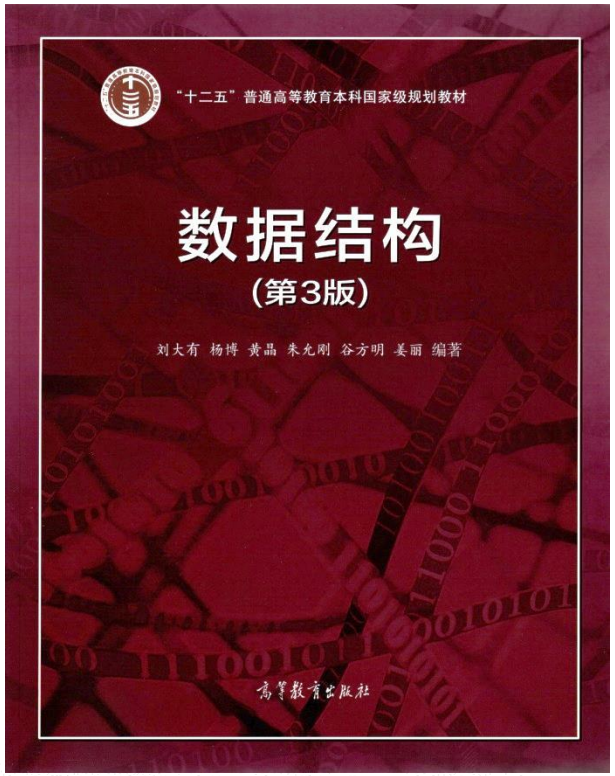
计算机学院王湘浩班

2024级




数据之法

结构之美

算法之道

# 堆排序

- 堆的概念及基本操作
- 堆排序算法

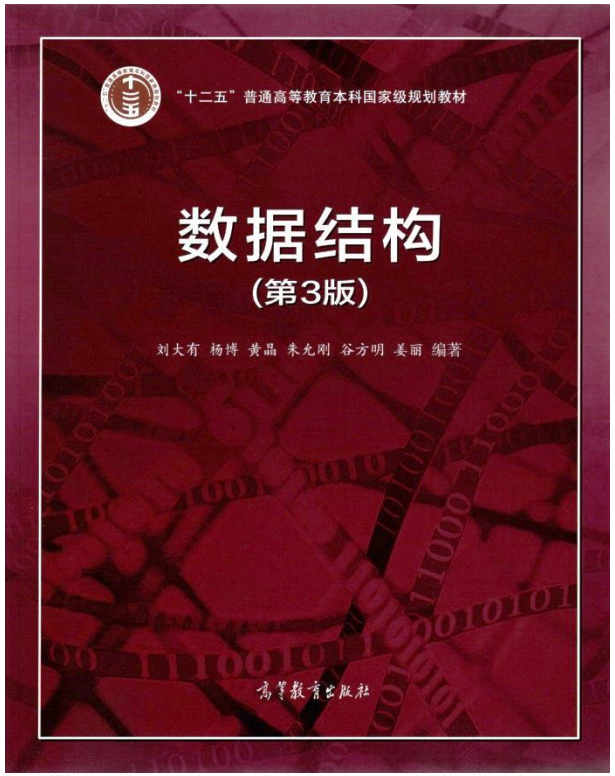
zhuyungang@jlu.edu.cn

Last updated on 2025.5


楼天城
清华大学博士
2次获ACM/ICPC全球总决赛金牌亚军
2次获Google全球编程挑战赛冠军
2次获Facebook骇客杯世界编程大赛季军
2次获百度之星程序设计大赛冠军
小马智行创始人兼CTO
身价75亿元人民币（胡润财富榜）

第一名可能会让你望尘莫及，直到最后也没办法追上。在此之前，要有一个正确的心态，不能自暴自弃，更不能妄自菲薄。然后要很实实在在的努力，每次必须做的事要做好，先把距离咬住再说。

