FURIJEOVA TRANSFORMACIJA

Primena DSP u upravljanju

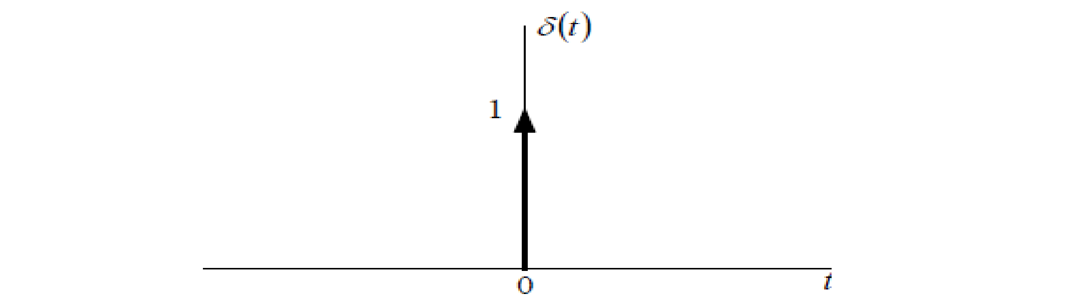
Aperiodični signali

- Aperiodičan signal je specijalni slučaj periodičnog kod kog perioda teži beskonačnosti
- $T_P \to \infty$ odatle sledi da $\omega_P = \frac{2\pi}{T_P} \to d\omega$, odnosno $\frac{1}{T_P} \to \frac{d\omega}{2\pi}$
- Razmak između susednih harmonika teži nuli
- Diskretni harmonici $n\omega_P$ prelaze u kontinualnu veličinu ω
- Spektar postaje kontinualna funkcija, a koeficijenti Furijeovog reda $d_n \to d(\omega)$ kompleksna funkcija

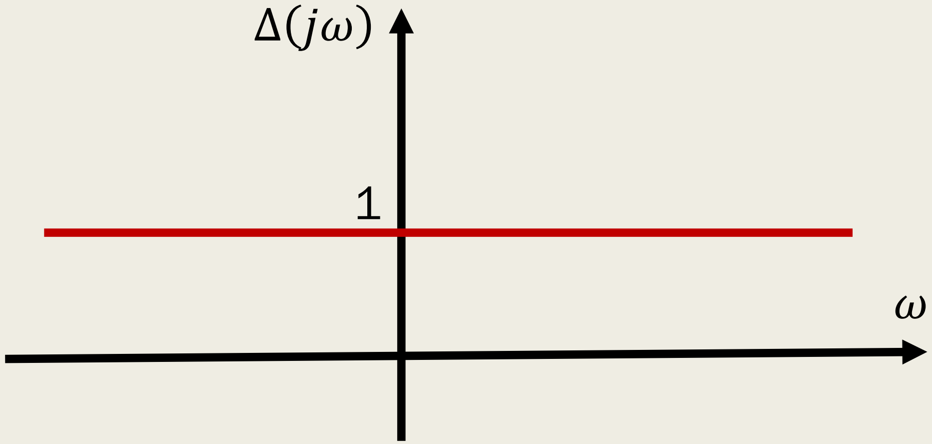
Furijeova transformacija

- Neka je *f(t)* aperiodični signal (funkcija), tada je razvoj u Furijeov red:
- $d_n = \frac{1}{T_P}\int_{-T_P/_2}^{T_P/_2} f(t)e^{-jn\omega_P t}dt \rightarrow d(\omega) = \frac{d\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
- Pošto je $\frac{1}{T_P} \rightarrow \frac{d\omega}{2\pi}$
- Ako izvršimo normalizaciju deljenjem obe strane jednakosti sa $\frac{d\omega}{2\pi}$ dobija se **Furijeova transformacija** signala *f(t)*
- $\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt}$ $F(j\omega) = \mathcal{F}\{f(t)\}$
- Dok je **inverzna Furijeova transformacija** spektra $F(j\omega)$ data izrazom:
- $\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega}$ $f(t) = \mathcal{F}^{-1}\{F(j\omega)\}$

