Chapter 10. Time-Correlation, Energy and Power Spectra

In this chapter, we go back to the concepts of energy and power and show that they can be subjected to spectral analysis too. The concept of “correlation” of two signals is introduced and its relationship with energy and power are explained.

This is a shortened version of this chapter, used in the academic year 2020/2021.

These topics are important per se but are perhaps more important in view of the introduction of stochastic processes, where conventional spectra have little meaning and instead such quantities as energy and power spectra or correlation often are the only ones that can be evaluated.

10.1 Finite Energy Signals: Correlations and Energy Spectra

10.1.1 The autocorrelation function

Definition:

Given a finite-energy signal $x(t)$, that is $x(t) \in L^2_{\mathbb{R}}$, its *autocorrelation function* is defined as follows:

$$R_x(\tau) \triangleq \left(x(t), x(t-\tau)\right) = \int_{-\infty}^{+\infty} x(t) x^*(t-\tau) dt$$

Eq. 10-1

where τ is a delay parameter that clearly has dimensions of time. Essentially, $R_x(\tau)$ is the inner product of a signal times a delayed version of itself.

We now explore the main properties of this quantity.

10.1.1.1 value at $\tau = 0$

We have:

$$R_x(0) = \left(x(t), x(t)\right) = \mathcal{E}\left\{x(t)\right\} = \left\|x(t)\right\|^2$$

So, for $\tau = 0$, the autocorrelation function is equal to the energy of the signal or, equivalently, to its norm squared.

10.1.1.2 symmetry properties

The autocorrelation enjoys the following symmetry relation:

$$R_x(-\tau) = R_x^*(\tau)$$

Eq. 10-2

Note that if $x(t) \in \mathbb{R}$ then $\mathrm{Im}\left\{R_x(\tau)\right\} = 0\left(\tau\right)$. We can therefore state that:

the autocorrelation function $R_x(\tau)$ of a real signal is real and even.

Proof:

We change sign to the delay parameter τ . We get:

$$R_x(-\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t+\tau)dt$$

We then change the integration variable: $\rho = t + \tau$, from which we also have $t = \rho - \tau$. We can then write:

$$R_x(-\tau) = \int_{-\infty}^{+\infty} x(\rho - \tau)x^*(\rho)d\rho = \int_{-\infty}^{+\infty} x(t-\tau)x^*(t)dt = R_x^*(\tau)$$

10.1.1.3 range of possible values of $R_x(\tau)$

The autocorrelation function has the following interesting property:

$$\left|R_x(\tau)\right| \le R_x(0)$$

Eq. 10-3

which can also be written as:

$$-R_x(0) \le R_x(\tau) \le R_x(0)$$

Eq. 10-4

This latter relation shows that the autocorrelation function is bounded between plus and minus its value at the origin. The above property can also be re-stated in terms of the energy or norm squared of the signal, since: $R_x(0) = \left\|x(t)\right\|^2 = \mathcal{E}\{x\}$.

$$-\mathcal{E}\{x\} \le R_x(\tau) \le \mathcal{E}\{x\}$$

This property can be proved in the frequency domain. We know that inner products yield the same value whether they are carried out in time or frequency. So, we can write:

$$R_x(\tau) = \left(x(t), x(t-\tau) \right) = \left(X(f), X(f) e^{-j2\pi f\tau} \right)$$
$$= \int_{-\infty}^{+\infty} X(f) X^*(f) e^{j2\pi f\tau} dt = \int_{-\infty}^{+\infty} \left| X(f) \right|^2 e^{j2\pi f\tau} dt$$

We can now take the absolute value of the leftmost and rightmost elements of this chain of equalities:

$$\left| R_x(\tau) \right| = \left| \int_{-\infty}^{+\infty} \left| X(f) \right|^2 e^{j2\pi f\tau} dt \right|$$

We then remember from calculus that:

$$\left| \int f(x) dx \right| \leq \int \left| f(x) \right| dx$$

Using this result, we get:

$$\left|R_x(\tau)\right| = \left|\int_{-\infty}^{+\infty} \left|X(f)\right|^2 e^{j2\pi f\tau} dt\right| \le \int_{-\infty}^{+\infty} \left|\left|X(f)\right|^2 e^{j2\pi f\tau}\right| dt$$
$$= \int_{-\infty}^{+\infty} \left|X(f)\right|^2 \left|e^{j2\pi f\tau}\right| dt = \int_{-\infty}^{+\infty} \left|X(f)\right|^2 dt = \mathcal{E}\left\{x(t)\right\}$$

That is:

$$\left|R_x(\tau)\right| \le \mathcal{E}\left\{x\right\} \quad \Rightarrow \quad -\mathcal{E}\left\{x\right\} \le R_x(\tau) \le \mathcal{E}\left\{x\right\}$$

10.1.1.4 the delay and sign invariance

Given the signal $x(t)$ and its delayed version $y(t) = x(t - t_d)$, they have the same autocorrelation function, that is $R_x(\tau) = R_y(\tau)$. This property is easy to prove by a simple change of variable in the integral.

$$R_y(\tau) = \int_{-\infty}^{+\infty} y(t)\, y^*(t-\tau)\, dt = \int_{-\infty}^{+\infty} x(t-t_d)\, x^*(t-t_d-\tau)\, dt$$

Using: $\rho = t - t_d$, so that $d\rho = dt$, we get:

$$R_y(\tau) = \int_{-\infty-t_d}^{+\infty-t_d} x(\rho)\, x^*(\rho-\tau)\, dt = \int_{-\infty}^{+\infty} x(\rho)\, x^*(\rho-\tau)\, dt = R_x(\tau)$$

Also, given the signal $x(t)$ and its version with opposite sign: $y(t) = -x(t)$, they have the same autocorrelation function, that is $R_x(\tau) = R_y(\tau)$. This property is obvious:

$$
\begin{aligned}
R_y(\tau) &= \int_{-\infty}^{+\infty} y(t)\, y^*(t-\tau)\, dt \\
&= \int_{-\infty}^{+\infty} \left[-x(t)\right]\left[-x^*(t-\tau)\right] dt \\
&= \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt
\end{aligned}
$$

10.2 The Energy Spectrum

We recall that the energy of a signal is defined as follows:

$$\mathcal{E}\left\{x(t)\right\} = \left\|x(t)\right\|^2 = \int_{-\infty}^{+\infty} \left|x(t)\right|^2 dt$$

We then showed that Parseval's relation carries over to the frequency-domain too so that:

$$\mathcal{E}\left\{x(t)\right\} = \mathcal{E}\left\{X(f)\right\} = \left\|X(f)\right\|^2 = \int_{-\infty}^{+\infty} \left|X(f)\right|^2 df$$

Eq. 10-5

We now define a new quantity called *energy spectral density* or *energy spectrum* of a finite-energy signal $x(t)$ as:

$$S_x(f) \triangleq \left|X(f)\right|^2$$

Eq. 10-6

Based on this definition we can re-write Eq. 10-5 as:

$$\mathcal{E}\left\{x(t)\right\} = \int_{-\infty}^{+\infty} S_x(f) df$$

Eq. 10-7

This formula shows that the overall energy of a signal comes from the "summation" (over the continuous index f) of many contributions.

