Digitalni ekvivalent

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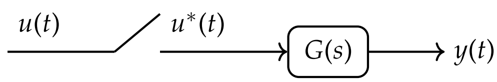
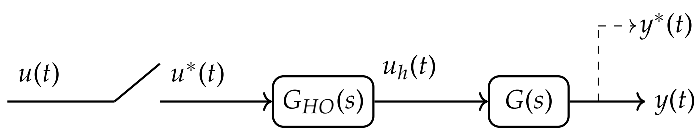
1 Uvod

Posmatrajmo sledeći sistem:



Podsećamo na dobro poznatu činjenicu u vezi sa digitalnim upravljačkim sistemima: znamo da su realni procesi najčešće kontinualni dok je upravljački signal diskretan (kao izlaz iz digitalnog upravljačkog uređaja), pa iz tog razloga mora postojati D/A konvertor, odnosno kolo zadrške nultog reda, pre samog procesa.



Ono što je bitno za nas jeste ponašanje sistema na izlazu u trenucima odabiranja.

Po teoremi sa predavanja:

$$Y^*(s) = (G_{HO}(s)G(s))^* U^*(s)$$

$$Y^*(s) = \underline{G_{HO}G(s)}^* U^*(s), \text{ gde je } \underline{G_{HO}G}^*(s) = \mathfrak{Z}\{\left(\mathcal{L}^{-1}\{G_{HO}G(s)\}\right)^*\}$$

Primer 1. Pronaći digitalni ekvivalent sledećeg sistema $G(s) = \frac{1}{s+1}$.
Rešenje:

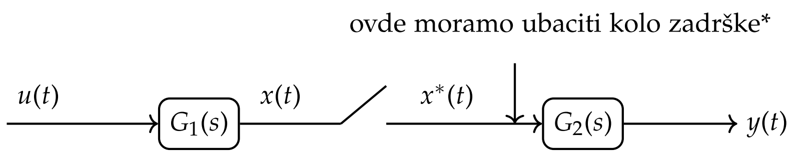
Znamo da je $\mathfrak{Z}\{1 - e^{-sT}\} = 1 - z^{-1}$

$$
\begin{aligned}
G_{DE}(z) = \frac{Y(z)}{U(z)} &= G_{HO}G(z) \\
&= \mathfrak{Z}\{\left(\mathcal{L}^{-1}\{G_{HO}G(s)\}\right)^*\} = \mathfrak{Z}\left\{\left(\mathcal{L}^{-1}\left\{\frac{1-e^{-sT}}{s}\frac{1}{s+1}\right\}\right)^*\right\} \\
&= \mathfrak{Z}\left\{\left(\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-sT}}{s(s+1)}\right\}\right)^*\right\} \\
&= (1-z^{-1})\left[\mathfrak{Z}\left\{\frac{1}{s}\right\} - \mathfrak{Z}\left\{\frac{1}{s+1}\right\}\right] \\
&= \frac{z-1}{\cancel{z}}\left[\frac{\cancel{z}}{z-1} - \frac{\cancel{z}}{z-e^{-sT}}\right] \\
&= \frac{\cancel{(z-1)}(\cancel{z} - e^{-T} - \cancel{z} + 1)}{\cancel{(z-1)}(z-e^{-T})} = \frac{1-e^{-T}}{z-e^{-T}}
\end{aligned}
$$

Implementirati u MATLAB-u.

Kakav je odnos odziva stvarnog sistema i njegovog digitalnog ekvivalenta?

Primer 2. Pronaći funkciju prenosa sledećeg sistema

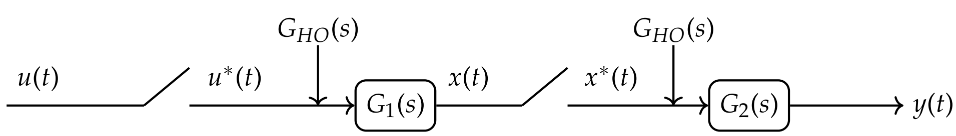


Rešenje:

$$Y^*(s) = X^*(s)(G_{HO}(s)G_2(s))^* = (U(s)G_1(s))^*(G_{HO}G_2(s))^*$$

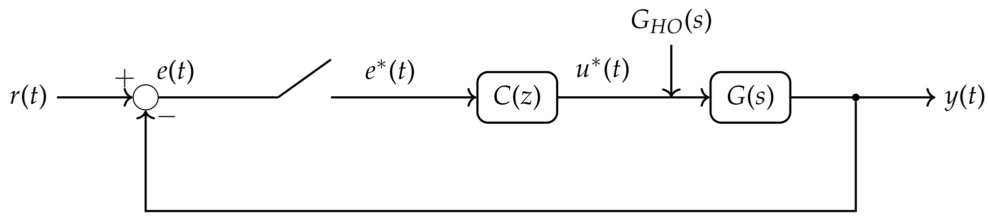
$$Y(z) = \underline{UG_1}(z)\underline{G_{HO}G_2}(z) \Rightarrow \text{Ne može se naći funkcija prenosa za dati sistem}\left(\frac{Y(z)}{U(z)}\right)$$