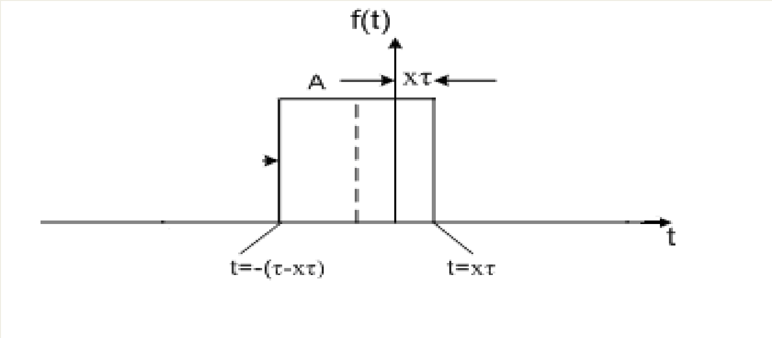
Primer 1: Furijeova transformacija Dirakovog impulsa



- $\Delta(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1\, e^{-j\omega 0} = 1$



Primer 2: Furijeova transformacija pravougaonog impulsa



$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{x\tau-\tau}^{x\tau} Ae^{-j\omega t}dt = A\frac{1}{j\omega}e^{-j\omega t}\Big|_{x\tau}^{x\tau-\tau} = A\frac{1}{j\omega}\left(e^{-j\omega(x\tau-\tau)} - e^{-j\omega x\tau}\right)$$

$$F(j\omega) = A\frac{2\tau}{2j\omega\tau}e^{-j\omega x\tau}\left(e^{j\omega\tau} - 1\right) = A\frac{2\tau}{2j\omega\tau}e^{-j\omega x\tau}e^{j\omega\tau/2}\cdot\left(e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right) = A\tau e^{j\omega\left(\frac{1}{2}-x\right)\tau}\frac{\sin(\frac{\omega\tau}{2})}{\frac{\omega\tau}{2}}$$