人生发展是个积累的过程，基本上99%的成功都是通过日日夜夜的积累完成的。努力是你唯一能依赖的东西。




# 堆排序

- 堆的概念及基本操作
- 堆排序算法

数据之法 结构之美 算法之道

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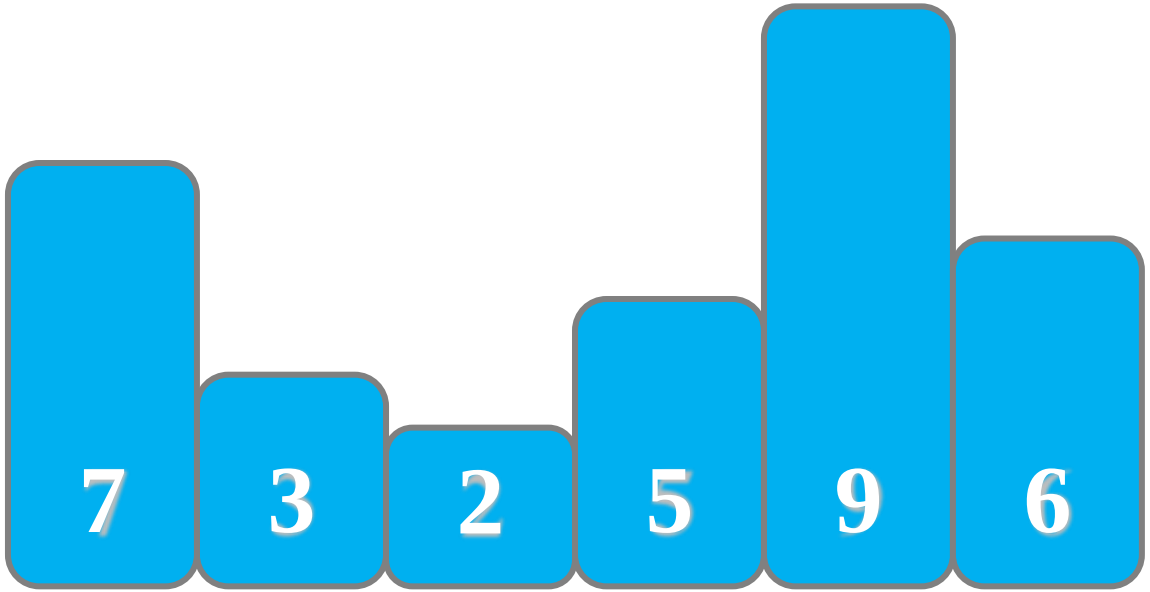
Last updated on 2025.5

# 动机

A

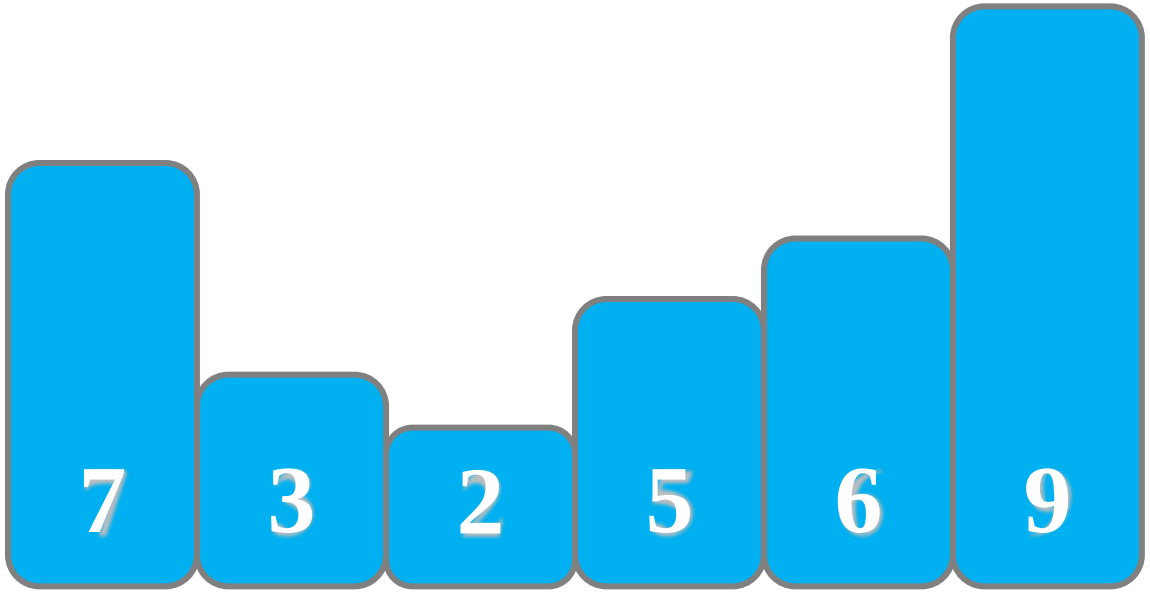
- 直接选择排序低效原因：每次找最大元素需花费O(*n*)时间，涉及大量重复比较。



