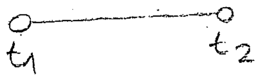
UVOD

– OPTIMALNO = NAJBOLJE REŠENJE POD ZADATIM USLOVIMA

KRITERIJUM OPTIMALNOSTI – REĆI ĆE NAM KOJE JE REŠENJE NAJBOLJE (NAJČEŠĆE JE TO BROJ – ŽELJA SE PRETVARA U KVANTITET)

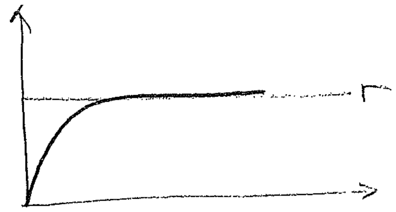
- F-JA CILJA
- CENA KOŠTANJA
- INDEKS PERFORMANSE
- COST FUNCTION

SINONIMI
– postavljaju se kriterijumi

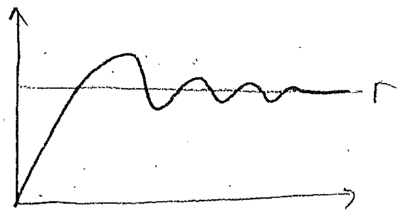


→ treba mi $\min(t_2 - t_1)$

$$\Rightarrow \boxed{I = \int_{t_1}^{t_2} dt} \rightarrow \text{ŽELIMO NEŠTO ZA NAJKRAĆE VREME}$$

1.


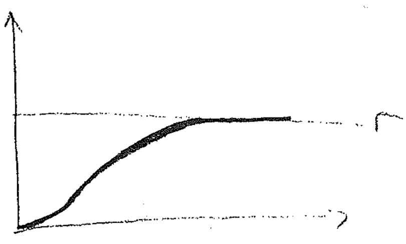
r - ŽELJENA VREDNOST

2.


$$e = r - y$$

$$I = \int_0^T |e|\,dt$$

$$\int_0^T e^2\,dt$$

3.


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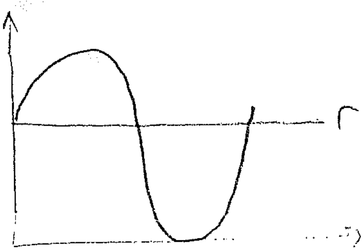


* STATIČKA OPTIMIZACIJA – KRITERIJUM OPTIMALNOSTI U OBLIKU SUME (I KOL)

* DINAMIČKA OPTIMIZACIJA – KRITERIJUM OPTIMALNOSTI U OBLIKU INTEGRALA (II KOL)

$\boxed{\text{STATIČKA OPTIMIZACIJA}}$

NAJLAKŠE PITANJE, 0% NA KOL. →

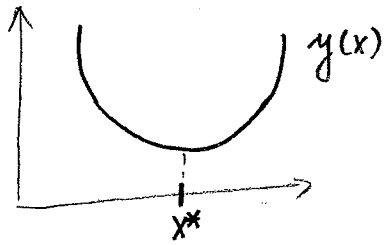
1. JEDNODIMENZIONALNA OPTIMIZACIJA – POTREBNI I DOVOLJNI USLOVI

– Po PRAVILU, KRITERIJUM OPTIMALNOSTI JE F-JA $y = y(x)$!

$y = y(x)$ → F-JA JEDNE PROMENLJIVE
→ ZA MIN

$y = -y(x) \left(\frac{1}{y(x)}\right)$ → ZA MAX

– PP: u tački x^* je MINIMUM!



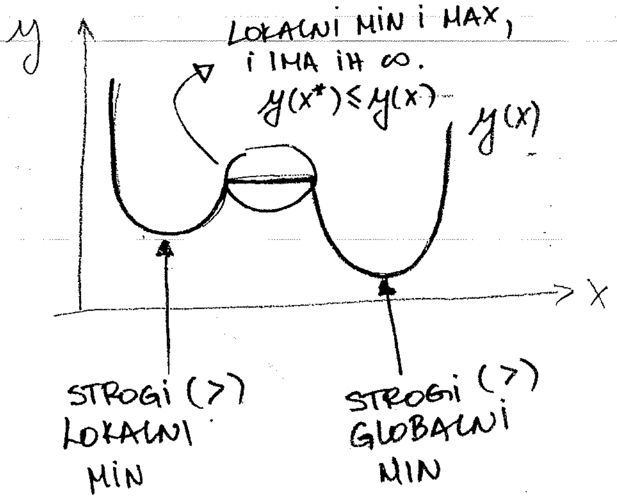
$y(x) \geq y(x^*)$

$y(x) - y(x^*) \geq 0, \ \|x - x^*\| < \varepsilon$ → $\boxed{\text{VAŽI ZA NEKO OKRUŽENJE PA JE TO LOKALNI MIN.}}$

$y(x) - y(x^*) \geq 0, \ \forall x$ → $\boxed{\text{VAŽI ZA SVAKO x, PA JE TO GLOBALNI MIN.}}$

$y(x) - y(x^*) > 0, \ \|x - x^*\| < \varepsilon$ → STROGI LOKALNI MIN (EKSTREM)

$y(x) - y(x^*) > 0, \ \forall x$ → STROGI GLOBALNI MIN (EKSTREM)



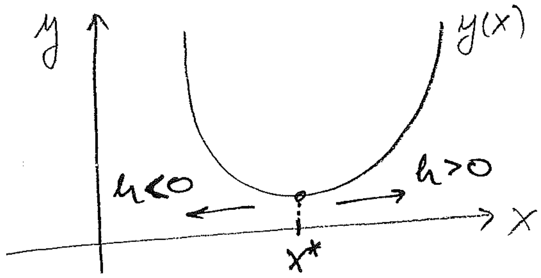
(*) AKO JE KRITERIJUM OPTIMALNOSTI SLOŽEN TJ. AKO F-JA NIJE KONVEKSNA I KONKAVNA, ONDA SE TO ZOVE NEKONVEKSNA OPTIMIZACIJA.

(*) KONVEKSNA OPTIMIZACIJA – ZNAČI DA SU I MIN I MAX UVEK STROGI, JER F-JE SU KONVEKSNE I KONKAVNE.

– PROBLEM KOJI MI REŠAVAMO $y = y(x)$

PP: u tački x^* se nalazi minimum

(fja je diferencijabilna do reda koji nam je potreban!)



– POSMATRAMO KAKAV JE ODNOS TAČKE x^* PREMA TAČKAMA LEVO I DESNO OD NJE. POMOĆU RAZVOJA U TEJLOROV RED ODREDIĆEMO KAKAV JE ODNOS.

$$y(x^*) \leq y(x)$$

RAZVOJ U TEJLOROV RED:

$$y(x) \approx y(x^*) + \underbrace{(x - x^*)}_{h \text{ - mali broj } (\varepsilon)} y'(x^*) + (x - x^*)^2 \frac{y''(x^*)}{2!} + \dots$$

$x - x^* = h$

DOMINANTNI ČLAN KOJI ODREĐUJE KARAKTER SISTEMA

$$\underbrace{y(x^* + h) - y(x^*)}_{\text{AKO JE OVO MIN}} = h\, y'(x^*) + \frac{h^2}{2!} y''(x^*) > 0$$

→ OVA RAZLIKA MORA DA BUDE VEĆA OD 0

$\Rightarrow$ $\boxed{y'(x^*) = 0}$ POTREBAN USLOV, JER h MOŽE DA BUDE + ILI – PA SE OVIM USLOVOM ELIMINIŠE UTICAJ h.

$$\lim_{h \to 0} \frac{y(x^* + h) - y(x^*)}{h \lessgtr} \overset{\text{MIN}}{\ } = \overset{>0}{\left(y'(x^*)\right)}$$

→ JOŠ JEDAN DOKAZ ZAŠTO JE POTREBAN USLOV $y'(x^*) = 0$.

! POTREBAN USLOV DA F-JA IMA MIN $y'(x^*) = 0$

Dovoljni uslovi:

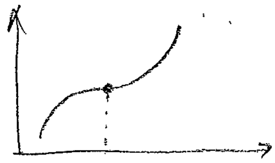
$$y(x^*+h) - y(x^*) = \frac{h^2}{2!} y''(x^*) > 0$$

$$\boxed{y''(x^*) > 0}$$

! h^2 uvek je pozitivan, pa je potreban uslov da $y''(x^*) > 0$.

MIN: $y'(x^*) = 0$, $y''(x^*) > 0$

MAX: $y'(x^*) = 0$, $y''(x^*) < 0$

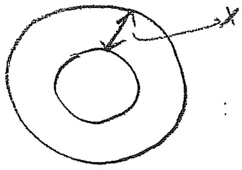


$\frac{h^3}{3!} y'''(x^*)$ — PREVOJNA TAČKA

$y = y(x)$

$y'(x^*)$	$y''(x^*)$	$y'''(x^*)$		$y^{(2k-1)}(x^*)$	$y^{(2k)}(x^*)$	$y^{(2k+1)}(x^*)$	
0	0	0	0	0	>0	$\neq 0$	MIN
0	0	0	0	0	<0	$\neq 0$	MAX
0	0	0	0	0	0	$\neq 0$	PREVOJNA TAČKA

PRIMER: KAKO DA ODREDIMO OPTIMALNU DEBLJINU IZOLACIJE TOPLOTNE CEVI, DA BISMO DOBILI NAJMANJU CENU?



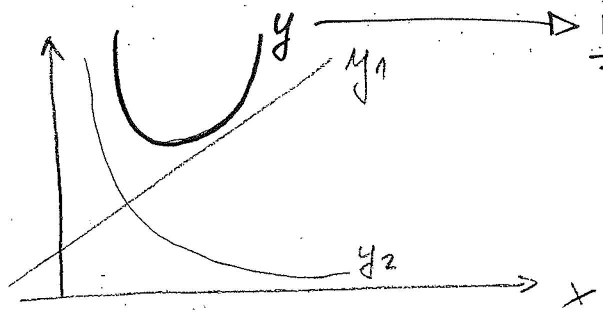
CENA KOŠTANJA: $y_1 = ax + b \, , \quad a, b > 0$

y_1 – din; a – din/m koeficijent; x – m izolacija; b – din fiksni troškovi

TROŠKOVI EKSPLOATACIJE: $y_2 = \dfrac{c}{x} + d \, , \quad c, d > 0$

y_2 – din; c – din·m; x – m; d – din fiksni troškovi

! y_1 i y_2 SU KONTRADIKTORNI ZAHTEVI.



y → **KOMPROMIS** IZMEĐU 2 SUPROTSTAVLJENA ZAHTEVA.

$y = y_1 + y_2 = ax + b + \dfrac{c}{x} + d$ → KRITERIJUM OPTIMALNOSTI

$$y'(x^*) = 0$$

$$y'(x^*) = a - \frac{c}{x^{*2}} = 0$$

$$x^* = \pm\sqrt{\frac{c}{a}}$$ → UZIMA SE SAMO + REŠENJE!

$$y''(x^*) = \frac{2c}{x^{*3}} > 0 \quad \text{MIN}$$

* FIKSNE VREDNOSTI KOJE NE ZAVISE OD PROMENLJIVE NE UTIČU NA KARAKTER REŠENJA.

NAJTEŽE PITANJE, 10% NA KOL → 2. VIŠEDIMENZIONALNA OPTIMIZACIJA, BEZ OGRANIČENJA

16.10.2017.

$$y = y(x_1, \dots, x_n) = y(\underset{\text{VEKTOR}}{\underline{X}}) \equiv y(x) \quad , \quad \underline{X} = [x_1, \dots, x_n]$$

→ KRITERIJUM OPTIMALNOSTI ZAVISI OD VIŠE PROMENLJIVIH

TAČKA OKO KOJE RAZVIJAMO U RED

$$y(x) \approx y(x^*) + \Delta y + \frac{\Delta^2 y}{2!} + \dots$$

1D OPT.: $y(x) = y(x^*) + \overbrace{(x - x^*)}^{h} y'(x^*)$

$$\Delta y = (x_1 - x_1^*)\left(\frac{\partial y}{\partial x_1}\right)^* + (x_2 - x_2^*)\left(\frac{\partial y}{\partial x_2}\right)^* + \dots + (x_n - x_n^*)\left(\frac{\partial y}{\partial x_n}\right)^*$$ → PRVI ČLAN RAZVOJA U RED

$$\left(\frac{\partial y}{\partial x_i}\right)^* = \frac{\partial y}{\partial x_i}\bigg|_{x_i = x_i^*}$$ → PRVI IZVOD ZA SVAKU PROMENLJIVU POSEBNO

$$\Delta^2 y = (x_1 - x_1^*)^2 \left(\frac{\partial^2 y}{\partial x_1^2}\right)^* + 2(x_1 - x_1^*)(x_2 - x_2^*)\left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right) + \dots + (x_n - x_n^*)^2 \left(\frac{\partial^2 y}{\partial x_n^2}\right)^*$$

OPŠTI SLUČAJ:

$$\Delta^k y = \sum \frac{k!}{p_1! \, p_2! \dots p_n!} (x_1 - x_1^*)^{p_1} (x_2 - x_2^*)^{p_2} \dots (x_n - x_n^*)^{p_n} \cdot \left(\frac{\partial^k y}{\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_n^{p_n}}\right)^*$$

$$\sum p_i = k$$

$$y = y(x_1, x_2)$$ → RADIMO F-JU SA 2 PROMENLJIVE (RAZVOJ U RED)

$k = 1$

p_1	p_2
1	0
0	1

$$\Delta y = \frac{1!}{0! \cdot 1} (x_1 - x_1^*)^{1 = p_1} \cdot \cancel{(x_2 - x_2^*)^0}^{\,1} \cdot \left(\frac{\partial y}{\partial x_1^{1 = p_1}}\right) + (x_2 - x_2^*)\left(\frac{\partial y}{\partial x_2}\right)^*$$

$$\Delta y = (x_1 - x_1^*)\left(\frac{\partial y}{\partial x_1}\right)^* + (x_2 - x_2^*)\left(\frac{\partial y}{\partial x_2}\right)^*$$

$k=2$

p_1	p_2
2	0
1	1
0	2

$$\Delta^2 y = (x_1-x_1^*)^2\left(\frac{\partial^2 y}{\partial x_1^2}\right)^* + 2(x_1-x_1^*)(x_2-x_2^*)\left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^* + (x_2-x_2^*)^2\left(\frac{\partial^2 y}{\partial x_2^2}\right)^*$$

$$\boxed{x_i - x_i^* = h_i}$$

$$\Rightarrow \boxed{\begin{aligned} \Delta y &= h_1\left(\frac{\partial y}{\partial x_1}\right)^* + h_2\left(\frac{\partial y}{\partial x_2}\right)^* \\ \Delta^2 y &= h_1^2\left(\frac{\partial^2 y}{\partial x_1^2}\right)^* + 2h_1 h_2\left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^* + h_2^2\left(\frac{\partial^2 y}{\partial x_2^2}\right)^* \end{aligned}}$$

*)

PP: u tački x^* se nalazi MAX

POTREBAN USLOV: $\left.\frac{\partial y}{\partial x_i} = 0\right|_{x_i = x_i^*}$

$$y(x) - y(x^*) = \frac{\Delta^2 y}{2!} < 0$$

(*) AKO JE $\Delta^2 y < 0$, ZA PROIZVOLJNE NETRIVIJALNE VREDNOSTI h_1 i h_2, ONDA KAŽEMO DA JE F-JA NEGATIVNO DEFINITNA.

AKO JE $\Delta^2 y > 0$, ZA PROIZVOLJNE NETRIVIJALNE VREDNOSTI h_1 i h_2, ONDA KAŽEMO DA JE F-JA POZITIVNO DEFINITNA.

AKO JE $\Delta^2 y \leq 0$, ZA PROIZVOLJNE NETRIVIJALNE VREDNOSTI h_1 i h_2, ONDA SE F-JA ZOVE NEGATIVNO SEMIDEFINITNA.

AKO JE $\Delta^2 y \geq 0$, ZA PROIZVOLJNE NETRIVIJALNE VREDNOSTI h_1 i h_2, ONDA KAŽEMO DA JE F-JA POZITIVNO SEMIDEFINITNA

AKO JE $\underbrace{\Delta^2 y \gtrless 0}_{\text{MENJA ZNAK}}$ ZA PROIZVOLJNE NETRIVIJALNE VREDNOSTI h_1 i h_2, ONDA KAŽEMO DA JE F-JA NEDEFINITNA.

DA F-JA IMA MAX, DOVOLJAN USLOV JE DA JE $\Delta^2 y < 0$ - NEGATIVNO DEFINITNA.

DA F-JA IMA MIN, DOVOLJAN USLOV JE DA JE $\Delta^2 y > 0$ - POZITIVNO DEFINITNA.

* ISPITIVANJE DEFINITNOSTI = TRAŽENJE MIN i MAX

PP: X^* MAX (NASTAVAK)

$$\Delta^2 y = \underset{+}{h_1^2}\left(\frac{\partial^2 y}{\partial x_1^2}\right)^* + \underbrace{2h_1h_2}\left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^* + \underset{+}{h_2^2}\left(\frac{\partial^2 y}{\partial x_2^2}\right)^* < 0$$

OVO MOŽE
DA PRAVI PROBLEM

PA ZATO MORAMO DA NAMESTIMO
DA SE SVAKO h_i NALAZI POD ZNAKOM KVADRATA.

IZRAZ ĆEMO DA NAMEŠTAMO NA: $(a+b)^2 = a^2 + 2ab + b^2$

$$\Delta^2 y = \left(\frac{\partial^2 y}{\partial x_1^2}\right)^* \left[\underset{a^2}{h_1^2} + 2\underset{a}{h_1} h_2 \underbrace{\frac{\left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^*}{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^*}}_{b} + h_2^2 \frac{\left(\frac{\partial^2 y}{\partial x_2^2}\right)^*}{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^*}\right] < 0$$

$+b^2 - b^2$ DODAJEMO I ODUZIMAMO b^2

ŠTA OSTANE OD $-b^2$

$$\Delta^2 y = \left(\frac{\partial^2 y}{\partial x_1^2}\right)^* \left[h_1 + h_2 \frac{\left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^*}{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^*}\right]^2 + h_2^2\left[\left(\frac{\partial^2 y}{\partial x_2^2}\right)^* - \frac{\left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^{*2}}{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^*}\right] < 0$$

* DA BI $\Delta^2 y < 0$ POTREBNO JE: $a + b < 0$ TJ. $a < 0$ i $b < 0$!

DOVOLJNI USLOVI:

$$\left(\frac{\partial^2 y}{\partial x_1^2}\right)^* < 0$$

$$\frac{\left(\frac{\partial^2 y}{\partial x_1^2}\right)^*\left(\frac{\partial^2 y}{\partial x_2^2}\right)^* - \left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^* \; {>0}}{\left(\frac{\partial^2 y}{\partial x_1^2}\right) < 0} < 0 \qquad \left.\begin{matrix} >0 \\ <0 \end{matrix}\right\} \text{MAX!}$$

* DA BI F-JA IMALA:

MAX:	MIN:
$\left(\frac{\partial^2 y}{\partial x_1^2}\right)^* < 0$	$\left(\frac{\partial^2 y}{\partial x_1^2}\right)^* > 0$
$\left(\frac{\partial^2 y}{\partial x_1^2}\right)^*\left(\frac{\partial^2 y}{\partial x_2^2}\right)^* - \left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^* > 0$	$\left(\frac{\partial^2 y}{\partial x_1^2}\right)^*\left(\frac{\partial^2 y}{\partial x_2^2}\right)^* - \left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^* > 0$

$y = y(x_1, x_2)$

* POTREBAN USLOV EKSTREMA: $\frac{\partial y}{\partial x_1} = 0$, $\frac{\partial y}{\partial x_2} = 0$

* DOVOLJNI USLOVI EKSTREMA:

$$H_2 = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} \\ \frac{\partial^2 y}{\partial x_1 \partial x_2} & \frac{\partial^2 y}{\partial x_2^2} \end{bmatrix}$$

D_1, D_2

MINORI – PODSKUPI MATRICA

(*) FORMIRAMO SVE MINORE (D_1, D_2) OKO GLAVNE ~~DIG~~ DIJAGONALE.

$$D_1 = \frac{\partial^2 y}{\partial x_1^2}, \quad D_2 = \frac{\partial^2 y}{\partial x_1^2} \cdot \frac{\partial^2 y}{\partial x_2^2} - \left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^2$$

F-JA IMA:

MIN: $D_1, D_2 > 0$

MAX: $D_1 < 0$, $D_2 > 0$ } AKO JE OBRNUTO ILI $D_1, D_2 < 0$ NIŠTA SE NE DOBIJA.

