unit - I

Relations →

A relation R from a set A to a set B is a subset of $A \times B$.

2 Finite set A, B

$R : A \to B$

eg → $A = \{1, 2\}$

$B = \{a, b, c\}$

$R_1 = \{(1, a)\ (2, c)\}$ ✓

$R_2 = \{(2, a)\ (2, 2)\}$ ✗ $\quad A \times B = (a, b) \neq (b, a)$

$R_3 = \{(1, a)(1, b)(2, b)\ (2, a)\}$ ✓

$A \times B = \{(1, a)\ (1, b)\ (1, c)\ (2, a)\ (2, b)\ (2, c)\}$

$2 \times 3 = 6$ pairs

$(1, a) \in R$ (belongs to), $(2, 2) \notin R$ (not belongs to).

$R : A \to A$

eg → $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$

which are the relations defined from A to B.

1) $\{(1, 6)\ (3, 4)\ (5, 2)\}$ ✗
2) $\{(1, 5)\ (2, 6)\ (3, 4)\ (3, 6)\}$ ✓
3) $\{(4, 2)\ (4, 3)\ (5, 1)\}$ ✗
4) $A \times B$ ✓ $\quad A \times B \subseteq A \times B$

$R = A \times B = \{(1, 4)\ (1, 5)\ (1, 6)\ (2, 4)\ (2, 5)\ (2, 6)\ (3, 4)\ (3, 5)\ (3, 6)\}$

$R : A \longrightarrow B$

$R \subseteq A \times B$

eg → $R : A \to A$, $R \subseteq A \times A$

$A = \{1, 2\}$

$R = A \times A = \{1, 2\} \times \{1, 2\} = \{(1, 1)\ (1, 2)\ (2, 1)\ (2, 2)\}$

$R_1 = \{(2, 1)\ (2, 2)\}$ ✓

Domain & Range of a Relation →

Domain : Let R be a relation from set A to set B then the set of all the first elements of the ordered pairs which belongs to R is called the domain of R

Range : and the set of all the second element of the ordered pairs which belongs to R, is called the Range of R.

* Every Relation have domain & range.

eg→ $R_1 = \{ (a,b)\ (c,d)\ (e,f) \}$

$R : A \to B$

$D = $ Domain $= \{ a, c, e \}$
$R = $ Range $= \{ b, d, f \}$

Set Builder Form :

Domain $= \{ x \mid x \in A \text{ and } (x, y) \in R \}$

Range $= \{ y : y \in B \text{ and } (x, y) \in R \}$

Inverse Relation →

$R: A \to B$ [Relation]

Domain → {A} $(a,b) \in R$

Range → {B}

$R^{-1}: B \to A$ [Inverse Relation]

Domain → {B} $(b,a) \in R^{-1}$

Range → {A}

→ If A & B are 2 sets, & R be the relation from set A to set B, then the inverse of R, denoted by R^{-1}, is a relation from set B to set A.

$$\boxed{R^{-1} = \{(b,a) \mid (a,b) \in R\}}$$

→ Also domain of R = Range of R^{-1}
& Range of R = Domain of R^{-1}

eg → If $A = \{1,2,3\}$, $B = \{a,b,c,d\}$ are 2 sets &
$R = \{(1,a),(1,c),(2,d),(2,c)\}$
is a relation from set A to B then find R^{-1}.

Sol → $R^{-1} = \{(a,1),(c,1),(d,2),(c,2)\}$

$\boxed{\text{Total no. of Relations}} \rightarrow 2^{mn}$

→ Let A & B are 2 non-empty finite sets having m and n element respectively, then A × B consist of (m, n) ordered pair

$A = \{1, 2\}$
$B = \{a, b, c\}$
$2 \times 3 = 6$ = ordered Pair = $m \times n$
$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

→ therefore total no.s of subsets of A × B is $2^{m \times n} = 2^{mn}$.

$R \subseteq A \times B$

$2^{2 \times 3} = 2^6 = 64$ ← max^m no. of Relatⁿ

φ   A × B

$\boxed{\text{NOTE}}$ → In this R included the 2 trivial relation also given by the null set & the set A × B.