# 动机

A

- 直接选择排序低效原因：每次找最大元素需花费O(*n*)时间，涉及大量重复比较。
- 能否借助树形结构，减少元素的重复比较次数。



# 动机

A

我们要做什么？

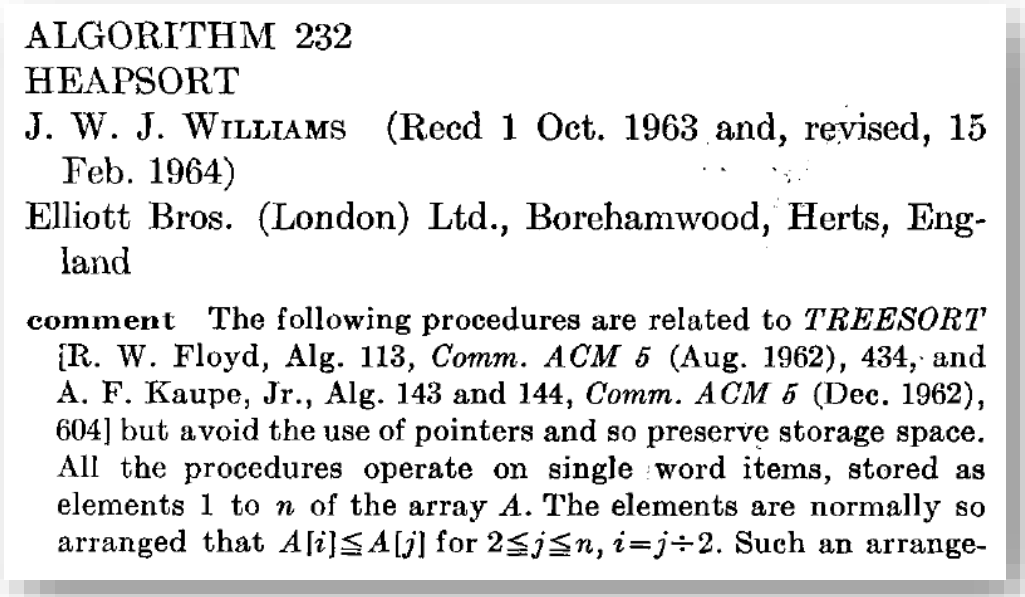
在若干元素中找最大的元素

我们想要多大时间空间？

时间O(log*n*)，空间O(1)

