For all $\ell \geqslant 1$, denote

$$[\ell] = \{1, 2, \ldots, \ell\}$$

and for all $a \leqslant b$, denote

$$[a, b] = \{x \in \mathbb{Z} : a \leqslant x \leqslant b\}.$$

If $a > b$, we understand $[a, b] = \emptyset$.

Definition 0.1. *Fix $m \geqslant 1$. Given an $m \times m$ matrix A and an integer $k \in [2m-1]$, let the **k-th diagonal** of A be those entries A_{ij} of A such that $m - k = j - i$*

For example, the m-th diagonal is the main diagonal and the 1-st diagonal is the entry in the upper-right corner of the matrix.

Definition 0.2. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $k' = \max K$. A $p \times p$ permutation matrix P, where $k' \leqslant 2p - 1$, **realizes K** if for all $k \in K$, the k-th diagonal of P has exactly one 1, and for all $\ell \in [k'] \setminus K$, the ℓ-th diagonal has zero 1's. If there exists at least one matrix realizing K, then the set K is called **realizable**.*

It is important to notice that if K is realizable, then so is $K' \subseteq K$ whenever $K' \neq \emptyset$. (Why?)

Definition 0.3. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $n = \#K$. The set K is called **blocking** if it is realizable and for all $p \times p$ matrices P realizing K, where $p > n$, there exists an $n \times n$ matrix A realizing K and a $(p-n) \times (p-n)$ permutation matrix B such that*

$$P = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

The reader may verify that the following are all examples of blocking sets:

$$\{1\}, \quad \{1, 3\}, \quad \{1, 3, 5\}, \quad \{2, 3, 4\}, \ \{2, 3, 5, 6\}, \ \{2, 4, 5, 6, 8\}.$$

For example, let P be any $p \times p$ matrix realizing $\{2, 3, 4\}$, where $p \geqslant 4$. The set $\{2, 3, 4\}$ is blocking because, working from the upper-right of the matrix P to the lower-left, we have two possible entries to place a 1 on the 2-nd diagonal; after that choice is made, the entries on the 3-rd and 4-th diagonal are all determined. For both possibilities, the 3×3 submatrix in the upper-right corner of P is a permutation matrix. These two matrices are shown below:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In fact, the sets $(2\mathbb{N} + 1) \cap [n]$ are blocking for all $n \geqslant 1$. The reader may also verify that if K_1, K_2 are both blocking sets, then $K_1 \cup (K_2 + 2\#K_1)$ is blocking as well.

By contrast, the following sets are all realizable, but none are blocking:

$$\{2\}, \quad \{2,3\}, \quad \{1,4\}, \quad \{3,4,5,6\}.$$

For example, let P be any $p \times p$ matrix realizing $\{2\}$, where $p \geqslant 2$. The singleton $\{2\}$ is not blocking because the 1×1 submatrix in the upper-right corner of the matrix P is $[0]$; the matrix $[0]$ is not a permutation matrix, so $\{2\}$ is not blocking.

Proposition 0.4. *Fix a nonempty finite set $K \subseteq \mathbb{N}$. If K is realizable, then for all $i \in [n-1]$, if $\{k_1, k_2, \ldots, k_i\}$ is blocking, then $[k_i + 1, 2i] \cap K = \emptyset$.*

Proposition 0.4 reveals the double meaning of the term "blocking". In one sense of the word, matrices that realize blocking sets are block matrices. In another sense, blocking sets "block off" later numbers from being part of the set.

Proof of Proposition 0.4. Let P be a $p \times p$ matrix realizing K; without loss of generality, assume that $p \geqslant k_n$. Fix $i \in [n-1]$ and denote $K' = \{k_1, \ldots, k_i\}$. If K' is blocking, then

$$P = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

for an $i \times i$ matrix A realizing K' and a $(p-i) \times (p-i)$ permutation matrix B. For all $\ell \in [k_i + 1, 2i]$, the ℓ-th diagonal of P does not intersect with the submatrix of P corresponding to B. Since A realizes K' and $\ell \notin K'$, then the intersection of the ℓ-th diagonal of P with the submatrix corresponding to A must have only zeroes. Therefore, the ℓ-th diagonal of P must have only zeroes, so $\ell \notin K$. □

The following proposition will help prove that several finite subsets of $\mathbb{N}$ are realizable.

Proposition 0.5. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $n = \#K$. Write $K = \{k_1, k_2, \ldots, k_n\}$, where $k_1 < k_2 < \cdots < k_n$. If for all $i \in [n]$, we have $2i - 1 \leqslant k_i$, then K is realizable.*

The bound $2i - 1 \leqslant k_i$ in Proposition 0.5 is not necessary. For example, the set $\{3,4,5,6,7\}$ does not satisfy the bound, but it is realizable (and in fact blocking).

In order to prove Proposition 0.5, we need the following Lemma 0.6 and its corollary Lemma 0.7.

Lemma 0.6. *Fix $m \geqslant 1$. Let P' denote an upper triangular, 0-1-valued $m \times m$ matrix with at most one 1 per column and at most one 1 per row. There exists an $m \times m$ permutation matrix P such that for all $i, j \in [m]$, we have*

$$\begin{aligned} P'_{ij} = 1 \quad &\Longrightarrow \quad P_{ij} = 1 \\ \big(P_{ij} = 1 \text{ and } P'_{ij} = 0\big) \quad &\Longrightarrow \quad i \geqslant j. \end{aligned}$$

Informally speaking, all of the "new" 1's occur in the lower triangular portion of P.

Proof. Working from the top row to the bottom, arbitrarily choose an entry for 1 in the lower triangular portion of P, as long as your choice does not collide with any 1's in the upper triangular portion or with any previously chosen 1's. $\square$

Lemma 0.7. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $k' = \max K$. If K is realizable, then for all $\ell \geqslant k'$, there exists an $\ell \times \ell$ matrix realizing K.*

Proof. Let P be a $p \times p$ matrix realizing K.

