For all $\ell \geqslant 1$, denote

$$[\ell] = \{1, 2, \ldots, \ell\}$$

and for all $a \leqslant b$, denote

$$[a, b] = \{x \in \mathbb{Z} : a \leqslant x \leqslant b\}.$$

If $a > b$, we understand $[a, b] = \emptyset$.

Definition 0.1. *Fix $m \geqslant 1$. Given an $m \times m$ matrix A and an integer $k \in [2m-1]$, let the **k-th diagonal** of A be those entries A_{ij} of A such that $m - k = j - i$*

For example, the m-th diagonal is the main diagonal and the 1-st diagonal is the entry in the upper-right corner of the matrix.

Definition 0.2. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $k' = \max K$. A $p \times p$ permutation matrix P, where $k' \leqslant 2p - 1$, **realizes K** if for all $k \in K$, the k-th diagonal of P has exactly one 1, and for all $\ell \in [k'] \setminus K$, the ℓ-th diagonal has zero 1's. If there exists at least one matrix realizing K, then the set K is called **realizable**.*

It is important to notice that if K is realizable, then so is $K' \subseteq K$ whenever $K' \neq \emptyset$. (Why?)

Definition 0.3. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $n = \#K$. The set K is called **blocking** if it is realizable and for all $p \times p$ matrices P realizing K, where $p > n$, there exists an $n \times n$ matrix A realizing K and a $(p-n) \times (p-n)$ permutation matrix B such that*

$$P = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

The reader may verify that the following are all examples of blocking sets:

$$\{1\}, \quad \{1, 3\}, \quad \{1, 3, 5\}, \quad \{2, 3, 4\}, \ \{2, 3, 5, 6\}, \ \{2, 4, 5, 6, 8\}.$$

For example, let P be any $p \times p$ matrix realizing $\{2, 3, 4\}$, where $p \geqslant 4$. The set $\{2, 3, 4\}$ is blocking because, working from the upper-right of the matrix P to the lower-left, we have two possible entries to place a 1 on the 2-nd diagonal; after that choice is made, the entries on the 3-rd and 4-th diagonal are all determined. For both possibilities, the 3×3 submatrix in the upper-right corner of P is a permutation matrix. These two matrices are shown below:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In fact, the sets $(2\mathbb{N} + 1) \cap [n]$ are blocking for all $n \geqslant 1$. The reader may also verify that if K_1, K_2 are both blocking sets, then $K_1 \cup (K_2 + 2\#K_1)$ is blocking as well.

By contrast, the following sets are all realizable, but none are blocking:

$$\{2\}, \quad \{2,3\}, \quad \{1,4\}, \quad \{3,4,5,6\}.$$

For example, let P be any $p \times p$ matrix realizing $\{2\}$, where $p \geqslant 2$. The singleton $\{2\}$ is not blocking because the 1×1 submatrix in the upper-right corner of the matrix P is $[0]$; the matrix $[0]$ is not a permutation matrix, so $\{2\}$ is not blocking.

Proposition 0.4. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $n = \#K$. Write $K = \{k_1, k_2, \ldots, k_n\}$, where $k_1 < k_2 < \cdots < k_n$. If for all $i \in [n-1]$, we have $2i - 1 < k_i$, then K is realizable. Conversely, if K is realizable, then the implication*

$$\{k_1, k_2, \ldots, k_i\} \text{ blocking} \implies [k_i + 1, 2i] \cap K = \emptyset, \tag{1}$$

is satisfied for all $i \in [n-1]$.

This proposition reveals the double meaning of the term “blocking”. In one sense of the word, matrices that realize blocking sets are block matrices. In another sense, blocking sets “block off” later numbers from being part of the set.

The bound $2i - 1 \leqslant k_i$ in Proposition 0.4 is not necessary. For example, the set $\{3,4,5,6,7\}$ does not satisfy the bound, but it is realizable (and in fact blocking).

In order to prove Proposition 0.4, we need the following.

Lemma 0.5. *Fix $m \geqslant 1$. Let P' denote an upper triangular, 0-1-valued $m \times m$ matrix with at most one 1 per column and at most one 1 per row. There exists an $m \times m$ permutation matrix P such that for all $i, j \in [m]$, we have*

$$\begin{aligned} P'_{ij} = 1 \quad &\implies \quad P_{ij} = 1 \\ \big(P_{ij} = 1 \text{ and } P'_{ij} = 0\big) \quad &\implies \quad i \geqslant j. \end{aligned}$$

Proof. Working from the top row to the bottom, arbitrarily choose an entry for 1 in the lower triangular portion of P, as long as your choice does not collide with any 1’s in the upper triangular portion or with any previously chosen 1’s. □

Proof of Proposition 0.4. We first prove the sufficiency. Notice that the bound $2i - 1 < k_i$ implies that the condition (1) holds for all $i \in [n-1]$.

Since all singletons are realizable, then assume that $n > 1$. Let $i_0 \in [n-1]$ be the maximal number such that there exists a $p \times p$ matrix P realizing $\{k_1, \ldots, k_{i_0}\}$, where $p \geqslant k_n$. Again since singletons are realizable, then i_0 exists. Towards a contradiction, assume that $i_0 < n$. Denote $K' = \{k_1, \ldots, k_{i_0}\}$.