Ukoliko je x=0, odnosno impuls simetričan oko t=0

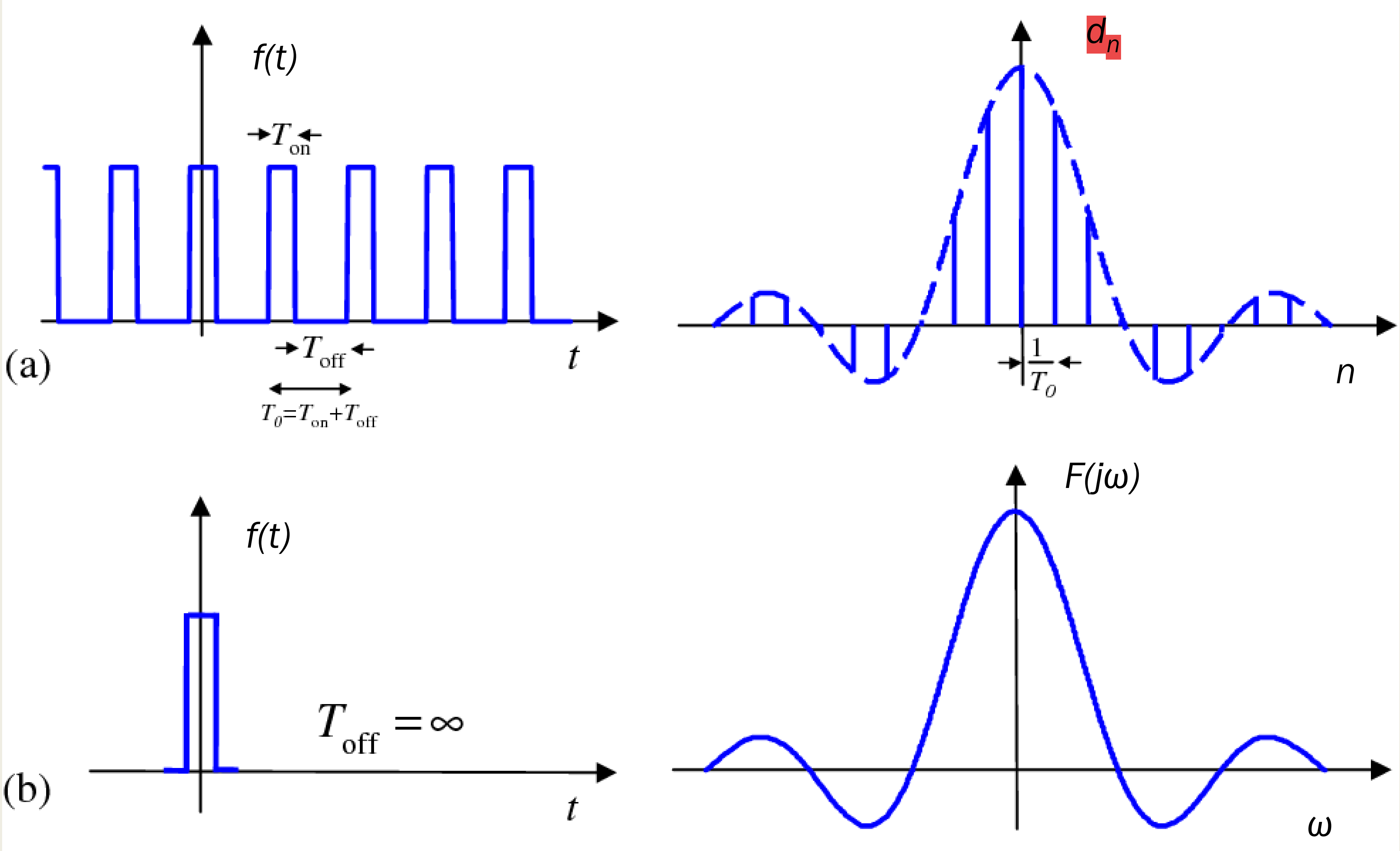
$$F(j\omega) = A\tau\frac{\sin(\frac{\omega\tau}{2})}{\frac{\omega\tau}{2}}$$

b)

Podsetimo se vrednosti koeficijenata Furijeovog reda za periodičnu povorku pravougaonih impulsa

$$d_n = \frac{A\tau}{T_P}\frac{\sin(\frac{n\omega_0\tau}{2})}{\frac{n\omega_0\tau}{2}}$$

a)