(*) <u>SILVESTEROVA TEOREMA</u> – FORMIRANJE MATRICE H_2 DA BI SE ISPITALA DEFINITNOST.

→

* ŽA F-JE VIŠE PROMENLJIVIH: $y = y(x_1, \ldots, x_n)$

POTREBAN USLOV: $\dfrac{\partial y}{\partial x_i} = 0$, $i = 1, \ldots, n$

DOVOLJNI USLOVI:

H MATRICA – HESEOVA MATRICA (HESIJAN)

$$H = \begin{array}{ccccc} D_1 & D_2 & D_3 & & D_4 \\ \left[\begin{array}{ccccc} \dfrac{\partial^2 y}{\partial x_1^2} & \dfrac{\partial^2 y}{\partial x_1 \partial x_2} & \dfrac{\partial^2 y}{\partial x_1 \partial x_3} & \cdots & \dfrac{\partial^2 y}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 y}{\partial x_1 \partial x_2} & \dfrac{\partial^2 y}{\partial x_2^2} & \dfrac{\partial^2 y}{\partial x_2 \partial x_3} & \cdots & \dfrac{\partial^2 y}{\partial x_2 \partial x_n} \\ \dfrac{\partial^2 y}{\partial x_1 \partial x_3} & \dfrac{\partial^2 y}{\partial x_2 \partial x_3} & \dfrac{\partial^2 y}{\partial x_3^2} & \cdots & \dfrac{\partial^2 y}{\partial x_3 \partial x_n} \\ \vdots & & & & \\ \dfrac{\partial^2 y}{\partial x_1 \partial x_n} & \dfrac{\partial^2 y}{\partial x_2 \partial x_n} & \dfrac{\partial^2 y}{\partial x_3 \partial x_n} & \cdots & \dfrac{\partial^2 y}{\partial x_n^2} \end{array}\right]_{n \times n} \end{array}$$

$$\boxed{\begin{array}{ll} \text{MIN:} & D_i > 0, \quad i = 1, \ldots, n \\ \text{MAX:} & D_{1,3,5,7,9\ldots} < 0 \\ & D_{2,4,6,8,10\ldots} > 0 \end{array}}$$

* AKO JE $D_i \geq 0$ MOŽE DA BUDE MIN, ALI NE MORA DA ZNAČI.
ZNAMO DA SIGURNO NIJE MAX.
NAJČEŠĆE SE DESI DA JE $D_1 = 0$.
PO PRAVILU NE ZNAMO ŠTA JE AKO JE $D_i \geq 0$.

3. OPTIMIZACIJA UZ OGRANIČENJE TIPA =, METOD SMENE I METOD OGRANIČENE VARIJACIJE

18.10.2017.

$y = y(x_1, \dots, x_n)$ → KRITERIJUM OPTIMALNOSTI KOJI ZAVISI OD n PROMENLJIVIH
→ U NAJVEĆEM BROJU SLUČAJEVA PROMENLJIVE $x_1, \dots, x_n$ TRPE NEKO DODATNO OGRANIČENJE

$$\left.\begin{aligned} g_1(x_1, \dots, x_n) &= 0 \\ &\vdots \\ g_m(x_1, \dots, x_n) &= 0 \end{aligned}\right\}$$

DODATNA OGRANIČENJA – UKUPAN BROJ OGRANIČENJA m
→ OVA OGRANIČENJA SU ALGEBARSKA I PP JE DA SU TIPA =

$\boxed{m < n}$ → m MORA BITI STROGO MANJE OD n JER MORA DA POSTOJI BAR JEDAN STEPEN SLOBODE

1. METOD SMENE
2. METOD OGRANIČENE VARIJACIJE

} 3. ISPITNO PITANJE

* 3. METOD LANGRANŽEVIH MNOŽITELJA
4. METOD KAZNENIH F-JA

} 4. ISPITNO PITANJE

1. METOD SMENE

– IZ SISTEMA OGRANIČENJA $g_k(x_1, \dots, x_n)$, $k = 1, \dots, m$ IZRAZIMO m PROMENLJIVIH, ZAMENIMO KRITERIJUM OPTIMALNOSTI I DALJE REŠAVAMO SLOBODNOM OPTIMIZACIJOM PO $n-m$ PROMENLJIVIH

PRIMER:

$y = 4x_1^2 + 5x_2^2$ → KRITERIJUM OPTIMALNOSTI SA 2 PROMENLJIVE

$2x_1 + 3x_2 = 6$ → 1 OGRANIČENJE

$2x_1 + 3x_2 = 6 \Rightarrow 2x_1 = 6 - 3x_2$

$x_1 = \frac{6 - 3x_2}{2}$

$\Rightarrow y = 4\left(\frac{6-3x_2}{2}\right)^2 + 5x_2^2$

$y = \cancel{4} \cdot \frac{(6-3x_2)^2}{\cancel{4}} + 5x_2^2$

$y = 36 - 36x_2 + 9x_2^2 + 5x_2^2$

$\boxed{y = 14x_2^2 - 36x_2 + 36}$

1° $\boxed{\frac{\partial y}{\partial x_2} = 0}$

$\frac{\partial y}{\partial x_2} = 28x_2 - 36 = 0$

$28x_2 = 36$

$x_2^* = \frac{36}{28} = 1,2861$

$x_1^* = \frac{6 - 3 \cdot 1,2861}{28} = 1,071$

2° $\boxed{\frac{\partial^2 y}{\partial x_2^2} \; ? \; 0}$

$\frac{\partial^2 y}{\partial x_2^2} = 28 > 0 \Rightarrow$ MIN

$y = x_2 - x_1$

$x_1^2 + x_2^2 = 1 \rightarrow$ OGRANIČENJE JE KRUŽNICA

I način: $x_1^2 + x_2^2 = 1 \Rightarrow x_2^2 = 1 - x_1^2$

$$x_2 = \pm\sqrt{1 - x_1^2}$$

$+$

$\Rightarrow y = \sqrt{1 - x_1^2} - x_1^2$

$\vdots$

$\boxed{x_1^* = 0}$

$-$

$y = -\sqrt{1 - x_1^2} - x_1^2$

$\vdots$

$\boxed{x_1^* = \pm 0,866}$

II način: $x_1^2 + x_2^2 = 1 \Rightarrow x_1^2 = 1 - x_2^2$

$\Rightarrow y = x_2^2 + x_2 - 1$

$$\frac{\partial y}{\partial x_2} = 2x_2 + 1 = 0$$

$\boxed{x_2^* = -0.5}$

$x_1^2 = 1 - (-0.5)^2$

$\boxed{x_1^* = \pm 0,866}$

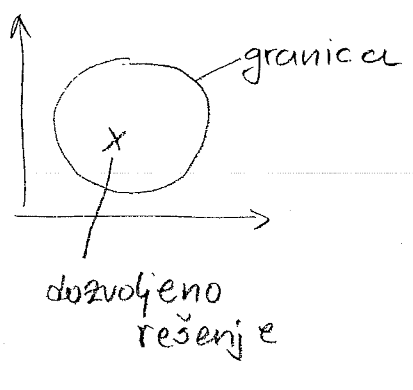
$x_1^* = 0$

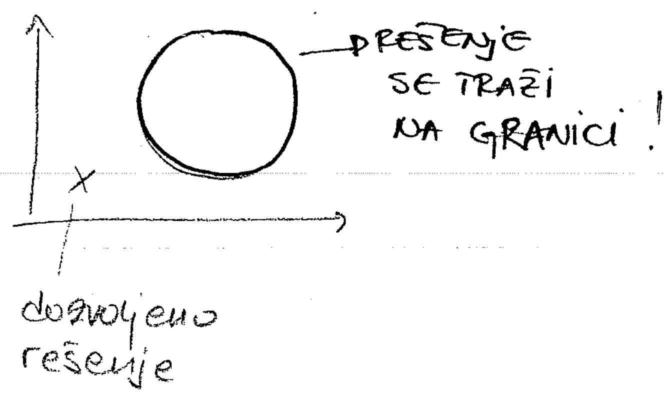
$x_2^* = \pm 1 \rightarrow$ NAJVEĆA VREDNOST KOJU x_2 MOŽE DA IMA I TO JE REŠENJE NA GRANICI.

$\Rightarrow$ KARAKTER OGRANIČENJA UTIČE NA METOD SMENE:

- LINEARNO $\rightarrow$ OK!
- NELINEARNO $\rightarrow$ NO! (NEĆEMO VIDETI SVA REŠENJA)

$\Rightarrow$ U OPTIMIZACIJI SU MOGUĆA 2 REŠENJA:
ILI JE REŠENJE ISTO KAO DA NE POSTOJI OGRANIČENJE
ILI JE REŠENJE NA GRANICI.





2. METOD OGRANIČENE VITRIJACIJE

$$y = y(x_1, \dots, x_n)$$

$$g_k(x_1, \dots, x_n) = 0,\ k = 1, \dots, m \longrightarrow \text{OGRANIČENJE}$$

$\boxed{m < n}$! MORA

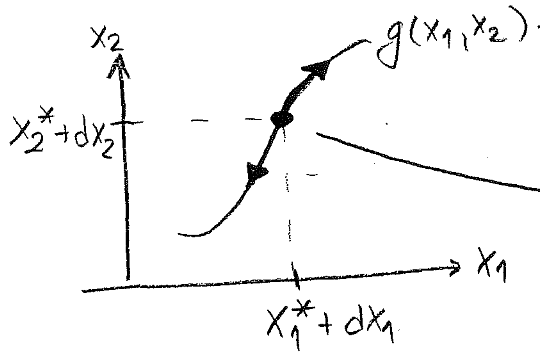
$y = y(x_1, x_2) \longrightarrow$ ZAKONOMERNOST NA 2 PROMENLJIVE (PA POSLE NA VIŠE)

$g(x_1, x_2) = 0 \longrightarrow$ OGRANIČENJE

PP: u tački x^* se nalazi ekstrem.

POTREBAN USLOV: $dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 = 0$

NISU PROIZVOLJNE VREDNOSTI



$\longrightarrow$ OGRANIČENJE UVEK MORA DA BUDE ZADOVOLJENO!

$\longrightarrow$ SAMO DUŽ OGRANIČENJA MOŽE DA SE POMERA:

$\boxed{g(x_1^* + dx_1,\ x_2^* + dx_2) = 0}$!

— RAZVIJAMO U RED OKO x_1^*:

$$g(x_1^* + dx_1, x_2^* + dx_2) = g(x_1^*, x_2^*) + \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2$$

$(x_1 - x_1^*)$

$x_1 + dx_1 - x_1^*$

ZA MALI POMERAJ KADA PREĐE IZ x_1 U x_2

$$dg = g(x_1^* + dx_1, x_2^* + dx_2) - g(x_1^*, x_2^*) = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2$$

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0$$

$$\Rightarrow dx_2 = -\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} dx_1$$

$$\Rightarrow dy = \left(\frac{\partial y}{\partial x_1} \cdot \frac{\partial g}{\partial x_2} - \frac{\partial y}{\partial x_2} \cdot \frac{\partial g}{\partial x_1} \right) dx_1 = 0 \longrightarrow \text{POTREBAN USLOV EKSTREMA!}$$

$$J_2\left(\frac{y, g}{x_2, x_1}\right) = \begin{vmatrix} \frac{\partial y}{\partial x_2} & \frac{\partial y}{\partial x_1} \\ \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_1} \end{vmatrix} = 0 \rightarrow$$ DAJU REŠENJE OGRANIČENJA ZA x_1, x_2

$+ \; g(x_1, x_2) = 0$

PRIMER:

$$y = 4x_1^2 + 5x_2^2$$

$$2x_1 + 3x_2 = 6 \longrightarrow \text{OGRANIČENJE}$$

$$2x_1 + 3x_2 - 6 = 0$$

$$g(x) = 0$$

$$J_2 = \begin{vmatrix} 10x_2 & 8x_1 \\ 3 & 2 \end{vmatrix} = 20x_2 - 24x_1 = 0$$

$$20x_2 - 24x_1 = 0$$

$$20x_2 = 24x_1$$

$$x_2 = \frac{6}{5}x_1$$

$$2x_1 + 3 \cdot \frac{6}{5}x_1 = 6$$

$$\frac{10x_1 + 18x_1}{5} = 6$$

$$28x_1 = 30$$

$$x_1^* = \frac{30}{28}$$

$$\boxed{x_1^* = 1.071}$$

$$x_2^* = \frac{6}{5} \cdot 1.071$$

$$\boxed{x_2^* = 1.286}$$

$y = y(x_1, \ldots, x_n)$ – KRITERIJUM ~~[illegible]~~ OPTIMALNOSTI VIŠE PROMENLJIVIH

$$\left.\begin{aligned} &g_1(x_1, \ldots, x_n) = 0 \\ &\vdots \\ &g_m(x_1, \ldots, x_n) = 0 \end{aligned}\right\} \text{ m OGRANIČENJA}$$

$J \rightarrow n - m$

J determinanti formiram: $m+1, m+2, m+3, \ldots, n$

$$J_k\left(\frac{y, g_1, g_2, \dots, g_m}{x_k, x_1, x_2, \dots, x_n}\right) = \begin{vmatrix} \frac{\partial y}{\partial x_k} & \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \dots & \frac{\partial y}{\partial x_n} \\ \frac{\partial g_1}{\partial x_k} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & & & \\ \frac{\partial g_m}{\partial x_k} & \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \dots & \frac{\partial g_m}{\partial x_n} \end{vmatrix}_{(m+1)\times(m+1)} = 0$$

$k = m+1, \dots, n$

---

4. OPTIMIZACIJA UZ OGRANIČENJE TIPA =, METOD LANGRANŽEVIH MNOŽITELJA I KAZNENE F-JE

$y = y(x_1, \dots, x_n)$

$g_k(x_1, \dots, x_n) = 0$, $k = 1, \dots, m \to m$ OGRANIČENJA

METOD LANGRANŽEVIH MNOŽITELJA: (UVEK RADI!)

ALGORITAM:

$\boxed{1.}$ FORMIRAMO PROŠIRENI KRITERIJUM OPTIMALNOSTI – LANGRANŽEV. F-JA ILI LANGRANŽIJAN F:

$$F \equiv L = \underbrace{y(x_1, \dots, x_n)}_{\text{STARI KRITERIJUM OPTIMALNOSTI}} + \sum_{k=1}^{m} \lambda_k \cdot g_k(x_1, \dots, x_n)$$

λ_k = LANGRANŽEVI MNOŽITELJI – PREDSTAVLJAJU DODATNE PROMENLJIVE KOJIMA SE POVEZUJU STARI ~~KKK~~ KRITERIJUM OPTIMALNOSTI I OGRANIČENJA. NOVIH PROMENLJIVIH IMA ONOLIKO KOLIKO IMA OGRANIČENJA.

(*) SVE PROMENLJIVE SE TRETIRAJU NA ISTI NAČIN KOD METODA LANGRANŽEVIH MNOŽITELJA. UKUPNO IMA $n+m$ PROMENLJIVIH.

2. $$\frac{\partial F}{\partial x_i} = \frac{\partial y}{\partial x_i} + \sum_{k=0}^{m} \lambda_k \frac{\partial g_k}{\partial x_i} = 0 \quad , i = 1, \dots, n$$ (SVI PRVI IZVODI JEDNAKI 0)

3. $$\frac{\partial F}{\partial \lambda_k} = g_k(x_1, \dots, x_n) = 0 \quad , k = 1, \dots, m$$

4. PROVERA DA LI SU ZADOVOLJENA OGRANIČENJA POSLE REŠAVANJA SISTEMA 2. I 3.
AKO JE OGRANIČENJE ZADOVOLJENO, REŠENJE JE PRIHVATLJIVO!

5. FORMIRAMO MATRICU Q ZA ISPITIVANJE DOVOLJNIH USLOVA,

$$Q = \begin{bmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} & \dots & \frac{\partial^2 F}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & \frac{\partial^2 F}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 F}{\partial x_n^2} \end{bmatrix}$$

PA ISPITUJEMO NJENU DEFINITNOST. AKO Q POKAŽE DA SE RADI O MIN/MAX TADA JE REŠENJE MIN/MAX.
AKO NE POKAŽE DA SE RADI O MIN/MAX, RADI SE KORAK 6.

6. MATRICA P – IZVODI PO SVAKOM OGRANIČENJU

$$P = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \dots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \dots & \frac{\partial g_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \dots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}_{m \times n}$$, PA ZATIM ⇒ $$\Delta = \begin{bmatrix} \emptyset_{m \times m} & P_{m \times n} \\ P^T_{n \times m} & Q - \mu I \end{bmatrix}_{(n+m) \times (n+m)} = 0$$

$I = eye(n)$ (jedinična matrica)

$\mu \geq 0 \Rightarrow$ MIN

$\mu \leq 0 \Rightarrow$ MAX

- [illegible]

$y = y(x_1, x_2)$ → KRITERIJUM OPTIMALNOSTI

$g(x_1, x_2) = 0$ → 1. OGRANIČENJE

$$\left.\begin{aligned} dy &= \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 = 0 \\ dg &= \frac{\partial g}{\partial x_1} dx + \frac{\partial g}{\partial x_2} dx_2 = 0 \end{aligned}\right\}$$ 3. ISPITNO PITANJE

SMENA: λ

$$dx_2 = -\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} dx_1 \Rightarrow dy = \left(\frac{\partial y}{\partial x_1}\left(-\frac{\frac{\partial y}{\partial x_2}}{\frac{\partial g}{\partial x_2}}\right) \cdot \frac{\partial g}{\partial x_1}\right) dx_1$$

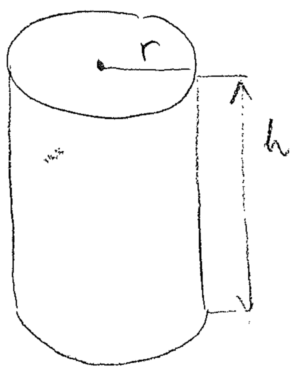
$$\Rightarrow dy = \left(\frac{\partial y}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1}\right) dx_1 = 0$$

$$-\frac{\frac{\partial y}{\partial x_2}}{\frac{\partial g}{\partial x_2}} = \lambda$$

$$\frac{\partial y}{\partial x_2} + \lambda \cdot \frac{\partial g}{\partial x_2} = 0$$

ODGOVARA $\boxed{2.}$ KORAKU

PRIMER: OD SVIH CILINDARA ZADATE ZAPREMINE ODREDITI ONAJ MINIMALNE POVRŠINE.



$A = 2r^2\pi + 2r\pi h$ → KRITERIJUM OPTIMALNOSTI JE POVRŠINA CILINDRA

$V = r^2\pi h$ → OGRANIČENJE JE ZAPREMINA

1. $F = 2r^2\pi + 2r\pi h + \lambda(V - r^2\pi h)$

2. $\frac{\partial F}{\partial r} = \cancel{4}^{2} r\cancel{\pi} + \cancel{2}\cancel{\pi}h - \cancel{2}\lambda r\cancel{\pi}h = 0$

$\frac{\partial F}{\partial h} = \cancel{2}r\cancel{\pi} - \lambda r^2\cancel{\pi} = 0$

$$\left.\begin{aligned} \Rightarrow 2r + h - \lambda r h &= 0 \\ r(2 - \lambda r) &= 0 \end{aligned}\right\} \Rightarrow \underbrace{2r = h}_{\text{PREČNIK}}$$ → KANDIDAT ZA EKSTREM

Ⓧ $\boxed{r = 0 \perp}$ $\lambda = \frac{2}{r}$

⊛ $r=0$ ⊥ ⟶ CILINDAR KOJI NE POSTOJI JE NAJMANJI CILINDAR NE MOŽE DA DOĐE U OBZIR JER NEZADOVOLJAVA OGRANIČENJE.

$H_2 \equiv Q$ ⟶ 2. ISPITNO PITANJE

$$Q = \begin{bmatrix} \frac{\partial^2 F}{\partial r^2} & \frac{\partial^2 F}{\partial r \partial h} \\ \frac{\partial^2 F}{\partial h \partial r} & \frac{\partial^2 F}{\partial h^2} \end{bmatrix}$$

(D_1, D_2)

$D_1 = -2$
$D_2 = -4$
} NIJE NIŠTA JER JE ZA MAX POTREBNO DA JE $D_1 < 0 \wedge D_2 > 0$, A ZA MIN $D_{1,2} > 0$

4. NASTAVAK – KAZNENE F-JE

23.10.2017.

