Finite Topologies

1 Definitions

TODO for Tori. Topology, continuous map, homeomorphism, basis, minimal basis, T0, inventory.

Definition 1. This is a definition environment.

2 Basic results

Remark. When the word "finite" is in bold "**finite**", the finiteness condition is required. When the word "finite" is surrounded by parentheses "(finite)", the finiteness condition is optional.

Lemma 2. Let X, Y, and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then their composition $g \circ f : X \to Z$ is continuous. Moreover, if f and g are homeomorphisms, then $g \circ f$ is a homeomorphism too.

Proof. □

Proposition 3. Let (X, τ) be a **finite** topological space. Then, for each $x \in X$, the set $U_x = \bigcap\{U \in \tau \mid x \in U\}$ is open, and the collection $\mathcal{B} = \{U_x \in \tau \mid x \in X\}$ is contained in every basis on X. (This justifies why we have been calling $\mathcal{B}$ the minimal basis.)

Proof. □

Proposition 4. Let (X, τ) be a **finite** topological space. Then, X is a T_0 space (also called a Kolmogorov space) if and only if for all $x, y \in X$, we have that $U_x = U_y$ implies that $x = y$.

Proof. □

Proposition 5. Let (X, τ) be a **finite** topological space. Then, the following are equivalent:

- X is a T_1 space (also called a Fréchet space)
- X is a T_2 space (usually called a Hausdorff space)
- The elements in the minimal basis are singletons.

Proof. □

Lemma 6. Let (X, τ) be a (finite) topological space. Define a relation on X by $x \sim y$ if and only if for every $U \in \tau$, we have $x \in U$ if and only if $y \in U$. Then, $\sim$ is an equivalence relation on X. Let $\tilde{X}$ be the set of equivalence classes defined by $\sim$. Let $q : X \to \tilde{X}$ be the map defined by $q(x) = [x]$, i.e., q maps x to the equivalence class of x. Let $\tilde{\tau}$ be the collection of subsets V of $\tilde{X}$ such that $q^{-1}(V) \in \tau$. Then, $\tilde{\tau}$ is a T_0 topology on $\tilde{X}$, and the map $q : X \to \tilde{X}$ is continuous.

Proof. □

Proposition 7. If (X, τ) and (Y, τ') are homeomorphic (finite) topological spaces, then the topologies $(\tilde{X}, \tilde{\tau})$ and $(\tilde{Y}, \tilde{\tau}')$, as constructed in the preceeding lemma, are homeomorphic T_0 topological spaces.

Proof. □

Lemma 8. Let (X, τ) be a **finite** topological space. Fix a basis of open sets $\mathcal{B} = \{V_x \in \tau \mid x \in X\}$ such that $x \in V_x$ for all $x \in X$. The collection $\mathcal{B}$ is the minimal basis of (X, τ) if and only if for all $x, y \in X$, if $x \in V_y$, then $V_x \subseteq V_y$.

Proof. Let us first prove the necessity; assume that $\mathcal{B}$ is the minimal basis of (X, τ). Fix $x, y \in X$ such that $x \in V_y$. Towards a contradiction, assume that there exists $z \in V_x$ such that $z \notin V_y$. Then $z \notin V_x \cap V_y$, hence $V_x \cap V_y$ is a proper subset of V_x, contrary to the assumption that $\mathcal{B}$ is minimal.

Conversely, assume that for all $x, y \in X$, if $x \in V_y$, then $V_x \subseteq V_y$. Fix $x \in X$ and an open set $U \in \tau$ such that $x \in U$. It suffices to prove that $V_x \subseteq U$. Since $\mathcal{B}$ is a basis, then there exists $y \in X$ such that $x \in V_y \subseteq U$. By assumption, $V_x \subseteq V_y \subseteq U$, as desired. □

2.1 Bounds on inventories

For all integers $n \geq 1$, denote the set $[n] = \{1, 2, \ldots, n\}$.

For all integers n, m, ℓ, k such that $n \geq 1$, $1 \leq m \leq n$, $1 \leq \ell \leq \binom{n}{m}$ and $1 \leq k \leq m$, define

$$\begin{aligned} S(n, m, \ell, k) = \min \Big\{ s \geq 1 \;\Big|\; & \big(\forall \mathcal{B} \subseteq \mathcal{P}([n]) \text{ such that } \#\mathcal{B} \geq s \text{ and } \forall A \in \mathcal{B},\ \#A = m\big) \\ & \big(\exists \text{ distinct collections } \{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_\ell, B_\ell\} \text{ such} \\ & \text{that } \forall i \in [\ell],\ \{A_i, B_i\} \subseteq \mathcal{B} \text{ and } A_i \neq B_i\big) \\ & \forall i \in [\ell],\ \#(A_i \cap B_i) \geq k \Big\}. \end{aligned}$$

Informally, $S(n, m, \ell, k)$ is the minimum number of subsets of size m of $[n]$ to guarantee that there are at least ℓ distinct intersections of size at least k. For example, $S(n, m, 1, 1) = \lfloor \frac{n}{m} \rfloor + 1$ for all n, m.

Lemma 9. Let (X, τ) be a **finite** topological space of size n. If $f : [n] \to \mathbb{N}$ is the inventory of (X, τ), then for all $m \in [n]$,

$$f(m) < \min_{1 \leq k < m} S\big(n, m, f(k), k\big).$$

Proof. TODO (for Michael). $\square$

TODO: find nice bounds on $S(n, m, \ell, k)$.