Quadrilateral Quality

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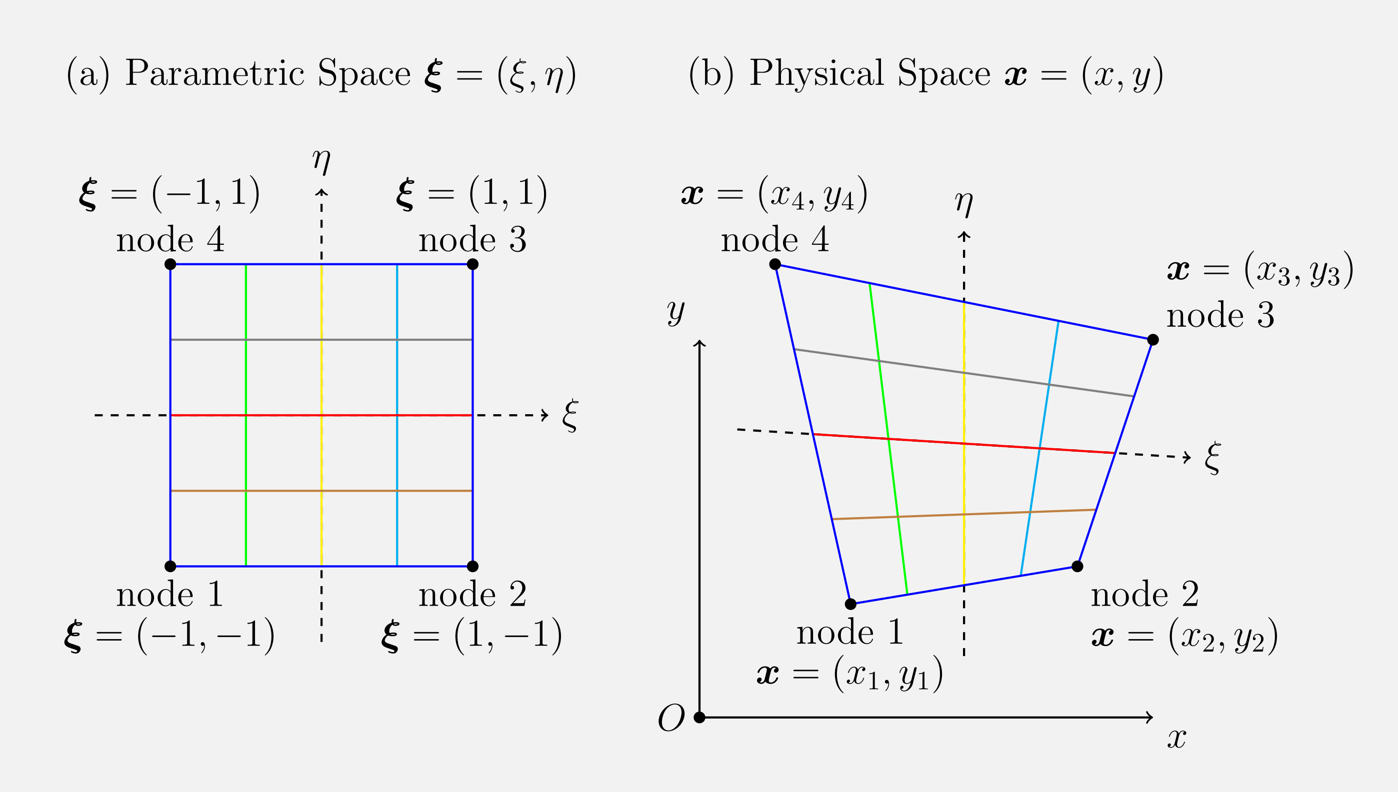

Figure 1: Parametric mapping $\boldsymbol{x} = f(\boldsymbol{\xi})$ from parametric space to physical space.