KAZNENE F-JE

$$y = y(x_1, \ldots, x_n)$$

$$g_1(x_1, \ldots, x_n) = 0$$

$$\vdots$$

$$g_m(x_1, \ldots, x_n) = 0 \qquad k = 1, \ldots, m$$

UVOD

$y = a x_1^2 + b x_2^2$, TRAŽIM MIN

a	b	
1	1	TEŽINSKI FAKTORI
0,1	10	⟶ ŠTO MANJI x_2

* FORMIRAM NOVI KRITERIJUM OPTIMALNOSTI:

$$\phi = y(x_1, \ldots, x_n) + \sum_{k=1}^{m} P_k \cdot g_k^2 \rightarrow MIN$$

$$\phi = y(x_1, \ldots, x_n) - \sum_{k=1}^{m} P_k \cdot g_k^2 \rightarrow MAX$$

↓

STARI KRITERIJUM OPTIMALNOSTI

P_k – KAZNENA F-JA $\rightarrow \infty$ (IMA SMISLA TRAŽENJE MIN SAMO AKO JE $g_k = 0$, JER U DRUGOM SLUČAJU $\rightarrow \infty$)

↓

TO JE BROJ (PARAMETAR) KOJI IMA VELIKU VREDNOST

PRIMER: $y = 4x_1^2 + 5x_2^2$ $n=2$

$2x_1 + 3x_2 = 6 \Rightarrow 2x_1 + 3x_2 - 6 = 0$ $m=1 \Rightarrow P_k = P_1$

TRAŽI SE MIN:

$$\phi = y + p\,g^2 = 4x_1^2 + 5x_2^2 + p(2x_1 + 3x_2 - 6)^2$$

$$\frac{\partial\phi}{\partial x_1} = 8x_1 + 2p(2x_1 + 3x_2 - 6)\cdot 2 = 8x_1 + 4p(2x_1 + 3x_2 - 6) = 0$$

$$\frac{\partial\phi}{\partial x_2} = 10x_2 + 2p(2x_1 + 3x_2 - 6)\cdot 3 = 10x_2 + 6p(2x_1 + 3x_2 - 6) = 0$$

$$(1)\ 8x_1 + 4p(2x_1 + 3x_2 - 6) = 0 \,/ : 4$$

$$(2)\ 10x_2 + 6p(2x_1 + 3x_2 - 6) = 0 \,/ : 2$$

$$2x_1 + p(2x_1 + 3x_2 - 6) = 0$$

$$5x_2 + 3p(2x_1 + 3x_2 - 6) = 0$$

$$2x_1 = -p(2x_1 + 3x_2 - 6)$$

$$5x_2 = -3p(2x_1 + 3x_2 - 6)$$

$$2x_1 = -p(2x_1 + 3x_2 - 6)$$

$$\frac{5}{3}x_2 = -p(2x_1 + 3x_2 - 6)$$

ISTI IZRAZ PA IZJEDNAČIM OVU SA LEVE STRANE

$$2x_1 = \frac{5}{3}x_2 \Rightarrow \boxed{x_1 = \frac{5}{6}x_2} = \frac{5}{6}\cdot 1{,}266 = 1{,}055$$

$$(1)^* \Rightarrow 8\cdot\frac{5}{6}x_2 + 4p\left(2\cdot\frac{5}{6}x_2 + 3x_2 - 6\right) = 0 \,/ : 4$$

$$\cancel{2}\cdot\frac{5}{\cancel{6}_3}x_2 + p\left(\frac{10}{6}x_2 + 3x_2 - 6\right) = 0$$

$$\frac{5}{3}x_2 + p\left(\frac{10+18}{6}x_2 - 6\right) = 0$$

$$\frac{5}{3}x_2 + \frac{28}{6}x_2 p - 6p = 0$$

$$\frac{5}{3}x_2 + \frac{14}{3}x_2 p = 6p \,/ \cdot 3$$

$$x_2(5 + 14p) = 18p$$

$$\boxed{x_2 = \frac{18p}{5+14p}},\ p \to \infty \Rightarrow x_2 = \frac{18}{\frac{5}{p} + 14} = \frac{18}{14} = 1{,}266$$

$$\boxed{A(1{,}055,\ 1{,}266)}$$

5. OGRANIČENJE TIPA NEJEDNAKOSTI, UVOĐENJE DODATNE PROMENLJIVE; METODI SMENE, OGRANIČENE VARIJACIJE I LANGRANŽEVIH MNOŽITELJA

$$y = y(x_1, \dots, x_n)$$
$$g_1(x_1, \dots, x_n) \leq 0$$
$$\vdots$$
$$g_m(x_1, \dots, x_n) \leq 0$$

$\boxed{m \lesseqgtr n}$ → NE POSTOJI OGRANIČENJE NA BROJ PROMENLJIVIH.

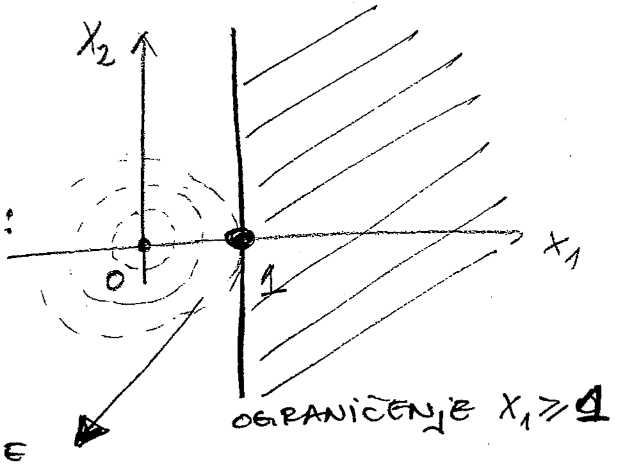
NAJJEDNOSTAVNIJI PRIMER: (NE RADIMO NA VEŽBAMA)

$$y = 4x_1^2 + 5x_2^2$$

$x_1 \geq 1$ → OGRANIČENJE

* RADIMO KAO DA OGRANIČENJE NE POSTOJI!

$$\left.\begin{aligned} \frac{\partial y}{\partial x_1} &= 8x_1 = 0 \\ \frac{\partial y}{\partial x_2} &= 10x_2 = 0 \end{aligned}\right\} \boxed{x_1^* = x_2^* = 0 \perp}$$



REŠENJE SE NALAZI NA GRANICI ZBOG $x_1 \geq 1$

* BIRAM REŠENJE KOJE SE NALAZI NA GRANICI: $\boxed{x_1^* = 1}$

$$\Rightarrow y = 4 + 5x_2^2$$

$$\frac{\partial y}{\partial x_2} = 10x_2 = 0 \Rightarrow \boxed{x_2^* = 0}$$

$$\frac{\partial^2 y}{\partial x_2^2} = 10 > 0 \Rightarrow \text{MIN}$$

---

* DA BISMO KORISTILI NEKI RANIJI METOD MORAMO DA PREBACIMO OGRANIČENJE TIPA $\leq$ U TIP $=$: $\leq \;\rightarrow\; =$

- UVEK SMATRAMO DA JE OGRANIČENJE $\leq$
- NA SVAKO OGRANIČENJE DODAJEM F-JU S KOJA IMA DODATNE PROMENLJIVE (BROJ DODATNIH PROMENLJIVIH = BROJ OGRANIČENJA)

$$\phi_k = g_k + S_k(x_{n+k}) = 0 \quad , \; k = 1, \dots, m$$

$\leq$

$S_k(x_{n+k}) \geq 0$
- $S_k = 0$ – REŠENJE NA GRANICI
- $S_k > 0$ – REŠENJE U DOZVOLJENOJ OBLASTI

$S_k = x_{n+k}^2$ → PARAZITSKA PROMENLJIVA = DODATNA PROMENLJIVA JE UVEK POD ZNAKOM KVADRATA I IMA ZADATAK DA SVEDE $\leq$ NA $=$!

$1 \ldots N$ (BEZ OGRANIČENJA) → MIN TRAŽIM

$1 \leftarrow 1 \ldots \boxed{100}$ → OGRANIČENJE

→ REŠENJE KAO DA OGRANIČENJE NE POSTOJI

$5 \leftarrow 5 \ldots \boxed{100}$

→ REŠENJE NA GRANICI

METOD SMENE – PRIMERI:

1) $y = 4x_1^2 + 5x_2^2$ $(n=2)$

→ PARAZITSKA PROMENLJIVA (DODATNA)

$(k=1)$ $x_1 \geq 1 \Rightarrow 1 - x_1 + x_3^2 = 0 \Rightarrow \boxed{x_1 = 1 + x_3^2}$

$\leqslant$ $\qquad n+k = 2+1 = 3$

$$\Rightarrow y = 4(1+x_3^2)^2 + 5x_2^2$$

$$\left.\begin{aligned}\frac{\partial y}{\partial x_2} &= 10x_2 = 0 \\ \frac{\partial y}{\partial x_3} &= 8(1+x_3^2)\cdot 2x_3 = 16x_3(1+x_3^2) = 0\end{aligned}\right\} \Rightarrow \begin{aligned}x_2^* &= 0 \\ x_3^* &= 0\end{aligned} \Rightarrow \boxed{x_1^* = 1}$$

→ REŠENJE NA GRANICI

$$Q = \begin{bmatrix} \frac{\partial^2 y}{\partial x_2^2} & \frac{\partial^2 y}{\partial x_2 \partial x_3} \\ \frac{\partial^2 y}{\partial x_3 \partial x_2} & \frac{\partial^2 y}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 16 \end{bmatrix}$$

$16 + \cancel{2x_3}^{\to 0} = 16$

$$\left.\begin{aligned}D_1 &= 10 > 0 \\ D_2 &= 160 > 0\end{aligned}\right\} \Rightarrow MIN$$

---

2) $y = 4x_1^2 + 5x_2^2$

$x_1 \leq 1$

$x_1 - 1 \leq 0$

$x_1 - 1 + x_4^2 = 0$

$x_1 = 1 - x_4^2$

$y = 4(1-x_4^2)^2 + 5x_2^2$

$\frac{\partial y}{\partial x_2} = 10x_2 = 0 \Rightarrow x_2^* = 0$

$(-16x_4 + 16x_4^3)'$

$-16 + 16\cdot 3x_4^2$

$\frac{\partial y}{\partial x_4} = -16x_4(1-x_4^2) = 0$

$x_4^* = 0$ X $\qquad \boxed{x_4^* = \pm 1}$ → BIRAMO OVO DA BI ZADOVOLJILI OGRANIČENJE

DOKAZ DA JE $x_4^* = \pm 1$ SU DOVOLJNI USLOVI:

$$Q = \begin{bmatrix} \frac{\partial^2 y}{\partial x_2^2} & \frac{\partial^2 y}{\partial x_2 \partial x_4} \\ \frac{\partial^2 y}{\partial x_4 \partial x_2} & \frac{\partial^2 y}{\partial x_4^2} \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & -16(1-3x_4^2) \end{bmatrix}$$

$D_1 = 10 > 0$

$D_2 = -160(1-3x_4^2)$

$x_4^* = 0 \Rightarrow D_2 < 0$ ⊥ NIJE MIN

$x_4^* = \pm 1 \Rightarrow \boxed{D_2 = +320 > 0}$

$$\left.\begin{aligned}D_1 &> 0 \\ D_2 &> 0\end{aligned}\right\} \Rightarrow MIN$$

! OVAJ PRIMER JE TEŽINE ZADATKA ZA TEST 1 !

$y = x_2^2 - 2x_1 - x_1^2$

$g: x_1^2 + x_2^2 \leq 1$ – OGRANIČENJE

$\leq \rightarrow =$

(3) $x_1^2 + x_2^2 + x_3^2 - 1 = 0$

$n = 3$

$m = 1$

$n - m = 2$ determinante J

$$J_2\left(\frac{y, g}{x_2, x_1}\right) = \begin{vmatrix} \frac{\partial y}{\partial x_2} & \frac{\partial y}{\partial x_1} \\ \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_1} \end{vmatrix} = \begin{vmatrix} 2x_2 & -2-2x_1 \\ 2x_2 & 2x_1 \end{vmatrix} = 0$$

m+1

$$J_3\left(\frac{y, g}{x_3, x_1}\right) = \begin{vmatrix} \frac{\partial y}{\partial x_3} & \frac{\partial y}{\partial x_1} \\ \frac{\partial g}{\partial x_3} & \frac{\partial g}{\partial x_1} \end{vmatrix} = \begin{vmatrix} 0 & -2-2x_1 \\ 2x_3 & 2x_1 \end{vmatrix} = 0$$

m+2

$J_2 = 4x_1x_2 - (-4x_2 - 4x_1x_2) = 4x_1x_2 + 4x_2 + 4x_1x_2 = 0$ (1)

$J_3 = -(-2-2x_1) \cdot 2x_3 = 4x_3 + 4x_1x_3 = 0$ (2)

(1) $8x_1x_2 + 4x_2 = 0$ /:4

(2) $4x_3(1 + x_1) = 0$ /:4

$x_2(1 + 2x_1) = 0 \Rightarrow x_2 = 0 \wedge 1 + 2x_1 = 0 \Rightarrow x_1 = -\frac{1}{2} = -0{,}5$

$x_3(1 + x_1) = 0 \Rightarrow x_3 = 0 \wedge x_1 = -1$

(3) $x_1^2 + x_2^2 + x_3^2 = 1$

	A	B	C	D
x_1	+1	−1	−0,5	0,5
x_2	0	0	$+\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
x_3	0	0	0	0

METOD LANGRANŽEVIH MNOŽITELJA:

$$y = y(x_1, \dots, x_n)$$
$$g_k \leq 0 \ , \ k = 1, \dots, m$$

ALGORITAM:

1) $\Phi_k = g_k + x_{n+k}^2 = 0$ → DODATNA PROMENLJIVA

2) FORMIRAM LANGRANŽIJAN:

$$F = y(x_1, \dots, x_n) + \sum_{k=1}^{m} \lambda_k (g_k + x_{n+k}^2)$$

3) PRVI IZVOD SAMO ORIGINALNIH PROMENLJIVIH:

$$\frac{\partial F}{\partial x_i} = \frac{\partial y}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_i} = 0 \ , \ i = 1, \dots, n$$

4) $\boxed{\frac{\partial F}{\partial x_{n+k}} = 2\lambda_k x_{n+k} = 0}$ UVEK! , $k = 1, \dots, m$

$n+m$ SISTEM JEDNAČINA

$\lambda_k = 0$ $\quad$ $x_{n+k} = 0$

4a) $\lambda_k = 0 \ \wedge \ x_{n+k} \neq 0$

$g_k - x_{n+k}$

$g_k < 0$ → TADA JE REŠENJE UVEK U DOZVOLJENOJ OBLASTI TJ. KAO DA NE POSTOJI REŠENJE

4b) $\lambda_k \neq 0 \ \wedge \ x_{n+k} = 0$

$g_k = 0$ → TADA JE REŠENJE NA GRANICI

4c) $\lambda_k = 0 \ \wedge \ x_{n+k} = 0$

$g_k = 0$ → TADA JE REŠENJE NA GRANICI

PRIMER:

$$y = x_2 - 2x_1 - x_1^2 \qquad n = 2$$
$$x_1^2 + x_2^2 \leq 1$$ → OGRANIČENJE JE KRUŽNICA $\quad m = 1 \to \lambda_1$

1) $x_1^2 + x_2^2 + x_3^2 - 1 = 0$ $\quad (1)$

x_3 → DODATNA PROMENLJIVA

2) $F = y + \lambda(g + x_3^2) = x_2 - 2x_1 - x_1^2 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1)$

→

3) $\frac{\partial F}{\partial x_1} = -2-2x_1+2\lambda x_1 = 0$ (1) /:(-2) $n+m = \cancel{3}+1 = \cancel{4}$ JED.

$\frac{\partial F}{\partial x_2} = 2x_2 + 2\lambda x_2 = 0$ (2) /:2

4) $\frac{\partial F}{\partial x_3} = 2\lambda x_3 = 0$ /:2 (3)

MORAM DOBITI OBLIK:
$2\lambda_k x_{n+k} = 0$

(1) $\Rightarrow$ $1 + x_1 - \lambda x_1 = 0$

(2) $\Rightarrow$ $x_2(1+\lambda) = 0$ $\to x_2 = 0$, $\to \lambda = -1$

(3) $\Rightarrow$ $\lambda x_3 = 0$ $\to x_3 = 0$, $\to \lambda = 0$

(4) $x_1^2 + x_2^2 + x_3^2 = 1$

	A	B	C	D
x_1	$+1$	-1	$-0{,}5$	$-0{,}5$
x_2	0	0	$+\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
x_3	0	0	0	0
λ	2	0	-1	-1
y	-3	1	$1{,}5$	$1{,}5$

ZA $-\lambda \Rightarrow y$ MAX

ZA $+\lambda \Rightarrow y$ MIN

! λ UKAZUJE NA KARAKTER REŠENJA.

MIN (A)

MAX (C, D)

NAJČEŠĆE PITANJE NA KOL.

6. KUN TAKEROVI USLOVI

(KARUŠ, KT USLOVI)

1939

20.10.2014.

$$y = y(x_1, \dots, x_n)$$

$$g_1(x_1 \dots x_n) \leq 0$$

$$\vdots$$

$$g_m(x_1, \dots, x_n) \leq 0 \;,$$

* POČINJEMO ISTO KAO 5. ISPITNO PITANJE!

1) OD OGRANIČENJA TIPA $\leq \rightarrow =$

$$\phi_k = g_k + x_{n+k}^2 = 0 \;, \; k = 1, \dots, m$$

2) NOVA LANGRANŽEVA F-JA:

$$F = y(x_1, \dots, x_n) + \sum_{k=1}^{m} \lambda_k \left(g_k(x_1, \dots, x_n) + x_{n+k}^2 \right)$$

3) PRVI IZVOD PO ORIGINALNIM PROMENJIVAMA:

$$\frac{\partial F}{\partial x_i} = \frac{\partial y}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_i} = 0 \;, \; i = 1, \dots, n$$

4) PRVI IZVOD PO DODATNOJ PROMENJIVOJ:

$$\boxed{\frac{\partial F}{\partial x_{n+k}} = 2 \lambda_k x_{n+k} = 0}$$ UVEK MORA BITI OVAKO!

$\lambda_k x_{n+k} = 0 \rightarrow$ NAJBOLJE MESTO ZA REŠAVANJE OPTIMIZACIONOG PROBLEMA

1° $\lambda_k = 0 \wedge x_{n+k} \neq 0$

$$g_k = -x_{n+k}^2$$

$g_k < 0 \rightarrow$ REŠENJE JE ISTO KAO DA NEMA OGRANIČENJA TJ. REŠENJE JE UNUTAR DOZVOLJENE OBLASTI.

2° $\lambda_k \neq 0 \wedge$ $x_{n+k} = 0$, $g_k = 0 \rightarrow$ REŠENJE JE NA GRANICI

3° $\lambda_k = 0 \wedge$ $x_{n+k} = 0$, $g_k = 0 \rightarrow$ REŠENJE JE NA GRANICI

$$\lambda_k x_{n+k} = 0 \iff \lambda_k g_k = 0$$

OVAJ IZRAZ ZAMENIMO SA $\lambda_k g_k = 0$ DA NE BISMO POVEĆAVALI DIMENZIONALNOST.

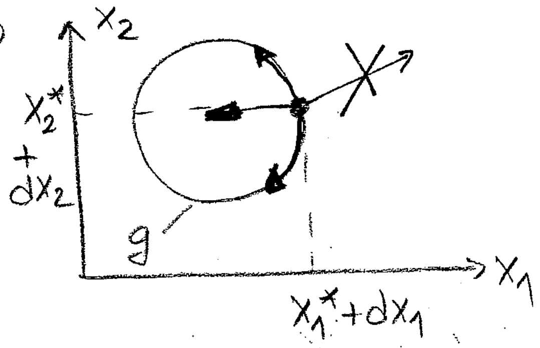
$$\left.\begin{array}{l}\lambda > 0 \to MIN \\ \lambda < 0 \to MAX\end{array}\right\} ?$$

DOKAZ: $y = y(x_1, .., x_n)$, crtamo za 2 promenljive (*)

$g(x_1, \dots, x_n) \le 0$, PP samo jedno ograničenje

PP: u tački ekstrema X^*, $\lambda > 0 \Rightarrow x_{n+1} = 0 \Rightarrow g_k = 0$

(*)



$\boxed{F = y + \lambda g}$ NAPOMENA $(g + \cancel{x_{n+1}^2}^{\,0})$

POTREBAN USLOV:

$$\frac{\partial F}{\partial x_i} = \frac{\partial y}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0 \quad , i = 1, \dots, n$$

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} + \lambda \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} = 0$$

$$\sum_{i=1}^{n} \frac{\partial y}{\partial x_i} = -\lambda \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \quad / \cdot dx_i$$

$$\sum_{i=1}^{n} \frac{\partial y}{\partial x_i} dx_i = -\lambda \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i$$

$$dy = -\lambda dg$$

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_n} dx_n$$

$$\boxed{\begin{array}{c} dy = -\lambda dg > 0 \ MIN \\ \lambda > 0 \quad dg < 0 \end{array}}$$

$g(x_1^*, x_2^*) = 0$

$g(x_1^* + dx_1, x_2^* + dx_2) \Rightarrow dg < 0$

POMERANJE KA UNUTRA:

$dg \downarrow$, $\lambda > 0 \Rightarrow dy > 0 \Rightarrow MIN$

(*) IZ TAČKE X^* $dg \le 0$, POŠTO MOŽE DA OSTANE NA GRANICI

$\Rightarrow dy \ge 0$ (NEOPADAJUĆA F-JA) ALI TO JE I DALJE USLOV F-JE ZA MIN.

