Grafovi III

Algoritam svih najkraćih putanja

Motivacija

- Određivanje svih najkraćih putanja bi se moglo rešiti algoritmima za određivanje najkraće putanje koji bi se pozvao |V| puta (jednom za svaki čvor kao početak).
- Npr. Dijkstra algoritam

- Da li postoji bolja metoda za pronalazenje najkraćih putanja?

Definicija problem

Neka je dat usmeren graf $G = (V,E)$ sa težinama grana $\mathrm{w}(\mathrm{u},\mathrm{v})$, $(\mathrm{u},\mathrm{v}) \in E$. Potrebno je za svaki par čvorova *u* i *v* odrediti najkraću putanju od *u* do *v*.

Matrica W_{nxn} je matrica težina sa elementima w_{ij}

$$w_{i,j} = \begin{cases} 0 & i = j \\ težina\ direktne\ grane\ (i,j) & i \neq j\ \land\ (\mathrm{i},\mathrm{j}) \in E \\ \infty & i \neq j\ \land\ (\mathrm{i},\mathrm{j}) \notin E \end{cases}$$

Dinamičko programiranje i množenje matrica

Algoritam dinamičkog programiranja:

1. Odrediti strukturu optimalnog rešenja.
2. Rekurzivno definisati vrednost optimalnog rešenja.
3. Izračunati vrednost optimalnog rešenja na način odozdo prema gore.

1. Struktura najkraće putanje

- Sve potputanje najkraćeg puta su najkraći putevi
- Neka je graf predstavljen matricom susedstva $W = (w_{ij})$
- Neka je p najkraća putanja između čvorova *i* i *j* i pretpostavimo da *p* sadrži maksimalno *m* grana.
- Ako je i=j tada je p=0 u suprotnom je putanja *p* može podeliti na

$$i \xrightarrow{p'} k \to j$$

gde putanja *p'* sadrži najviše m-1 čvorova

- Putanja *p'* je nakraća putanja od *i* do *k*

$$\delta(i, j) = \delta(i, k) + w_{kj}$$

2. Rekurzivno rešavanje problema najkraćih putanja

- Neka je $l_{ij}^{(m)}$ minimalna težina putanje od čvora *i* do čvora *j* koja sadrži najviše *m* čvorova.
- m = 0 <=> i = j $$l_{ij}^{(0)} = \begin{cases} 0 & if \quad i = j \\ \infty & if \quad i \neq j \end{cases}$$
- Za m>0

$$l_{ij}^{(m)} = \min\left(l_{ij}^{(m-1)}, \min_{1 \le k \le n}\left\{l_{ik}^{(m-1)} + w_{kj}\right\}\right) = \min_{1 \le k \le n}\left\{l_{ik}^{(m-1)} + w_{kj}\right\}$$

3. Proračun optimalnog rešenja odozdo prema gore

```
EXTEND-SHORTEST-PATHS(L, W)
n = L.rows
let L' = (l'_ij) be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        l'_ij = ∞
        for k = 1 to n
            l'_ij = min(l'_ij, l_ik + w_kj)
return L'
```

Veza sa množenjem matrica

- Izračunati matrični proizvod $C = A \cdot B$ gde su A i B dimenzija $n \times n$.

- Tada, za $i, j = 1, 2, \ldots, n$, se izračunava

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

$$l_{ij}^{(m)} = \min_{1 \le k \le n} \left\{ l_{ik}^{(m-1)} + w_{kj} \right\}$$

- Ako se uvede smena

$$l^{(m-1)} \to a$$

$$w \to b$$

$$l^{(m)} \to c$$

$$\min \to \sum \ (+)$$

$$+ \to \cdot$$

Veza sa množenjem matrica

Ako se uvedu ove smene u EXTEND-SHORTEST-PATHS i ∞ zameni sa 0, dobija se procedura ($\Theta(n^3)$) za množenje matrica:

SQUARE-MATRIX-MULTIPLY(A, B)

```
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

EXTEND-SHORTEST-PATHS(L, W)

```
n = L.rows
let L' = (l'_ij) be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        l'_ij = ∞
        for k = 1 to n
            l'_ij = min(l'_ij, l_ik + w_kj)
return L'
```

$l^{(m-1)} \to a$

$w \to b$

$l^{(m)} \to c$

$\min \to \sum \quad (+)$

$+ \to \cdot$

Problem najkraćih putanja svih parova

- Proračun putanja minimalnih težina se vrši proširenjem najkraće putanje granu po granu. Ako je sa $A \cdot B$ obeležen *„proizvod“* vraćen sa EXTEND-SHORTEST-PATHS(*A*, *B*), računa se sekvenca od n - 1 matrica

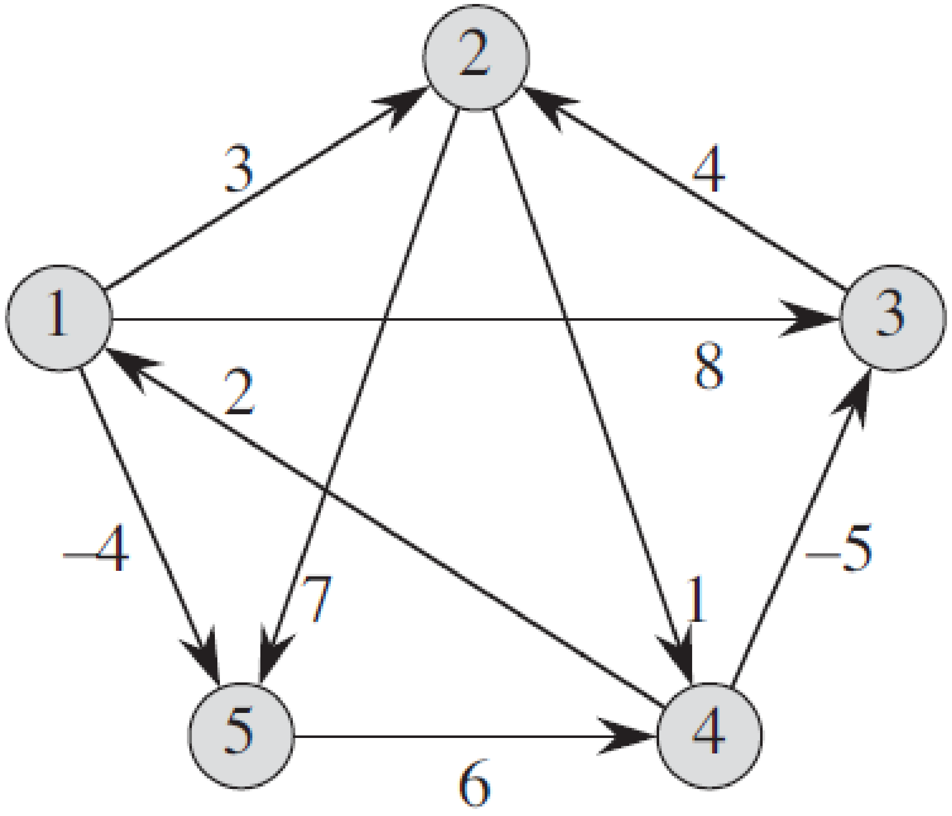
$$
\begin{aligned}
L^{(1)} &= L^{(0)} \cdot W = W, \\
L^{(2)} &= L^{(1)} \cdot W = W^2, \\
L^{(3)} &= L^{(2)} \cdot W = W^3, \\
&\ldots \\
L^{(n-1)} &= L^{(n-2)} \cdot W = W^{n-1},
\end{aligned}
$$

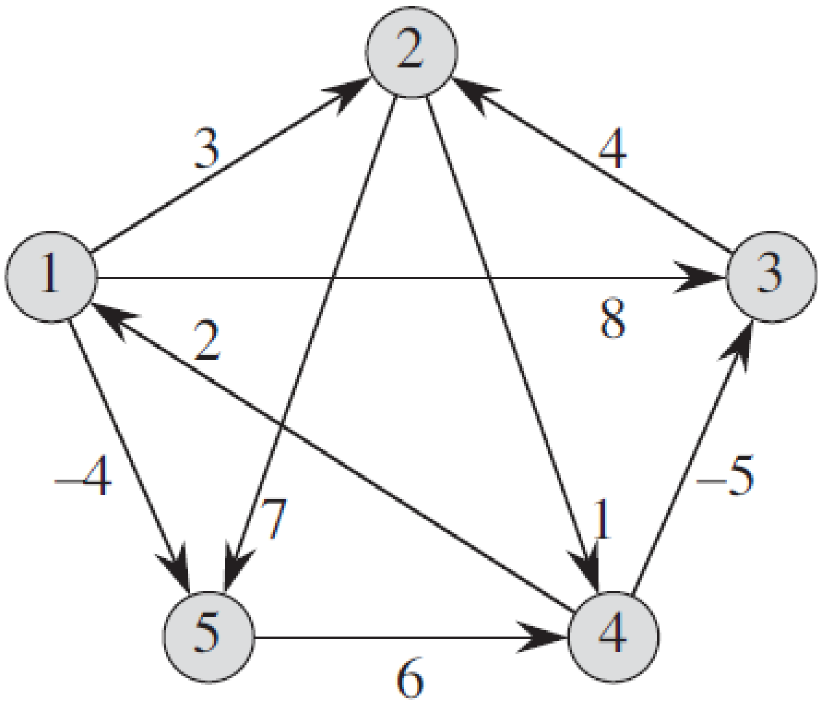
SLOW-ALL-PAIRS-SHORTEST-PATHS(W)

