 0} = 1$$

3. Odrediti Furijeovu transformaciju prostoperiodičnog signala $c(t) = \cos(\omega_0 t)$.

$$C(j\omega) = \int_{-\infty}^{\infty} c(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} e^{-j\omega t} dt =$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t} dt$$

U prethodnom zadatku smo pokazali da su $\delta(t)$ i $\Delta(j\omega) = 1$ Furijeov transformacioni par, pa je:

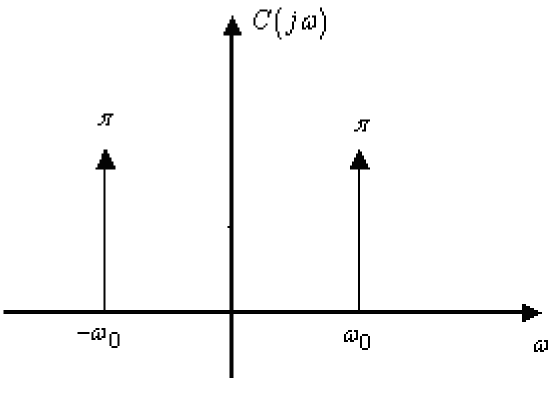
$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Delta(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega \Rightarrow \int_{-\infty}^{\infty} e^{j\omega t} d\omega = 2\pi \delta(t)$$

Ako uvedemo smenu $\tau = -t$, dobijamo:

$$C(j\omega) = -\frac{1}{2} \int_{\infty}^{-\infty} e^{j(\omega-\omega_0)\tau} d\tau - \frac{1}{2} \int_{\infty}^{-\infty} e^{j(\omega+\omega_0)\tau} d\tau = \frac{1}{2} \int_{-\infty}^{\infty} e^{j(\omega-\omega_0)\tau} d\tau + \frac{1}{2} \int_{-\infty}^{\infty} e^{j(\omega+\omega_0)\tau} d\tau =$$

o

$$= \frac{1}{2} \cdot 2\pi \delta(\omega - \omega_0) + \frac{1}{2} \cdot 2\pi \delta(\omega + \omega_0) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$



Slika 2 - Spektar prostoperiodičnog signala $c(t) = \cos(\omega_0 t)$

4. Dokazati sledeće osobine autokorelacije:

a) parna je funkcija i
b) njena Furijeova transformacija je kvadrat spektralne gustine amplituda.
c) Nacrtati autokorelaciju i spektralnu gustinu energije pravougaonog impulsa trajanja T i amplitude E.

Rešenje:

a) $$R_{xx}(\tau)=\int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

$$R_{xx}(-\tau)=\int_{-\infty}^{\infty} x(t)x(t-\tau)dt$$

Ako uvedemo smenu $\mu = t-\tau$, dobijamo:

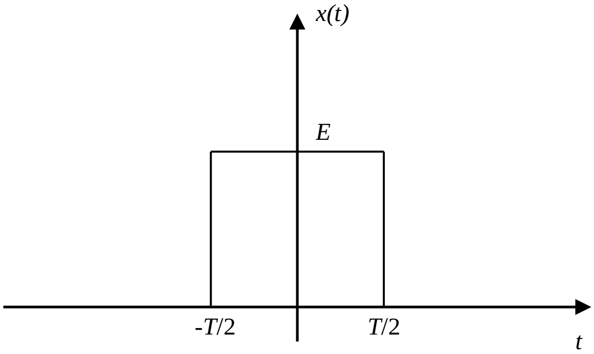
$$R_{xx}(-\tau)=\int_{-\infty}^{\infty} x(\mu+\tau)x(\mu)d\mu = R_{xx}(\tau) \Rightarrow \text{parna funkcija}$$

b)

$$F\{R_{xx}(\tau)\}=\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x(t+\tau)dt\cdot e^{-j\omega\tau}d\tau\, e^{j\omega t}e^{-j\omega t}=$$

$$=\int_{-\infty}^{\infty} x(t)e^{j\omega t}\left(\int_{-\infty}^{\infty} x(t+\tau)e^{-j\omega(t+\tau)}d\tau\right)d\tau = X(j\omega)\cdot X(-j\omega)=\left|X(j\omega)\right|^2$$

c)



Slika 3 - Pravougaoni impuls trajanja T i amplitude E

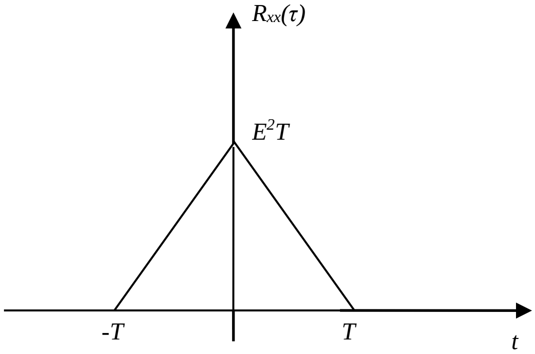
$$R_{xx}(\tau)=\int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