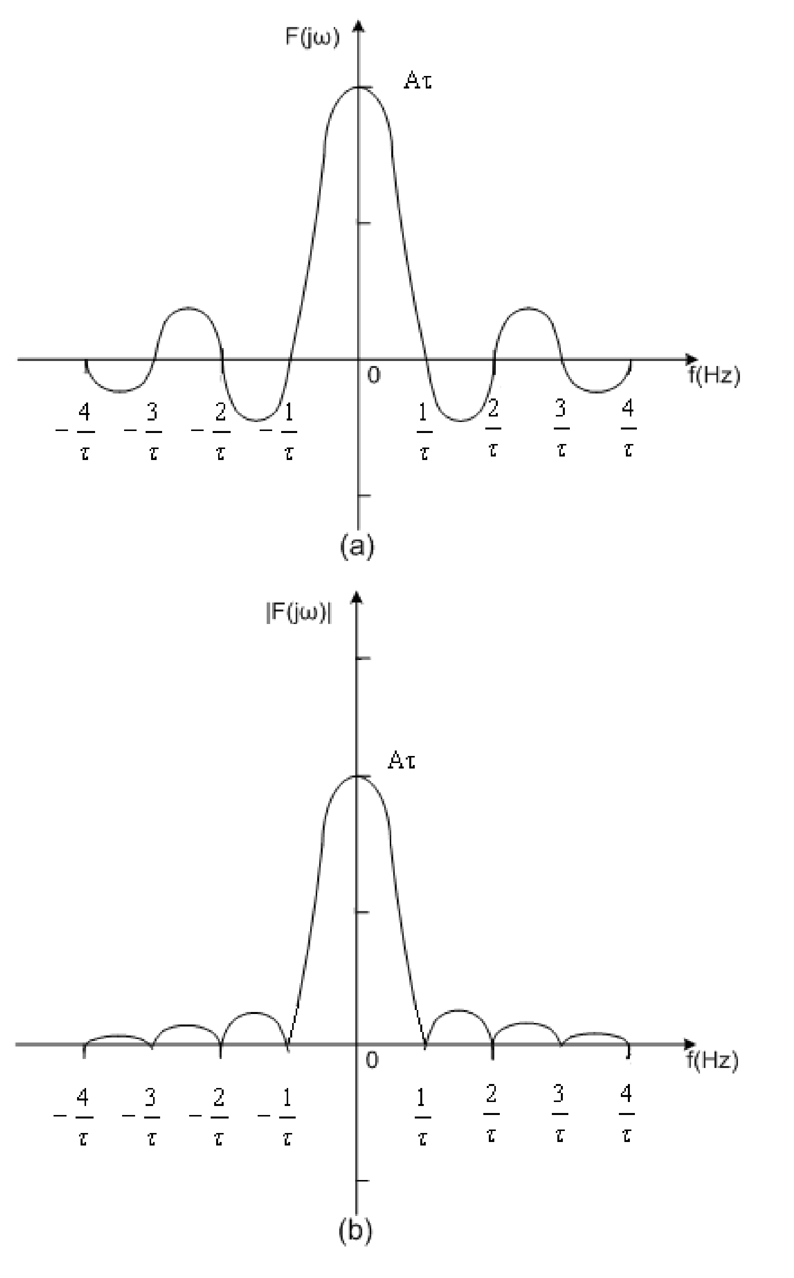
f(t)
T_{on}
T_{off}
$T_0=T_{on}+T_{off}$
(a)
t
d_n
$\frac{1}{T_0}$
n
f(t)
$T_{off}=\infty$
(b)
t
F(jω)
ω

Furijeova transformacija periodičnih signala

- $f(t) = \sum_{n=-\infty}^{\infty} d_n e^{jn\omega_P t}$
- $d_n = \frac{1}{T_P} \int_{-T_P/2}^{T_P/2} f(t) e^{-jn\omega_P t} dt$
- $\mathcal{F}\{f(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} d_n e^{jn\omega_P t}\right\} = \sum_{n=-\infty}^{\infty} d_n \mathcal{F}\left\{e^{jn\omega_P t}\right\} = \sum_{n=-\infty}^{\infty} d_n 2\pi\delta(\omega - n\omega_P)$

- $F(j\omega) = A\tau \frac{\sin(\frac{\omega\tau}{2})}{\frac{\omega\tau}{2}}$

- $F(j\omega) = 0 \; za \; \omega = \frac{2k\pi}{\tau}$ odnosno $f = \frac{k}{\tau}$



Osobine: Linearnost

- $\boldsymbol{\mathcal{F}\{af(t)+bg(t)\}=a\mathcal{F}\{f(t)\}+b\mathcal{F}\{g(t)\}}$

- $\int_{-\infty}^{\infty}[af(t)+bg(t)]e^{-j\omega t}dt=\int_{-\infty}^{\infty}af(t)e^{-j\omega t}dt+\int_{-\infty}^{\infty}bg(t)e^{-j\omega t}dt=$
- $=a\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt+b\int_{-\infty}^{\infty}g(t)e^{-j\omega t}dt=a\mathcal{F}\{f(t)\}+b\mathcal{F}\{g(t)\}$

Osobine: Vremensko pomeranje

- $\boldsymbol{\mathcal{F}\{f(t-\tau)\} = \mathcal{F}\{f(t)\}e^{-j\omega\tau}}$
- $\mathcal{F}\{f(t-\tau)\} = \int_{-\infty}^{\infty}[f(t-\tau)]e^{-j\omega t}dt$ uvedimo smenu: u=t-τ, te je t=u+τ
- $\mathcal{F}\{f(t-\tau)\} = \int_{-\infty}^{\infty}[f(u)]e^{-j\omega(u+\tau)}du = e^{-j\omega\tau}\int_{-\infty}^{\infty}[f(u)]e^{-j\omega u}du = e^{-j\omega\tau}F(j\omega)$