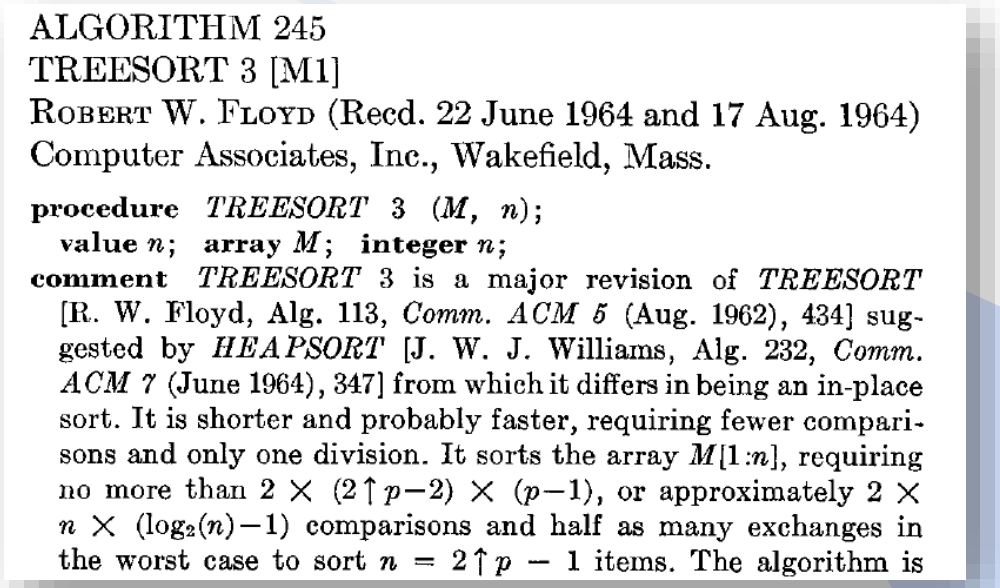
# 堆排序

- 由J.W.J. Williams、Robert Floyd提出。



ALGORITHM 232
HEAPSORT
J. W. J. WILLIAMS (Recd 1 Oct. 1963 and, revised, 15 Feb. 1964)
Elliott Bros. (London) Ltd., Borehamwood, Herts, England

**comment** The following procedures are related to *TREESORT* [R. W. Floyd, Alg. 113, *Comm. ACM 5* (Aug. 1962), 434, and A. F. Kaupe, Jr., Alg. 143 and 144, *Comm. ACM 5* (Dec. 1962), 604] but avoid the use of pointers and so preserve storage space. All the procedures operate on single word items, stored as elements 1 to $n$ of the array $A$. The elements are normally so arranged that $A[i] \leqq A[j]$ for $2 \leqq j \leqq n$, $i = j \div 2$. Such an arrange-



ALGORITHM 245
TREESORT 3 [M1]
ROBERT W. FLOYD (Recd. 22 June 1964 and 17 Aug. 1964)
Computer Associates, Inc., Wakefield, Mass.

**procedure** *TREESORT* 3 (*M*, *n*);
**value** *n*; **array** *M*; **integer** *n*;
**comment** *TREESORT* 3 is a major revision of *TREESORT* [R. W. Floyd, Alg. 113, *Comm. ACM 5* (Aug. 1962), 434] suggested by *HEAPSORT* [J. W. J. Williams, Alg. 232, *Comm. ACM 7* (June 1964), 347] from which it differs in being an in-place sort. It is shorter and probably faster, requiring fewer comparisons and only one division. It sorts the array $M[1:n]$, requiring no more than $2 \times (2 \uparrow p-2) \times (p-1)$, or approximately $2 \times n \times (\log_2(n)-1)$ comparisons and half as many exchanges in the worst case to sort $n = 2 \uparrow p - 1$ items. The algorithm is


[1] **J. W. J. Williams**. Heapsort, *Communications of the ACM*, 7(6):378-348, 1964.
[2] **Robert W. Floyd**. Treesort 3, *Communications of the ACM*, 7(12):701, 1964.


Coca-Cola
Estrella Damm
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S'EREM
3 3
WE WANT TO VOTE!
REPSOL
TARRAGONA T'ESPERA
TOTS A PLAÇA DE NOU