If $p > k'$, then let P' denote the the $k' \times k'$ submatrix in the upper-right corner of P. Now P' is merely a 0-1 valued matrix with at most one 1 per column and at most one 1 per row; furthermore, for all $k \in K$, the k-th diagonal of P' has exactly one 1 and for all $\ell \in [k'] \setminus K$, the ℓ-th diagonal has exactly zero 1's. Apply Lemma 0.6 to P' to obtain a $k' \times k'$ matrix P'' realizing K.

If $p < k'$, then adjoin any $(k'-p) \times (k'-p)$ permutation matrix Q to P in the fashion

$$P'' = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$$

to obtain a $k' \times k'$ matrix P'' realizing K.

Having so obtained a $k' \times k'$ matrix P'' realizing K, fix $\ell > k'$ and again adjoin any $(\ell - k') \times (\ell - k')$ permutation matrix Q to P'' in the fashion

$$\begin{pmatrix} 0 & P'' \\ Q & 0 \end{pmatrix}$$

to obtain an $\ell \times \ell$ matrix realizing K. $\square$

Proof of Proposition 0.5. Choose $i_0 \in [n]$ maximally so that $K' = \{k_1, \dots, k_{i_0}\}$ is realizable. Towards a contradiction, assume that $i_0 < n$ and let P be a $k_{i_0+1} \times k_{i_0+1}$ matrix realizing K', which exists by Lemma 0.7. Since $2i - 1 \leqslant k_i$ for all $i \in [n]$, then

$$\#\{k \in K : k < i\} \leqslant \lfloor i/2 \rfloor$$

for all $i \in [n]$. So for all $i \in [i_0]$, there exist at most $2\lfloor i/2 \rfloor$ entries in the i-th diagonal of P that lie in either the same row or column of any of nonzero entries lying in the first through $(i-1)$-th diagonals of P, inclusive; stated more formally,

$$\#\Big\{ j \in [i] : \exists k \in K \cap [i-1] \text{ such that } P_{j, i_0-k+j} = 1 \text{ or } P_{k-j+1, i_0-j+1} = 1 \Big\} \leqslant 2\lfloor i/2 \rfloor.$$

If k_{i_0+1} is odd, then $2\lfloor \frac{k_{i_0+1}}{2} \rfloor < k_{i_0+1}$, so there exists some entry on the (k_{i_0+1})-th diagonal of P that does not lie in either the same row or column of any the nonzero entries lying on any of the K'-diagonals; therefore, by Lemma 0.6, there exists a matrix realizing $\{k_1, \dots, k_{i_0+1}\}$, contrary to the maximality of i_0.

If k_{i_0+1} is even, then the bound $2i - 1 \leqslant k_i$ implies that the minimum value of k_{i_0+1} is $2i_0 + 2$. Using a similar argument as in the previous paragraph, the proof is complete.

$\square$

Example 0.8. *The sets $[n, 2n-1]$ are realizable for all $n \geqslant 1$.*

Proof. The sets $[n, 2n-1]$ satisfy the sufficient condition of Proposition 0.5. $\square$

Example 0.9. *For all $n \geqslant 2$ and all $a \in [n, 2n-2]$, the sets $K_1 = [n, 2n-2]$ and $K_2 = [n, 2n-1] \setminus \{a\}$ are realizable and satisfy the bounds $\#K_1 \leqslant \left\lfloor \frac{\max K_1}{2} \right\rfloor$ and $\#K_2 \leqslant \left\lfloor \frac{\max K_2}{2} \right\rfloor$.*

A proof of Conjecture 0.13 below would imply that the sets K_1 and K_2 of Example 0.9 are "symmetrically realizable", so to speak.

Emperical evidence suggests the following.

Conjecture 0.10. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $n = \#K$. Write $K = \{k_1, k_2, \ldots, k_n\}$, where $k_1 < k_2 < \cdots < k_n$. If K is realizable, then for all $i \in [n-1]$, we have $\frac{3}{2}i - \frac{1}{2} \leqslant k_i$.*

The bound $\frac{3}{2}i - \frac{1}{2} \leqslant k_i$ is tight for several realizable sets, such as $\{3, 4, 5, 6, 7\}$. It is not sufficient however; $\{2, 4, 5, 6, 7\}$ is a non-realizable set that satisfies the bound and the necessary condition of Proposition 0.4.

I suspect that the following is true, although I have not thoroughly emperically tested it.

Conjecture 0.11. *Fix a nonempty finite set $K \subseteq \mathbb{N}$. The set K is realizable if and only if there exists a finite set $K' \subseteq \mathbb{N}$ such that $\max K < \min K'$ and $K \cup K'$ is blocking.*

For example, the realizable set $\{2\}$ is contained in the blocking set $\{2, 3, 4\}$, the realizable set $\{3\}$ in the blocking set $[3, 7]$, and the realizable set $\{4\}$ in the blocking set $[4, 10]$.

Conjecture 0.12. *For all $n \geqslant 1$, the set $[n, 3n-2]$ is blocking and for all $n < k < 3n-2$, the set $[n, k]$ is not blocking.*

The following conjecture is of great practical interest:

Conjecture 0.13. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $n = \#K$ and $k' = \max K$. If K is realizable and $n \leqslant \left\lfloor \frac{k'}{2} \right\rfloor$, then there exists a $(k'+1) \times (k'+1)$ symmetric matrix realizing K.*

It is desirable to characterize the possible cycle decompositions arising from permutation matrices that realize a given K.

Realizable sets do not quite form a matroid, because the swap property does not hold. But can matroid theory be used regardless?