Let β be the Salem number whose minimal polynomial is

$$Q(x) = x^6 - 10x^5 - 40x^4 - 59x^3 - 40x^2 - 10x + 1.$$

Let $\{c_n\}_{n=0}^{\infty}$ be the coefficients of the β-expansion of β; that is, $0 \leqslant c_n \leqslant \lfloor \beta \rfloor$ for all $n \geqslant 0$ and

$$\beta = \sum_{n=0} \beta^{-n} c_n.$$

Define $T(x) = \{\beta x\}$, where $\{\cdot\}$ denotes the fractional part. The Salem number β is particularly troublesome because the iterates $T^n(1)$ frequently get extremely close to 0. Let me clarify what is meant by "extremely close".

Define the polynomial

$$P_n(x) = x^n - \sum_{k=1}^{n} c_k x^{n-k}$$

and the polynomial

$$B_n(x) = P_n(x) \bmod Q(x).$$

(I mean that $B_n(x)$ is the degree $\leqslant 5$ representative of the equivalence class $P_n(x) \bmod Q(x)$.)

Let ϵ be the "machine epsilon". (Very) roughly speaking, if $|x - y| \geqslant \epsilon$, then the computer can distinguish between x and y, and if $|x - y| < \epsilon$, then the computer cannot distinguish between them. The default is about $\epsilon = 5 \times 10^{-16}$ on most computers. Assume that our machine epsilon ϵ is fixed, and choose β_0 such that $|\beta_0 - \beta| < \epsilon$. Define $S(x) = \{\beta_0 x\}$.

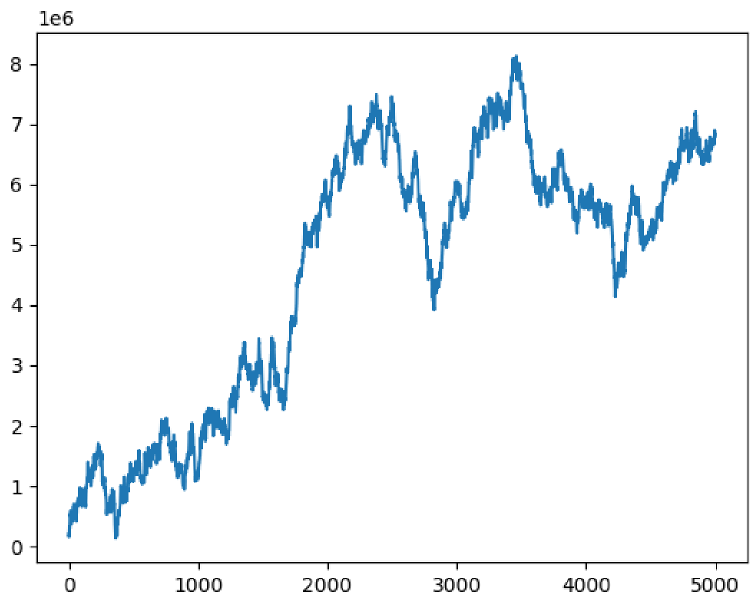
Boyd claims, without proof, that if

$$\big\{S^n(1)\big\} \geqslant \epsilon \left.\frac{d}{dx}\right|_{\beta_0+\epsilon} \big(xB_n(x)\big) \qquad (1)$$

then $c_n = \lfloor S^n(1) \rfloor$, which means that our computer calculation is accurate. (The inequality above is somewhat dubious because it seems like there should be an absolute-value on the right-hand-side. I did put an absolute value in in my code, but it's not in Boyd's original paper.)

The inequality (1) fails at approximately $n = 2.04$ billion when $\epsilon = 10^{-64}$, which is twice the maximal precision Boyd had. Since then, I have set $\epsilon = 10^{-128}$ and I restarted the calculation. I stopped the calculation at $n = 5$ billion, which took approximately 5.5 days. I have the $\epsilon = 10^{-128}$ orbit saved to disk, which in its (uncompressed) form inhabits approximately 261 GB.

The graph below shows the maximal coefficient of B_n in absolute value. Both of the axes have units of 10^6.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