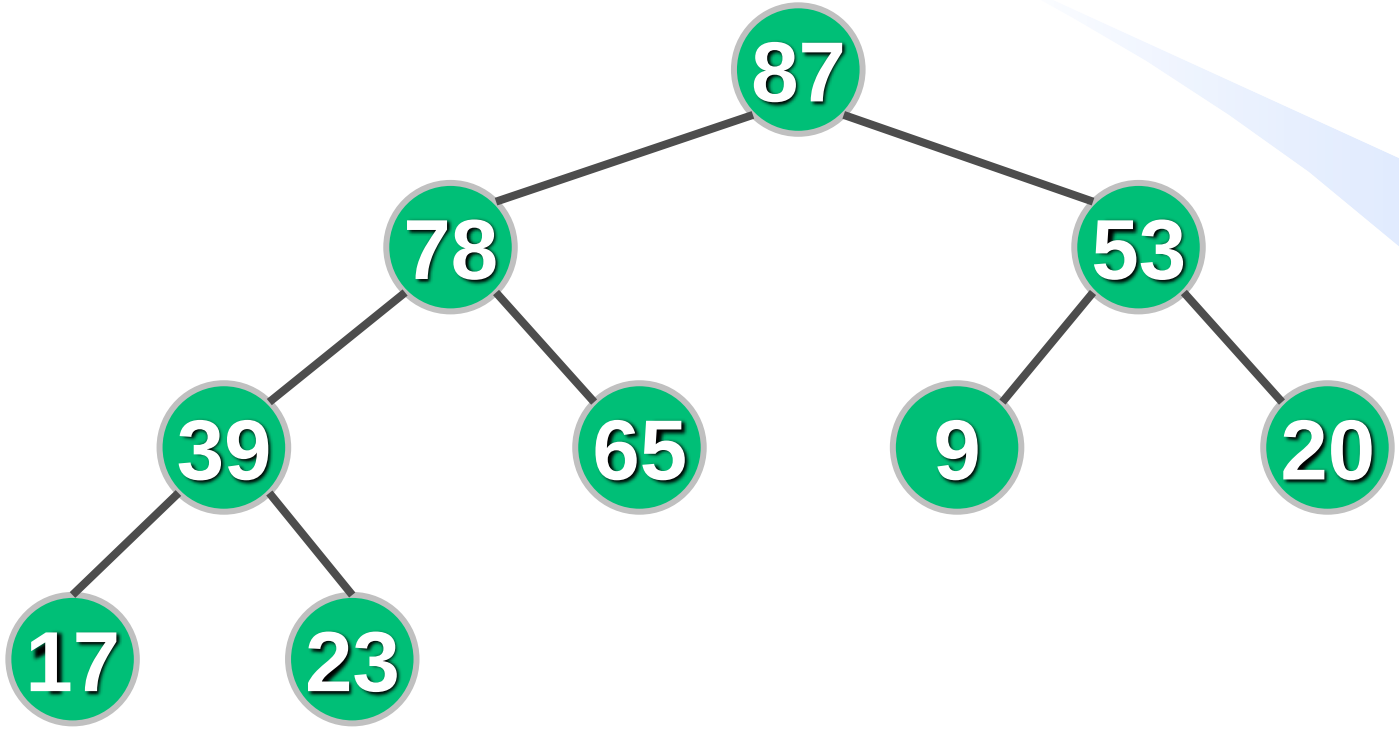
# 二叉堆（Binary Heap），简称堆（Heap）

A

- 最大堆（大根堆）：一棵完全二叉树，其中任意结点的关键词大于等于它的两个孩子的关键词。
  - ✓ 结构性：完全二叉树。
  - ✓ 堆序性：任意结点的关键词大于等于其孩子的关键词。
- 最小堆（小根堆）：一棵完全二叉树，其中任意结点的关键词小于等于它的两个孩子的关键词。
- 堆的优势：最大堆中根结点的关键词最大，最小堆中根结点的关键词最小。