```
n = W.rows
L^(1) = W
for m = 2 to n − 1
    let L^(m) be a new n × n matrix
    L^(m) = EXTEND-SHORTEST-PATHS(L^(m−1), W)
return L^(n−1)
```

- Matrica $L^{(n-1)} = W^{n-1}$ sadrži putanju najmanjih težina sa kompleksnošću $\Theta(n^4)$

Problem





$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

$$L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad L^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$\begin{aligned} l_{ij}^{(m)} &= \min\left(l_{ij}^{(m-1)}, \min_{1 \le k \le n} \left\{l_{ik}^{(m-1)} + w_{kj}\right\}\right) \\ &= \min_{1 \le k \le n} \left\{l_{ik}^{(m-1)} + w_{kj}\right\} . \end{aligned}$$

Poboljšani algoritam

- Vreme izvršavanja $\Theta(n^4)$ je mnogo lošije nego korišćenjem Dijkstra algoritma.
- Može li se poboljšati?

- Interesuje nas samo $L^{(n-1)}$, svi ostali članovi su pomoćni $L^i \quad \left(1 < i < n-2\right)$
- Imamo $2^{\lceil \lg(n-1) \rceil} > n-1$, pa je $L^{2^{\lceil \lg(n-1) \rceil}} = L^{(n-1)}$

Poboljšani algoritam

$L^{(16)} \to L^{(15)}$ $n=15$ $L^{(n)} = 2^{\overbrace{\lceil \log_2 15 \rceil}^{4}}$

Može se izračunati $L^{2^{\lceil \lg(n-1) \rceil}}$ korišćenjem “uzastopnog kvadriranje” za dobijanje

$$L^{(2)}, L^{(4)}, L^{(8)}, \dots, L^{(2^{\lceil \log_2 n \rceil})}$$

FASTER-ALL-PAIRS-SHORTEST-PATHS(W)