Assume that K' is blocking, and choose a $i_0 \times i_0$ matrix A realizing K' and a $(p - i_0) \times (p - i_0)$ permutation matrix B such that

$$P = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

Notice that $k_{i_0+1} > 2i_0$ by assumption, hence the (k_{i_0+1})-th diagonal of P does not intersect the submatrix of P corresponding to A. If B' is any $(p-i_0)\times(p-i_0)$ permutation matrix with a 1 in the $(k_{i_0+1}-2i_0)$-th diagonal of B', then the $p\times p$ permutation matrix

$$P' = \begin{pmatrix} 0 & A \\ B' & 0 \end{pmatrix}$$

realizes $\{k_1,\dots,k_{i_0+1}\}$, contrary to the maximality of i_0.

On the other hand, assume that K' is not blocking. Let P be a $p\times p$ matrix realizing K' that does not decompose in the fashion

$$P = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix},$$

where A is $i_0\times i_0$ and B is $(p-i_0)\times(p-i_0)$. Without loss of generality, assume that $p > k_{i_0}$. Assume for now that $2i_0 < k_{i_0}$. With this assumption, either

1. $\exists i' \in [k_{i_0}]$ such that $\forall j \in [p]$, we have $p+i'-j \notin K'$ or
2. $\exists j' \in [p-k_{i_0}+1,p]$ such that $\forall i \in [p]$, we have $p+i-j' \in K'$.

In the first case, define the matrix P' by

$$P'_{ij} = \begin{cases} 1 & p+i-j \in K' \\ 1 & i=i' \text{ and } j = p+i'-k_{i_0+1} \\ 0 & \text{otherwise.} \end{cases}$$

In the second case, the matrix P' is defined similarly. Apply lemma 0.5 to the matrix P' to obtain a matrix P realizing $\{k_1,\dots,k_{i_0+1}\}$, contrary to the maximality of i_0.

If $2i_0 \geqslant k_{i_0}$, then $2i_0 = k_{i_0}$ by our assumption $2i_0 - 1 < k_{i_0}$. We can use the same argument as we did above, unless $K' = \{2,4,\dots,2i_0\}$, but clearly $\{2,4,\dots,2i_0\}$ is realizable.

We now prove the necessity. Let P be a $p\times p$ matrix realizing K; without loss of generality, assume that $p \geqslant k_n$. Fix $i \in [n-1]$ and denote $K' = \{k_1,\dots,k_i\}$. If K' is blocking, then

$$P = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

for an $i\times i$ matrix A realizing K' and a $(p-i)\times(p-i)$ permutation matrix B. For all $\ell \in [k_i+1, 2i]$, the ℓ-th diagonal of P does not intersect with the submatrix of P corresponding to B. Since A realizes K' and $\ell \notin K'$, then the intersection of the ℓ-th diagonal of P with the submatrix corresponding to A must have only zeroes. Therefore, the ℓ-th diagonal of P must have only zeroes, so $\ell \notin K$. $\square$

Emperical evidence suggests the following.

Conjecture 0.6. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $n = \#K$. Write $K = \{k_1, k_2, \ldots, k_n\}$, where $k_1 < k_2 < \cdots < k_n$. If K is realizable, then for all $i \in [n-1]$, we have $\frac{3}{2}i - \frac{1}{2} \leqslant k_i$.*

The bound $\frac{3}{2}i - \frac{1}{2} \leqslant k_i$ is tight for several realizable sets, such as $\{3, 4, 5, 6, 7\}$. It is not sufficient however; $\{2, 4, 5, 6, 7\}$ is a non-realizable set that satisfies the bound and the necessary condition (1).

I suspect that the following is true, although I have not thoroughly emperically tested it.

Conjecture 0.7. *Fix a nonempty finite set $K \subseteq \mathbb{N}$. The set K is realizable if and only if there exists a finite set $K' \subseteq \mathbb{N}$ such that $\max K < \min K'$ and $K \cup K'$ is blocking.*

For example, the realizable set $\{2\}$ is contained in the blocking set $\{2, 3, 4\}$, the realizable set $\{3\}$ in the blocking set $[3, 7]$, and the realizable set $\{4\}$ in the blocking set $[4, 10]$.

Conjecture 0.8. *For all $n \geqslant 1$, the set $[n, 3n-2]$ is blocking and for all $n < k < 3n-2$, the set $[n, k]$ is not blocking.*

The following conjecture is of great practical interest:

Conjecture 0.9. *Fix a nonempty finite set $K \subseteq \mathbb{N}$ and let $n = \#K$ and $k' = \max K$. If K is realizable and $n \leqslant \left\lfloor \frac{k'}{2} \right\rfloor$, then there exists a $(k'+1) \times (k'+1)$ symmetric matrix realizing K.*

It is desirable to characterize the possible cycle decompositions arising from permutation matrices that realize a given K.

Realizable sets do not quite form a matroid, because the swap property does not hold. But can matroid theory be used regardless?