* ALGORITAM ZA REŠAVANJE KUN TAKERA.

$$y = y(x_1, \dots, x_n) \text{ – KRITERIJUM OPTIMALNOSTI}$$

$$g_1(x_1, \dots, x_n) \leq 0$$

$$\vdots$$

$$g_m(x_1, \dots, x_n) \leq 0$$

SAMO KOD KUN TAKERA

1) FORMIRAMO PROŠIRENI KRITERIJUM OPTIMALNOSTI **BEZ** UVOĐENJA NOVE PROMENLJIVE:

$$F = y(x_1, \dots, x_n) + \sum_{k=1}^{m} \lambda_k g_k(x_1, \dots, x_n) \quad , k = 1, \dots, m$$

2) $$\frac{\partial F}{\partial x_i} = \frac{\partial y}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_i} = 0 \quad , i = 1, \dots, n$$

3) NE IZVODIMO (TO JE ZAMENA ZA $\lambda_k x_{n+k} = 0 \equiv \lambda_k g_k = 0$)

$$\lambda_k g_k = 0 \longrightarrow \text{SAMO SE NAPIŠE.}$$

4) REŠITI SISTEM 2)-3) I PROVERITI OGRANIČENJA.

5) DISKUTOVATI ZNAK λ_k:

AKO JE: $\lambda_k \geq 0 \Rightarrow \text{MIN}$ (MORA BITI BAR JEDNO $\lambda > 0$ DA BI BIO MIN)

$\lambda_k \leq 0 \Rightarrow \text{MAX}$

ZADATAK TEŽINE ISPITNOG: METODOM KUN TAKERA REŠITI ZADATAK:

$$y = x_1^3 + 2x_2^2 - 10x_1 + 6 + 2x_2^3 \qquad n = 2, \; m = 3$$

g_1: $x_1 x_2 \leq 10$

g_2: $x_1 \geq 0$

g_3: $x_2 \geq 0$

BEZ OBZIRA ŠTO SU OVA OGRANIČENJA JEDNOSTAVNA, MORAMO DA IH UVRSTIMO ZBOG λ DA GA NE BISMO IZGUBILI. λ ODREĐUJE KARAKTER SISTEMA.

$$x_1 x_2 - 10 \leq 0$$

$$-x_1 \leq 0$$

$$-x_2 \leq 0$$

1) $F = y + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3$! BEZ DODATNE PROMENLJIVE

$$F = x_1^3 + 2x_2^2 - 10x_1 + 6 + 2x_2^3 + \lambda_1(x_1 x_2 - 10) + \lambda_2(-x_1) + \lambda_3(-x_2)$$

2) $\dfrac{\partial F}{\partial x_1} = 3x_1 - 10 + \lambda_1 x_2 - \lambda_2 = 0$

$\dfrac{\partial F}{\partial x_2} = 4x_2 + 6x_2^2 + \lambda_1 x_1 - \lambda_3 = 0$

3) $\lambda_k g_k = 0$ (USLOVI KOJE NE IZVODIMO)!

$$\left.\begin{array}{l} (1)\ \lambda_1 (x_1 x_2 - 10) = 0 \\ (2)\ \lambda_2 (-x_1) = 0 \\ (3)\ \lambda_3 (-x_2) = 0 \end{array}\right\} \text{NAJBITNIJE!}$$

1° (1) $\Rightarrow \lambda_1 \neq 0$

$x_1 x_2 = 10 \;\; \begin{array}{l} \nearrow x_1 \neq 0 \\ \searrow x_2 \neq 0 \end{array} \;\; \Rightarrow \lambda_2 = 0 \wedge \lambda_3 = 0$

$$\left.\begin{array}{r} 3x_1^2 - 10 + \lambda_1 x_2 = 0 \\ 4x_2 + 6x_2^2 + \lambda_1 x_1 = 0 \\ x_1 x_2 = 10 \end{array}\right\} \text{Reši sistem}$$

2° $\lambda_1 = 0$ ($\Rightarrow x_1 x_2 - 10 = 0$)

(*) $3x_1^2 - 10 - \lambda_2 = 0$ $\qquad \lambda_2 (-x_1) = 0 \Rightarrow \boxed{\lambda_2 = 0}$

$4x_2 + 6x_2^2 - \lambda_3 = 0$ $\qquad \lambda_3 (-x_2) = 0$

$4 \cdot 0 + 6 \cdot 0 - \lambda_3 = 0$ $\qquad x_2 = 0$

$\boxed{\lambda_3 = 0}$

(*) $3x_1^2 - 10 - \lambda_2 = 0$

Ako je $\lambda_2 = 0 \Rightarrow x_1 = \pm\sqrt{\frac{10}{3}}$

Ako je $x_1 = 0 \Rightarrow \lambda_2 = -10 < 0$, uslov za MAX.

PRIMER: METODOM KUN TAKERA REŠITI PROBLEM:

$$y = y(x_1, \ldots, x_n)$$
$$g(x_1, \ldots, x_n) \leq 0$$
$$h(x_1, \ldots, x_n) = 0$$

LANGRANŽEV MNOŽITELJ ZA JEDNAKOST.

$$F = y + \lambda g + \nu h$$

$$\frac{\partial F}{\partial x_i} = \frac{\partial y}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} + \nu \frac{\partial h}{\partial x_i} = 0$$

$\lambda g = 0$ → ONO ŠTO SE NE IZVODI
[OVO JE POSLEDICA ALGORITMA ZAMENA ZA $\lambda_k g_k = 0$]

! $\boxed{\frac{\partial F}{\partial \nu} = 0 = h = 0}$ $\boxed{\nu h \cancel{=} 0 \text{ POGREŠNO}}$

$\boxed{\nu > 0 \text{ MIN — OVO NE POSTOJI POŠTO JE TIP} =}$

! PITANJA NA SLAJDOVIMA:

7. LINEARNO PROGRAMIRANJE, GRAFIČKI METOD
8. LINEARNO PROGRAMIRANJE, PRINCIPI SIMPLEX METODE
9. LINEARNO PROGRAMIRANJE, SIMPLEX METOD
10. PRINCIPI MREŽNOG PROGRAMIRANJA
11. TRANSPORTNI PROBLEM
12. KVADRATNO PROGRAMIRANJE

13.-18. JEDNODIMENZIONE NUMERIČKE METODE

6.11.2014.

$y = y(x)$

- NUMERIČKE METODE - KORISTIMO ZBOG PROBLEMA KOJI NE MOŽEMO ANALITIČKI REŠITI. KORISTIMO ZA F-JE KOJE IMAJU PREKIDE.

* 3 TIPA 1D NUMERIČKIH METODA:

1) GRADIJENTNE METODE → NJUTN-RAPSON, → SEČICA

$y'(x) = 0$

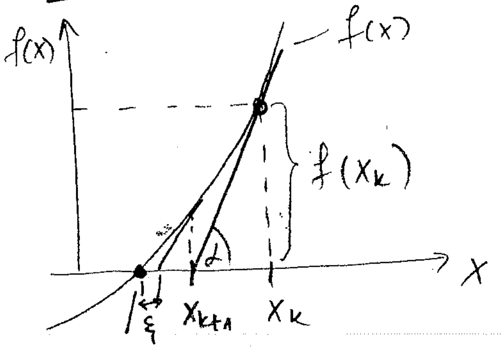
2) METODE PRETRAGE → FIBONAČI, → ZLATNI PRESEK

- SUŽAVAMO INTERVAL DOK NE BUDE IZVESNO DA JE TO MIN F-JE. NE MORA BITI DIFERENCIJABILNA.

3) METODE APROKSIMACIJE POLINOMA → METOD PARABOLE, → KUBNI METOD

- OPTIMIZACIJA POLINOMA II i III REDA

1.a) NJUTN - RAPSONOV METOD - SLUŽI ZA TRAŽENJE $f'(x) = 0$



$f(x) = 0$ (NULA F-JE)

- GEOMETRIJSKA INTERPRETACIJA

$$f'(x_k) = \overbrace{\frac{f(x_k)}{x_k - x_{k+1}}}^{tg\alpha}$$

$$\Rightarrow \boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

$$f'(x_k) = 0$$

$$\boxed{x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}}$$

NULE PRVOG IZVODA

- RAZVOJ U RED U OKOLINI TAČKE x_k.

$$f(x) \approx f(x_k) + (x - x_k) f'(x_k) + \dots$$

$$g(x) = f(x_k) + (x - x_k) f'(x_k) = 0$$

- TRAŽIM NULE F-JE $g(x) = 0$

$$x = x_k - \frac{f(x_k)}{f'(x_k)}$$

- NULE I IZVODA: $g'(x) = 0$

$$x = x_k - \frac{f'(x_k)}{f''(x_k)}$$

MATLAB:

$$\underbrace{[x, fx, n]}_{n \to \text{BROJ ITERACIJA}} = \text{newton}(\underset{f(x)}{\text{fun}}, \underset{f'(x)}{\text{dfun}}, \underset{f''(x)}{\text{d2fun}}, \underset{x_0}{a}, \underset{\xi}{\text{tol}})$$

- F-JA MORA BITI DIFERENCIJABILNA DO II REDA
- INICIJALIZACIJA PROBLEMA = POČETNO POGAĐANJE
- RELATIVNA GREŠKA: $\left| \frac{x_{k+1} - x_k}{x_{k+1}} \right| < \xi$ → RELATIVNI METOD - AKO ZNAM DA KRAJNJA TAČKA NIJE (
- APSOLUTNA GREŠKA: $\underbrace{|x_{k+1} - x_k| < \xi}_{\text{KRITERIJUM ZAUSTAVLJANJA}}$ → AKO JE KONAČNO REŠENJE 0 IMAMO PROBLEM - ZATO SE KORISTI APSOLUTNI METOD

! ZA MAX: $\min(f) = \max(-f)$

- N-R METOD JE IZUZETNO OSETLJIV NA POČETNO POGAĐANJE.
  AKO JE IZABRANO DOBRO POGAĐANJE → KONVERGIRA
  AKO NIJE IZABRANO DOBRO POGAĐANJE → DIVERGIRA

1. b) METOD SEČICE

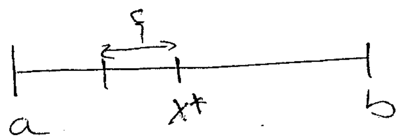
APROKSIMACIJE IZ DUS-A

$$f''(x_k) \approx \frac{f'(x_k) - f'(x_{k-1})}{x_{k-1} - x_k} \quad \Bigg| \quad \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$

– KOMPENZACIJA: MORA DA SE POZNAJE PRVI IZVOD U 2 TAČKE.

8.11.2017. – NASTAVAK

2) METODE DIREKTNOG PRETRAŽIVANJA



→ SUŽAVAMO INTERVAL U OKOLINI MIN

2.a) FIBONAČIJEV METOD

IZVOĐENJE:

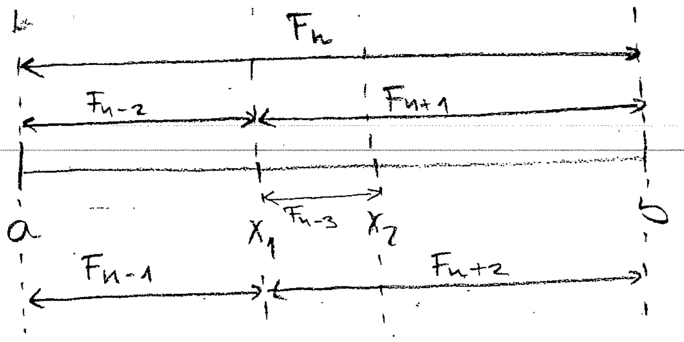
1. GENERIŠEMO FIBONAČIJEVE BROJEVE

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	......
1	1	2	3	5	8	13	21	

– INTERVAL $[a,b]$ JE INTERVAL U KOM SE NALAZI UNIMODALNA F-JA

– PP: TRAŽIM MIN

– $[a,b]$, L_0 – DUŽINA INTERVALA



– SMATRAMO DA CEO INTERVAL ODGOVARA FIBONAČIJEVIM BROJEVIMA

$$F_n = F_{n-1} + F_{n-2}$$

$$F_{n-1} = F_{n-2} + F_{n-3}$$

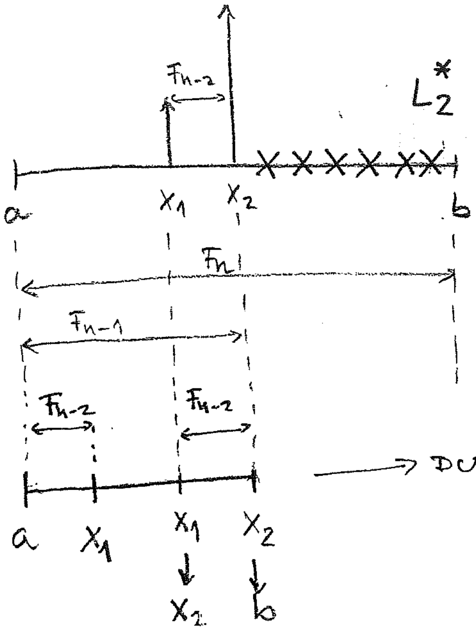
$$x_1 = a + \frac{F_{n-2}}{F_n} \cdot L_0$$

$a - x_1$

$$x_2 = a + \frac{F_{n-1}}{F_n} \cdot L_0 = b - \frac{F_{n-2}}{F_n} \cdot L_0 = a + b - x_1$$

2. IZRAČUNATI VREDNOST F-JE i POREDIMO IH
$f(x_1)$ i $f(x_2)$

PP:

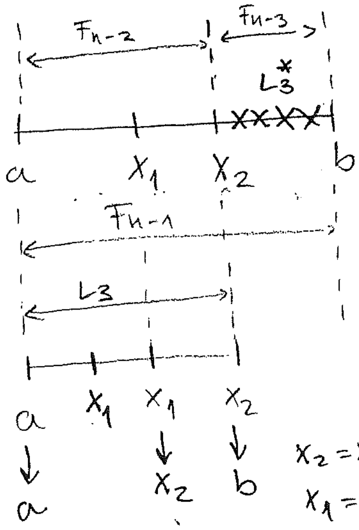


$L_2^* = \frac{F_{n-2}}{F_n} \cdot L_0$ – OBLAST KOJU ELIMINIŠEMO.

⟶ DUŽINA INTERVALA $L_2 = L_0 - L_2^* = \frac{F_{n-1}}{F_n} \cdot L_0$

$$x_{1\_NOVA\_TAČKA} = a + \frac{F_{n-3}}{F_n} \cdot L_0 = a + \frac{F_{n-3}}{F_{n-1}} L_2$$

3. $f(x_1)$ i $f(x_2)$ IZRAČUNATI



KADA SE SREDI IZ PREDHODNOG KORAKA, ELIMINIŠEM OBLAST L_3^*

$$L_3^* = \frac{F_{n-3}}{F_{n-1}} \cdot L_2 = \frac{F_{n-3}}{F_n} \cdot L_0$$

$$L_3 = L_2 - L_3^* = \frac{F_{n-2}}{F_{n-1}} \cdot L_2 = \frac{F_{n-2}}{F_n} \cdot L_0$$

$x_2 = x_1$
$x_1 = a + b - x_1$

U OPŠTIM BROJEVIMA:

$$L_k^* = \frac{F_{n-k}}{F_{n-(k-2)}} \cdot L_{k-1}$$

$$L_k = \frac{F_{n-(k-1)}}{F_n} \cdot L_0$$

* AKO JE $k=n \Rightarrow L_n = \frac{F_1}{F_n} \cdot L_0 = \frac{1}{F_n} \cdot (b-a)$

⟶ DUŽINA INTERVALA POSLE n ITERACIJA

— ŽELIM POSLE N ITERACIJA DA DUŽINA INTERVALA BUDE NAJVIŠE 0.1 U KOME JE MIN?

$b = 1$

$a = 0$

$F_n > \frac{1-0}{0.1 = L_n} = 10$; POŠTO SE 10 NALAZI IZMEĐU F_6 i F_7 ONDA JE $\underline{\underline{n=7}}$ (TOLIKO ITERACIJA TREBA)

$$\frac{F_{n-2}}{F_n} = \frac{F_{n-1}}{F_n} \cdot \frac{F_{n-2}}{F_n} \Big|_{n\to\infty} = \frac{1}{\varphi^2}$$

φ - FIBONAČIJEV BROJ

$$\frac{F_{n-3}}{F_n} = \frac{1}{\varphi^3}$$

za $k = n$ $\quad \frac{F_1}{F_n} = \frac{1}{\varphi^n}$

---

3) METOD APROKSIMACIJE POLINOMA

3.a) METOD PARABOLE

- AKO NE MOGU DA ODREDIM MIN/MAX F-JE APROKSIMIRAM POLINOMOM 2. REDA → METOD PARABOLE

$$ax^2 + bx + c \to 3 \text{ TAČKE}$$

* F-JA MORA DA BUDE UNIMODALNA

3.b) KUBNI METOD

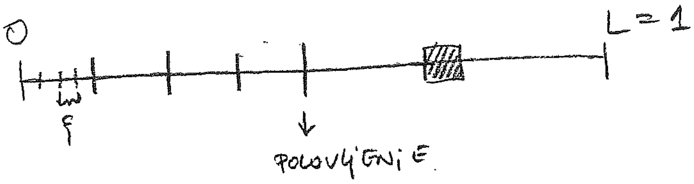
- APROKSIMACIJA POLINOMOM 3. REDA

$$ax^3 + bx^2 + cx + d \to 4 \text{ TAČKE}$$

* FJA MORA DA BUDE UNIMODALNA

13.11.2017.

Rapaja ♡



ZADATO: ξ = TAČNOST

PITANJA KOJA SMETE DA POSTAVITE: DA LI JE TO TAČKA?