```
n = W.rows
L^(1) = W
m = 1
while m < n − 1
    let L^(2m) be a new n × n matrix
    L^(2m) = EXTEND-SHORTEST-PATHS(L^(m), L^(m))
    m = 2m
return L^(m)
```

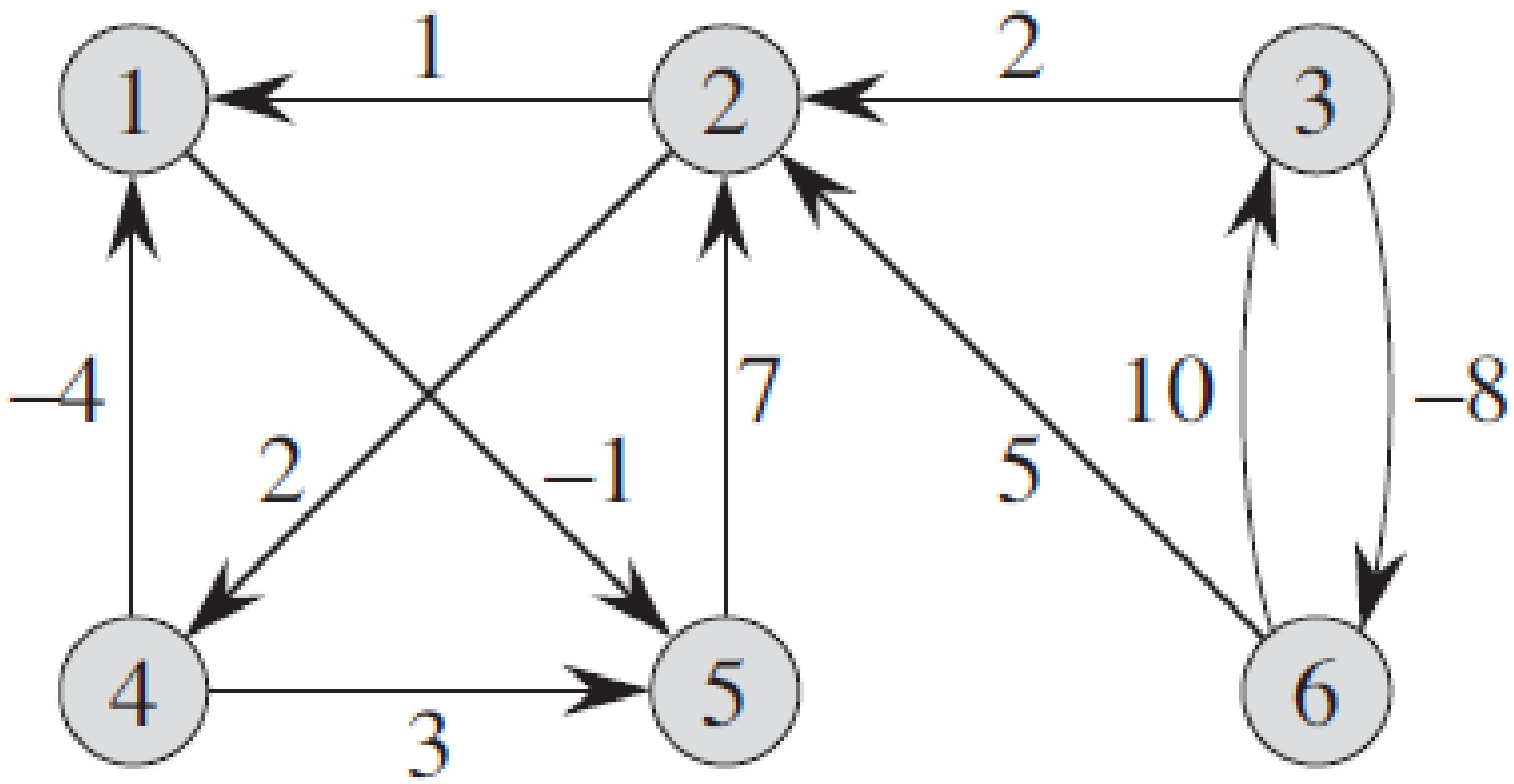
$$
\begin{aligned}
L^{(1)} &= W, \\
L^{(2)} &= W^2 &&= W \cdot W, \\
L^{(4)} &= W^4 &&= W^2 \cdot W^2 \\
L^{(8)} &= W^8 &&= W^4 \cdot W^4, \\
&\vdots \\
L^{(2^{\lceil \lg(n-1) \rceil})} &= W^{2^{\lceil \lg(n-1) \rceil}} &&= W^{2^{\lceil \lg(n-1) \rceil - 1}} \cdot W^{2^{\lceil \lg(n-1) \rceil - 1}}
\end{aligned}
$$

W $L^{(m-1)}$ W

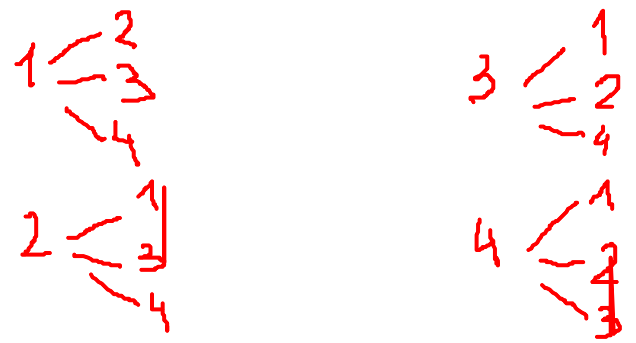
$L^{(2m)} \leftarrow (L^{(m)}, L^{(m)})$

Vreme izvršavanja FASTER-ALL-PAIRS-SHORTEST-PATHS je $\Theta(n^3 \lg n)$

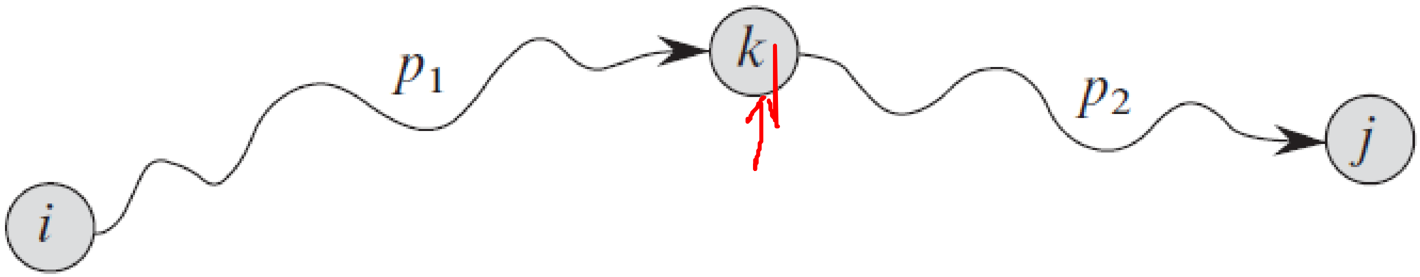
Primer



Floyd-Warshall algoritam



- Koristi drugačiju formulaciju dinamičkog programiranja za pronalaženje svih najkraćih putanja grafa G(V,E)
- Izvršava se u $\Theta(n^3)$ gde je n – broj čvorova
- Algoritam pretražuje čvorove unutar određene najkraće putanje, tj. deli putanju (i,j) na potputanje (i,k) i (k,j)



- Koristi se princip odozdo na gore i računaju matrice $D^{(k)}$

- Koristi se princip odozdo na gore i računaju matrice $D^{(k)}$