In practice, each frequency-component $X(f)$ of the signal adds a little bit of energy in the form of $|X(f)|^2$ to the total energy of the signal. This is similar to what we saw for signals in L^2_{I}, decomposed according to the discrete Fourier basis (i.e., in Fourier series), where what we called the "energy spectrum" was provided by $\left\{|s_n|^2\right\}_{n=-\infty}^{\infty}$, with $|s_n|^2$ being the energy carried by the signal component at frequency $n \cdot f_0$.

Note that, formally, according to Eq. 10-7 we can view $S_x(f)$ as:

$$S_x(f) = \frac{d\mathcal{E}}{df}$$

because:

$$\int_{-\infty}^{+\infty} \frac{d\mathcal{E}}{df} df = \int_{-\infty}^{+\infty} S_x(f) df = \mathcal{E}$$

In physics, those quantities that provide infinitesimal contributions such as $S_x(f) = d\mathcal{E}/df$, are called "densities". For instance, the linear mass density $\mu(l) = \frac{dm}{dl}$ of a steel bar provides infinitesimal contributions to mass, along a spatial coordinate. It delivers the total bar mass through an integral over the bar length:

$$\int_{-\infty}^{+\infty} \frac{dm}{dl} dl = \int_{-\infty}^{+\infty} \mu(l) dl = m$$

By similarity, then, $S_x(f) = d\mathcal{E}/df$ is called "energy density over frequency" or equivalently "energy spectral density", because it represents infinitesimal contributions to energy, over frequency. For brevity, we call it "energy spectrum".

Note that, given the energy spectral density $S_x(f)$, we can actually look at the energy contribution of a *range* of frequencies, rather than the total energy collected by looking at all frequencies. Specifically, given a frequency range $\left[f_1, f_2\right]$, the energy contributed by just these frequencies is found as:

$$\mathcal{E}_{[f_1, f_2]} = \int_{f_1}^{f_2} \frac{d\mathcal{E}}{df} df = \int_{f_1}^{f_2} S_x(f) df$$

10.2.1 Relationship between autocorrelation and energy spectrum

The energy spectrum and the autocorrelation function are closely related. We first recall that $R_x(\tau)$ is essentially an inner product. We then remark that inner products can be carried out either in "time-domain" or in "frequency-domain", with the same result. So:

$$R_x(\tau) = \left(x(t), x(t-\tau)\right) = \left(X(f), X(f)e^{-j2\pi f\tau}\right)$$

Note that the factor $e^{-j2\pi f\tau}$ accounts for the delay present in $x(t-\tau)$. Expanding the rightmost side of the equality:

$$\left(X(f), X(f)e^{-j2\pi f\tau} \right) = \int_{-\infty}^{+\infty} X(f)X^*(f)e^{j2\pi f\tau}df$$

$$= \int_{-\infty}^{+\infty} \left|X(f)\right|^2 e^{j2\pi f\tau}df = \int_{-\infty}^{+\infty} S_x(f)e^{j2\pi f\tau}df$$

We then recognize the last integral to be the inverse-Fourier transform of the energy spectrum $S_x(f)$, so in conclusion:

$$R_x(\tau) \quad \overset{\mathrm{F}}{\longleftrightarrow} \quad S_x(f)$$

Eq. 10-8

In other words, *the energy spectrum* $S_x(f)$ *and the autocorrelation function* $R_x(\tau)$

of the same finite-energy signal $x(t)$ are strictly related through a Fourier transformation.

10.2.1.1 problem

Given the signal:

$$x(t) = \Pi_T \left(t - \tfrac{T}{2}\right)$$

compute $R_x(\tau)$ and $S_x(f)$.

By definition, $R_x(\tau)$ is:

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t-\tau)dt$$

Eq. 10-9

Since we have:

$$x(t) = \Pi_T\left(t - \tfrac{T}{2}\right)$$

$$x(t-\tau) = \Pi_T\left(t - \tau - \tfrac{T}{2}\right)$$

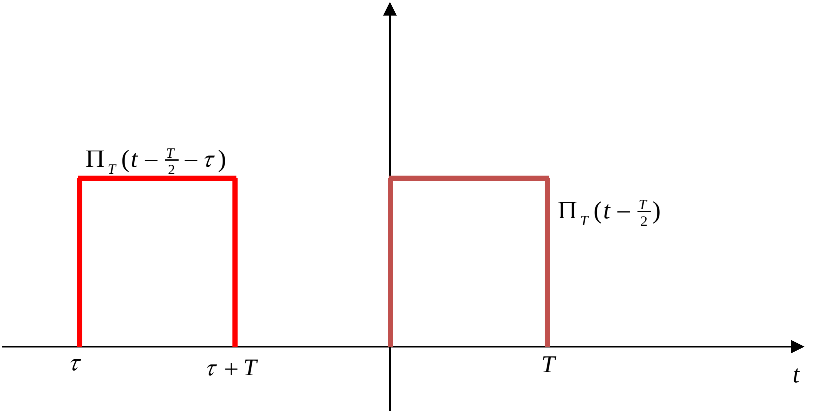
by direct substitution into the integral in Eq. 10-9, we get:

$$R_x(\tau) = \int_{-\infty}^{\infty} \Pi_T\left(t - \tfrac{T}{2}\right)\Pi_T\left(t - \tfrac{T}{2} - \tau\right)$$

Similarly to what happens in convolution products, one function is stationary, namely: $\Pi_T\left(t - \frac{T}{2}\right)$, and the other function $\Pi_T\left(t - \frac{T}{2} - \tau\right)$ actually "shifts" along the integration variable axis t, by a delay τ. We then have four possible cases.