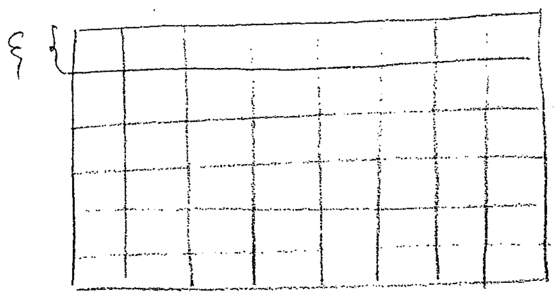
- OČEKIVANI BROJ PITANJA KOJA MORAMO POSTAVITI $= \frac{1}{2\xi} = N$

PRIMER: $\xi = 10^{-6}$

$N = 0.5 \cdot 10^{6}$

VREME POTREBNO DA ODGOVORIMO NA JEDNO PITANJE $= t_1 = 1ms$

UKUPNO VREME $= N \cdot t_1 = 0.5 \cdot 10^6 \cdot 1ms = 500s = t$



$N = \frac{1}{2}\left(\frac{1}{\xi}\right)^2$

$N = \frac{1}{2} \cdot 10^{12}$

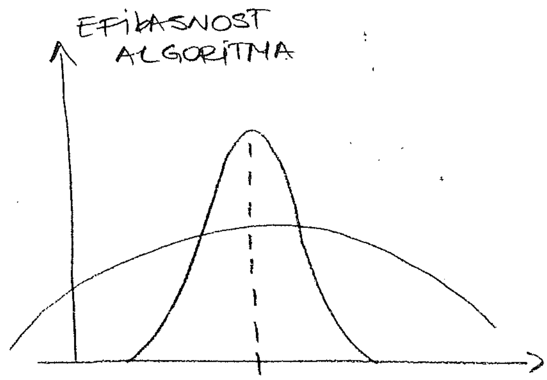
$t = N \cdot t_1 = \frac{1}{2} \cdot 10^9$

- U OPŠTEM SLUČAJU: $t = \frac{1}{2}\left(\frac{1}{\xi}\right)^d \cdot t_1$, d = broj dimenzija

PRIMER: $d = 100$

$t = \frac{1}{2} \cdot 10^{600} \cdot 1ms$

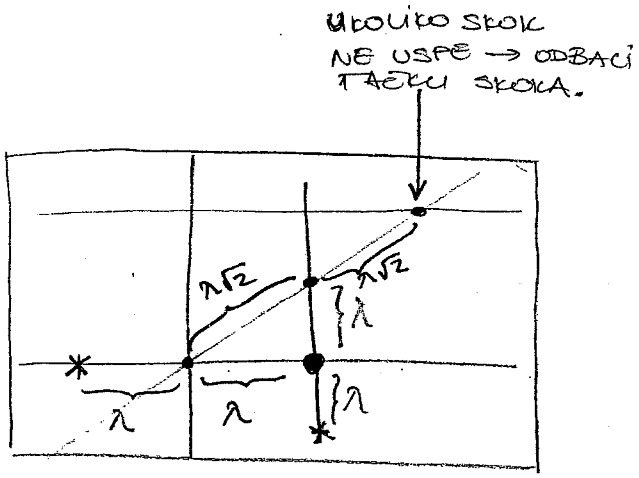
$\log_2 \frac{1}{\xi}$



* DA LI JE $f(x)$ MANJE OD $f(y)$?

* $2^d - 1$ - OVAKO RASTE BROJ PITANJA KADA IMAMO POLOVLJENJE U 2 DIMENZIJE

18. HUK-DŽIVSOV METOD
(HOOK-JEVES)



* $\lambda\sqrt{n}$, n - BROJ DIMENZIJA ODNOSNO PO KOLIKO DIMENZIJA SE PROMENI TAČKA

0. INICIJALIZACIJA

- POČETNA TAČKA x_0
- POČETNA DUŽINA KORAKA λ

1. PRETRAGA U OKOLINI TEKUĆE TAČKE

- PRETRAGA LEVO-DESNO
- UKOLIKO NE USPE - SMANJI KORAK

2. SKOK

3. PRETRAGA „KRNJA" U OKOLINI SKOKA

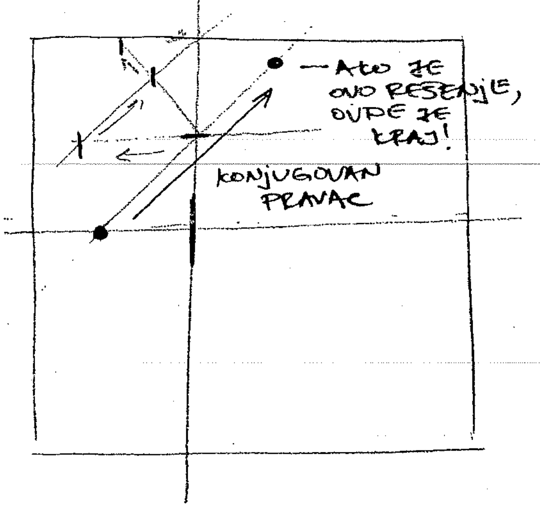
(SAMO DO PRVOG NAPREDKA, NIJE POTPUNA PRETRAGA)

KRITERIJUM ZAUSTAVLJANJA: $\lambda < \varepsilon$

- PREDNOSTI: JEDNOSTAVNA JE.
- MANE: PRAVCI PRETRAGE SU KOORDINATNE OSE I MEĐUUGLOVI, A TREBA NAM ALGORITAM KOJI PRETRAŽUJE PO SVIM PRAVCIMA.

PAUELOV METOD (POWELL)

- META METOD - MEHANIZAM ZA GENERISANJE KODA



- PROBLEM - LINEARNI PRAVCI
- ZGODNO JE NAKON n ITERACIJA RESTARTOVATI PRAVCE.

0. INICIJALIZACIJA

- POČETNA TAČKA x_0
- ZADAJE SE 1D OPTIMIZACIONA METODA

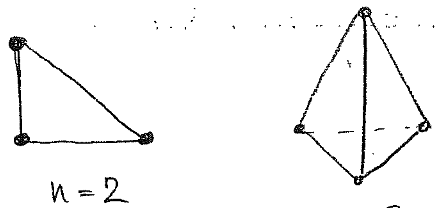
1. 1D OPTIMIZACIJA PO KOORDINATNIM OSAMA

2. 1D OPTIMIZACIJA PO KONJUGOVANOM PRAVCU

3. ZAMENI JEDAN POLAZNI PRAVAC KONJUGOVANIM

NELDER-MID-OV ALGORITAM (NELDER-MEAD)

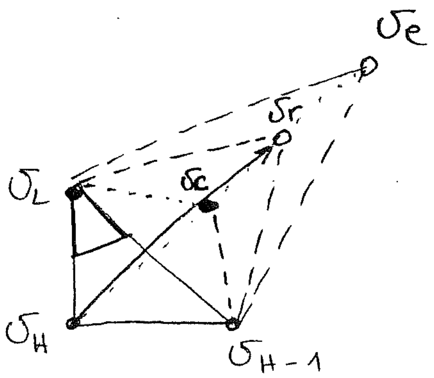
(SIMPLEX SEARCH)



n=2   n=3

⊛ U PROSTORU OD n DIMENZIJA TELO KOJE ČINI n-LINEARNO NEZAVISNIH VEKTORA NAZIVA SE SIMPLEX.

ZA n=2



σ_L - NAJBOLJA TAČKA

σ_H - NAJGORA TAČKA

σ_{H-1} - DRUGA NAJGORA TAČKA „DO-NAJGORA" TAČKA

σ_r - REFLEKTOVANA TAČKA

σ_e - TAČKA NAKON EKSPANZIJE (AKO JE σ_r BOLJA OD NAJBOLJE)

σ_c - KONTRAKCIJA

⊛ REFLEKSIJA SE SMATRA USPELOM AKO JE BOLJA OD DO-NAJGORE

⊛ SKUPLJANJE SIMPLEXA – BACIM SIMPLEX, OSTAVIM SAMO NAJBOLJU TAČKU I SE OSE SKRATIM ZA POLA.

21. GENETSKI ALGORITAM

15. 11. 2017.

Rapaja

0° INICIJALIZACIJA

1° UKRŠTANJE I MUTACIJA – IZBOR PAROVA, RODITELJA I GENERISANJE POTOMAKA (PODRAZUMEVA SE I MUTACIJA)

2° SELEKCIJA – IZBOR SLEDEĆE GENERACIJE

3° TEST KRAJA

TERMINOLOGIJA:

- POPULACIJA – SKUP SVIH REŠENJA U TEKUĆOJ ITERACIJI
- GENERACIJA – ITERACIJA
- JEDINKA – REŠENJE
- PRILAGOĐENOST – KRITERIJUM OPTIMALNOSTI
- GEN – DEO REŠENJA NAD KOJIM SE NEPOSREDNO VRŠI MUTACIJA
- KODIRANJE – REPREZENTACIJA REŠENJA

BINARNO

0	1	1	0	0	0	1	1

GEN

– REŠENJE U OBLIKU BINARNOG BROJA

REALNO – U OBLIKU REALNOG BROJA

$(\underbrace{12.6}_{\text{GEN}}, 18, 32, 8.01)$

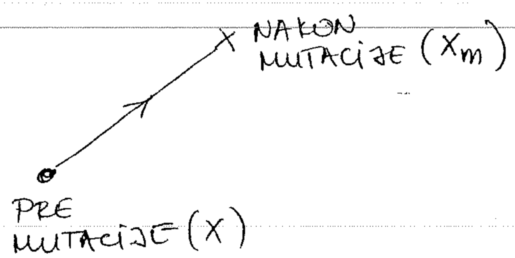
IMPLEMENTACIJA MUTACIJE

BINARNO KODIRANA REŠENJA

0	1	0	0	1	0	0	0
↓	↓	↓	↓	X	↓	↓	↓
0	1	0	0	0	0	0	0

– VEROVATNOĆA MUTACIJE

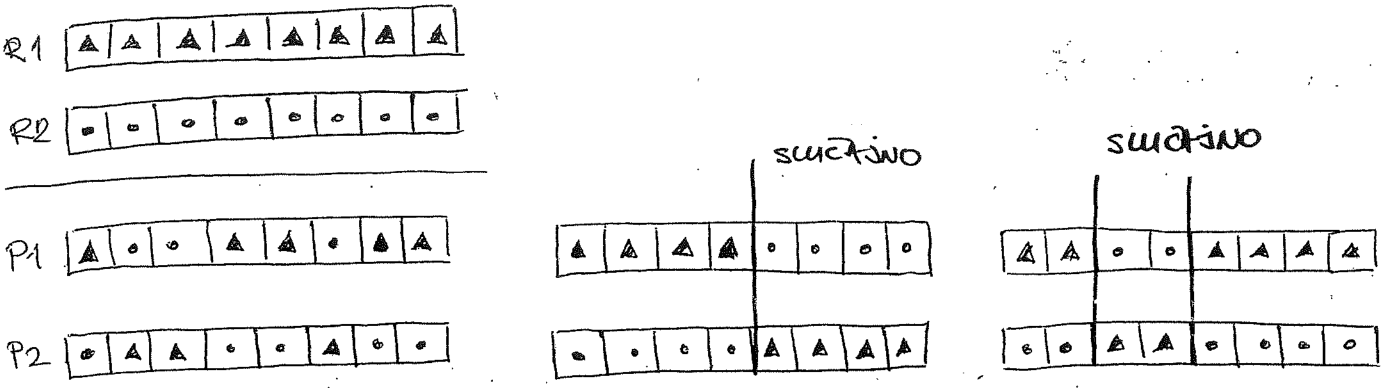
REALNO KODIRANJE



$$X_m = X + r * rand(size(x))$$

IMPLEMENTACIJA ...

BINARNO KODIRANJE

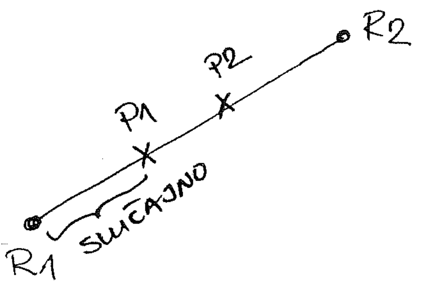


– POTPUNO SLUČAJNO

– UKRŠTANJE U JEDNOJ TAČKI

– UKRŠTANJE U 2 TAČKE

REALNO KODIRANJE



IMPLEMENTACIJA SELEKCIJE RODITELJA

1° VEROVATNOĆA IZBORA $\sim f(x)$ AKO TRAŽIM MAX
$\frac{1}{f(x)}$ AKO TRAŽIM MIN } RULETSKA SELEKCIJA

↓ DOVODI DO POJAVE „SUPER REŠENJA"

2° VEROVATNOĆA IZBORA $\sim$ RANG REŠENJA (REDNI BROJ REŠENJA KADA SE POREĐAJU OD NAJGOREG KA NAJBOLJEM) – RULETSKA SELEKCIJA SA RANGOM

SELEKCIJA NOVE GENERACIJE

– ELITIZAM – DOPUŠTAMO NEKOM BROJU RODITELJA DA PREŽIVI AKO SU BOLJI OD POTOMAKA

ga – F-JA U MATLABU

20.11.2017.

20. POSTUPCI INDIREKTNE PRETRAGE
(GRADIJENT, NJUTN, ...)

0. INICIJALIZACIJA
   - IZBOR POČETNOG REŠENJA
   - PODEŠAVANJE META PARAMETARA

1. KORAK

$$X_{k+1} = X_k + d_k$$

d_k → KORAK - MENJA SE OD ALGORITMA DO ALGORITMA

X_{k+1} ↓ NOVO POGAĐANJE

X_k ↓ STARO POGAĐANJE

2. KRITERIJUM ZAUSTAVLJANJA

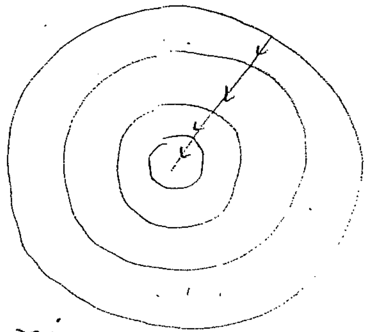
$$|d_k| < \xi$$

* GRADIJENTNI POSTUPAK

$$\nabla y = \left[\frac{\partial y}{\partial x_1} \quad \frac{\partial y}{\partial x_2} \quad \cdots \quad \frac{\partial y}{\partial x_n} \right]^T$$

- GRADIJENT DEFINIŠE PRAVAC U KOM F-JA NAJBRŽE RASTE

$\nabla y \equiv$ PRAVAC NAJBRŽEG RASTA

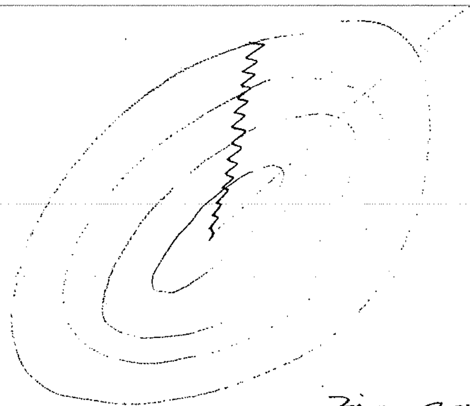


* GRADIJENTNI POSTUPAK:

$$d_k = -\mu \nabla y(x_k)$$

μ – „DUŽINA KORAKA" („KORAK")

$X_{k,i} \equiv$ i-TA KOMPONENTA U k-TOJ ITERACIJI



Zig-zag

* NJUTNOV POSTUPAK

(SKUPO)

$$y(x_{k+1}) \approx y(x_k) + \sum_{k=1}^{n} \frac{\partial y}{\partial x_{k,i}} \cdot d_k + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 y}{\partial x_{k,i} \partial x_{k,j}} d_{k,i} \cdot d_{k,j}$$

- $d_k = x_{k+1} - x_k \equiv$ POMERAJ U ~~KAT~~ K-ATOJ ITERACIJI

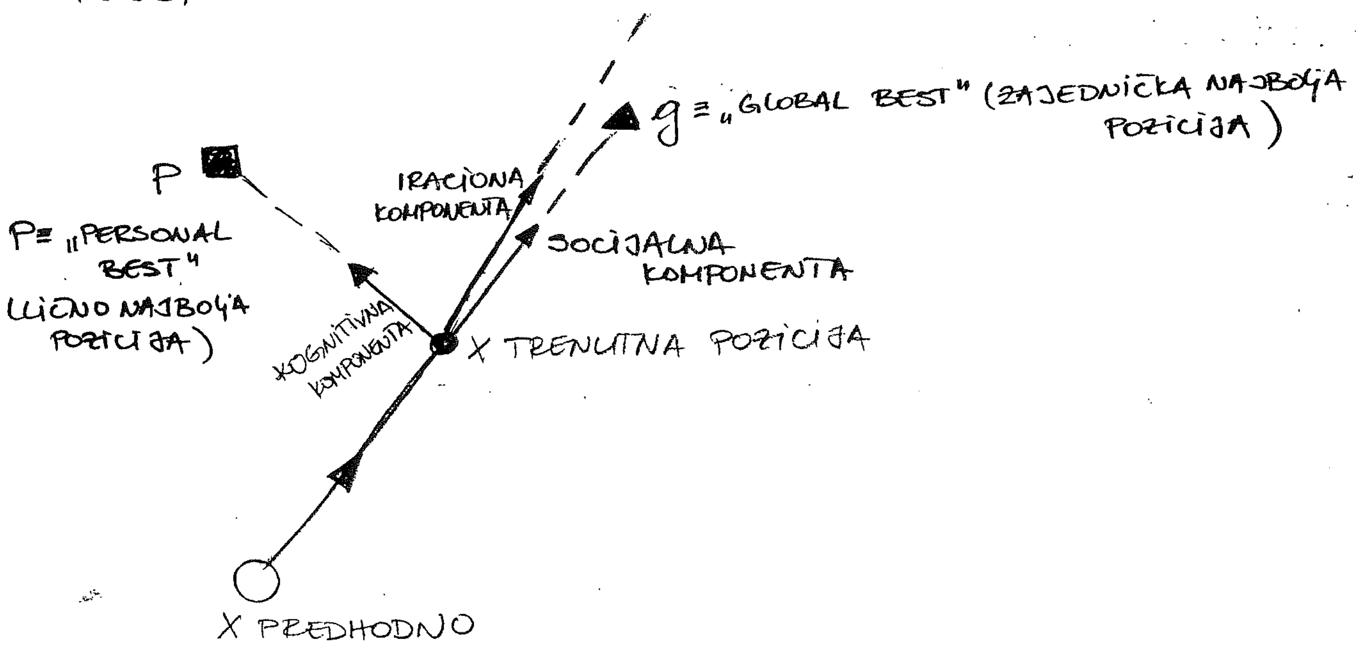
$d_{k,i} \equiv$ i-TA KOMPONENTA OD d_k

$$\boxed{y(x_{k+1}) = y(x_k) + \nabla y^T(x_k) \cdot d_k + \frac{1}{2} d_k^T \nabla^2 y(x_k) d_k}$$

NJUTNOV METOD: $d_k = -[\nabla^2 y(x_k)]^{-1} \cdot \nabla y(x_k)$

22. PARTICLE SWARM OPTIMIZATION (PSO)

- OPTIMIZACIJA BROJEM ČESTICA
- 1995.



- PAMĆENJE POJEDINAČNE ČESTICE: - P
  - PREDHODNA BRZINA (VEKTOR)

$$\boxed{\begin{aligned} v_{k+1} &= v_k + c_p(p - x_k) + c_g(g - x_k) \\ x_{k+1} &= x_k + v_{k+1} \end{aligned}}$$

ORIGINALNA VARIJANTA ALGORITMA

MODERNIJA VARIJANTA ALGORITMA 1:

$$v_{k+1} = v_k + c_p r_p (p - x_k) + c_g \cdot r_g (g - x_k)$$

$r_p, r_g \in U(0,1)$ – SLUČAJNI BROJEVI KOJI SE BIRAJU UNIFORMNO IZ $[0,1]$.

MODERNA VARIJANTA ALGORITMA 2: (STVARNA VARIJANTA)

$$v_{k+1} = \omega \cdot v_k + c_p r_p (p - x_k) + c_g r_g (g - x_k)$$

$\omega \in (0,1)$ – FAKTOR INERCIJE

c_p, c_g – FAKTORI UBRZANJA

- BIRANJE PARAMETARA:
  - $\omega: 0.9 \to 0.4$
  - $c_p: 2.5 \to 0.5$
  - $c_g: 0.5 \to 2.5$

0. INICIJALIZACIJA – SVAKA ČESTICA SE NA SLUČAJAN NAČIN POSTAVI UNUTAR OBLASTI PRETRAGE

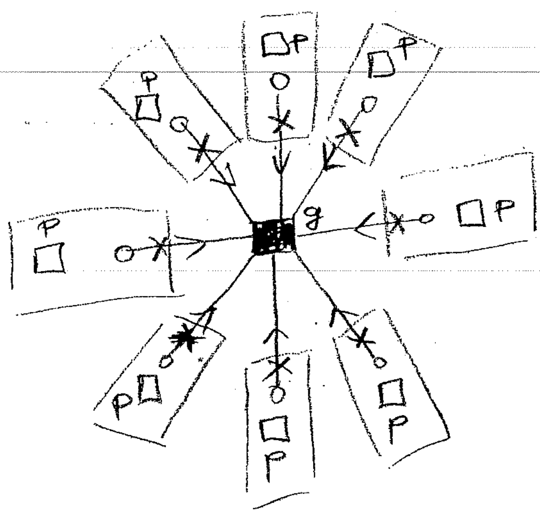
1. $v_{k+1} = \dots$

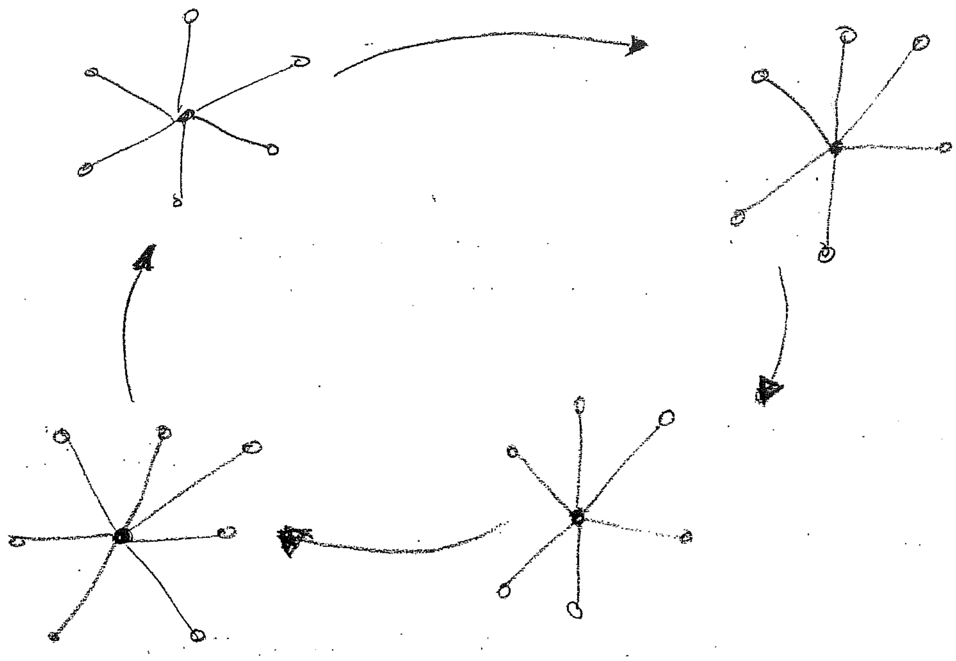
   $x_{k+1} = \dots$

2. SRAČUNATE NOVO p i g → NAJGORA TAČKA

   ZA SVAKU ČESTICU

TOPOLOGIJA ROJA – ZVEZDASTA TOPOLOGIJA (STAR TOPOLOGY)





23. ANT COLONY OPTIMIZATION (ACO) → NIJE ISPITNO PITANJE, IZBAČENO JE!

II KOLOKVIJUM 27.11.2014.