Primer 3. Pronaći funkciju prenosa sledećeg sistema



Rešenje:

$$
\begin{aligned}
Y^*(s) &= X^*(s)(G_{HO}(s)G_2(s))^* = X^*(s)(\underline{G_{HO}G_2}^*(s)) \\
X^*(s) &= U^*(s)\underline{G_{HO}G_1}^*(s) \\
Y^*(s) &= U^*(s)\underline{G_{HO}G_1}^*(s)\underline{G_{HO}G_2}^*(s) \\
\frac{Y^*(s)}{U^*(s)} &= \underline{G_{HO}G_1}^*(s)\underline{G_{HO}G_2}^*(s)
\end{aligned}
$$



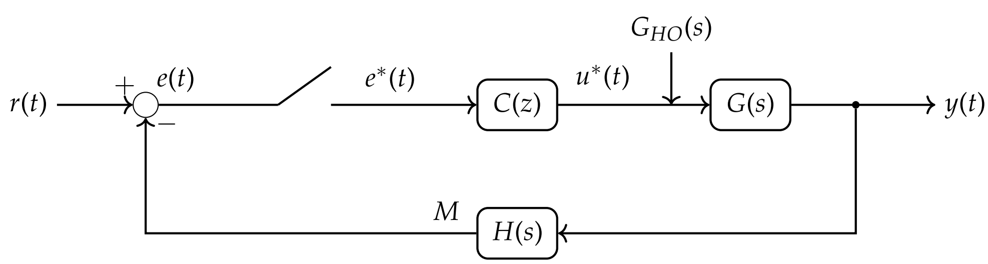
Primer 4. Pronaći funkciju prenosa sledećeg sistema:
Rešenje:

$$Y^*(s) = U^*(s)(G_{HO}G(s))^* = E^*(s)C(z)(\underline{G_{HO}G^*}(s)) = (R^*(s) - Y^*(s))^*C(z)\underline{G_{HO}G^*}(s)$$

$$Y^*(s)\left[1 + C(z)(\underline{G_{HO}G^*}(s))\right] = R^*(s)C(z)\underline{G_{HO}G^*}(s)$$

$$\frac{Y^*(s)}{U^*(s)} = \frac{C(z)(\underline{G_{HO}G^*}(s))}{1 + C(z)\underline{G_{HO}G^*}(s)}$$

Primer 5. Pronaći funkciju prenosa sledećeg sistema:



Rešenje:

$$Y(s) = G(s)G_{HO}(s)U^*(s)$$

$$U^*(s) = C(z)E^*(s)$$

$$E(s) = R(s) - H(s)Y(s) = R(s) - H(s)G(s)G_{HO}(s)U^*(s)\Big/^*$$

$$E^*(s) = R^*(s) - \underline{HGG_{HO}}^*(s)U^*(s)$$

$$E^*(s) = R^*(s) - \underline{HGG_{HO}}^*(s)C(z)E^*(s)$$

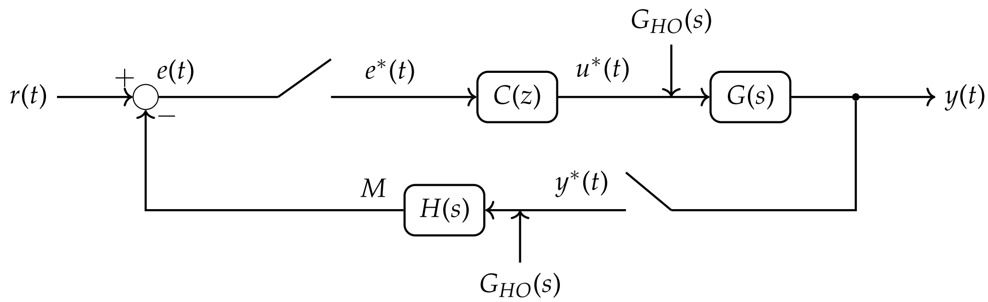
$$E^*(s) = \frac{R^*(s)}{1 + \underline{HGG_{HO}}^*(s)C(z)}$$

$$U^*(s) = \frac{C(z)R^*(s)}{1 + \underline{HGG_{HO}}^*(s)C(z)}$$

$$Y^*(s) = \frac{\underline{GG_{HO}}^*(s)C(z)R^*(s)}{1 + \underline{HGG_{HO}}^*(s)C(z)}$$