Chapter 1

Quality

1.1 Isoparametric Mapping

Let the parametric mapping $f : \boldsymbol{\xi} \in [-1, 1] \times [-1, 1] \mapsto \boldsymbol{x} \in \mathbb{R}^2$ be defined as

$$x(\xi, \eta) = \sum_{a=1}^{4} N_a(\xi, \eta)\, x_a, \tag{1.1}$$

$$y(\xi, \eta) = \sum_{a=1}^{4} N_a(\xi, \eta)\, y_a, \tag{1.2}$$

where a nodal **shape function** is defined for each of the four nodes

$$N_1(\xi,\eta) \triangleq \tfrac{1}{4}(1-\xi)(1-\eta), \tag{1.3}$$

$$N_2(\xi,\eta) \triangleq \tfrac{1}{4}(1+\xi)(1-\eta), \tag{1.4}$$

$$N_3(\xi,\eta) \triangleq \tfrac{1}{4}(1+\xi)(1+\eta), \tag{1.5}$$

$$N_4(\xi,\eta) \triangleq \tfrac{1}{4}(1-\xi)(1+\eta). \tag{1.6}$$

1.2 Jacobian

For the quadrilateral element, the Jacobian $\boldsymbol{J}$ is calculated as the matrix of partial derivatives of $\boldsymbol{x} = (x, y)$ with respect to $\boldsymbol{\xi} = (\xi, \eta)$,

$$\boldsymbol{J}(\xi, \eta) \triangleq \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right] = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}. \tag{1.7}$$

Substituting $x(\xi, \eta)$ and $y(\xi, \eta)$ with shape function equations (1.1)-(1.2) and expanding terms, the Jacobian takes the form

$$\boldsymbol{J}(\xi, \eta) = \frac{1}{4} \begin{bmatrix} -1+\eta & 1-\eta & 1+\eta & -1-\eta \\ -1+\xi & -1-\xi & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}. \tag{1.8}$$

The determinant of the Jacobian, $\det(\boldsymbol{J})$, can be found to be

$$\det(\boldsymbol{J}\,(\xi, \eta)) = c_0 + c_1 \xi + c_2 \eta, \tag{1.9}$$

where

$$c_0 = \frac{1}{8}\left[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)\right], \tag{1.10}$$

$$c_1 = \frac{1}{8}\left[(x_3 - x_4)(y_1 - y_2) - (x_1 - x_2)(y_3 - y_4)\right], \tag{1.11}$$

$$c_2 = \frac{1}{8}\left[(x_2 - x_3)(y_1 - y_4) - (x_1 - x_4)(y_2 - y_3)\right]. \tag{1.12}$$

1.3 Quality

We follow *The Verdict Geometry Quality Library* documentation[1] and implementation[2] for the definitions of quality metrics. The SNL Cubit help manual is also helpful.[3]

1.3.1 Preliminaries

Let the four edge vectors and their respective lengths be defined as

$$\boldsymbol{e}_1 \triangleq \boldsymbol{x}_2 - \boldsymbol{x}_1, \qquad \ell_1 \triangleq \| \ \boldsymbol{e}_1 \ \|, \tag{1.13}$$

$$\boldsymbol{e}_2 \triangleq \boldsymbol{x}_3 - \boldsymbol{x}_2, \qquad \ell_2 \triangleq \| \ \boldsymbol{e}_2 \ \|, \tag{1.14}$$

$$\boldsymbol{e}_3 \triangleq \boldsymbol{x}_4 - \boldsymbol{x}_3, \qquad \ell_3 \triangleq \| \ \boldsymbol{e}_3 \ \|, \tag{1.15}$$

$$\boldsymbol{e}_4 \triangleq \boldsymbol{x}_1 - \boldsymbol{x}_4, \qquad \ell_4 \triangleq \| \ \boldsymbol{e}_4 \ \| \ . \tag{1.16}$$

The two (non-normalized) *principal axes* are the defined though vector addition of the two opposing side lengths

$$\boldsymbol{X} \triangleq \boldsymbol{e}_1 - \boldsymbol{e}_3 = (\boldsymbol{x}_2 - \boldsymbol{x}_1) - (\boldsymbol{x}_4 - \boldsymbol{x}_3), \tag{1.17}$$

$$\boldsymbol{Y} \triangleq \boldsymbol{e}_2 - \boldsymbol{e}_4 = (\boldsymbol{x}_3 - \boldsymbol{x}_2) - (\boldsymbol{x}_1 - \boldsymbol{x}_4). \tag{1.18}$$

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[1] Knupp PM, Ernst CD, Thompson DC, Stimpson CJ, Pebay PP. The verdict geometric quality library. Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA (United States); 2006 Mar 1. OSTI https://www.osti.gov/servlets/purl/901967.