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eg → $A = \{1\}$ $B = \{a, b\}$

$1 \times 2 = 2$

$A \times B = \{ (1, a) \ (1, b) \}$

$$2^{mn} = 2^{1\times 2} = 4$$

$R_1 = \{(1, a)\}$
$R_2 = \{(1, b)\}$
$R_3 = \{(1, a) (1, b)\}$
$R_4 = \{\phi\}$

Types of Relation →

① **Identity Relation** → the identity relation of set A is that relation on A, in which each element of A is related to itself.

- No extra pair element

→ It is normally denoted by I_A.

eg → $A = \{1, 2, 3\}$

$R : A \to A$

$I_A = \{(1,1) \ (2,2) \ (3,3)\}$

$\boxed{(1, a) \quad 1 R a}$

$\boxed{1 R 1}$

$A \times A = \{ (1,1) \ (1,2) \ (1,3)$
$(2,1) \ (2,2) \ (2,3)$
$(3,1) \ (3,2) \ (3,3) \}$

$I_A' = \{(1,1), (2,2)\}$ ← yes ✓ Identity R or not ?

$$I_A = \{ (a, a) \mid a \in A \}$$

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(2) **Reflexive Relation** → A relation R on a set A is said to be reflexive if every element of A is related to itself.

- extra pair element can be present

NOTE → A relation R on a set A is not reflexive if there exist an element $a \in A$ such that $(a,a) \notin R$

eg → $A = \{1,2,3\}$ $R: A \to A$

Reflexive → $R_1 = \{(1,1)\ (2,2)\ (3,3)\ (1,3)\ (2,1)\} = \cancel{I_A}$

Not Reflexive → $R_2 = \{(1,1)\ (3,3)\ (2,1)\ (3,2)\} = \cancel{I_A}$

(3) **Symmetric Relation** → A relation R on a set A is said to be a symmetric Relation, iff $(a,b) \in R \Leftrightarrow (b,a) \in R$, $a,b \in A$

$R: A \to A$

$A = \{1,2,3\}$

Symm ✓ $R_1 = A \times A$

$A \times A = \{1,2,3\} \times \{1,2,3\}$

No Sy. ✗ $R_2 = \{(1,2)\ (1,3)\ (3,1)\ (2,3)\ (3,2)\}$

Sym R. ✓ $R_3 = \{(1,2)\ (2,1)\ (1,3)\ (3,1)\ (2,3)\ (3,2)\}$

Symm R. ✓ $R_4 = \{(1,1)\ (2,2)\}$

eg → let L be the set of all lines in a plane. Let R be the Relation defined as $L_1 \, R \, L_2 \Rightarrow L_1$ is perpendicular to L_2

$R : L \to L$

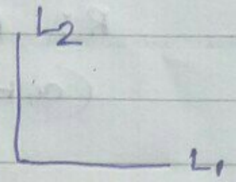
$R = (L_1, L_2)$

$L_1 \perp L_2$

$L_2 \perp L_1$ ✓

$(L_1, L_2) \in R$

$(L_2, L_1) \in R$ ✓



Relation is Symmetric

④ **Transitive Relation** →

let A be any set, then a Relation R on set A is said to be a transitive relation iff

$(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$
where $a, b, c \in A$

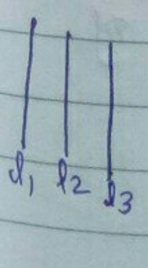
eg → Let R be the set of all lines in a plane defined by $(l_1, l_2) \in R \Rightarrow$ line l_1 is parallel to line l_2. So that R is a transitive relation.

$l_1, l_2, l_3 \in L$

$(l_1, l_2) \in R$ and $(l_2, l_3) \in R \Rightarrow (l_1, l_3) \in R$

↳ line l_1 is parallel to line L_1

line l_2 is parallel to line l_3



So Relation R is Transitive.