Primer 6. Pronaću funkciju prenosa sledećeg sistema:
Rešenje:

$M^*(s) = \underline{G_{HO}H^*}(s)Y^*(s)$



$$
\begin{aligned}
&Y^*(s) = U^*(s)(G_{HO}G(s))^* = E^*(s)C(z)\underline{G_{HO}G^*}(s) = (R^*(s) - M^*(s))C(z)\underline{G_{HO}G^*}(s)\\
&Y^*(s) = R^*(s)C(z)\underline{G_{HO}G^*}(s) - Y^*(s)\underline{G_{HO}H^*}(s)^*C(z)\underline{G_{HO}G^*}(s)\\
&Y^*(s)\left[1 + \underline{G_{HO}H^*}(s)C(z)(\underline{G_{HO}G^*}(s))\right] = R^*(s)C(z)\underline{G_{HO}G^*}(s)\\
&\frac{Y^*(s)}{U^*(s)} = \frac{C(z)\underline{G_{HO}G^*}(s)}{1 + \underline{G_{HO}H^*}(s)C(z)\underline{G_{HO}G^*}(s)} = G_{DE}(z)
\end{aligned}
$$

Koja je funkcija prenosa od r do e?

$$
\begin{aligned}
&E^*(s) = R^*(s) - M^*(s) = R^*(s) - \underline{G_{HO}H^*}(s)Y^*(s) = R^*(s) - \underline{G_{HO}H^*}(s)U^*(s)\underline{G_{HO}G^*}(s)\\
&E^*(s) = R^*(s) - \underline{G_{HO}H^*}(s)E^*(s)C(z)\underline{G_{HO}G^*}(s)\\
&E^*(s)(1 + \underline{G_{HO}H^*}(s)C(z)\underline{G_{HO}G^*}(s)) = R^*(s)\\
&\frac{E^*(s)}{R^*(s)} = \frac{1}{1 + \underline{G_{HO}H^*}(s)C(z)\underline{G_{HO}G^*}(s)}
\end{aligned}
$$

Primer 7. Ako je $C(z) = K$, $G(s) = \frac{1}{s+1}$, $H(s) = 2$, naći $G_{DE}(z)$ iz primera 6, implementirati u MATLAB-u i uporediti sa realnim sistemom.
Rešenje:

$$
\frac{Y^*(s)}{U^*(s)} = \frac{C(z)\underline{G_{HO}G^*}(s)}{1 + \underline{G_{HO}H^*}(s)C(z)\underline{G_{HO}G^*}(s)} = G_{DE}(z)
$$

$$
G_{HO}G^*(s) = \mathfrak{Z}\{G_{HO}G(s)\} = \mathfrak{Z}\left\{\frac{1-e^{-sT}}{s}\frac{1}{s+1}\right\} = \frac{1-e^{-T}}{z-e^{-T}}
$$

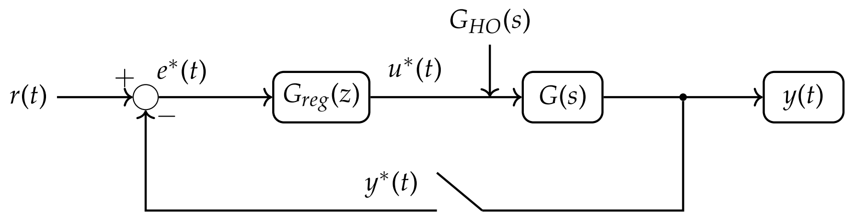
$$
G_{HO}H^*(s) = \mathfrak{Z}\left\{\frac{1-e^{-sT}}{s}\cdot 2\right\} = (1-z^{-1})\mathfrak{Z}\left\{\mathcal{L}^{-1}\left\{\frac{2}{s}\right\}\right\} = \frac{z-1}{z}\cdot 2\cdot\frac{z}{z-1} = 2
$$

$$
G_{DE}(z) = \frac{K\frac{1-e^{-T}}{z-e^{-T}}}{1+2K\frac{1-e^{-T}}{z-e^{-T}}} = \frac{K(1-e^{-T})}{z-e^{-T}+2K-2Ke^{-T}}
$$

Zadatak 1.

Proces je opisan funkcijom prenosa $G(s) = \frac{1}{s+2}$. Formirati diskretni regulator koji upravlja procesom. Simulirati u MATLAB-u ovaj primer.

Rešenje:



Slika 1: Blok dijagram.

Blok dijagram je dat na Slici 1.