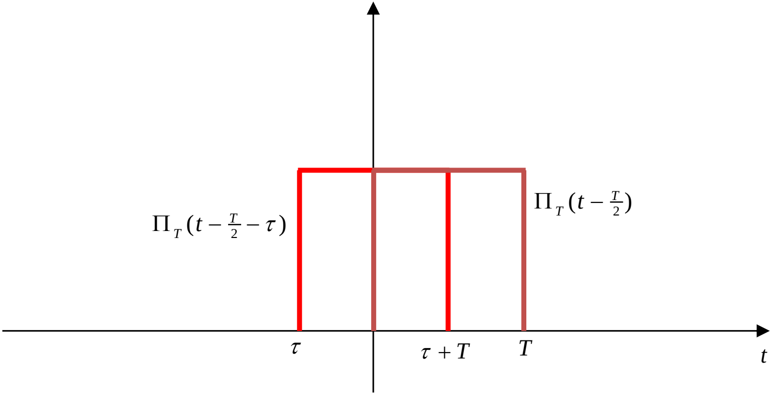
Note: in the following figures, the symbol $P_T(t)$ stands for $\Pi_T(t)$.

Case $\tau + T < 0$



As shown above, when $\tau + T < 0$ the two functions making up the overall integrand have disjoint extensions and therefore the integral is zero.

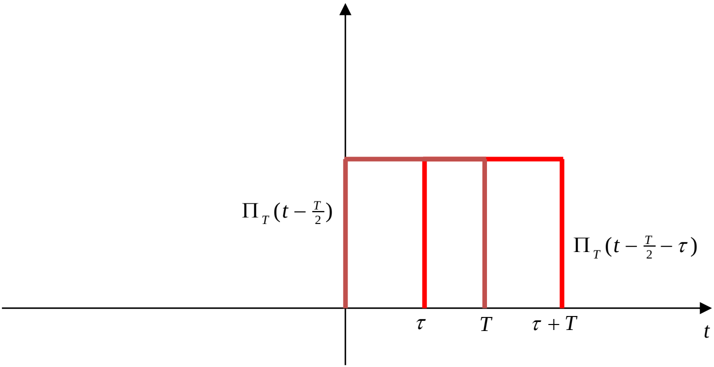
Case $0 < \tau + T < T$



In this case, the extensions of the integrand functions overlap partially and so:

$$R_x(\tau) = \int_0^{\tau+T} 1(t)dt = T + \tau$$

Case $0 < \tau < T$



Here too we have partial overlap, though on the other side of the "stationary" blue rectangle. The integration limits change and the result is:

$$R_x(\tau) = \int_\tau^T 1(t)dt = T - \tau$$

Case $\tau > T$

When $\tau > T$ the extensions of the functions are disjoint and therefore the overall integrand function is zero everywhere. As a consequence, the integral is also zero everywhere.

Summarizing, we get:

$$R_x(\tau) = \begin{Bmatrix} 0 & \tau >| T | \\ \tau + T & -T < \tau < 0 \\ -\tau + T & 0 < \tau < T \end{Bmatrix} = T \cdot \Lambda_T\left(\tau\right)$$

Eq. 10-10

To obtain the energy spectral density, we transform the autocorrelation function:

$$S_x(f) = \mathrm{F}\left\{R_x(\tau)\right\} = \mathrm{F}\left\{T \cdot \Lambda_T\left(\tau\right)\right\} = T \cdot \mathrm{F}\left\{\Lambda_T\left(\tau\right)\right\}$$

Eq. 10-11

Remembering that:

$$\mathrm{F}\left\{\Lambda_T\left(\tau\right)\right\} = T \cdot \mathrm{Sinc}^2\left(fT\right)$$

we then have:

$$S_x(f) = T^2 \cdot \mathrm{Sinc}^2\left(fT\right) = \frac{\sin^2\left(\pi f T\right)}{\pi^2 f^2}$$

Clearly, the property required of all energy spectra:

$$S_x(f) = \mathrm{F}\left\{R_x(\tau)\right\} = |X(f)|^2 \geq 0$$

is indeed satisfied.

The energy of the signal is:

$$\mathcal{E}\{x(t)\} = \int_{-\infty}^{+\infty} |\Pi_T(t)|^2 \, dt = \int_{-\infty}^{+\infty} S_x(f)\, df = R_x(0) = T$$

10.2.1.2 problem

Given the signal:

$$x(t) = e^{-at}u(t)$$

find $R_x(\tau)$ *and* $S_x(f)$.

To compute the autocorrelation function we need to evaluate the following integral

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)dt = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-a(t-\tau)}u(t-\tau)dt$$

Eq. 10-12

Note the presence of $u(t)$: as a result, the integrand function is zero for $t < 0$. Therefore, we can directly set the lower limit of the integral to zero, and remove $\mathrm{u}(t)$:

$$R_x(\tau) = e^{a\tau} \int_0^{+\infty} e^{-2at}u(t-\tau)dt$$

We have also extracted the factor $e^{a\tau}$ because it does not depend on the integration variable t.

We now discuss the factor $u(t-\tau)$. It is $u(t-\tau) = 0$, for $t-\tau < 0$, that is, for

$t < \tau$. So, this means that the integrand function is certainly zero whenever $t < \tau$. Consequently, the integral lower limit should be equal to τ. However, we have also previously found that the integral lower limit cannot be lower than zero. Putting the two requirements together, it turns out that the lower integration limit t_{low} must be such that:

$$\left(t \geq \tau \quad \text{and} \quad t \geq 0 \right)$$

where "and" means in fact that both have to be true together. This condition is equivalent to the following:

$$t_{\text{low}} = \max\left\{0, \tau\right\}$$

that is:

$$R_x(\tau) = e^{a\tau} \int_{\max\{0,\tau\}}^{+\infty} e^{-2at} dt$$

There are then two possible cases. For $\tau < 0$,

$$R_x(\tau) = e^{a\tau}\int_0^{+\infty} e^{-2at}dt = e^{a\tau}\left[-\frac{e^{-2at}}{2a}\right]_0^{+\infty} = \frac{1}{2a}e^{a\tau}$$

Instead, for $\tau > 0$ the integral becomes:

$$R_x(\tau) = \int_\tau^{+\infty} e^{-2at}e^{a\tau}dt = e^{a\tau}\left[-\frac{e^{-2at}}{2a}\right]_\tau^{+\infty} = \frac{1}{2a}e^{a\tau}e^{-2a\tau} = \frac{1}{2a}e^{-a\tau}$$

Putting the two together, we get:

$$R_x(\tau) = \begin{cases} \dfrac{1}{2a} e^{-a\tau} & \tau > 0 \\ \dfrac{1}{2a} e^{a\tau} & \tau \leq 0 \end{cases} = \frac{1}{2a} e^{-a|\tau|}$$

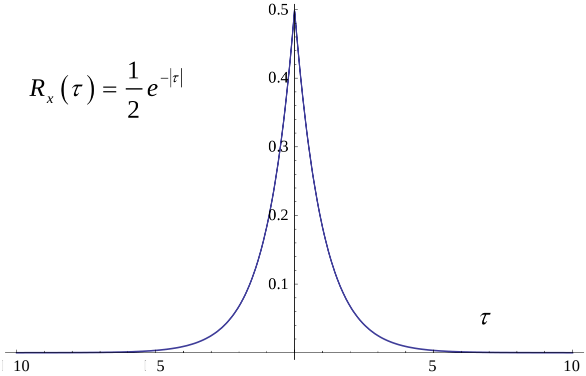


Fig. 10-1 Autocorrelation function of the decreasing unilateral exponential for $a = 1$.