$$D^{(0)} = \left[w_{ij}\right]$$

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}) \qquad \text{za } k = 1, \ldots, n$$

- Za formiranje najkraćih putanja koristi se i matrice prethodnih čvorova Π sa elementima $\pi_{ij}^{(k)}$

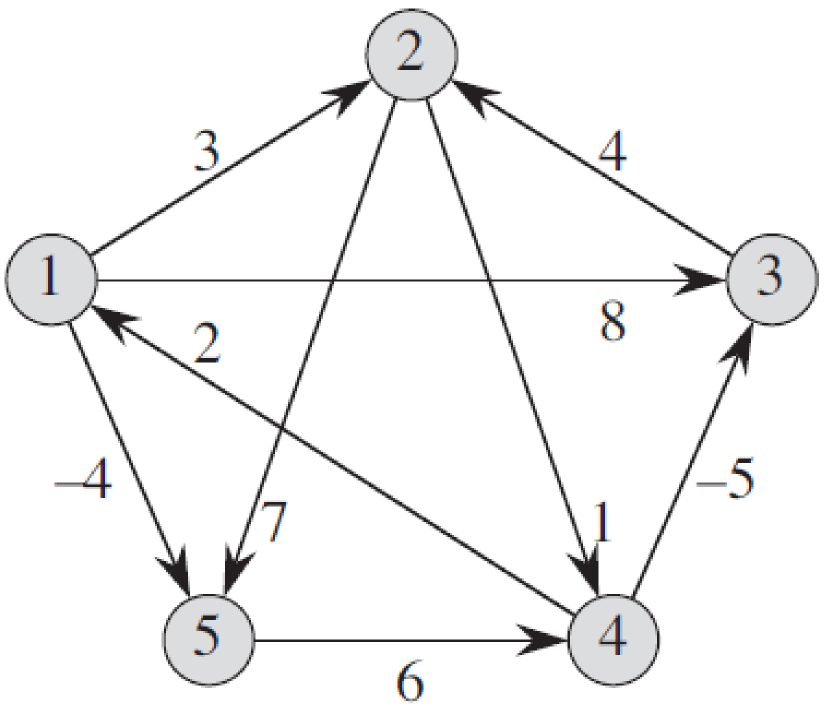
$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty , \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty . \end{cases}$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \end{cases}$$

Floyd-Warshall Algoritam

FLOYD-WARSHALL(W)

$n = W.rows$
$D^{(0)} = W$
for $k = 1$ **to** n
    let $D^{(k)} = \left(d_{ij}^{(k)}\right)$ be a new $n \times n$ matrix
    for $i = 1$ **to** n
        for $j = 1$ **to** n
            $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
return $D^{(n)}$, $\Pi^{(n)}$



$$D^{(0)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad \Pi^{(0)} = \begin{bmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & NIL & 4 & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)}, \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}. \end{cases}$$

- k=1

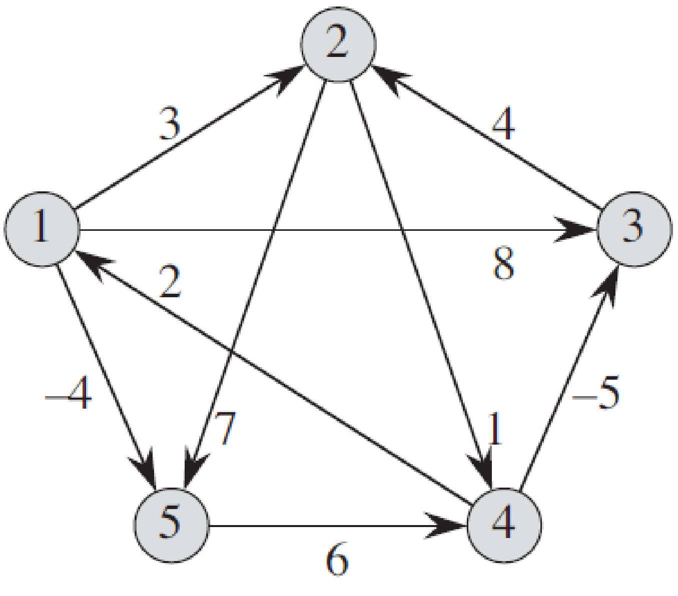
$$d_{42}^{(1)} = \min\left(\infty, \overset{2}{d_{41}^{(0)}} + \overset{3}{d_{12}^{(0)}}\right) = 5$$