Funkcija spregnutog prenosa sistema se računa na sledeći način:

$$\frac{Y^*(s)}{R^*(s)} = \frac{G_{reg}(z)[G_{HO}G(s)]^*}{1 + G_{reg}(z)[G_{HO}G(s)]^*}$$

$$\frac{Y^*(s)}{R^*(s)} = \frac{G_{reg}(z)\mathfrak{Z}\{G_{HO}G(s)\}}{1 + G_{reg}(z)\mathfrak{Z}\{G_{HO}G(s)\}}$$

$$\mathfrak{Z}\{G_{HO}G(s)\} = \mathfrak{Z}\left\{\frac{1-e^{-sT}}{s}\frac{1}{s+2}\right\} = (1-z^{-1})\mathfrak{Z}\left\{\frac{1}{s(s+2)}\right\} = (1-z^{-1})\mathfrak{Z}\left\{\frac{1/2}{s} - \frac{1/2}{s+2}\right\}$$

$$= \frac{z-1}{\cancel{z}}\frac{1}{2}\left(\frac{\cancel{z}}{z-1} - \frac{\cancel{z}}{z-e^{-2T}}\right) = \frac{1}{2}\frac{\cancel{(z-1)}(\cancel{z}-e^{-2T}-\cancel{z}+1)}{\cancel{(z-1)}(z-e^{-2T})}$$

$$= \frac{1}{2}\frac{1-e^{-2T}}{z-e^{-2T}} = \frac{a}{z-b}$$

gde je $a = \frac{1}{2}\left(1-e^{-2T}\right), \quad b = e^{-2T}$. Smenu uvodimo zbog jednostavnijeg računa u nastavku.

- **Regulator**

Proces je opisan funckijom prenosa prvog reda, pa ćemo koristiti PI regulator za upravljanje ovim procesom.

$$G_{pi}(s) = K_p + K_i\frac{1}{s}$$

Za diskretizaciju regulatora ćemo koristiti Ojler 2 aproksimaciju:

$$s = \frac{z-1}{zT} \Rightarrow G_{pi}(z) = K_p + K_i\frac{zT}{z-1} = \frac{K_p(z-1) + K_i zT}{(z-1)} = \frac{z(K_p + K_i T) - K_p}{(z-1)}$$

- **Odabir periode**

$$G = \frac{1}{s+2}$$
$$T_r = 2T_d = 2 \cdot \frac{1}{2} = 1$$
$$T_o \in \left[\frac{T_r}{10}, \frac{T_r}{4}\right] \Rightarrow \boxed{T = 0.1}$$

Funkcija spregnutog prenosa sistema:

$$\omega_{sp}(z) = \frac{\frac{z(K_p+K_iT)-K_p}{z-1}\frac{a}{z-b}}{1+\frac{z(K_p+K_iT)-K_p}{z-1}\frac{a}{z-b}} = \frac{za(K_p+K_iT)-K_pa}{(z-1)(z-b)+za(K_p+K_iT)-aK_p}$$

Karakteristični polinom je dat izazom:

$$f(z) = (z-1)(z-b) + za(K_p + K_iT) - aK_p$$
$$f(z) = z^2 + z(aK_p + aK_iT - b - 1) + b - aK_p$$

Za ispitivanje stabilnosti koristićemo Jurijev kriterijum koji je već rađen. Uslovi za stabilnost:

1.

$$f(1) > 0$$
$$\cancel{1} + \cancel{aK_p} + aTK_i - \cancel{b} - \cancel{1} + \cancel{b} - \cancel{aK_p} > 0$$
$$aTK_i > 0$$
$$\boxed{K_i > 0}$$

2.

$$(-1)^2 f(-1) > 0$$
$$1 - aK_p - aTK_i + b + 1 + b - aK_p > 0$$
$$2 - 2aK_p - aTK_i + 2b > 0$$
$$2 - 2aK_p + 2b > aTK_i$$
$$\boxed{K_i < \frac{2(1 - aK_p + b)}{aT}}$$

3.

$$|b - aK_p| < 1$$
$$-1 < b - aK_p < 1$$
$$-1 - b < -aK_p < 1 - b$$
$$\frac{1+b}{a} > K_p > \frac{b-1}{a} \quad b = e^{-2T}, a = \frac{1}{2}(1 - e^{-2T})$$
$$\boxed{-2 < K_p < 20.06}$$

Ako uzmemo npr. da je $K_p = 10$, onda sledi da jr $K_i < 201$.

Formiramo digitalni regulator za upravljanje procesom:

$$\begin{aligned}
G_{pi} &= K_p + K_i \frac{zT}{z-1} = \frac{U}{E} \\
U &= K_p E + K_i \frac{zT}{z-1} E \\
(z-1)U &= (z-1)EK_p + K_i zTE \\
U &= z^{-1}U + EK_p - z^{-1}eK_p + K_i TE
\end{aligned}$$

najzad, dobijamo diferencnu jednačinu za upravljanje:

$$u[k] = u[k-1] + e[k](K_p + K_i T) - K_p E[k-1]$$