We now calculate the energy spectrum. We know that:

$$\begin{aligned}
S_x(f) &= \mathrm{F}\{R_x(\tau)\} = \frac{1}{2a}\mathrm{F}\{e^{-a|\tau|}\} = \frac{1}{2a}\mathrm{F}\{e^{-a\tau}u(\tau)+e^{a\tau}u(-\tau)\} \\
&= \frac{1}{2a}\mathrm{F}\{e^{-a\tau}u(\tau)\} + \frac{1}{2a}\mathrm{F}\{e^{a\tau}u(-\tau)\} \\
&= \frac{1}{2a}\left[\frac{1}{a+j2\pi f}+\frac{1}{a-j2\pi f}\right] \\
&= \frac{1}{2a}\left[\frac{a-j2\pi f}{a^2+4\pi^2 f^2}+\frac{a+j2\pi f}{a^2+4\pi^2 f^2}\right] = \frac{1}{a^2+4\pi^2 f^2}
\end{aligned}$$

So, in the end:

$$S_x(f) = \frac{1}{a^2+4\pi^2 f^2}$$

This spectrum has its own name: it is called a Lorentzian spectrum. It appears quite often in Physics, for instance regarding laser emission.

$\boxed{\text{On your own:}}$ calculate the energy spectral density of the signal $x(t) = \Lambda_T(t)$.

Answer:

$S_x(f) = T^2 \cdot \mathrm{Sinc}^4(f \cdot T)$

10.2.2 Cross-correlation function

The cross-correlation function between two finite-energy signals $x(t)$ and $y(t)$ is defined as follows:

$$R_{xy}(\tau) \triangleq \left(x(t), y(t-\tau) \right)$$

Eq. 10-13

In the following, the main properties of this function are discussed.

10.2.3 Energy cross-spectral-density

Recalling that the energy spectrum was defined as:

$$S_X(f) = X(f)X^*(f)$$

it comes as a natural extension to define a *cross energy spectrum* or *cross energy spectral density* as:

$$S_{XY}(f) \triangleq X(f)Y^*(f)$$

Eq. 10-14

It turns out that:

$$S_{XY}(f) = \mathcal{F}\left\{R_{xy}(\tau)\right\}$$

Eq. 10-15

10.3 Finite Average Power Signals Correlations and Power Spectra

For finite-average-power signals, the energy integral diverges:

$$E\{x\} = \int_{-\infty}^{\infty} |x(t)|^2 dt \to \infty$$

Also, the quantity $R_x(\tau)$ defined in the previous section for finite energy signals cannot be used because it does not converge. We then need to define different quantities.

10.3.1 Autocorrelation Function

We recall from Chapter 3 that the average power of a finite-average power signal is given by:

$$\mathcal{P}\left\{x(t)\right\} \triangleq \left\langle P_s(t)\right\rangle_{\mathbb{R}} = \lim_{T_0 \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left|x(t)\right|^2 dt$$

The presence of the renormalization factor $1/T$ is key for ensuring the convergence of the integral.

As the integration range goes to infinity, the integral would diverge but $1/T$ tends to go to zero. For finite (non-zero) average power signals these two trends balance each other out and $\mathcal{P}\left\{x(t)\right\}$ converges to a finite positive number.

A similar renormalization is then used to define a suitable **autocorrelation function for finite-average-power signals**:

$$\Phi_x(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, x^*(t-\tau)\, dt$$

Eq. 10-16

Note that for $\tau = 0$, the above equation is simply $\mathcal{P}\{x(t)\}$. In other words:

$$\Phi_x(0) = \mathcal{P}\{x(t)\}$$

Eq. 10-17

10.3.1.1 Other properties of $\Phi_x(\tau)$

$\Phi_x(\tau)$ has most of the properties of finite-energy-signal autocorrelation functions. In particular, the symmetry relation is preserved:

$$\Phi_x(\tau) = \Phi_x^*(-\tau)$$

and all the properties derived from it. It is also possible to show that, similar to the case of finite-energy signals, here too:

$$\left|\Phi_x(\tau)\right| \le \Phi_x(0) \qquad \forall \tau \in \mathbb{R}$$

Of course, the meaning of the value at the origin is different.

For finite-energy-signals autocorrelations, it is the energy of the signal. For finite-average-power-signals autocorrelations **it is the average power of the signal**.

10.3.2 Power spectral density

For finite-average power signals, with power greater than zero, the energy spectral density does not converge. We are then interested in a quantity that represents the **spectral density of power**. That is a “density of power versus frequency,” which we could write as:

$$G_x(f) = \frac{d\mathcal{P}\{x\}}{df}$$

Eq. 10-18

It can be shown that such quantity is provided by Fourier transform of the autocorrelation function for finite-average-power signals:

$$\mathrm{G}_x(f) = F\{\Phi_x(\tau)\}$$

Eq. 10-19

Note that thanks to Eq. 10-19 we can write:

$$\mathcal{P}\{x\} = \Phi_x(0) = \Phi_x(\tau)\Big|_{\tau=0} = \mathrm{F}^{-1}\{G_x(f)\}\Big|_{\tau=0} =$$

$$= \int_{-\infty}^{+\infty} G_x(f)\, e^{j2\pi f\tau} df \Bigg|_{\tau=0} = \int_{-\infty}^{+\infty} G_x(f)\, df$$

Looking at the two extremes of the chain of equalities:

$$\mathcal{P}\{x\} = \int_{-\infty}^{+\infty} G_x(f)\, df$$

Eq. 10-20

which shows that indeed $G_x(f) = \dfrac{d\mathcal{P}\{x\}}{df}$, that is, $G_x(f)$ is the spectral density of the power of the signal.