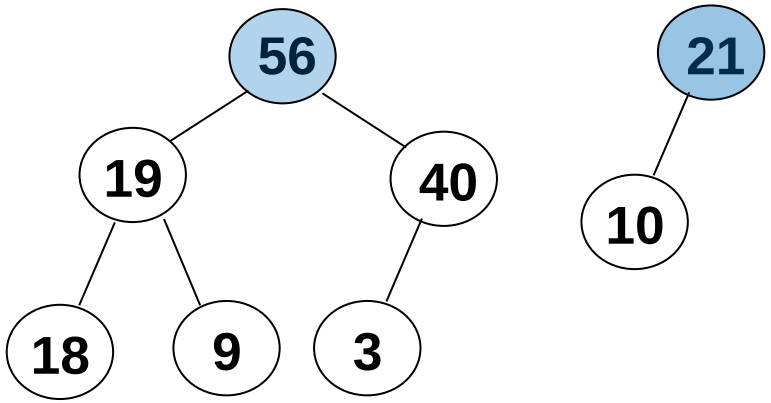

A

# 最大堆

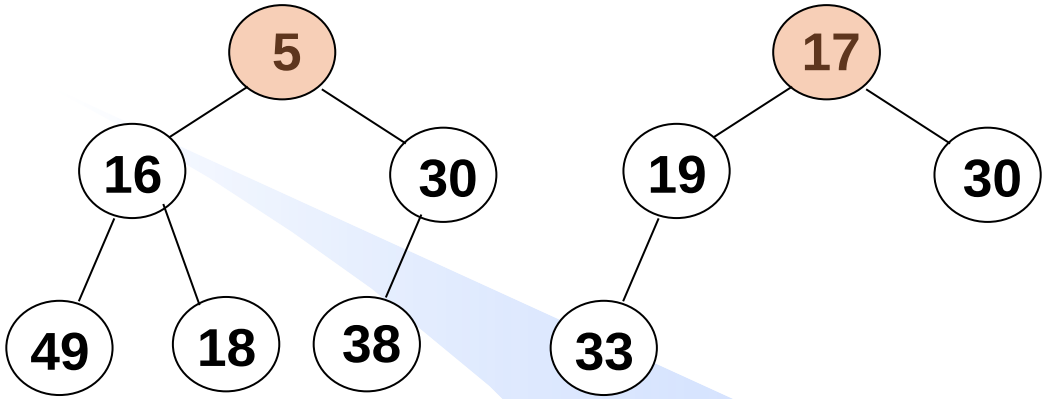


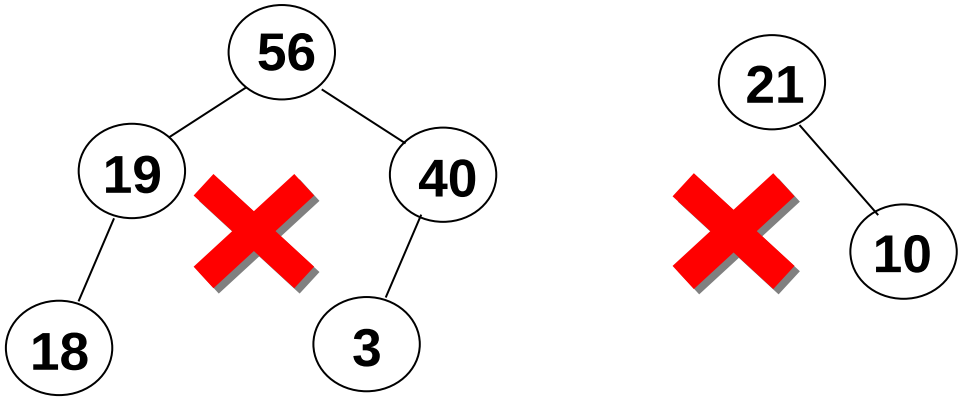

A

## 最大堆



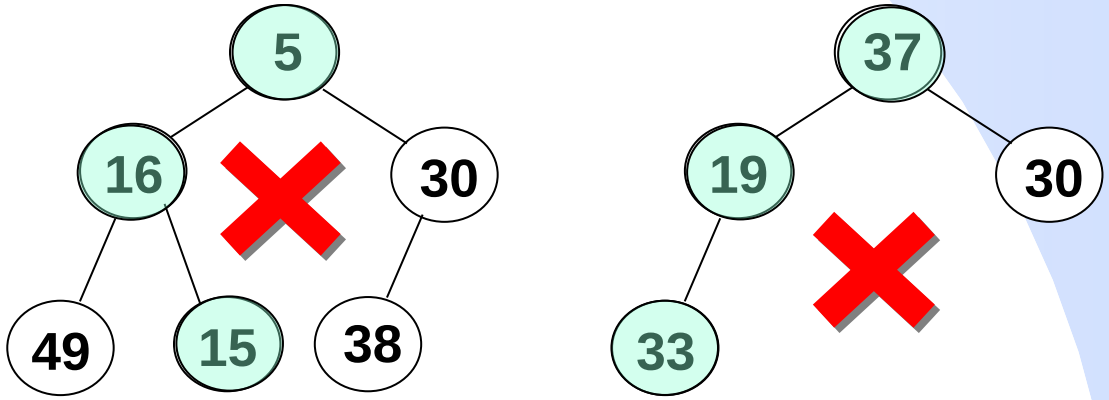

## 最小堆




不满足结构性

## 不是堆

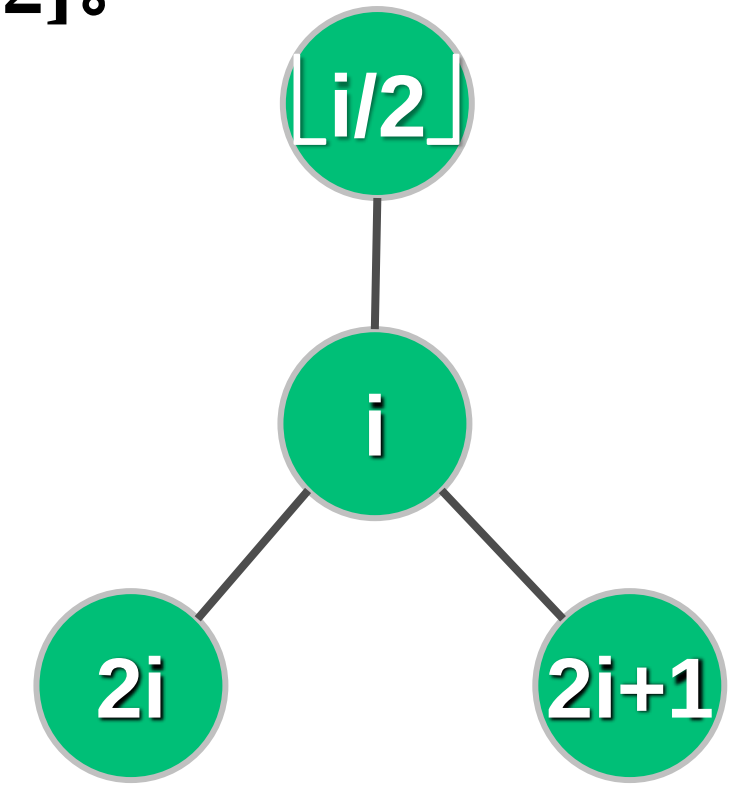



不满足堆序性