DINAMIČKA OPTIMIZACIJA

– PROMENA TOKOM VREMENA

$X = X(t)$ – REŠENJE VIŠE NIJE OPTIMALNA TAČKA, SAD JE REŠENJE OPTIMALNA F-JA

25. VARIJACIJA F-JE I NJENA SVOJSTVA ! JAKO ČESTO PITANJE.

• OSNOVNI ZADATAK KOJI POKUŠAVAMO DA REŠIMO:

$$I = \int_a^b F(t, x, \dot{x})\,dt \longrightarrow \text{KRITERIJUM OPTIMALNOSTI}$$

F – FUNKCIONAL – F-JA ČIJI SU ARGUMENTI F-JE

t – VREME, NEZAVISNA PROMENLJIVA

$x = x(t)$

$\dot{x} = \dot{x}(t) = \dfrac{dx}{dt}$

PP: $\left.\begin{array}{l} x(a) = x_a \\ x(b) = x_b \end{array}\right\}$ POZNATE VREDNOSTI NA POČETKU I NA KRAJU INTERVALA

OSNOVNI ZADATAK JE CILJ DA IZRAČUNAMO $x(t)$ KOJA PRIPADA ODREĐENOJ KLASI (ZNAČI DA ZADOVOLJAVA POČETNE I GRANIČNE USLOVE), A ODREĐENOM INTEGRALU I SAOPŠTAVA MIN ODNOSNO MAX VREDNOST.

$X = X(t)$ – EKSTREMALA

(*) OSNOVNI MATEMATIČKI APARAT: VARIJACIJA F-JE δx.

– VARIJACIJA F-JE JE JAKO MALA PROMENA F-JE I NJENIH IZVODA BEZ PROMENE NEZAVISNE PROMENLJIVE t.

$$\boxed{\delta x \overset{def}{=} \bar{x}(t) - x(t) = \xi \cdot \phi(t)}$$ VARIJACIJA F-JE

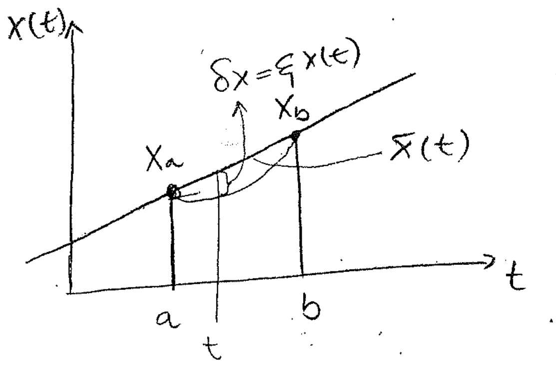
$\bar{x}(t)$ - VARIJACIJA - IZMENJENA F-JA (VARIKA NA ... OPTIMALNOG REŠENJA)

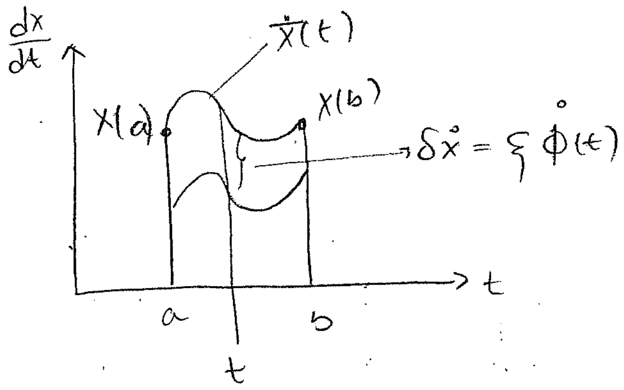
$x(t)$ - EKSTREMALA - NAJBOLJE OPTIMALNO REŠENJE

ξ - NAGLAŠAVA MALU RAZLIKU

$\phi(t)$ - UKAZUJE NA PROIZVOLJNOST

$$\boxed{\delta\dot{x} = \dot{\bar{x}}(t) - \dot{x}(t) = \xi\dot{\phi}(t)}$$ VARIJACIJA PRVOG IZVODA





- AKO JE POZICIJA 0, NE ZNAČI DA JE I BRZINA 0.

OSOBINE VARIJACIJE:

1. $\delta t \overset{def}{=} 0$ (VARIJACIJA PO VREMENU)

2. $$\delta\left(\frac{dx}{dt}\right) = \frac{d\bar{x}}{dt} - \frac{dx}{dt} \underset{\downarrow}{=} \frac{d(\bar{x}-x)}{dt} = \frac{d}{dt}\delta x$$

IZVOD LINEARNI OPERATOR

$\underbrace{d\delta = \delta d}$ KOMUTATIVNOST!

3. $$\delta\int_a^b x(t)\,dt = \int_a^b \bar{x}(t)\,dt - \int_a^b x(t)\,dt = \int_a^b (\bar{x}-x)\,dt = \int_a^b \delta x\,dt$$

$\delta\int = \int\delta$ → I OVDE VAŽI KOMUTATIVNOST!

26. POTREBNI I DOVOLJNI USLOVI EKSTREMA, ! JAKO VAŽNO!
OJLER-LAGRANŽEVE JEDNAČINE.

PODSEĆANJE ①,②:

$$y(x)-y(x^*) \overset{>0}{\approx} \dots$$

$$y(x) \approx y(x^*)+y'(x^*)(x-x^*)+\dots$$

$$I=\int_a^b F(t,x,\dot{x})\,dt\,, \quad \left.\begin{matrix} x(a)=x_a \\ x(b)=x_b \end{matrix}\right\} \text{POZNATO!}$$

– APARAT KOJI KORISTIMO: $\delta\dot{x}=\dot{\bar{x}}-\dot{x}=\xi\dot{\phi}$

$$y(x)-y(x^*) \qquad I(x)=\int_a^b F(t,x,\dot{x})dt$$

$$\downarrow \qquad \downarrow$$

$$I(\bar{x})-I(x) \qquad I(\bar{x})=\int_a^b F(t,\bar{x},\dot{\bar{x}})dt$$

$I(\bar{x})$ ↓ U OKOLINI OPTIMALNOG REŠENJA

$I(x)$ ↓ OPTIMALNO REŠENJE

RAZVOJ U RED NEOPTIMALNOG REŠENJA $\bar{x}$

$$F(t,\bar{x},\dot{\bar{x}})\approx F(t,x,\dot{x})+\frac{\partial F}{\partial x}(\bar{x}-x)+\frac{\partial F}{\partial \dot{x}}(\dot{\bar{x}}-\dot{x})+\underbrace{\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{x}-x)^2+\frac{\partial^2 F}{\partial x\partial\dot{x}}(\bar{x}-x)(\dot{\bar{x}}-\dot{x})}_{A}$$

$$\underbrace{+\frac{1}{2}\frac{\partial^2 F}{\partial \dot{x}^2}(\dot{\bar{x}}-\dot{x})^2}_{A}$$

U OKOLINI OPTIMALNOG

A ——— ČLANOVI VIŠEG REDA

$$I(\bar{x})-I(x)=\delta I+\delta^2 I$$

$$\delta I=\int_a^b\Big[\frac{\partial F}{\partial x}\underbrace{(\bar{x}-x)}_{\delta x}+\frac{\partial F}{\partial \dot{x}}\underbrace{(\dot{\bar{x}}-\dot{x})}_{\delta\dot{x}}\Big]dt=\int_a^b\Big[\frac{\partial F}{\partial x}\underset{\xi\cdot\phi(t)}{\underset{\parallel}{\delta x}}+\frac{\partial F}{\partial \dot{x}}\underset{\xi\cdot\dot{\phi}(t)}{\underset{\parallel}{\delta\dot{x}}}\Big]dt$$

$$\delta I=\xi\int_a^b\Big[\frac{\partial F}{\partial x}\phi+\frac{\partial F}{\partial \dot{x}}\dot{\phi}\Big]dt$$

$$\delta^2 I = \frac{1}{2}\int_a \Big[\frac{\partial^2 F}{\partial x^2}(\underbrace{\hat{x} - x}_{\delta x = \xi\phi})^2 + 2\frac{\partial^2 F}{\partial x \partial \dot{x}}(\underbrace{\hat{x}-x}_{\delta x = \xi\phi})(\underbrace{\dot{\hat{x}}-\dot{x}}_{\delta \dot{x} = \xi\dot{\phi}}) + \frac{\partial^2 F}{\partial \dot{x}^2}(\underbrace{\dot{\hat{x}}-\dot{x}}_{\delta\dot{x} = \xi\dot{\phi}})^2 \Big] dt$$

$$\delta^2 I = \frac{1}{2}\xi^2 \int_a^b \Big[\frac{\partial^2 F}{\partial x^2}\phi^2 + 2\frac{\partial^2 F}{\partial x \partial \dot{x}}\phi \cdot \dot{\phi} + \frac{\partial^2 F}{\partial \dot{x}^2}\dot{\phi}^2 \Big] dt$$

> POTREBAN USLOV ZA MIN I MAX: $\delta I = 0$
>
> DOVOLJAN USLOV: $\delta^2 I > 0 \Rightarrow$ MIN
>
> $\delta^2 I < 0 \Rightarrow$ MAX

DOKAZ: $\delta I = 0$

$$\delta I = \xi \int_a^b \Big[\frac{\partial F}{\partial x}\phi + \underbrace{\frac{\partial F}{\partial \dot{x}}}_{x}\underbrace{\dot{\phi}}_{\frac{dy}{dt}} \Big] dt = 0$$

$$\left[\frac{d}{dt}(x \cdot y) = \frac{dx}{dt}\cdot y + x\frac{dy}{dt} \right]$$

$\rightarrow$ TOTALNI DIFERENCIJAL PROIZVODA, KORISTIM DA $\dot{\phi} \rightarrow \phi$

$$\frac{\partial F}{\partial \dot{x}}\dot{\phi} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\cdot\phi\right) - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\cdot\phi \longrightarrow \text{UBACUJEM U } \delta I$$

~~$\Rightarrow \delta I = \frac{\partial F}{}$~~

IZADE ISPRED $\int$

$$\Rightarrow \delta I = \frac{\partial F}{\partial \dot{x}}\xi\phi\Big|_a^b + \xi\int_a^b \Big[\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\Big]\phi\, dt = 0$$

$$\delta I = \underbrace{\frac{\partial F}{\partial \dot{x}}\delta x\Big|_a^b}_{(*)} + \int_a^b \Big[\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\Big]\delta x\, dt = 0$$

$$\Rightarrow \boxed{\frac{\partial F}{\partial x} - \frac{d}{dt}\cdot\frac{\partial F}{\partial \dot{x}} = 0}$$

OJLER–LAGRANŽOVA JED.

$\Rightarrow$ AKO REŠIMO O–L DIF. JED. U STANJU DA DOĐEMO DO OPTIMALNOG REŠENJA

* O–L JED. JE DIF. JED. II REDA. POTREBNA 2 USLOVA: $x(a) = x_a$, $x(b) = x_b$ } $x(a) \vee x(b)$ – REŠAVANJE U NUMERICI. —

(*) $\delta x(b) = 0$, $x(b) = x_b$; $\delta x(a) = 0$, $x(a) = x_a$ } NEMA VARIJACIJE, JER SU NAM POZNATE POČETNE I KRAJNJE TAČKE

SVESTI GA ILI NA JEDNO ILI NA DRUGO.

*** DVOTAČKASTI GRANIČNI PROBLEM**

- NJEGOVO REŠAVANJE JE MOGUĆE SAMO U OGRANIČENOM BROJU SLUČAJEVA (VREDNOST POZNATA I NA POČETKU I NA KRAJU!)

SVUDA OSIM U OVOM BITNOM [illegible]

$$I = \int_a^b F(t, x, \dot{x})\, dt \qquad \left(\frac{\partial F}{\partial t} \cancelto{0}{\delta t} \right)$$ JAKO BITNO!

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial x} \delta x + \frac{\partial F}{\partial \dot{x}} \delta \dot{x} \right] dt$$

$$\frac{\partial F}{\partial \dot{x}} \delta \dot{x} = \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \delta x \right) - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} \delta x$$

VRATIM U $\int$ — ISPRED $\int$

$$\Rightarrow \delta I = \frac{\partial F}{\partial \dot{x}} \delta x \Big|_a^b + \int_a^b \left[\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} \right] \delta x \, dt = 0$$

JAKO BITNO!

- Dovoljni uslovi:

$$\delta^2 I > 0 \Rightarrow MIN \longrightarrow \frac{\partial^2 F}{\partial \dot{x}^2} \geq 0$$

$$\delta^2 I < 0 \Rightarrow MAX \longrightarrow \frac{\partial^2 F}{\partial \dot{x}^2} \leq 0$$

LEŽANDROVI USLOVI

27. STRUKTURA O-L JEDNAČINE ZA NEKE SPECIJALNE SLUČAJEVE (LAKO)

$$I = \int_a^b F(t, x, \dot{x})\, dt$$ – KRITERIJUM OPTIMALNOSTI

$x = x(t)$ – EKSTREMALA (REŠENJE KOJE ŽELIMO DA IZRAČUNAMO)

POTREBAN USLOV: $\delta I = 0$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \longrightarrow$$ O-L DIF. JED. II REDA

PP: $x(a) = x_a$, $x(b) = x_b$ } DVOTAČKASTI GRANIČNI PROBLEM – POZNATE VREDNOSTI NA POČETKU I NA KRAJU

* O-L DIF. JED. II REDA $\xrightarrow{?}$ DIF. JED. I REDA

↳ MOŽE SE SPUSTITI RED AKO POSTOJE PRVI INTEGRALI

1) $I = \int_a^b F(t, \dot{x})\, dt$ (AKO F NE ZAVISI EKSPLICITNO OD x)

$$\Rightarrow \cancel{\frac{\partial F}{\partial x}}^{\,0} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \Rightarrow \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \Rightarrow \boxed{\frac{\partial F}{\partial \dot{x}} = const.}$$ DIF. JED. I REDA

(ZAKON ODRŽANJA KRETANJA)

2) $I = \int_a^b F(x, \dot{x})\, dt$ (AKO F NE ZAVISI EKSPLICITNO OD t)

$$(\#)\quad \frac{dF}{dt} = \frac{\partial F}{\partial \dot{x}}\ddot{x} + \frac{\partial F}{\partial x}\dot{x} + \cancel{\frac{\partial F}{\partial t}}^{\,0}$$ (TOTALNI DIFERENCIJAL)

$$\Rightarrow \text{O-L:} \quad \frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \Rightarrow \underbrace{\frac{\partial F}{\partial x} = \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}}_{\text{UBACUJEM U } (\#)}$$

$$\Rightarrow \frac{dF}{dt} = \frac{\partial F}{\partial \dot{x}}\ddot{x} + \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\dot{x}$$

$$\frac{dF}{dt} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\dot{x}\right)$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\dot{x} - F\right) = \cancel{const.}\ 0$$

$$\boxed{\frac{\partial F}{\partial \dot{x}}\dot{x} - F = const.}$$ DIF. JED. I REDA

(ZAKON ODRŽANJA ENERGIJE SISTEMA)

28. UOPŠTENJE, FUNKCIONAL ZAVISI OD VIŠE F-JA (NAJLAKŠE)

28. UOPŠTENJA, FUNKCIONAL ZAVISI OD VIŠE F-JA

$$I = \int_a^b F(t, x_1, x_2, x_3, \dots, x_n, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)\,dt$$

$$I = \int_a^b F(t, x_i, \dot{x}_i)\,dt, \quad i = 1, \dots, n$$

* POTREBAN USLOV: $\delta I = 0$

$$\delta I = \int_a^b \sum_{i=1}^{n} \left[\frac{\partial F}{\partial x_i} \delta x_i + \frac{\partial F}{\partial \dot{x}_i} \delta \dot{x}_i \right] dt$$

$\delta \dot{x}_i$ ŽELIM DA SVEDEM δx_i

$$\frac{\partial F}{\partial \dot{x}_i} \delta \dot{x}_i = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i} \delta x_i \right) - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_i} \delta x_i$$

ISPRED $\int$

TOTALNI DIFERENCIJAL:

$$\frac{d}{dt}(x \cdot y) = \frac{d}{dt} x \cdot y + x \frac{d}{dt} y$$

$$\Rightarrow \delta I = \sum_{i=1}^{n} \frac{\partial F}{\partial \dot{x}_i} \delta x_i \Big|_a^b + \int_a^b \sum_{i=1}^{n} \Big[\underbrace{\frac{\partial F}{\partial x_i} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_i}}_{=0} \Big] \delta x_i \, dt = 0$$

$$\boxed{\frac{\partial F}{\partial x_i} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_i} = 0}$$

DOBIJAMO SISTEM DIF. JED.

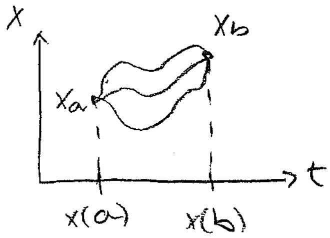
29. PRIRODNI GRANIČNI USLOVI – USLOVI TRANSFERZALNOSTI

$$I = \int_a^b F(t, x, \dot{x})\, dt$$

$$\delta I = 0$$

$$\delta I = \frac{\partial F}{\partial \dot{x}} \delta x \Big|_a^b + \int_a^b \left[\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} \right] \delta x\, dt = 0 \quad \Big\}\ \text{POTREBAN USLOV}$$

1. POZNATE VREDNOSTI: $x(a) = x_a$, $x(b) = x_b$ $\Big\} \Rightarrow$ $\delta x(a) = 0$, $\delta x(b) = 0$



$$\delta I = \cancel{\frac{\partial F}{\partial \dot{x}} \delta x \Big|_a^b}^{\,0} + \int_a^b \left[\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} \right] \delta x\, dt = 0$$

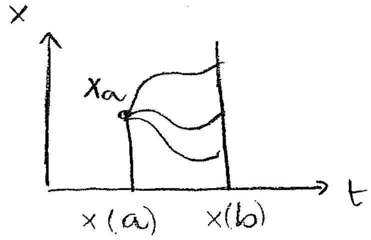
$\Rightarrow$ REŠAVAMO SAMO O-L DIF. JED.

$$\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = 0$$

2. POZNATA VREDNOST: $x(a) = x_a \Rightarrow \delta x(a) = 0$ (BOLJI NAČIN OD 1.)

NEPOZNATA VREDNOST: $x(b) = ? \Rightarrow \frac{\partial F}{\partial \dot{x}} = 0 \Big|_{t=b}$

$\hookrightarrow$ USLOV TRANSFERZALNOSTI TJ. PRIRODAN USLOV KOJI SE SAM NAMEĆE

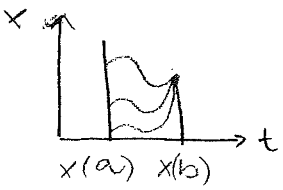


$\Rightarrow$ REŠAVAMO O-L DIF. JED.

$$\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = 0$$

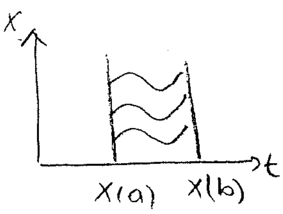
3. NEPOZNATA VREDNOST: $x(a) = ? \Rightarrow \frac{\partial F}{\partial \dot{x}} = 0 \Big|_{t=a}$

POZNATA VREDNOST: $x(b) = x_b \Rightarrow \delta x(b) = 0$



$\Rightarrow$ REŠAVAMO O-L DIF. JED.: $\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = 0$

4. NEPOZNATE POČETNE VREDNOSTI: $x(a) = ? \Rightarrow \frac{\partial F}{\partial \dot{x}} = 0 \Big|_{t=a}$, $x(b) = ? \Rightarrow \frac{\partial F}{\partial \dot{x}} = 0 \Big|_{t=b}$ $\Big\}$ PRIRODNI GRANIČNI USLOVI



$\Rightarrow$ REŠAVAMO O-L DIF. JED.: $\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = 0$

- PRIRODNI GRANIČNI USLOVI
- NE MORAMO NIGDE DA SPECIFICIRAMO POČETNE USLOVE.