10.3.3 Cross-correlations and cross-spectra for finite-average

power signals

Similarly to what is done with finite-energy signals, also for finite average power signals it is possible to define cross-correlations and cross spectra.

$$\Phi_{xy}(\tau) \triangleq \lim_{T_0 \to \infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)\, y^*(t-\tau)\, dt$$

Eq. 10-21

Given such definitions, it is possible to prove that, similarly to what happens between $R_{xy}(\tau)$ and $S_{xy}(f)$ for finite-energy signals:

$$\Phi_{xy}(\tau) \xleftrightarrow{\mathrm{F}} \mathrm{G}_{xy}(f)$$

10.3.4 Autocorrelation and power spectrum of periodic signals

Periodic signals are finite-average power signals. We know that they can be represented as (see Chapter 6):

$$x(t) = \sum_{n=-\infty}^{+\infty} x_T(t - nT_0)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} q(t - nT_0)$$

They can also be represented using Fourier series:

$$x(t) = \sqrt{f_0} \sum_{n=-\infty}^{+\infty} x_{T_n} e^{j2\pi n f_0 t}$$

See Chapter 6 for the details on these three representations.

The corresponding three forms of the Fourier transform of the periodic signal are:

$$X(f) = f_0 \sum_{n=-\infty}^{+\infty} X_{T_0}(nf_0)\delta(f - nf_0)$$

$$X(f) = f_0 \sum_{n=-\infty}^{+\infty} Q(nf_0)\delta(f - nf_0)$$

$$X(f) = \sqrt{f_0} \sum_{n=-\infty}^{+\infty} x_{T_0,n}\, \delta(f - nf_0)$$

It turns out that the power spectral density of periodic signals is very similar to their Fourier transform. It can indeed be obtained from $X(f)$ by **simply taking the absolute value squared of the coefficients that multiply each delta.**

In fact:

$$
\begin{aligned}
G_x(f) &= f_0^2 \sum_{n=-\infty}^{+\infty} \left| X_{T_0}(nf_0) \right|^2 \delta(f - nf_0) \\
&= f_0^2 \sum_{n=-\infty}^{+\infty} \left| Q(nf_0) \right|^2 \delta(f - nf_0) \\
&= f_0 \sum_{n=-\infty}^{+\infty} \left| x_{T_0,n} \right|^2 \delta(f - nf_0)
\end{aligned}
$$

Eq. 10-23

By taking the inverse Fourier transform of each form of $G_x(f)$, we can find directly the autocorrelation function:

$$\Phi_x(\tau) = f_0^2 \sum_{n=-\infty}^{+\infty} \left| X_T(nf_0) \right|^2 e^{j2\pi n f_0 \tau}$$
$$= f_0^2 \sum_{n=-\infty}^{+\infty} \left| Q(nf_0) \right|^2 e^{j2\pi n f_0 \tau}$$
$$= f_0 \sum_{n=-\infty}^{+\infty} \left| x_{T_0,n} \right|^2 e^{j2\pi n f_0 \tau}$$

Eq. 10-24

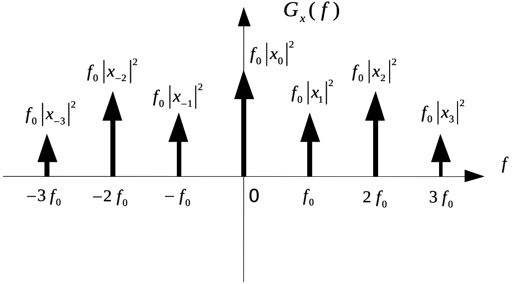


Fig. 10-2 Example of Power Spectral Density of a periodic signal. The power carried by each delta coincides with the coefficient multiplying it, in this case expressed as the discrete-Fourier basis

components of the truncated signal $x_{T_0}(t)$.

If we now evaluate the overall power of the signal, we get the three equivalent results:

$$\mathcal{P}\{x\} = \int_{-\infty}^{+\infty} G_x(f)\,df = f_0^2 \sum_{n=-\infty}^{+\infty} \left|X_{T_0}(nf_0)\right|^2$$
$$= f_0^2 \sum_{n=-\infty}^{+\infty} \left|Q(nf_0)\right|^2 = f_0 \sum_{n=-\infty}^{+\infty} \left|x_{T_0,n}\right|^2$$

This clearly shows that each spectral line provides a specific contribution to the total signal power. Specifically, the n-th spectral line contributes in the amount of exactly $f_0^2\left|X_{T_0}(nf_0)\right|^2$, $f_0^2\left|Q(nf_0)\right|^2$, or $f_0\left|x_{T_0,n}\right|^2$.

10.3.4.1 alternative form of the autocorrelation

It is easy to show that the autocorrelation can also be written in a different way. Given:

$$x(t) = \sum_{n=-\infty}^{+\infty} q(t - nT_0)$$

and given the autocorrelation of the finite-energy signal $q(t)$:

$$R_q(\tau) = \left(q(t), q(t-\tau)\right)$$

then this alternative form of the writing of $\Phi_x(\tau)$ exists:

$$\Phi_x(\tau) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} R_q(\tau - nT_0)$$

Eq. 10-25

Even though formally very different, this formula is completely equivalent to Eq. 10-24.

We now take Eq. 10-25 as a result to prove. To do it, we will prove that starting from it we can arrive at a known correct equation for $G_x(f)$.

We first remark that Eq. 10-25 is of the form:

$$\Phi_x(\tau) = \sum_{n=-\infty}^{+\infty} w(\tau - nT_0)$$

Eq. 10-26

This is one of the ways to write a periodic signal. In fact, $w(\tau) = f_0 \, R_q(\tau)$ is an elementary signal that generates the periodic signal $\Phi_x(\tau)$ of period T_0 by repetition. We typically call the elementary signal $q(t)$, but we have already used a signal called $q(t)$ to generate $x(t)$, so here we change letter to w.

Then, from the theory of periodic signals, we know that the Fourier transform of the periodic signal $\Phi_x(\tau)$ in the form Eq. 10-26 is given by:

$$F\left\{\Phi_x(\tau)\right\} = f_0 \sum_{n=-\infty}^{+\infty} W(nf_0)\,\delta(f - nf_0)$$

We now remark that $F\left\{\Phi_x(\tau)\right\} = G_x(f)$, so we can immediately write:

$$G_x(f) = f_0 \sum_{n=-\infty}^{+\infty} W(nf_0)\,\delta(f - nf_0)$$

Eq. 10-27

To get ahead in the proof, we need to figure out what $W(f)$ is. We know that

$$w(\tau) = f_0\, R_q(\tau)$$

Therefore:

$$W(f) = F\{w(\tau)\} = f_0\ F\{R_q(\tau)\}$$

But we also know what the Fourier transform of the autocorrelation $R_q(\tau)$ is:

$$F\{R_q(\tau)\} = |Q(f)|^2$$

So:

$$W(f) = f_0\ F\{R_q(\tau)\} = f_0\ |Q(f)|^2$$

Substituting this into Eq. 10-27, we get:

$$G_x(f) = f_0^2 \sum_{n=-\infty}^{+\infty} |Q(nf_0)|^2\ \delta(f - nf_0)$$

Eq. 10-28

This is precisely one of the known forms of $G_x(f)$, as it can be seen in Eq.