(5) **Anti-Symmetric Relatⁿ** → A Relatⁿ R on a set A is said to be anti-symmetric Relation IFF $(a,b) \in R$ and $(b,a) \in R \Rightarrow a=b$, $\forall a,b \in A$

$R: A \to A$
$(a,b) \in R$, $a \neq b$, $(b,a) \notin R$, $\forall a,b \in A$

eg → $R: A \to A$
$A = \{a,b\}$, $R = A \times A = \{(a,b)(a,a)(b,a)(b,a)\}$
$R_1 = \{(a,a)(b,b)\}$
$R_2 = \{(a,b)(b,a)\}$ ✗ A.S.R.

$A = \{1,2,3,4\}$
$R_1 = \{(1,2)(2,3)\}$ ✓
$R_2 = \{(1,2)\}$ ✓
$R_3 = \{(1,2)(2,1)\}$ ✗

(6) **Equivalence Relation** A Relatⁿ R on a set A is said to be equivalence R. IFF,
i) it is Reflexive $(a,a) \in R$, $\forall a \in A$
ii) It is symmetric $(a,b) \in R$, $(b,a) \in R$
iii) It is Transitive $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$

eg → $R: A \to A$, $A = \{$
I) Let R be a Relation on the set of all lines

$$L = \{l_1, l_2, l_3, l_4 \ldots\}$$

Proove → $R : L \to L$

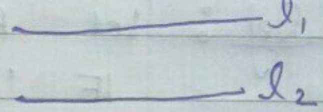
$$l \in L$$

$$(l_1, l_2) \in R \quad , \quad l_1 \parallel l_2$$

I) Reflexive → $l_1 \in L$

$$(l_1, l_1) \in R \quad , \quad l_1 \parallel l_1$$

II) Symmetric → $l_1, l_2 \in L$

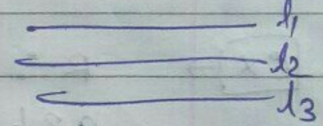


$$(l_1, l_2) \overset{\in R}{\Rightarrow} \cancel{(l_2, l_1) \in R} \quad l_1 \parallel l_2 \qquad l_1 \parallel l_2$$

$$(l_2, l_1) \in R \Rightarrow l_2 \parallel l_1$$

III) Transitive →

$$l_1, l_2, l_3 \in L$$



$$(l_1, l_2) \in R \Rightarrow l_1 \parallel l_2$$

$$(l_2, l_3) \in R \Rightarrow l_2 \parallel l_3$$

$$(l_1, l_3) \in R \Rightarrow l_1 \parallel l_3$$

∴ R is Equivalence.

(7) # $\boxed{\text{Partial Order Relation}}$ → A Relat^n R on a set A is called partial order IF R is Reflexive, Anti-symmetric & Transitive.

Equivalence Classes →

A = {1, 2, 3, 4}

$A_1 = \{2\}$ $A_2 = \{3\}$ $\{1, 4\}$

$A_1 \cap A_2 \cap A_3 = \phi$

$A_1 \cup A_2 \cup A_3 = A = \{1, 2, 3, 4\}$

Def: Let R be an equivalence relation on set A. If 'a' is any element of set A; $a \in A$, then the equivalence class of set A is denoted by [a] or $\overline{a}$, and defined as

$$[a] = \{ x \mid x \in A \text{ and } xRa \}$$

$(x, a) \in R$

$\boxed{ex}$ → R : Z (integer set) $Z = \{0, \pm 1, \pm 2 - - \}$

aRb

$(a,b) \in R \Rightarrow (a-b)$ is divisible by 4.

A = {1, 2}

R = { (1,1) (1,2) (3,3) }

$(a,c) \in R \Rightarrow (a-c)$ is divisible by 4.

$R : Z \to Z$

$(a,b) \in R \Rightarrow (a,b)$ is divisible by 4.

I Reflexive property →

$\forall a \in A$, $a \in Z$

$(a,a) \in R \Rightarrow (a-a)$ is divisible by 4.