30. PROBLEMI BOLCA TIPA (VAŽNO ISPITNO PITANJE)

* KRITERIJUM OPTIMALNOSTI KOJI U SEBI POSEDUJE ALGEBARSKI ČLAN I KRITERIJUM OPTIMALNOSTI => BOLCOV PROBLEM.

$$I = \underbrace{\Psi}_{\text{TEŽINSKI FAKTOR}} [\underbrace{x(a), x(b)}_{\text{ALGEBARSKI ČLAN}}] + \underbrace{\int_a^b F(t,x,\dot{x})dt}_{\text{KRIT. OPT.}}$$

* POTREBAN USLOV: $\delta I = 0$

$$\delta I = \underbrace{\frac{\partial \Psi}{\partial x(a)}\Big|_{t=a} \delta x(a) + \frac{\partial \Psi}{\partial x(b)}\Big|_{t=b} \delta x(b)}_{\text{ZA ALGEBARSKI ČLAN}} + \underbrace{\overbrace{\frac{\partial F}{\partial \dot{x}} \delta x \Big|_a^b}^{(*)} + \int_a^b [\underbrace{\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}}_{=0}] \delta x\, dt}_{\text{VARIJACIJA } \int} = 0$$

- REŠAVAMO O-L JED: $\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0$
- $(*) = \frac{\partial F}{\partial \dot{x}}\Big|_{t=b} \delta x(b) - \frac{\partial F}{\partial \dot{x}}\Big|_{t=a} \delta x(a)$

=> 1° $\frac{\partial \Psi}{\partial x(a)} - \frac{\partial F}{\partial \dot{x}} = 0$ ZA $t=a$ $(\delta x(a))$ (#)

2° $\frac{\partial \Psi}{\partial x(b)} + \frac{\partial F}{\partial \dot{x}} = 0$ ZA $t=b$ $(\delta x(b))$

- SLUČAJAN ZADATAK SA ISPITA:

$$I = \Psi[t, x(t)] + \int_a^b F(t,x,\dot{x})dt$$

$$\downarrow$$

$$\Psi(b) - \Psi(a) + \frac{\partial F}{\partial \dot{x}}\Big|_b - \frac{\partial F}{\partial \dot{x}}\Big|_a$$

U (#) UMESTO ~~+~~ − IDE + !

31. O-L JEDNAČINA ZA DISKRETNE SISTEME

Ојлер Лагранжева једначина за дискретне системе (TEŠKO PITANJE)

$$I = \sum_{k=k_0}^{k_f-1} \phi[x(k), x(k+1), k]$$

$$\delta I = 0$$

(*) IZRAZ KOJI SREĆU KVARI!

$$\delta I = \sum_{k=k_0}^{k_f-1} \left[\frac{\partial \phi}{\partial x(k)} \delta x(k) + \boxed{\frac{\partial \phi}{\partial x(k+1)} \delta x(k+1)} \right]$$

SVODIM $\delta x(k+1) \rightarrow \delta x(k)$

SMENA

$$\sum_{k=k_0}^{k_f-1} \frac{\partial \phi[x(k), x(k+1), k]}{\partial x(k+1)} \delta x(k+1) = [k+1 = m]$$

$$= \sum_{m=k_0+1}^{k_f} \frac{\partial \phi[x(m-1), x(m), m-1]}{\partial x(m)} \delta x(m)$$

$$= \sum_{m=k_0+1}^{k_f} \frac{\partial \phi[x(k-1), x(k), k-1]}{\partial x(k)} \delta x(k) + \underline{\qquad}\Big|_{k=k_0} - \underline{\qquad}\Big|_{k=k_f} \quad (*)$$

IZ PRVOG DELA

$$\Rightarrow \delta I = \sum_{k=k_0}^{k_f-1} \underbrace{\left[\frac{\partial \phi[x(k), x(k+1), k]}{\partial x(k)} + \frac{\partial \phi[x(k-1), x(k), k-1]}{\partial x(k)} \right]}_{\text{O-L JED. ZA DISKRETNE SISTEME}} \delta x(k) + \frac{\partial \phi[x(k-1), x(k), k-1]}{\partial x(k)} \Bigg|_{k}^{k}$$

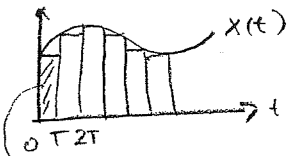
IZ - IZRAZA

USLOVI ZA O-L JED. ZA DISKRETNE SISTEME

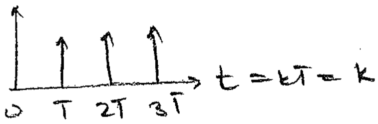
DUS

$$I = \int_a^b F(t, x, \dot{x})\, dt$$

$x = x(t) \rightarrow x = x(k)$



$$x(0)\cdot T + x(T)\cdot T + x(2T)\cdot T + \sum_{i=0}^{k-1} x(i)$$



$$\dot{x} = \frac{x(k) - x(k-1)}{T}$$

$$\dot{x} = \frac{x(k+1) - x(k)}{T}$$

=0

O-L JED ZA DISKRETNE SISTEME:

$$\frac{\partial \phi[x(k), x(k+1), k]}{\partial x(k)} + \frac{\partial \phi[x(k-1), x(k), k-1]}{\partial x(k)} = 0$$

$k = k_0$
$k = k_0 + 1$
$k = k_0 + 2$
$\vdots$
$k = k_f - 1$

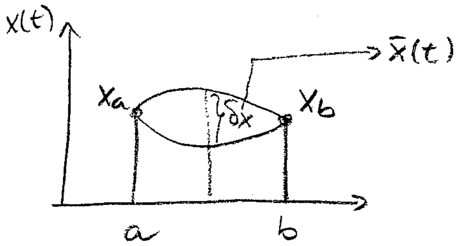
} NA OVAJ NAČIN SE IZRAČUNAVA

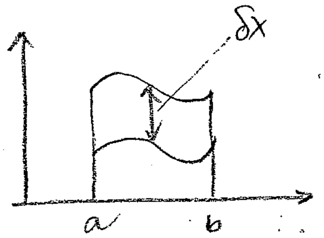
IZ OVOGA NE IDU ZADACI NA PISMENI DEO!

32. ASINHRONO VARIRANJE (NAJTEŽI ZADACI I NAJTEŽE TEORIJSKO PITANJE)

- OBIČNA VARIJACIJA:

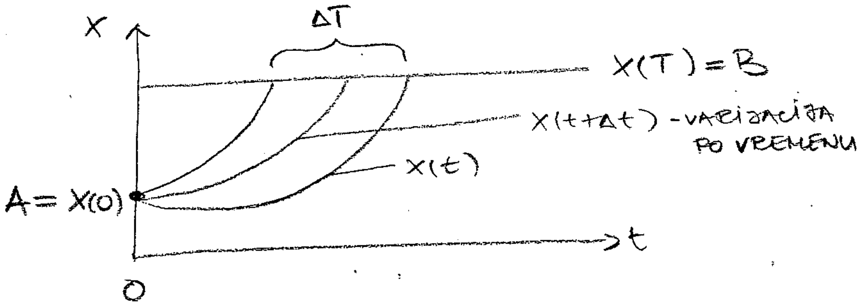
$$\delta x = \bar{x}(t) - x(t)$$
$$\delta \dot{x} = \dot{\bar{x}}(t) - \dot{x}(t)$$





- a i b SU POZNATI NA OBA GRAFIKA. INTERVAL NAM JE UVEK BIO POZNAT!

* a, b ? - INTERVAL JE NEPOZNAT!



→ ZNAM GDE TREBA DA ZAVRŠIM (U TAČKI B), ALI VREME (INTERVAL) MI NIJE POZNAT.

* PROBLEMI KOD KOJIH VREMENSKA GRANICA NIJE POZNATA REŠAVAJU SE ASINHRONIM VARIRANJEM.

$\bar{x}(t+\Delta t) - x(t)$? (OVO SE RAČUNA)

$$x(t+\Delta t) = \underbrace{\bar{x}(t)}_{\delta x + x(t)} + \underbrace{\dot{\bar{x}}}_{\delta \dot{x} + \dot{x}(t)}\Delta t \implies \bar{x}(t+\Delta t) - x(t) = \delta x + [\delta \dot{x} + \dot{x}(t)]\Delta t = \delta x + \dot{x}\Delta t$$

$$\delta \dot{x} \Delta t \approx 0$$

$$\implies \bar{x}(t+\Delta t) - x(t) = \delta x + \dot{x}\Delta t$$

$$\boxed{\Delta x \overset{def}{=} \bar{x}(t+\Delta t) - x(t) = \delta x + \dot{x}\Delta t}$$ ASINHRONA VARIJACIJA

* ZA PROIZVOLJNU F-JU, ASINHRONA VARIJACIJA SE RAČUNA:

$$\Delta f = \delta f + \dot{f}\Delta t$$

* OSOBINE ...

$$\left.\begin{aligned}(\Delta x)^{\circ} &= \delta\dot{x} + \ddot{x}\Delta t + \dot{x}(\Delta t)^{\circ} \\ \Delta\dot{x} &= \delta\dot{x} + \ddot{x}\Delta t\end{aligned}\right\}$$ $\Delta d \neq d\Delta$ – KOMUTATIVNOST <u>NE VAŽI</u> KOD ASINHRONE VARIJACIJE

* t_0, t_1 – NEPOZNATO

VARIJACIJA ODREĐENOG INTEGRALA: $\Delta\int_{t_0}^{t_1} \underbrace{F dt}_{\text{F-JA KOJU POSMATRAM}} = \delta\int_{t_0}^{t_1} F dt + F\Delta t\Big|_{t_0}^{t_1}$

GLAVNA DEFINICIJA ASINHRONOG VARIRANJA

$$I = \int_{t_0}^{t_1} F(t, x, \dot{x})\, dt, \quad t_0, t_1 = ?$$

– KAKO SE RAČUNA EKSTREMALA $x = x(t)$?

$\boxed{\Delta I = 0}$ POTREBAN USLOV ASINHRONE VARIJACIJE

$$\Delta I = \delta\int_{t_0}^{t_1} F(t, x, \dot{x})\, dt + F\Delta t\Big|_{t_0}^{t_1}$$ (PITANJA 26, 27, 28)

$$\Delta I = \int_{t_0}^{t_1}\left[\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right]\delta x\, dt + \underbrace{\frac{\partial F}{\partial \dot{x}}\delta x\Big|_{t_0}^{t_1} + F\Delta t\Big|_{t_0}^{t_1}}_{\#}$$

$$\boxed{\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0}$$ O-L JEDNAČINA KAO POTREBAN USLOV

→ ODAVDE RAČUNAM $x(t_0) = ?$ i $x(t_1) = ?$

$\boxed{\frac{\partial F}{\partial \dot{x}}\delta x\Big|_{t_0}^{t_1} + F\Delta t\Big|_{t_0}^{t_1} = 0}$ → ODAVDE RAČUNAM VREMENSKI INTERVAL

$x(t_0) = ?$

$x(t_1) = ?$

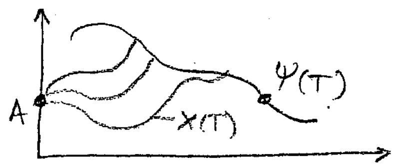
$\Delta x = \delta x + \dot{x}\Delta t$

$\delta x = \Delta x - \dot{x}\Delta t$

$$\Rightarrow \frac{\partial F}{\partial \dot{x}}\Delta x\Big|_{t_0}^{t_1} + \left(F - \frac{\partial F}{\partial \dot{x}}\dot{x}\right)\Delta t\Big|_{t_0}^{t_1} = 0$$

$$I = \int_0^T F(t, x, \dot{x})\,dt \quad , \quad x(0) = A \text{ - POZNATO}$$

$$T = ? \qquad\qquad x(T) = \psi(T)$$



- REŠAVAM O-L JED.: $\dfrac{\partial F}{\partial x} - \dfrac{d}{dt}\dfrac{\partial F}{\partial \dot{x}} = 0$
- IZRAZ SE KOMPLIKUJE SA: $\dfrac{\partial F}{\partial \dot{x}} \Delta x \Big|_0^T + \left(F - \dfrac{\partial F}{\partial \dot{x}} \dot{x}\right) \Delta t \Big|_0^T = 0$

$\Delta x(0) = 0 \qquad\qquad \Delta t(0) = 0$

$$\frac{\partial F}{\partial \dot{x}} \Delta x \Big|_{t=T} + \left(F - \frac{\partial F}{\partial \dot{x}} \dot{x}\right) \Delta T = 0 \quad ; \text{ ZA } t = T$$

VEZA IZMEĐU $\Delta x(T)$ I ΔT:

$$\Delta x(T) = \underbrace{\delta x(T)}_{0} + \dot{x}(T)\Delta T \quad \Rightarrow \quad \boxed{\Delta x(T) = \dot{\psi}(T)\Delta T}$$

(JER GA OČEKUJEM
U POZNATOM $\psi(T)$
TRENUTKU.
$\psi(T) = x(T)$ MORAJU
BITI ISTI.)

$$\Rightarrow \boxed{F + (\dot{\psi} - \dot{x})\frac{\partial F}{\partial \dot{x}} = 0, \text{ ZA } t = T}$$

GRANIČNI
USLOV DA
ODREDIMO
VREME.

33. VARIJACIONI PROBLEM SA OGRANIČENJIMA
VARIJACIONI PROBLEM SA OGRANIČENJIMA
U VIDU ALGEBARSKIH JEDNAČINA

KRITERIJUM OPTIMALNOSTI: $I = \int_a^b F(t, x_i, \dot{x}_i)\,dt, \quad i = 1, \dots, n$

ALGEBARSKA OGRANIČENJA: $g_k(t, x_i) = 0, \quad k = 1, \dots, m$ $\qquad \boxed{m < n}$!

* AKO JE $m = n$, PROBLEM JE PREDEFINISAN.

FORMIRAMO NOVI KRITERIJUM OPTIMALNOSTI ϕ:

$$\phi = F(t, x_i, \dot{x}_i) + \sum_{i=1}^{m} \underbrace{\lambda_k(t)}_{\text{F-JA VREMENA!}} g_k(t, x_i)$$

$$\frac{\partial \phi}{\partial x_i} - \frac{d}{dt}\frac{\partial \phi}{\partial \dot{x}_i} = 0$$

$$\frac{\partial \phi}{\partial \lambda_k} - \cancel{\frac{d}{dt}}\frac{\partial \phi}{\partial \dot{x}_k}^{\nearrow 0} = 0 \quad \Rightarrow g_k = 0$$

$$\frac{\partial F}{\partial x_i} + \sum_{k=1}^{m} \lambda_k(t)\frac{\partial g_k}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} = 0$$

$$\bar{I} = \int_a^b \left[F(t, x_i, \dot{x}_i) + \sum_{k=1}^{m} \lambda_k(t)\, g_k(t, x_i)\right] dt, \quad \begin{matrix} i = 1, \dots, n \\ k = 1, \dots, m \end{matrix}$$

$$\delta \bar{I} = \int_a^b \left\{ \sum_{i=1}^{n} \Big[\underbrace{\frac{\partial F}{\partial \dot{x}_i}\delta \dot{x}_i}_{\circledast} + \frac{\partial F}{\partial x_i}\delta x_i + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_i}\delta x_i \Big] + \sum_{k=1}^{m} g_k \delta\lambda_k \right\} dt$$

$$\circledast \quad \frac{\partial F}{\partial \dot{x}_i}\delta \dot{x}_i = \underbrace{\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\delta x_i\right)}_{\text{ISPRED } \int} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i}\delta x_i \longrightarrow \text{VRAĆAM U } \delta \bar{I}$$

$$\Rightarrow \delta \bar{I} = \dots + \int_a^b \left\{ \sum_{i=1}^{n} \Big[\frac{\partial F}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} \Big]\delta x_i + \sum_{k=1}^{m} g_k \delta\lambda_k \right\} dt$$

54. VARIJACIONI PROBLEM SA OGRANIČENJIMA TIPA JEDNAKOSTI (JAKO LAKO PITANJE)

KRITERIJUM OPTIMALNOSTI: $I = \int_a^b F(t, x_i, \dot{x}_i)\,dt \quad , i = 1, \dots, n$

$$f_k(t, x_i, \dot{x}_i) = 0 \; , \quad k = 1, \dots, m \qquad \boxed{m < n} \; !$$

- NOVI PROŠIRENI KRITERIJUM OPTIMALNOSTI:

$$\Phi = F(t, x_i, \dot{x}_i) + \sum_{k=1}^{m} \lambda_k(t)\, f_k(t, x_i, \dot{x}_i)$$

$$\frac{\partial \Phi}{\partial x_i} - \frac{d}{dt}\underbrace{\cancel{\frac{\partial \Phi}{\partial \dot{x}_k}}}_{0} = 0 \Rightarrow f_k(t, x_i, \dot{x}_i) = 0$$

$$\frac{\partial F}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial \dot{x}_i} \right) = 0$$

35. VARIJACIONI PROBLEMI SA INTEGRALNIM OGRANIČENJIMA - IZOPERIMETRIJSKI PROBLEM

(ČESTO KAO ZADATAK; ~~TEŠ~~ TEŠKO TEORIJSKO)

$$I=\int_0^T F(t,x_i,\dot{x}_i)\,dt\,,\quad i=1,\dots,n$$

$$\int_0^T G_k(t,x_i,\dot{x}_i)\,dt=A_k\,,\quad k=1,\dots,m$$

└ REŠENJE JE SAMO BROJ!