10-23.

So, to conclude, starting from Eq. 10-25, which we took as a result to prove, we can arrive at a known correct expression of the power spectral density $G_x(f)$ shown in Eq. 10-28. This proves the correctness of Eq. 10-25.

10.3.5 Problem

Given the periodic signal:

$$x(t) = \sum_{n=-\infty}^{+\infty} q(t - nT_0) \qquad\qquad q(t) = e^{-t^2 / 2T_q^2}$$

compute the Fourier transform, autocorrelation function, power and power spectrum of $x(t)$.

First of all, we find the Fourier transform:

$$X(f) = \mathcal{F}\{x(t)\} = f_0 \sum_{n=-\infty}^{+\infty} Q(nf_0)\,\delta(f - nf_0)$$

$$Q(f) = \mathcal{F}\left\{ e^{-t^2 / 2T_q^2} \right\} = \sqrt{2\pi T_q^2}\, e^{-2\pi^2 f^2 T_q^2}$$

$$Q(nf_0) = \sqrt{2\pi T_q^2}\, e^{-2\pi^2 n^2 f_0^2 T_q^2}$$

So, pulling these results together:

$$X(f) = f_0 \sqrt{2\pi T_q^2} \sum_{n=-\infty}^{+\infty} e^{-2\pi^2 n^2 f_0^2 T_q^2}\, \delta(f - nf_0)$$

As for the power spectral density:

$$G_x(f) = 2\pi T_q^2 f_0^2 \sum_{n=-\infty}^{+\infty} e^{-4\pi^2 n^2 f_0^2 T_q^2} \delta(f - nf_0)$$

The total power is then:

$$\mathcal{P}\{x\} = \int_{-\infty}^{+\infty} G_x(f)\,df = 2\pi T_q^2 f_0^2 \sum_{n=-\infty}^{+\infty} e^{-4\pi^2 n^2 f_0^2 T_q^2}$$

The autocorrelation function is:

$$\Phi_x(\tau) = 2\pi T_q^2 f_0^2 \sum_{n=-\infty}^{+\infty} e^{-4\pi^2 n^2 f_0^2 T_q^2} e^{j2\pi n f_0 \tau}$$

On your own, find the Fourier transform, autocorrelation function and power spectral density of the following periodic signals:

$$x(t) = \sum_{n=-\infty}^{+\infty} q(t - nT_0)$$

using:

$$q(t) = e^{-at}\, u(t) \qquad\qquad a > 0$$

$$q(t) = \Pi(t - 1/2) - \Pi(t - 3/2), \qquad\qquad T_0 = 2$$

$$q(t) = te^{-2t}\mathrm{u}(t), \qquad\qquad T_0 = 1$$

$$q(t) = \Lambda(t - 1) - \Lambda(t - 3), \qquad\qquad T_0 = 4$$

When using the first of the three $q(t)$, all even-index coefficients of the spectral deltas must be zero. Can you explain why?

10.4 LTI Systems, Power Spectra and

Autocorrelation Functions

Given an LTI system such that:

$$y(t) = x(t) * h(t)$$
$$Y(f) = X(f) \cdot H(f)$$

we are interested in finding the autocorrelation and energy or power spectrum of the output, given those of the input. We are also interested in finding the cross-correlation and cross-spectra between input and output.

In the table below, we report a summary of all these results. The derivation of them is found in the following sections.

	Finite Average Power	Finite Energy

Spectral Density	$G_y(f) = G_x(f) \cdot \left\|H(f)\right\|^2$	$S_y(f) = S_x(f) \cdot \left\|H(f)\right\|^2$
Autocorrelation	$\Phi_y(\tau) = \Phi_x(\tau) * R_h(\tau)$	$R_y(\tau) = R_x(\tau) * R_h(\tau)$

Table 10-1: Correlations and spectra with LTI systems

10.4.1 Energy spectrum and auto-correlation of the output of an LTI system

Given an LTI system such that:

$$y(t) = x(t) * h(t)$$
$$Y(f) = X(f) \cdot H(f)$$

we want to find the energy spectral density or power spectral density of the output (depending on the nature of the output).

We will restrict the discussion to LTI systems such that their transfer function is bounded:

$$\left|H(f)\right| < \infty, \quad \forall f \in \mathbb{R}$$

Eq. 10-29

and such that $H(f)$ is a finite-energy signal, that is:

$$\mathcal{E}\left\{H(f)\right\} < \infty$$

Eq. 10-30

The reason for the restriction Eq. 10-29 is that a bounded frequency response cannot turn a finite-energy input signal into an infinite-energy output signal, and cannot turn a finite time-averaged power input signal into an infinite time-averaged power output signal. This is easy to prove and is left to demonstrate **optionally on your own**. The reason for the restriction Eq. 10-30 will be made explicit later on.

10.4.1.1 finite-energy input

If the input $x(t)$ is a finite-energy signal, we have:

$$S_y(f) = |Y(f)|^2 = |X(f)H(f)|^2 =$$
$$= |X(f)|^2 |H(f)|^2 = S_x(f)|H(f)|^2$$

that is:

$$S_y(f) = S_x(f)|H(f)|^2$$

Eq. 10-31

Note that $|H(f)|^2$ here acts as a sort of *energy transfer function* of the LTI system, between the input energy spectrum $S_x(f)$ and the output energy spectrum $S_y(f)$.