- Važno je primetiti da vremensko kašnjenje prouzrokuje pomeranje faze u frekvencijskom domenu
- Pomeranje faze je proporcionalno frekvenciji: $\Delta\varphi = -\omega\tau$

Osobine: Skaliranje

- $\boldsymbol{\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left[j\frac{\omega}{a}\right]; \quad a \in R}$
- $\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty}[f(at)]e^{-j\omega t}dt$ uvodimo smenu: *at=p*, te je *t=p/a*
- Za $a>0$ važi:
- $\mathcal{F}\{f(at)\} = \frac{1}{a}\int_{-\frac{\infty}{a}}^{\frac{\infty}{a}}[f(p)]e^{-j\omega\frac{p}{a}}dp = \frac{1}{|a|}\int_{-\infty}^{\infty}[f(p)]e^{-j\omega\frac{p}{a}}dp = \frac{1}{|a|}F\left[j\frac{\omega}{a}\right]$
- Za a<0 obrću se granice integraljenja, ali je i 1/a <0 pa važi:
- $\mathcal{F}\{f(at)\} = \frac{1}{a}\int_{-\frac{\infty}{a}}^{\frac{\infty}{a}}[f(p)]e^{-j\omega\frac{p}{a}}dp = -\frac{1}{|a|}\int_{\infty}^{-\infty}[f(p)]e^{-j\omega\frac{p}{a}}dp = \frac{1}{|a|}\int_{-\infty}^{\infty}[f(p)]e^{-j\omega\frac{p}{a}}dp =$
  $\frac{1}{|a|}F\left[j\frac{\omega}{a}\right]$
- Proširivanje signala dovodi do sužavanja spektra i obrnuto

Osobine: Simetrija

- Ako je $\boldsymbol{F(j\omega)} = \boldsymbol{\mathcal{F}\{f(t)\}}$ tada je $\boldsymbol{\mathcal{F}\{F(t)\}} = \boldsymbol{2\pi f(-j\omega)}$
- $F(j\omega) \ = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$, ako zamenimo promenljive $t \leftrightarrow \omega$ tada važi
- $F(t) \ = \int_{-\infty}^{\infty} f(j\omega)e^{-j\omega t}d\omega$
- Sa druge strane: $\mathcal{F}^{-1}\{2\pi f(-j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi f(-j\omega)e^{j\omega t}d\omega$ smenom –ω= ω
- $\mathcal{F}^{-1}\{2\pi f(-j\omega)\} = \int_{-\infty}^{\infty} f(j\omega)e^{-j\omega t}d\omega = F(t)$

Osobine: Konvolucija

- $\boldsymbol{\mathcal{F}\{f(t) * g(t)\} = F(j\omega)G(j\omega)}$
- $f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$
- $\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right]e^{-j\omega t}dt$
- $\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} g(t-\tau)e^{-j\omega t}dt\right]d\tau$; smena: $p = t - \tau$
- $\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} g(p)e^{-j\omega(p+\tau)}dp\right]d\tau$
- $\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau \int_{-\infty}^{\infty} g(p)e^{-j\omega p}dp = F(j\omega)G(j\omega)$

Parselvalova teorema

- $\boldsymbol{W = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega}$
- $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$ i $f(t)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)^* e^{-j\omega t} d\omega$
- $W = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} f(t) f(t)^* \, dt = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)^* e^{-j\omega t} d\omega\right] dt$
- $W = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)^* \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt\right] d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)^* F(j\omega) d\omega$
- $W = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$
- $|F(j\omega)|^2$ se naziva spektralna gustina energije

Osobine F.T. sumarno

TABLE 3.1 Properties of the Fourier Transform

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega) \quad n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\sin(\omega_0 t)$	$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration in the time domain	$\int_{-\infty}^{t} x(\lambda)\, d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)\, d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$