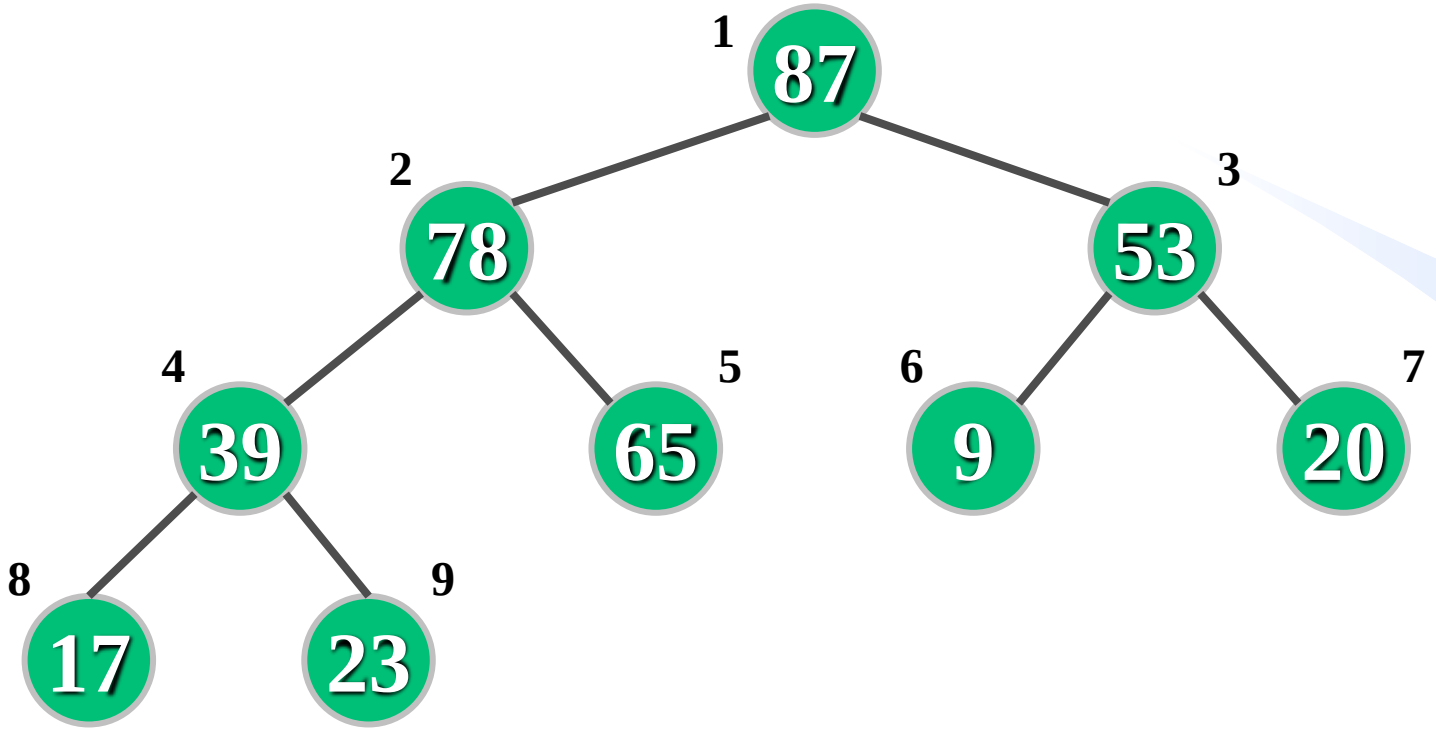
# 回顾： 完全二叉树的顺序存储

A

$R[1]$存储二叉树的根结点。结点$R[i]$的左孩子（若有的话）存放在$R[2i]$处，而$R[i]$的右孩子（若有的话）存放在$R[2i+1]$处。$R[i]$的父结点是$R[\lfloor i/2 \rfloor]$。



# 堆的顺序存储



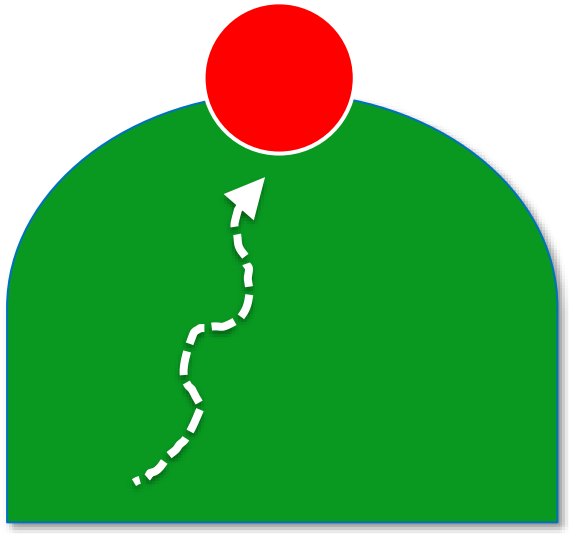

| $R$ | | 87 | 78 | 53 | 39 | 65 | 9 | 20 | 17 