$x(t)\neq 0$ za $-T/2 \le t \le T/2$

$x(t+\tau)\neq 0$ za $-T/2 \le t+\tau \le T/2 \Rightarrow -T/2-\tau \le t \le T/2-\tau \Rightarrow$

$x(t)x(t+\tau)\neq 0$ za $-T/2 \le t \le T/2-\tau$

$$R_{xx}(\tau)=\int_{-T/2}^{T/2-\tau} E^2 dt = E^2\left(\frac{T}{2}-\tau+\frac{T}{2}\right)=E^2(T-\tau)$$



Slika 4 - Autokorelacija pravougaonog impulsa trajanja *T* i amplitude *E*

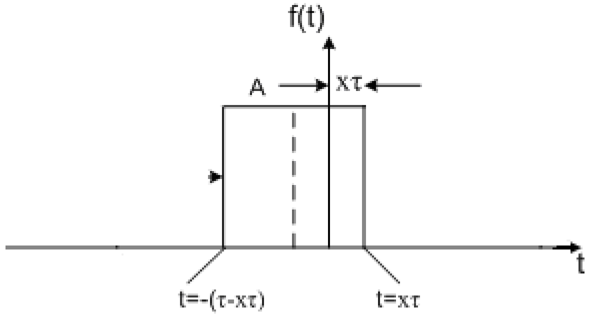
Furijeova transformacija signala $x(t)$ je:

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-T/2}^{T/2}Ee^{-j\omega t}dt=-\frac{E}{j\omega}e^{-j\omega t}\Bigg|_{-T/2}^{T/2}=ET\cdot\frac{\sin(\omega T/2)}{\omega T/2},$$

pa je spektralna gustina energije:

$$|X(j\omega)|^2=E^2T^2\cdot\left(\frac{\sin(\omega T/2)}{\omega T/2}\right)^2$$

5. Odrediti Furijeovu transformaciju signala sa slike i skicirati njegov amplitudski spektar.



Slika 5

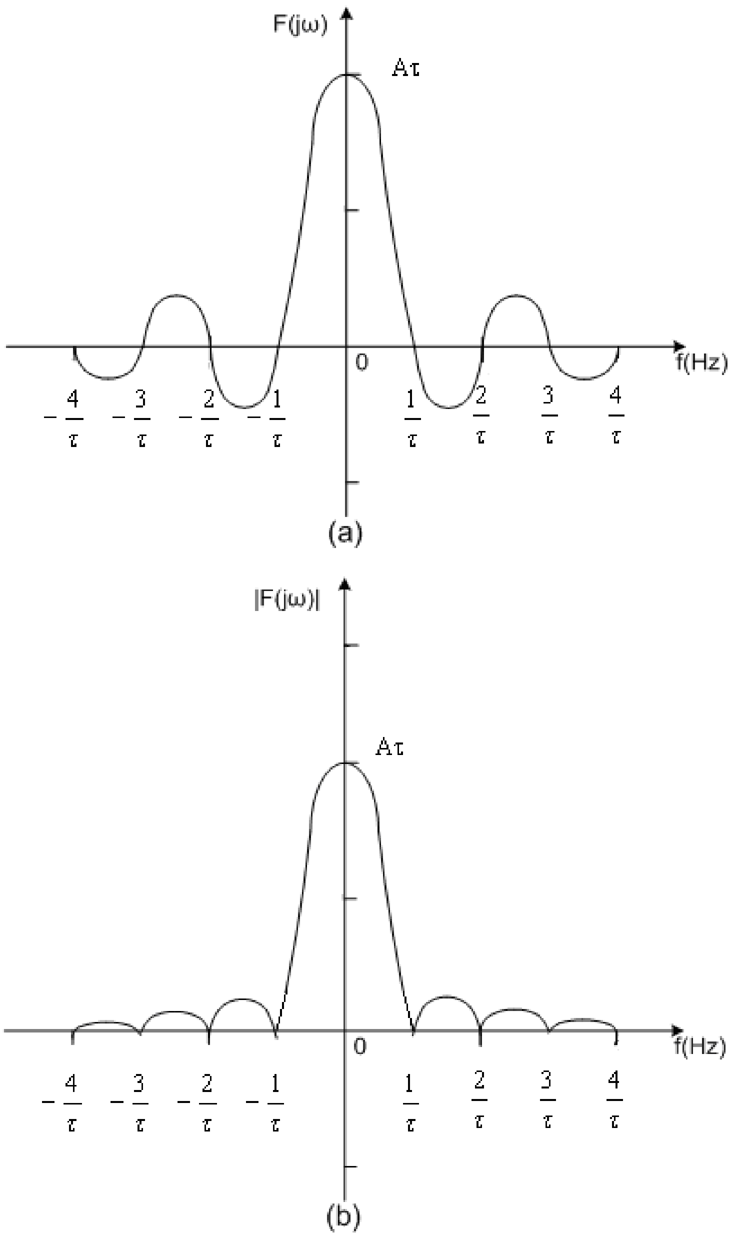
Rešenje:

$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt=\int_{x\tau-\tau}^{x\tau}Ae^{-j\omega t}dt=A\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}e^{j\omega(0.5-x)\tau}$$

Ako je impuls centriran ($x=0.5$) Furijeova transformacija je realna funkcija:

$$F(j\omega)=A\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}$$

i ima nule na učestanostima $\omega=\frac{2k\pi}{\tau}$, odnosno $f=\frac{k}{\tau}$, $k=\ldots-2,-1,1,2,\ldots$



Slika 6

6. Dat je signal $s(t) = Ae^{-t/\tau}h(t)$.

a) Izračunati Furijeovu transformaciju i nacrtati amplitudski i fazni spektar ovog signala.

b) Odrediti odnos energije signala do učestanosti f_c prema ukupnoj energiji signala. Faza signala na učestanosti $\pm f_c$ je $\pm\pi/4$.