$$\pi_{42}^{(1)} = \pi_{12}^{(0)} = 1$$

$$d_{45}^{(1)} = \min\left(\infty, \underset{2}{d_{41}^{(0)}} + \underset{-4}{d_{15}^{(0)}}\right) = -2$$

$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$\Pi^{(1)} = \begin{bmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$



$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad \Pi^{(1)} = \begin{bmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

- k=2

$$d_{14}^{(2)} = \min\left(\infty, d_{12}^{(1)} + d_{24}^{(1)}\right) = 3 + 1 = 4$$

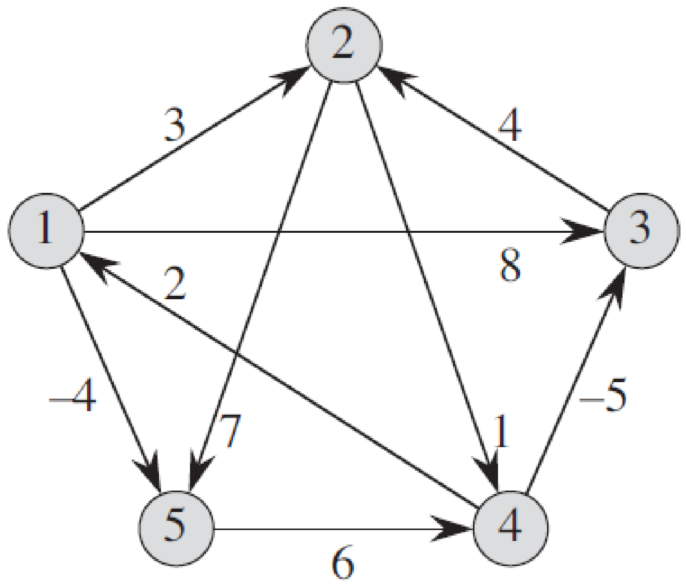
$\pi_{14}^{(2)} = \pi_{24}^{(1)} = 2$

$$d_{34}^{(2)} = \min\left(\infty, d_{32}^{(1)} + d_{24}^{(1)}\right) = 4 + 1 = 5$$

$$d_{35}^{(2)} = \min\left(\infty, d_{32}^{(1)} + d_{25}^{(1)}\right) = 4 + 7 = 11$$

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$\Pi^{(2)} = \begin{bmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$



$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad \Pi^{(2)} = \begin{bmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

- k=3

$$d_{42}^{(3)} = \min\left(d_{42}^{(2)}, d_{43}^{(2)} + d_{32}^{(2)}\right) = \min(4, -5+4) = -1$$