Based on Eq. 10-31 it is then easy to find the relationship between the autocorrelation function of the input $R_x(\tau)$ and the autocorrelation of the output $R_y(\tau)$. This can be done in frequency domain. In fact, by definition, we have:

$$R_y(\tau) = F^{-1}\left\{S_y(f)\right\} = F^{-1}\left\{\left|Y(f)\right|^2\right\}$$

Eq. 10-32

But we just proved that:

$$S_y(f) = S_x(f)\left|H(f)\right|^2$$

Substituting the above formula into Eq. 10-32, we get:

$$R_y(\tau) = F^{-1}\left\{\left|X(f)H(f)\right|^2\right\} = F^{-1}\left\{\left|X(f)\right|^2\left|H(f)\right|^2\right\} =$$

$$= F^{-1}\left\{S_X(f)\left|H(f)\right|^2\right\} = F^{-1}\left\{S_X(f)\right\} * F^{-1}\left\{\left|H(f)\right|^2\right\} =$$

$$= R_x(\tau) * F^{-1}\left\{\left|H(f)\right|^2\right\}$$

It can now be seen why we made the assumption that $H(f)$ is finite-energy. Thanks to such assumption, then $\left|H(f)\right|^2$ is the energy spectrum of $h(t)$ and we can write:

$$F^{-1}\left\{\left|H(f)\right|^2\right\} = F^{-1}\left\{S_h(f)\right\} = R_h(\tau)$$

So, in the end:

$$R_y(\tau) = R_x(\tau) * R_h(\tau)$$

Eq. 10-33

10.4.1.2 finite time-averaged power input

If the input $x(t)$ is a finite average-power signal, we can derive very similar formulas. We omit the derivation. The final results, are:

$$G_y(f) = G_x(f) \cdot \left| H(f) \right|^2$$

Eq. 10-34

$$\Phi_y(\tau) = \Phi_x(\tau) * R_h(\tau)$$

Eq. 10-35

Note that here the role of $|H(f)|^2$ is that of a *power transfer function*.

10.4.2 Problem

Here we continue Problem 10.3.5 by assuming that the signal $x(t)$ goes through an LTI system whose transfer function is:

$$H(f) = \Pi_B(f - 4f_0) + \Pi_B(f + 4f_0)$$
$$0 < B < 2f_0$$

We want to:

- find the signal $y(t)$ at the output of the LTI system, both in time-domain and in frequency-domain
- find the autocorrelation and the power spectral density of $y(t)$
- compare the average power of $x(t)$ and of $y(t)$: have we lost any power,

going through the LTI system?

We first find $Y(f)$. Clearly, we have:

$$Y(f) = X(f) \cdot H(f) =$$

$$= f_0 \sum_{n=-\infty}^{+\infty} Q(nf_0)\,\delta(f - nf_0) \cdot \left[\Pi_B(f - 4f_0) + \Pi_B(f + 4f_0)\right]$$

Only the fourth positive and negative spectral lines go through the filter. Therefore:

$$Y(f) = f_0 \cdot \left[Q(4f_0)\,\delta(f - 4f_0) + Q(-4f_0)\,\delta(f + 4f_0)\right] =$$

$$= f_0 \sqrt{2\pi T_q^2} \cdot \left[e^{-2\pi^2 4^2 f_0^2 T_q^2}\,\delta(f - 4f_0) + e^{-2\pi^2 (-4)^2 f_0^2 T_q^2}\,\delta(f + 4f_0)\right] =$$

$$= f_0 \sqrt{2\pi T_q^2} \cdot e^{-2\pi^2 4^2 f_0^2 T_q^2} \left[\delta(f - 4f_0) + \delta(f + 4f_0)\right]$$

To find the system output in time domain, we inverse-Fourier transform $Y(f)$:

$$y(t) = F^{-1}\{Y(f)\} =$$

$$= f_0\sqrt{2\pi T_q^2}\cdot e^{-2\pi^2 4^2 f_0^2 T_q^2} F^{-1}\{\delta(f-4f_0)+\delta(f+4f_0)\} =$$

$$= f_0\sqrt{2\pi T_q^2}\cdot e^{-2\pi^2 4^2 f_0^2 T_q^2}\left[e^{j2\pi 4 f_0 t}+e^{-j2\pi 4 f_0 t}\right]$$

$$= f_0\sqrt{2\pi T_q^2}\cdot e^{-2\pi^2 4^2 f_0^2 T_q^2}\, 2\cos(2\pi\cdot 4 f_0 t) = 2 f_0\sqrt{2\pi T_q^2}\cdot e^{-2\pi^2 4^2 f_0^2 T_q^2}\cos(8\pi f_0 t)$$

Eq. 10-36

The autocorrelation of $y(t)$ can be found in various ways. The simplest is to compute the power spectral density $G_y(f)$ first and then inverse-Fourier-transform. We know that $G_y(f)$ can be found easily from $Y(f)$ by simply squaring the overall coefficient of each delta. We can therefore immediately write:

$$G_y(f) = \sum_{n=-\infty}^{+\infty} f_0^2 \left|Y(nf_0)\right|^2 \delta(f - nf_0)$$

$$= f_0^2\, 2\pi T_q^2 \cdot e^{-4\pi^2 4^2 f_0^2 T_q^2} \left[\delta(f - 4f_0) + \delta(f + 4f_0)\right]$$

We can verify easily that the same power spectrum result is found by starting from the power spectrum of the input:

$$G_x(f) = 2\pi T_q^2 f_0^2 \sum_{n=-\infty}^{+\infty} e^{-4\pi^2 n^2 f_0^2 T_q^2} \delta(f - nf_0)$$

and then "filtering" it through $\left|H(f)\right|^2$, to yield:

$$G_y(f) = G_x(f) \cdot \left| H(f) \right|^2 =$$

$$= 2\pi T_q^2 f_0^2 \sum_{n=-\infty}^{+\infty} e^{-4\pi^2 n^2 f_0^2 T_q^2} \delta\left(f - nf_0\right) \cdot \left[\Pi_B\left(f - 4f_0\right) + \Pi_B\left(f + 4f_0\right) \right]$$

$$= f_0^2 \, 2\pi T_q^2 \cdot e^{-4\pi^2 4^2 f_0^2 T_q^2} \left[\delta\left(f - 4f_0\right) + \delta\left(f + 4f_0\right) \right]$$

$\boxed{\text{On your own}}$ write down $\Phi_y(\tau)$ as $\Phi_x(\tau) * \Phi_h(\tau)$. Notice that through this path the calculation is a lot more difficult (do not do it…).