Rešenje:

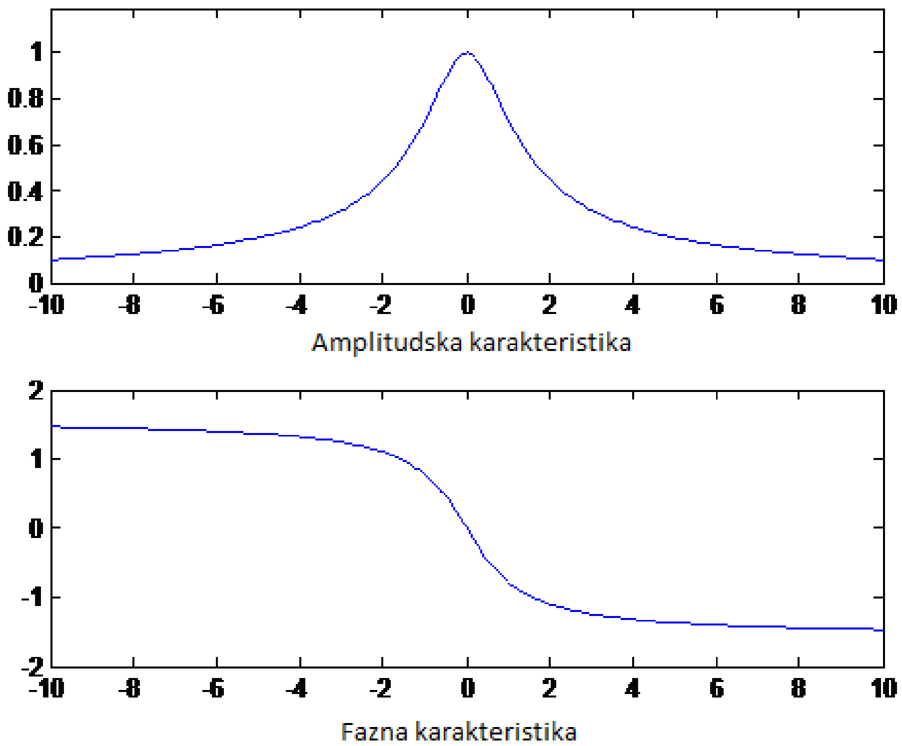
a)

$$S(j\omega)=\int_{-\infty}^{\infty}s(t)e^{-j\omega t}dt=\int_{0}^{\infty}Ae^{-t/\tau}e^{-j\omega t}dt=A\int_{0}^{\infty}e^{-(1/\tau+j\omega)t}dt=\frac{A}{-(1/\tau+j\omega)}e^{-(1/\tau+j\omega)t}\Big|_{0}^{\infty}=$$

$$=\frac{A\tau}{1+j\omega\tau}\Rightarrow$$

$$|S(j\omega)|=\frac{A\tau}{\sqrt{1+\omega^2\tau^2}}$$

$$\varphi(\omega)=arctg(-\omega\tau)=-arctg(\omega\tau)$$



Slika 7 – Amplitudska i fazna karakteristika

b) $\varphi(\omega_c)=-arctg(\omega_c\tau)=-\pi/4\Rightarrow\omega_c\tau=1\Rightarrow\omega_c=1/\tau$

Ukupna energija signala je:

$$W_{uk}=\int_{-\infty}^{\infty}s^2(t)dt=\int_{0}^{\infty}A^2e^{-2t/\tau}dt=-\frac{A^2\tau}{2}e^{-2t/\tau}\Big|_{0}^{\infty}=\frac{A^2\tau}{2}$$

Energija signala do učestanosti ω_c je:

$$W_{\omega_c}=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}|S(j\omega)|^2d\omega=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}\frac{A^2\tau^2}{1+\omega^2\tau^2}d\omega$$

Ako uvedemo smenu $\omega\tau=x$, dobijamo:

$$W_{\omega_c} = \frac{A^2\tau^2}{2\pi}\int_{-\omega_c\tau}^{\omega_c\tau}\frac{1}{1+x^2}\,dx = \frac{A^2\tau^2}{2\pi}\,arctg(x)\bigg|_{-\omega_c\tau}^{\omega_c\tau} = \frac{A^2\tau^2}{4} \Rightarrow$$

$$\frac{W_{\omega_c}}{W_{uk}} = \frac{1}{2}$$