$$D^{(3)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad \Pi^{(3)} = \begin{bmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

- k=4

$$D^{(3)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad \Pi^{(3)} = \begin{bmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

$$d_{13}^{(4)} = \min\left(d_{13}^{(3)}, d_{14}^{(3)} + d_{43}^{(3)}\right) = \min(8,4-5) = -1$$

$$d_{21}^{(4)} = \min\left(d_{21}^{(3)}, d_{24}^{(3)} + d_{41}^{(3)}\right) = \min(\infty, 1+2) = 3$$

$$d_{23}^{(4)} = \min\left(d_{23}^{(3)}, d_{24}^{(3)} + d_{43}^{(3)}\right) = \min(\infty, 1-5) = -4$$

$$d_{25}^{(4)} = \min\left(d_{25}^{(3)}, d_{24}^{(3)} + d_{45}^{(3)}\right) = \min(7,1-2) = -1$$

$$d_{31}^{(4)} = \min\left(d_{31}^{(3)}, d_{34}^{(3)} + d_{41}^{(3)}\right) = \min(\infty, 5+2) = 7$$

$$d_{35}^{(4)} = \min\left(d_{35}^{(3)}, d_{34}^{(3)} + d_{45}^{(3)}\right) = \min(11, 5-2) = 3$$

$$d_{51}^{(4)} = \min\left(d_{51}^{(3)}, d_{54}^{(3)} + d_{41}^{(3)}\right) = \min(\infty, 6+2) = 8$$

$$d_{52}^{(4)} = \min\left(d_{52}^{(3)}, d_{54}^{(3)} + d_{42}^{(3)}\right) = \min(\infty, 6-1) = 5$$

$$d_{53}^{(4)} = \min\left(d_{53}^{(3)}, d_{54}^{(3)} + d_{43}^{(3)}\right) = \min(\infty, 6-5) = 1$$