Finally, the input power to the system is:

$$P\left\{x(t)\right\} = \int_{-\infty}^{\infty} G_x(f)\, df = 2\pi T_q^2 f_0^2 \sum_{n=-\infty}^{+\infty} e^{-4\pi^2 n^2 f_0^2 T_q^2} \int_{-\infty}^{\infty} \delta\left(f - nf_0\right) df =$$

$$= 2\pi T_q^2 f_0^2 \sum_{n=-\infty}^{+\infty} e^{-4\pi^2 n^2 f_0^2 T_q^2}$$

Instead, for the output we have:

$$P\{y(t)\} = \int_{-\infty}^{\infty} G_y(f)\,df =$$

$$= 2\pi T_q^2 f_0^2 e^{-4\pi^2 4^2 f_0^2 T_q^2} \int_{-\infty}^{\infty} \left[\delta(f-4f_0)+\delta(f+4f_0)\right] df =$$

$$= 4\pi T_q^2 f_0^2 e^{-4\pi^2 4^2 f_0^2 T_q^2}$$

Clearly, substantial power is lost. In fact, the lost power is:

$$P\{x(t)\} - P\{y(t)\} = 2\pi T_q^2 f_0^2 \sum_{\substack{n=-\infty \\ n\neq \pm 4}}^{+\infty} e^{-4\pi^2 n^2 f_0^2 T_q^2}$$

$\boxed{\text{On your own}}$ Assume now that:

$$H(f) = j \cdot \Pi_B(f - 4f_0) - j \cdot \Pi_B(f + 4f_0)$$
$$0 < B < 2f_0$$

Redo all calculations and point out the differences with the previous results.

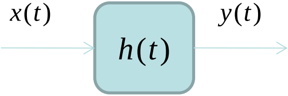
10.5 Problems

10.5.1

Let:

$$x(t) = -\Pi_T\left(t - \tfrac{T}{2}\right)$$

and



1. Find the autocorrelation function of $x(t)$, $R_x(\tau)$
2. Find the energy spectrum of $x(t)$, $S_x(f)$

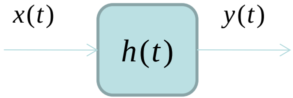
3. Assume that the impulse response $h(t) = e^{-a|t|}$ with $a \in \mathbb{R}^+$. Find the energy spectrum $S_y(f)$ of the signal $y(t)$ at the output of the LTI system.

10.5.2

Let:

$$x(t) = e^{-at}u(t) \qquad a \in \mathbb{R}^+$$

and



1. Find the autocorrelation function of $x(t)$, that is $R_x(\tau)$
2. Find the energy spectrum of $x(t)$, that is $S_x(f)$

3. Assuming that the impulse response $h(t) = \pi_T(t)$, find the energy spectral density of the signal $y(t)$, that is find $S_y(f)$

10.5.3

assume that the signal $x(t) = e^{-at}u(t)$ is passed through an LTI system whose impulse response is: $h(t) = e^{-at}u(t)$.

Find:

- the autocorrelation and energy spectrum of $x(t)$ and $h(t)$

- the output of the LTI system $y(t)$

- the energy spectrum of the output $y(t)$

- $\boxed{\text{Optional:}}$ the autocorrelation of the output $y(t)$; this last calculation is a bit lengthier, the result is:

$$R_y(\tau) = \frac{e^{-a|\tau|}}{4a^3} \cdot \left(1 + a|\tau|\right)$$

10.5.4

Consider the periodic signal:

$$x(t) = \sum_{n=-\infty}^{+\infty} q(t - nT_0) \qquad\qquad q(t) = e^{-at}u(t), \quad a > 0$$

Do the following: compute the

1. Fourier transform of $x(t)$
2. power spectral density of $x(t)$
3. autocorrelation function of $x(t)$

4. average power of $x(t)$ over $\mathbb{R}$

Answers:

1. $$X(f) = f_0 \sum_{n=-\infty}^{+\infty} Q(nf_0)\delta(f - nf_0) = \sum_{n=-\infty}^{+\infty} \frac{f_0}{a + j2\pi n f_0}\delta(f - nf_0)$$

2. $$G_x(f) = \sum_{n=-\infty}^{+\infty} \left| f_0 Q(nf_0) \right|^2 \delta(f - nf_0) = \sum_{n=-\infty}^{+\infty} \frac{f_0^{\,2}}{a^2 + 4\pi^2 n^2 f_0^{\,2}}\delta(f - nf_0)$$

3. $$\Phi_x(\tau) = \sum_{n=-\infty}^{+\infty} \left| f_0 Q(nf_0) \right|^2 e^{j2\pi n f_0 t} = \sum_{n=-\infty}^{+\infty} \frac{f_0^{\,2}}{a^2 + 4\pi^2 n^2 f_0^{\,2}} e^{j2\pi n f_0 t}$$

   Alternatively, this other equivalent formula can be used:

   $$\Phi_x(\tau) = f_0 \sum_{n=-\infty}^{+\infty} R_q(\tau - nT_0) = \sum_{n=-\infty}^{+\infty} \frac{f_0}{2a} e^{-a|\tau - nT_0|}$$

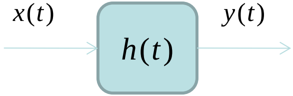
4. $$P\{x(t)\} = \int_{-\infty}^{\infty} G_x(f)\,df = \sum_{n=-\infty}^{+\infty} \frac{f_0^{\,2}}{a^2 + 4\pi^2 n^2 f_0^{\,2}} = \frac{f_0}{2a}\coth\left(\frac{a}{2f_0}\right)$$

10.5.5

Consider the periodic signal:

$$x(t) = \sum_{n=-\infty}^{+\infty} q(t - nT_0) \qquad\qquad q(t) = te^{-at}u(t), \quad a > 0$$

Consider the LTI system



whose transfer function is

$$H(f) = \Pi_B(f - 2f_0) + \Pi_B(f + 2f_0)$$
$$0 < B < f_0$$

where $f_0 = 1/T_0$. Do the following:

1. compute the Fourier transform, autocorrelation function, average power and power spectrum of $x(t)$
2. compute $y(t)$
3. compute the Fourier transform, autocorrelation function, average power and power spectrum of $y(t)$.

10.5.6

Consider the periodic signal:

$$x(t) = \sum_{n=-\infty}^{+\infty} \Lambda_{T_0/2}(t - nT_0)$$

Consider the LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

whose transfer function is

$$H(f) = \Pi_B(f - f_0) + \Pi_B(f + f_0)$$
$$0 < B < f_0$$

where $f_0 = 1/T_0$. Do the following:

1. compute the Fourier transform, autocorrelation function, average power and power spectrum of $x(t)$
2. compute $y(t)$
3. compute the Fourier transform, autocorrelation function, average power and power spectrum of $y(t)$.

10.5.7

Consider the periodic signal:

$$x(t) = \sum_{n=-\infty}^{+\infty} \Lambda_{T_0/2}(t - nT_0)$$

Consider the LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

whose transfer function is

$$H(f) = \Pi_{2B}(f)$$
$$0 < 2B < f_0$$

where $f_0 = 1/T_0$. Do the following:

4. compute the Fourier transform, autocorrelation function, average power and power spectrum of $x(t)$
5. compute $y(t)$
6. compute the Fourier transform, autocorrelation function, average power and power spectrum of $y(t)$.