$\Rightarrow$ 0 is divisible by 4.

II <u>Symmetric</u> → $(a,b) \in R \Rightarrow (b,a) \in R$

$a,b \in Z$

$(a,b) \in R \Rightarrow (a-b)$ is divisible by 4

~~(b,a)~~ ~~(a,b)~~

$(a-b) = 4n \;,\; n \in Z$

$\frac{16}{4}$ $16 = 4 \times 4$ $16 = 4n$

$(b,a) \in R \Rightarrow (b-a)$ is divisible by 4

$-(a-b) = -4n$

$b-a = 4(-n) \;,\; -n \in Z$

$\frac{-16}{4}$ $-16 = 4(-4)$

$\frac{b-a}{4} =$ $-n$

III <u>Transitive</u> → $a,b,c \in Z$

$(a,b) \in R \Rightarrow (a-b) = 4n$ —— ① $, n \in Z$

$(b,c) \in R \Rightarrow (b-c) = 4m$ —— ② $, m \in Z$

$(a,c) \in R \Rightarrow (a-c) = 4P \;,\; P \in Z$

① + ②

$(a-b) + (b-c) = 4n + 4m$

$a-c = 4(n+m) \;,\; n+m \in Z$

$a-c = 4P \;,\; P \in Z$

$\therefore (a,c) \in R$

ie. Relation is Equivalence Relation.

$$[a] = \{x \mid x \in A \ \& \ (x, a) \in R\}$$

$$\boxed{x - a = 4k}$$

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$$Z = \{0, \mp 1, \mp 2, \mp 3, \mp 4, \ldots\}$$

$$[0] = \bar{0} = \{x \mid x \in Z \text{ and } (x, 0) \in R\}$$

$$\{x \mid x \in Z \text{ and } (x-0) \text{ is divi by } 4\}$$

$$= \{x \mid x \in Z \text{ and } (x-0) = 4k, \ k \in Z\}$$

$$\{x \mid x \in Z \text{ and } x = 4k, \ k \in Z\}$$

$k = 0 \Rightarrow x = 0$

$k = 1 \Rightarrow x = 4$

$k = -1 \Rightarrow x = -4$

$x = 2 \Rightarrow x = 8$

$x = -2 \Rightarrow x = -8$

$x = 3 \Rightarrow x = 12$

$x = -3 \Rightarrow x = -12$

$\Rightarrow$

$$[0] = \{\ldots -12, -8, -4, 0, 4, 8, 12 \ldots\}$$

$$[1] = \bar{1} = \{x \mid x \in Z \text{ and } (x, 1) \in R\}$$

$$= \{x \mid x \in Z \text{ and } (x-1) = 4k, \ k \in Z\}$$

$$x = 4k + 1$$

$k = 0 \Rightarrow x = 1$

$k = 1 \Rightarrow x = 5$

$k = -1 \Rightarrow x = -3$

$k = 2 \Rightarrow x = 9$

$k = -2 \Rightarrow x = -7$

$k = 3 \Rightarrow x = 13$

$k = -3 \Rightarrow x = -11$

$$[1] \Rightarrow \{\ldots -11, -7, -3, 1, 5, 9, 13 \ldots\}$$

$[2] = \{ x \mid x \in \mathbb{Z}$ and $(x, 2) \in R \}$

$\{ x \mid x \in \mathbb{Z}$ and $(x-2) = 4K,\ K \in \mathbb{Z}$

$x = 4K + 2$

$K = 0 \quad x \Rightarrow 2$

$K = 1 \quad x = 6$

$K = -1 \quad x = -2$

$K = 2 \quad x = 10$

$K = -2 \quad x = -6$

$K = 3 \quad x = 14$

$[0]$

$[1] \quad K = -3 \quad x = -10$

$[2] = \{ \ldots -4, -10, -6, -2, 2, 6, 10, 14 \}$

$[3] = \{ \ldots -13, -9, -5, -1, 3, 7 \ldots \}$

$[4] = \{ \ldots 0, \pm 4, \pm 8, \pm 12 \ldots \}$

$[5]\ [8]$

$[4] \quad [5] \quad [6] \quad [7]$ 4 Equivalence classes, then they are repeat

$[0] \quad [1] \quad [2] \quad [3]$

$\mathbb{Z} = [0] \cup [1] \cup [2] \cup [3]$

$[0] \cap [1] \cap [2] \cap [3] = \phi$

partitions of set

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$[3] = \{ x \mid x \in Z \text{ and } (x, 3) \in Z \}$