$$D^{(4)} = \begin{bmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

$$\Pi^{(4)} = \begin{bmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{bmatrix}$$

- k=5

$$D^{(4)} = \begin{bmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \quad \Pi^{(4)} = \begin{bmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{bmatrix}$$

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

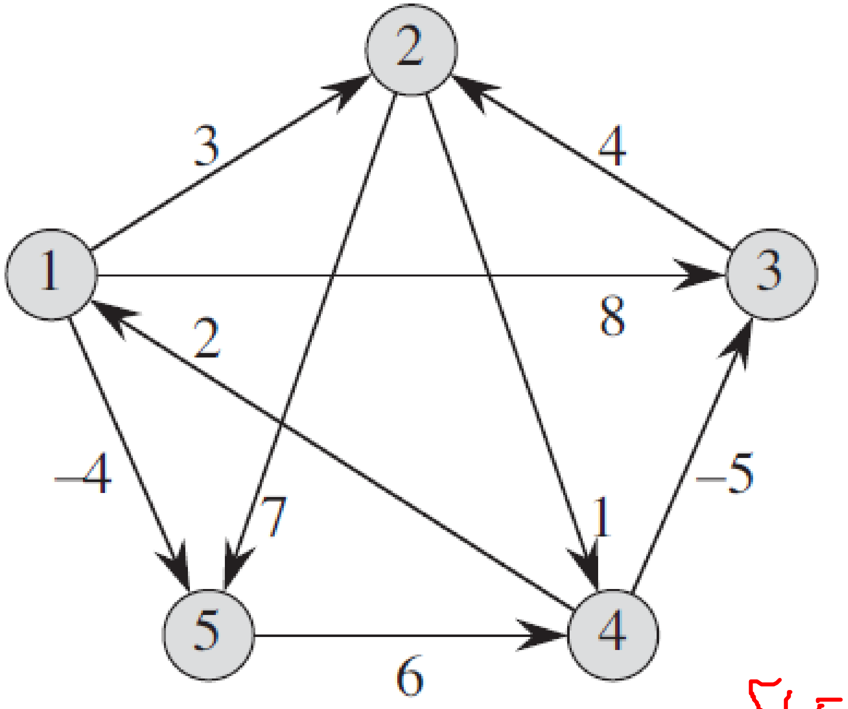
$$d_{12}^{(5)} = \min\left(d_{12}^{(4)}, d_{15}^{(4)} + d_{52}^{(4)}\right) = \min(3, -4+5) = 1$$

$$d_{13}^{(5)} = \min\left(d_{13}^{(4)}, d_{15}^{(4)} + d_{53}^{(4)}\right) = \min(-1, -4+1) = -3$$

$$d_{14}^{(5)} = \min\left(d_{14}^{(4)}, d_{15}^{(4)} + d_{54}^{(4)}\right) = \min(4, -4+6) = 2$$

$$\pi_{14}^{5} = \pi_{54}^{4} = 5$$

$$D^{(5)} = \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \quad \Pi^{(5)} = \begin{bmatrix} NIL & 3 & 4 & 5 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{bmatrix}$$



5,4

$\delta(5,1) = 8 = \delta(5,4) + \delta(4,1)$

$6 \quad + 2$

$$D^{(5)} = \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

$$\Pi^{(5)} = \begin{bmatrix} NIL & 3 & 4 & 5 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{bmatrix}$$