$\boxed{m \lessgtr n}$ POTPUNO PROIZVOLJNO OD X

$$\Rightarrow\quad \Phi=F(t,x_i,\dot{x}_i)+\sum_{k=1}^{m}\lambda_k G_k(t,x_i,\dot{x}_i)$$

$\lambda_k \downarrow$ const. (VIŠE NIJE F-JA OD t, DOKAZATI)

$$\frac{\partial\Phi}{\partial x_i}-\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_i}=0$$

$$\int_0^T G_k(t,x_i,\dot{x}_i)\,dt=A_k$$

$\boxed{*\text{ Cilj}}$: DOKAZATI DA JE $\lambda_k=\text{const}$:

$$I=\int_0^T F(t,x_i,\dot{x}_i)\,dt$$

$$\int_0^T G_k(t,x_i,\dot{x}_i)\,dt=A_k \longrightarrow \int_0^t G_k(t,x_i,\dot{x}_i)\,dt=Z_k(t)\ \Big/\frac{d}{dt}$$

$$\boxed{\begin{gathered}G_k(t,x_i,\dot{x}_i)=\dot{Z}_k(t)\\ Z_k(0)=0\\ Z_k(T)=A_k\end{gathered}}$$

$$\Phi=F(t,x_i,\dot{x}_i)+\sum_{k=1}^{m}\lambda_k(t)\left[G_k(t,x_i,\dot{x}_i)-\dot{Z}_k(t)\right]$$

$$\frac{\partial\Phi}{\partial x_i}-\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_i}=0$$

$$\underbrace{\frac{\partial\Phi}{\partial Z_k}}_{0}-\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{Z}_k}=0\quad\Rightarrow\quad\frac{d}{dt}\lambda_k=0\ \Big/\int$$

$$\Rightarrow\quad\boxed{\lambda_k=\text{const.}}$$

* NAPOMENE ZA REŠAVANJE ZADATAKA *

(*) $\ddot{x} + \lambda x = 0$

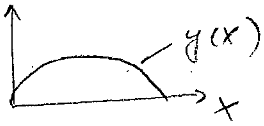
$\lambda < 0$ ← $\quad \lambda > 0$ →

$x(\frac{\pi}{2}) = 1 \longrightarrow$ OČEKIVANO REŠENJE: $\sin t, \cos t$

$x(1) = e^{1} \longrightarrow$ OČEKIVANO REŠENJE: e^{-t}, e^{t}

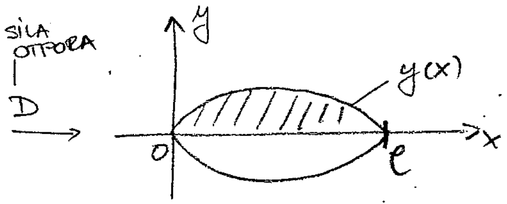
(*) $x = x(t)$

$y = y(x)$



$$I = \int_a^b F(x, y(x), y'(x))\,dx$$ — STRUKTURNA OPTIMIZACIJA

(*) ODREDITI OPTIMALNI OBLIK KRILA AVIONA TAKO DA JE OTPORNA SILA NAJMANJA, A POVRŠINA POPREČNOG PRESEKA JE POZNATA.



– ZBOG SIMETRIJE REŠAVAMO SAMO GORNJU KRIVU:

$$\int_0^l y(x)\,dx = \frac{A}{2}$$

$$D = k\int_0^l y'^2(x)\,dx$$ (SILA OTPORA ZAVISI OD KVADRATA BRZINE)

$$\phi = k y'^2(x) + \lambda y$$ – KRIT. OPT. + OGRANIČENJE.

$$\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y} = 0$$

$$\lambda - \frac{d}{dx}\frac{\partial \phi}{\partial y} = 0$$

$$\lambda - 2k y'' = 0 \Rightarrow y'' = \frac{\lambda}{2k} \;/\int$$

$$y' = \frac{\lambda}{2k}x + C \;/\int$$

$$\boxed{y = \frac{\lambda}{4k}x^2 + Cx + D}$$

$y(0)=0$
$y(\ell)=0$ } MORA BITI OGRANIČENJE!

$y(0)=0 \Rightarrow D=0$

$y(\ell)=0 \Rightarrow \frac{\lambda}{4k}\ell^2+C\ell=0 \Rightarrow \underbrace{C=-\frac{\lambda}{4k}\ell}_{\text{UBACIM U } y=\frac{\lambda}{4k}x^2+Cx+\cancel{D}^{\,0}}$

$\cancel{\ell=0}$

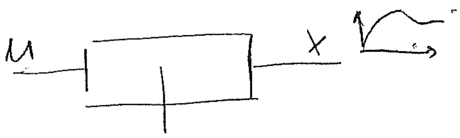
OTPADA ZBOG OGRANIČENJA!

$\Rightarrow C=\frac{3A}{\ell^2}$

$\lambda=-\frac{12A}{\ell^3}$

11.12.2017.

(OVO NIJE DEO ISPITNOG PITANJA)



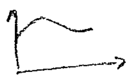
* KAKO DA MENJAMO ULAZ SISTEMA KOJI SE PONAŠA KAO $\dot{x}=\ldots$ DA BI DOBILI ŽELJENI IZLAZ?

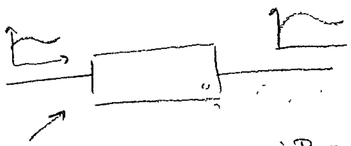
$\dot{x}=f(t,x,u)$

$I=\int_0^T F(t,x,u)\,dt$ — KVALITET IZLAZA

$u=u_{opt}(t) \longrightarrow$ TEORIJA OPTIMALNOG UPRAVLJANJA

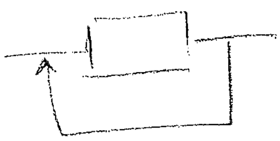
$u=u(t)$
$u=u(t,x)$ }





TEORIJE: PONTRJAGIN I BELMAN

* KOD BELMANA SE STALNO RAČUNA JER IMA POVRATNU SPREGU



— NE MOŽEMO DA ELIMINIŠEMO ONO ŠTO UNAPRED NISMO PREDVIDELI

30a. OPTIMALNO UPRAVLJANJE BEZ OGRANIČENJA NA KOMPONENTE UPRAVLJANJA (NIJE NA SPISKU PITANJA)

$\dot{x} = f(t, x, u)$ → SISTEM OPISAN DIF. JED.

t – VREME

x – PROMENLJIVA STANJA I IZLAZ

u → ULAZ - UPRAVLJANJE

$x(0) = \alpha$ – PONAŠANJE SISTEMA U POČETNOM TRENUTKU JE POZNATO

$u_{opt} = u_{opt}(t)$ – ODREDITI UPRAVLJANJE TAKO DA JE KRIT. OPT. I U MIN ILI MAX.

$$I = \int_0^T F(t, x, u)\,dt$$

– FORMIRAMO NOVI KRIT. OPT. SA LAGRANŽEVIM MNOŽITELJIMA:

$$\bar{I} = \int_0^T \{ [F(t,x,u) \underset{\text{VAŽNO}}{-} \underbrace{p(t)}_{\text{LAGRANŽEV MNOŽITELJ}} \overbrace{[\dot{x} - f(t,x,u)]}^{\text{OGRANIČENJE}} \}\,dt$$

– POTREBAN USLOV: $\delta \bar{I} = 0$

$$\delta \bar{I} = \int_0^T [\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial u}\delta u - \underbrace{1}_{\frac{\partial F}{\partial p}} \cdot \delta p[\dot{x} - f(t,x,u)] - \underbrace{p(t)[\overset{\frac{\partial F}{\partial x}}{1} \cdot \delta\dot{x}}_{\delta \dot{x} \xrightarrow{\text{SVODIM}} \delta x} - \frac{\partial F}{\partial x}\delta x - \frac{\partial F}{\partial u}\delta u]\,dt$$

$$-p(t)\delta\dot{x} = -\frac{d}{dt}(p\,\delta x) + \dot{p}\,\delta x$$

IZLAZI ISPRED $\int$

$$\delta\bar{I} = \underbrace{-p(t)\delta x(t)\Big|_0^T}_{=0 \;\circledast} + \int_0^T \{ \underbrace{[\frac{\partial F}{\partial x} + \dot{p} + p(t)\frac{\partial F}{\partial x}]}_{=0}\delta x - \delta p \underbrace{[\dot{x} - f(t,x,u)]}_{=0}$$

$$+ \underbrace{[\frac{\partial F}{\partial u} + p\frac{\partial F}{\partial u}]}_{=0}\delta u \}\,dt = 0$$

→ DIF. JED:

$$\dot{x} = f(t,x,u)$$
$$\dot{p} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial x}$$

→ ALGEBARSKA JED:

$$\frac{\partial F}{\partial u} + p\frac{\partial F}{\partial u} = 0$$

} SPECIJALAN SLUČAJ O-L JEDNAČINA

$\delta x(0)=0 \rightarrow$ DONJA GRANICA POZNATA · F-NUOI

$\delta x(T)\neq 0 \rightarrow$ GORNJA GRANICA NIJE POZNATA

(*) $\boxed{x(0)=\alpha \Rightarrow p(t)=0}$

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36. PRINCIP MAKSIMUMA (SIGURNO ZADATAK!)

13.12.2017.

$\dot{x}_i = f_i(t, x_i, u_k)$ – SISTEM JE OPISAN SISTEMOM DIF. JED.

$i=1,\dots,n$
$k=1,\dots,m$

$x_i(0)=\alpha_i$, $0\le t \le T$ – POZNAT INTERVAL U KOM SE ODVIJA PROCES

Cilj: ODREDITI OPTIMALNU PROMENU ULAZA TAKO DA JE KRIT. OPT U MIN ILI MA

$u_{kopt} = u_{kopt}(t)$?

$$I = \int_0^T F(t, x_i, u_k)\,dt$$

IZ (36a)

LAGRANŽEVI MNOŽITELJI

$$(*)\quad \bar{I} = \int_0^T \Big[F(t,x_i,u_k) - \sum_{i=1}^{n} p_i(t)\underbrace{[\dot{x}_i - f_i(t,x_i,u_k)]}_{\text{OGRANIČENJE}}\Big]dt$$

→ PONTRIJAGIN: OPTIMIZACIJA OVOG PROBLEMA I SE SVODI NA TRAŽENJE MIN HAMILTONOVE F-JE

$$\min_{u} H(t, x_i, u_k, p_i)$$

1° NEMA OGRANIČENJE NA UPRAVLJANJU $\frac{\partial H}{\partial u}=0$

2° IMA OGRANIČENJE: $|u|\le u_{max}$ (REŠENJE NA GRANICI)

$$(**)\quad H = H(t,x_i,u_k,p_i) \stackrel{def}{=} F(t,x_i,u_k) + \sum_{i=1}^{n} p_i f_i$$

→ DO OPTIMALNOG REŠENJA SE DOLAZI MINIMIZACIJOM ENERGIJE SISTEMA

→ BAVIMO SE SAMO 1° KADA U NEMA OGRANIČENJA

$$(*), (**) \Rightarrow \bar{I} = \int_0^T \Big[H(t,x_i,u_k,p_i) - \sum_{i=1}^{n} p_i \dot{x}_i\Big]dt$$

$k=1,\dots,m$ $\quad$ $i=1,\dots,n$

- POTREBAN USLOV SISTEMA: $\delta I = 0$

$$\delta I = \int_0^T \left\{ \sum_{i=1}^{n} \left[\frac{\partial H}{\partial x_i} \delta x_i + \frac{\partial H}{\partial p_i} \delta p_i - 1 \cdot \delta p_i \dot{x}_i - p_i \delta \dot{x}_i \right] + \sum_{k=1}^{m} \frac{\partial H}{\partial u_k} \delta u_k \right\} dt$$

$\frac{\partial}{\partial p_i}(p_i \dot{x}_i)\delta p_i$ 1

$1 \cdot \dot{x}_i \delta p_i$

SVODIM NA δx_i (*)

(*) $p_i \delta \dot{x}_i = \frac{d}{dt}(p_i \delta x_i) - \dot{p}_i \delta x_i$ (VRAĆAM U δI)

$\ominus$ $\oplus$

$\int$

$$\Rightarrow \delta I = -\sum_{i=1}^{n} p_i \delta x_i \Big|_0^T + \int_0^T \left\{ \sum_{i=1}^{n} \left[\left(\frac{\partial H}{\partial x_i} + \dot{p}_i \right) \delta x_i + \left(\frac{\partial H}{\partial p_i} - \dot{x}_i \right) \delta p_i \right] + \sum_{k=1}^{m} \frac{\partial H}{\partial u_k} \delta u_k \right\} dt = 0$$

* DA BI $\delta I = 0$, POD INTEGRALOM:

- PROMENLJIVE STANJA: $\dot{x}_i = \frac{\partial H}{\partial p_i}$
- GENERISANI IMPULSI (LAGRANŽOVI MNOŽITELJI): $\dot{p}_i = -\frac{\partial H}{\partial x_i}$
- ALGEBARSKA JED. (POTREBAN USLOV!): $\frac{\partial H}{\partial u_k} = 0$

- REŠAVA SE SISTEM OVIH JEDNAČINA. IMA 2n DIFERENCIJALNIH JED. I m ALGEBARSKIH JED.
- OVAJ SISTEM SE NAZIVA KANONSKI SISTEM JEDNAČINA!

- $\min_u H$ - TRAŽI SE MIN HAMILTONOVE F-JE PO UPRAVLJANJU

$\dot{x}_i = \frac{\partial H}{\partial p_i} = f_i(t, x_i, u_k) \rightarrow$ JEDNAČINA OGRANIČENJA

$\dot{p}_i = -\frac{\partial H}{\partial x_i} \rightarrow$ IZVOD LAGRANŽEVIH MNOŽITELJA

$x_i(0) = \alpha_i$

$-\sum_{i=1}^{n} p_i \delta x_i \Big|_0^T = 0$

$\rightarrow$ POŠTO JE $x_i(0) = \alpha_i \Rightarrow \boxed{\delta x_i(0) = 0}$!

$\rightarrow$ POŠTO $\delta x_i(T) \neq 0$ (NIJE POZNATO) $\Rightarrow \boxed{p_i(T) = 0}$!

$\left.\begin{array}{l} x_i(0) = \alpha_i \\ p_i(T) = 0 \end{array}\right\}$ DVOTAČKASTI GRANIČNI PROBLEM (PRIRODNI GRANIČNI USLOVI)

(*) POZNATO: $x_i(0)=\alpha_i$

$x_i(T)=\beta_i$

$0 \leq t \leq T$ (INTERVAL POZNAT)

$$\Rightarrow \delta x_i(0)=0$$

$$\delta x_i(T)=0$$

PA NAM ONDA NE TREBA USLOV $p_i(T)=0$!

(*) Dovoljni uslovi – UGLAVNOM SE NE ISPITUJU, ALI AKO SE ISPITUJU TADA:

$$\frac{\partial^2 H}{\partial u_k^2} > 0 \Rightarrow MIN$$

$$\frac{\partial^2 H}{\partial u_k^2} < 0 \Rightarrow MAX$$

$$\frac{\partial^2 H}{\partial u_k^2} = \begin{bmatrix} \frac{\partial^2 H}{\partial u_1^2} & \frac{\partial^2 H}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 H}{\partial u_1 \partial u_m} \\ \vdots & & & \\ \frac{\partial^2 H}{\partial u_m \partial u_1} & \frac{\partial^2 H}{\partial u_m \partial u_2} & \cdots & \frac{\partial^2 H}{\partial u_m^2} \end{bmatrix}$$

(*) $H(x_i, p_i, u_k)$ NE ZAVISI EKSPLICITNO OD VREMENA.

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial u}\dot{u} + \cancel{\frac{\partial H}{\partial t}}^{\,0}$$

AKO JE REŠENJE OPTIMALNO, TADA VAŽE KANONSKE JEDNAČINE:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \frac{\partial H}{\partial u} = 0$$

$$\Rightarrow \frac{dH}{dt} = \cancel{\frac{\partial H}{\partial x}\frac{\partial H}{\partial p}} - \cancel{\frac{\partial H}{\partial p}\frac{\partial H}{\partial x}} + 0 + 0$$

$$\frac{dH}{dt} = 0$$

$$\boxed{H = \text{const}}$$

OPTIMALNO UPRAVLJANJE, SLUČAJ NESPECIFICIRANOG

37. OPTIMALNO UPRAVLJANJE, SLUČAJ VREMENSKOG INTERVALA

$\dot{x}_i = f_i(t, x_i, u_k)$, $i = 1, \dots, n$
$k = 1, \dots, m$

$x_i(0) = \alpha_i$

$x_i(T) = \beta_i$

$T = ?$ ⟶ INTERVAL NEPOZNAT

$$I = \int_0^T F(t, x_i, u_k)\, dt$$

Cilj: $u_{opt} = u_{opt}(t)$?

IZ 32. ASINHRONO VARIRANJE: $\Delta x = \delta x + \dot{x} \Delta t$

$$\Delta \int_{t_0}^{t_1} F dt = \delta \int_{t_0}^{t_1} F dt + F \Delta t \Big|_{t_0}^{t_1}$$

$$H = F(t, x_i, u_k) + \sum_{i=1}^{n} p_i(t) \cdot f(t, x_i, u_k)$$

$$\bar{I} = \int_0^T \left\{ H(t, x_i, u_k, p_i) - \sum_{i=1}^{n} p_i(t) \dot{x}_i \right\} dt$$

$\boxed{\Delta \bar{I} = 0}$ POŠTO VREMENSKI INTERVAL NIJE POZNAT ASINHRONA VARIJACIJA JE $= 0$.

$$\Delta \bar{I} = \delta \bar{I} + (\text{PODINTEGRALNA F-JA}) \Delta t \Big|_0^T$$

$$\Delta \bar{I} = \underbrace{\int_0^T \left\{ \sum_{i=1}^{n} \left[\left(\frac{\partial H}{\partial p_i} - \dot{x}_i \right) \delta p_i + \left(\frac{\partial H}{\partial x_i} + \dot{p}_i \right) \delta x_i \right] + \sum_{k=1}^{m} \frac{\partial H}{\partial u_k} \delta u_k \right\} dt}_{\delta \bar{I} \text{ (36. pitanje)}} \underbrace{- \sum_{i=1}^{n} p_i \delta x_i \Big|_0^T + \left(H - \sum_{i=1}^{n} p_i \dot{x}_i \right) \Delta t \Big|_0^T}_{=0}$$

IZ DEFINICIJE ASINHRONE VARIJACIJE, TRAŽIM T ! (*)

- DA BI $\Delta T = 0$ => KANONSKE JEDNAČINE.

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = f_i(t, x_i, u_k)$$

- ZADATO: $x_i(0) = \alpha_i$

$x_i(T) = \beta_i$

- NEPOZNATO: T ?

$$\dot{p}_i = -\frac{\partial H}{\partial x_i}$$

$$\frac{\partial H}{\partial u_k} = 0$$

$$(*) \Rightarrow -\sum_{i=1}^{n} p_i(\underbrace{\delta x_i + \dot{x}_i \Delta t}_{\Delta x_i \text{ - ASINHRONA VARIJACIJA}})\Big|_0^T + H(t)\Delta t\Big|_0^T = 0$$

$$-\sum_{i=1}^{n} p_i \Delta x_i\Big|_0^T + H(t)\Delta t\Big|_0^T = 0$$

! DA BI ASINHRONA VARIJACIJA BILA 0 I VREME I VREDNOSTI MORAJU BITI POZNATI.

$\Delta x_i(0) = 0$ AKO JE: 1° $x_i(0) = \alpha_i$

2° $x_i(T) = \beta_i$

T-POZNATO

$$H \Delta t\Big|_0^T = 0$$

$\Delta T \neq 0$

T?

$\Delta t(0) = 0$

$\boxed{H(T) = 0}$ USLOV IZ KOG RAČUNAMO T!

* AKO H NE ZAVISI EKSPLICITNO OD VREMENA => $H = const.$

AKO JE $H = const.$ => $H(0) = 0$

$)\Delta t\Big|_0^T$

$= 0$

$)$

* DISKUSIJA MOGUĆIH ZADATAKA *

$\dot{x}_i = f_i(t, x_i, u_k)$ – SISTEM DIF. JED.

$I = \int_0^T F(t, x_i, u_k)\,dt$ – KRIT. OPT.

1) NEPOZNATO: $x_i(0) = ?$
$x_i(T) = ?$
POZNATO: T

$\Rightarrow$ $\delta x_i(0) \neq 0$, $\delta x_i(T) \neq 0$ PA SU NAM POTREBNI USLOVI:

1° $\boxed{p_i(0) = 0 \quad p_i(T) = 0}$

– 36. ISPITNO PITANJE: 2° REŠAVANJE KANONSKIH JEDNAČINA

$$\dot{x} = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x}$$

$$\frac{\partial H}{\partial u_k} = 0$$

2) POZNATO: $x_i(0) = \alpha_i$
NEPOZNATO: $x_i(T) = ?$
POZNATO: T

– KANONSKE JED.:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x}$$

$$\frac{\partial H}{\partial u_k} = 0$$

– USLOVI: $\boxed{x_i(0) = \alpha_i \quad p_i(T) = 0}$

3) POZNATO: $x_i(0) = \alpha_i$
$x_i(T) = \beta_i$
$0 \leq t \leq T$ (INTERVAL POZNAT)

$\Rightarrow$ $\boxed{\delta x_i(0) = 0 \quad \delta x_i(T) = 0}$

USLOVI

4) NEPOZNATO: $x_i(0) = ?$ $\Rightarrow$ $p_i(0) = 0$
POZNATO: $x_i(T) = \beta_i$ $\rightarrow$ $\delta x_i(T) = 0$
$\hookrightarrow T$

USLOVI

5) POZNATO: $x_i(0) = \alpha_i$
$x_i(T) = \beta_i$

$\Rightarrow$ $\boxed{\delta x_i(0) = 0 \quad \delta x_i(T) = 0}$

NEPOZNATO: T? $\Rightarrow$ $\boxed{H(T) = 0}$