[2] See https://github.com/Kitware/VTK/blob/master/ThirdParty/verdict/vtkverdict/ and in particular, the `quad_scaled_jacobian` function in the `V_QuadMetric.cpp` implementation.

[3] See https://cubit.sandia.gov/files/cubit/16.04/help_manual/WebHelp/cubithelp.htm

At each vertex, there is a normal vector and its respective normalized unit vector

$$\boldsymbol{N}_1 \triangleq \boldsymbol{e}_4 \times \boldsymbol{e}_1, \qquad \hat{\boldsymbol{n}}_1 \triangleq \boldsymbol{N}_1 / \| \boldsymbol{N}_1 \|, \tag{1.19}$$

$$\boldsymbol{N}_2 \triangleq \boldsymbol{e}_1 \times \boldsymbol{e}_2, \qquad \hat{\boldsymbol{n}}_2 \triangleq \boldsymbol{N}_2 / \| \boldsymbol{N}_2 \|, \tag{1.20}$$

$$\boldsymbol{N}_3 \triangleq \boldsymbol{e}_2 \times \boldsymbol{e}_3, \qquad \hat{\boldsymbol{n}}_3 \triangleq \boldsymbol{N}_3 / \| \boldsymbol{N}_3 \|, \tag{1.21}$$

$$\boldsymbol{N}_4 \triangleq \boldsymbol{e}_3 \times \boldsymbol{e}_4, \qquad \hat{\boldsymbol{n}}_4 \triangleq \boldsymbol{N}_4 / \| \boldsymbol{N}_4 \| . \tag{1.22}$$

At the center of the element, there is principal axis normal as well

$$\boldsymbol{N}_c \triangleq \boldsymbol{X} \times \boldsymbol{Y}, \qquad \hat{\boldsymbol{n}}_c \triangleq \boldsymbol{N}_c / \| \boldsymbol{N}_c \| . \tag{1.23}$$

There are four contributions to the quadrilateral area from each of the four nodal areas[4]

$$\alpha_1 \triangleq \boldsymbol{N}_1 \cdot \hat{\boldsymbol{n}}_c, \tag{1.24}$$

$$\alpha_2 \triangleq \boldsymbol{N}_2 \cdot \hat{\boldsymbol{n}}_c, \tag{1.25}$$

$$\alpha_3 \triangleq \boldsymbol{N}_3 \cdot \hat{\boldsymbol{n}}_c, \tag{1.26}$$

$$\alpha_4 \triangleq \boldsymbol{N}_4 \cdot \hat{\boldsymbol{n}}_c. \tag{1.27}$$

[4] It may be tempting to (erroneously) write $\hat{\boldsymbol{n}}_1 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_2 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_3 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_4 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_c$ and $\alpha_1 \xrightarrow{\text{2D}} \| \boldsymbol{N}_1 \|, \alpha_2 \xrightarrow{\text{2D}} \| \boldsymbol{N}_2 \|, \alpha_3 \xrightarrow{\text{2D}} \| \boldsymbol{N}_3 \|, \alpha_4 \xrightarrow{\text{2D}} \| \boldsymbol{N}_4 \|$ given that for the 2D quadrilateral case, all unit norms are in the same plane. The problem with such a construction is that it destroys the sign information carried by each normal vector. For the non-degenerate case, the foregoing simplification is true, since all give normals will carry the same sign. However, for degenerate cases, such as when a quadrilateral folds over onto itself, the sign information is no longer homogenous, and the sign information *must* be retained to accurately calculate Jacobian metrics that go negative.

1.3.2 Signed Area

The **signed area** SA is defined as the average of all nodal area contributions:

$$\mathrm{SA} \triangleq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4}. \tag{1.28}$$

The metric dimension is L^2 and the ideal (unit square) value is `1.0`.