$(x-3) = 4K \quad , \quad K \in Z$

$x = 4K + 3$

$K = 0 \qquad x = 3$

$K = 1 \qquad x = 7$

$K = -1 \qquad x = -1$

$K = 2 \qquad x = 11$

$K = -2 \qquad x = -5$

$K = 3 \qquad x = 15$

$K = -3 \qquad x = -9$

$[3] = \{ \ldots -13, -9, -5, -1, 3, 7, 11, 15 \}$

x — x — x

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Operations On Relation

$R_1 : A \to B$

$R_2 : A \to B$

$A = \{1, 2, 3\}$ $\quad B = \{3, 4, 5\}$

$R_1 = \{(1,3)\ (1,4)\ (3,3)\}$

$R_2 = \{(2,3)\ (2,4)\ (3,3)\ (2,5)\}$

① $\boxed{\text{union}}$ $R_1 \cup R_2 = \{(1,3)\ (1,4)\ (3,3)\ (2,3)\ (2,4)\ (2,5)\}$

② $\boxed{\text{Intersection}}$ $R_1 \cap R_2 = \{(3,3)\}$

③ Difference of 2 set

$R_1 - R_2 = \{(1,3)\ (1,4)\}$

$R_2 - R_1 = \{(2,3)\ (2,4)\ (2,5)\}$

④ Symmetric Difference of 2 set

$(A \oplus B)$ $\quad R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)$

$= \{(1,3)\ (1,4)\ (2,3)\ (2,5)\ (2,4)\}$

Closure of Relations →

① Reflexive closure →

$R: A \to A$ $\quad A = \{1, 2, 3\}$

Let say we have a relation R which is defined on set A to A

$$R = \{(1,1)\ (2,2)\ (2,3)\}$$

$$\forall a \in A,\ (a,a) \in R$$

The relation R is not reflexive.
The smallest reflexive relatn that contains R must include the ordered pair (3,3)

$$R_{New} = \{(1,1)\ (2,2)\ (3,3)\ (2,3)\}$$

Def: Reflexive closure of a relatn R on a set A is the smallest reflexive relatn of the set A that contains R. The Reflexive closure of R is usually denoted by R_r^+.

$$R_r^+ = R \cup \{(a,a) \mid a \in A\}$$

(2) **Symmetric closure** → of a relatn R on a set A is the smallest ~~reflexive~~ symmetric relation that contains R.

$$R_S^+ = R \cup \{(b,a) \mid (a,b) \in R\}$$

eg → Let R be the relatn on the set $A = \{0,1,2,3\}$ containing the ordered pair $R = \{(0,1)\ (1,1)\ (1,2)\ (2,0)\ (2,2)\ (3,0)\}$ Find the sym clos of R.

$$R_S^+ = \{(0,1)\ (1,0)\ (1,1)\ (1,2)\ (2,1)\ (2,0)\ (0,2)\ (2,2)\ (3,0)\ (0,3)\}$$

III **Transitive closure** → of a binary relation R on a set A is the smallest transitive relatn on a set A that contains R.

$$R_T^+ = R \cup \{(a,c) \mid (a,b) \in R \text{ and } (b,c) \in R\}$$

eg → Let R be the relation on a set $A = \{1,2,3\}$ containing the ordered pair (1,1) (2,3) and (3,1). Find the transi clo of R.

$$R = \{(1,1)\ (2,3)\ (3,1)\}$$

$$R_T^+ = \{(1,1)\ (2,3)\ (3,1)\ (2,1)\}$$