1.3.3 Aspect Ratio

The **aspect ratio** AR is defined as the maximum edge length ratios taken at the quadrilateral center.[5] This can be expressed in terms of the norms of the principal axes as

$$\mathrm{AR} = \max\left(\frac{\parallel \boldsymbol{X} \parallel}{\parallel \boldsymbol{Y} \parallel}, \frac{\parallel \boldsymbol{Y} \parallel}{\parallel \boldsymbol{X} \parallel},\right) \tag{1.29}$$

Alternatively, the perimeter length multiplied by the maximum side length, divided by four times the area to define a triangle aspect ratio that is meaningful for quadrilaterals,[6] with dimension L^0 and acceptable range $[1.0, 1.3]$.

[5] Robinson J. CRE method of element testing and the Jacobian shape parameters. Engineering Computations. 1987 Feb 1.

[6] Knupp 2006, *op. cit.* at 38.

1.3.4 Minimum Jacobian

The **Minimum Jacobian** J_{min} is defined as the minimum pointwise area of local map at the four corners and center of quadrilateral[7]

$$J_{\text{min}} \triangleq \min\left(\alpha_1, \alpha_2, \alpha_3, \alpha_4\right). \tag{1.30}$$

1.3.5 Minimum Scaled Jacobian

The **Minimum Scaled Jacobian** $\hat{J}_{\text{min}}$ is the minimum nodal area divided by the lengths of the two edge vector connecting that point[8]

$$\hat{J}_{\text{min}} \triangleq \min\left(\frac{\alpha_1}{\ell_4\ell_1}, \frac{\alpha_2}{\ell_1\ell_2}, \frac{\alpha_3}{\ell_2\ell_3}, \frac{\alpha_4}{\ell_3\ell_4}\right), \tag{1.31}$$

We warned previously in Footnote 4 for Jacobians that errors may result if sign information is not properly retained. We note a similar admonishment[9] for Scaled Jacobians, since the latter is a function of the former.

The dimension is L^0. The full range is $[-1.0, 1.0]$. The acceptable range is typically taken as $[0.3, 1.0]$ in the generous case and $[0.5, 1.0]$ in the more restricted case.

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[7] Knupp 2006, *op. cit.* at 42.

[8] Knupp 2006, *op. cit.* at 51.

[9] It may (again) be tempting to (erroneously) write for the 2D case $\hat{J}_{\text{min}} \xrightarrow{\text{2D}} \min\left(\sin\theta_1, \sin\theta_2, \sin\theta_3, \sin\theta_4\right)$, where θ_1 is the angle between $\boldsymbol{e}_4$ and $\boldsymbol{e}_1$, θ_2 with $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, θ_3 with $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, and θ_4 with $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$. Such a simplification will only work if $\boldsymbol{\theta}$ is retained as a vector quantity (thus retaining the sign). If θ is considered only as a scalar, errors will result when the Jacobian metric goes negative.

Appendix A

Computational Details

A.1 Mathematica

This section demonstrates the steps used in *Mathematica* to obtain the result in Eq. (1.9).

```
MatrixForm[A = {{-1 + b, 1 - b, 1 + b, -1 - b}, {-1 + a, -1 - a, 1 + a, 1 - a}}]
MatrixForm[B = {{x1, y1}, {x2, y2}, {x3, y3}, {x4, y4}}]
J = 1/4 * (A . B)
Expand[Det[J]]
result = Collect[Det[J] // Expand, {a, b}]
TeXForm[result]
```

produces

$$\det(\boldsymbol{J}(a,b)) = \frac{a}{8}(-x_1y_3 + x_1y_4 + x_2y_3 - x_2y_4 + x_3y_1 - x_3y_2 - x_4y_1 + x_4y_2)$$
$$+ \frac{b}{8}(-x_1y_2 + x_1y_3 + x_2y_1 - x_2y_4 - x_3y_1 + x_3y_4 + x_4y_2 - x_4y_3)$$
$$+ \frac{1}{8}(x_1y_2 - x_1y_4 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 + x_4y_1 - x_4y_3) \tag{A.1}$$