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|
| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

- ✓$R[1]$存根结点；
- ✓结点$R[i]$的左孩子（若有的话）存放在$R[2i]$处；
- ✓$R[i]$的右孩子（若有的话）存放在$R[2i+1]$处；
- ✓$R[i]$的父结点为$R[i/2]$。

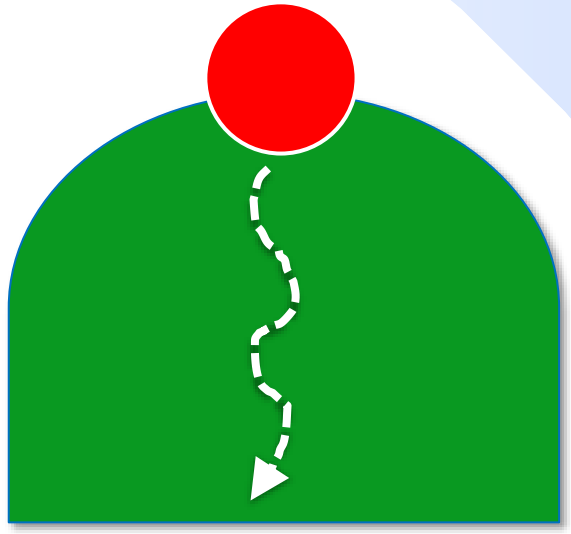
# 堆的两个基本操作

A

上浮操作



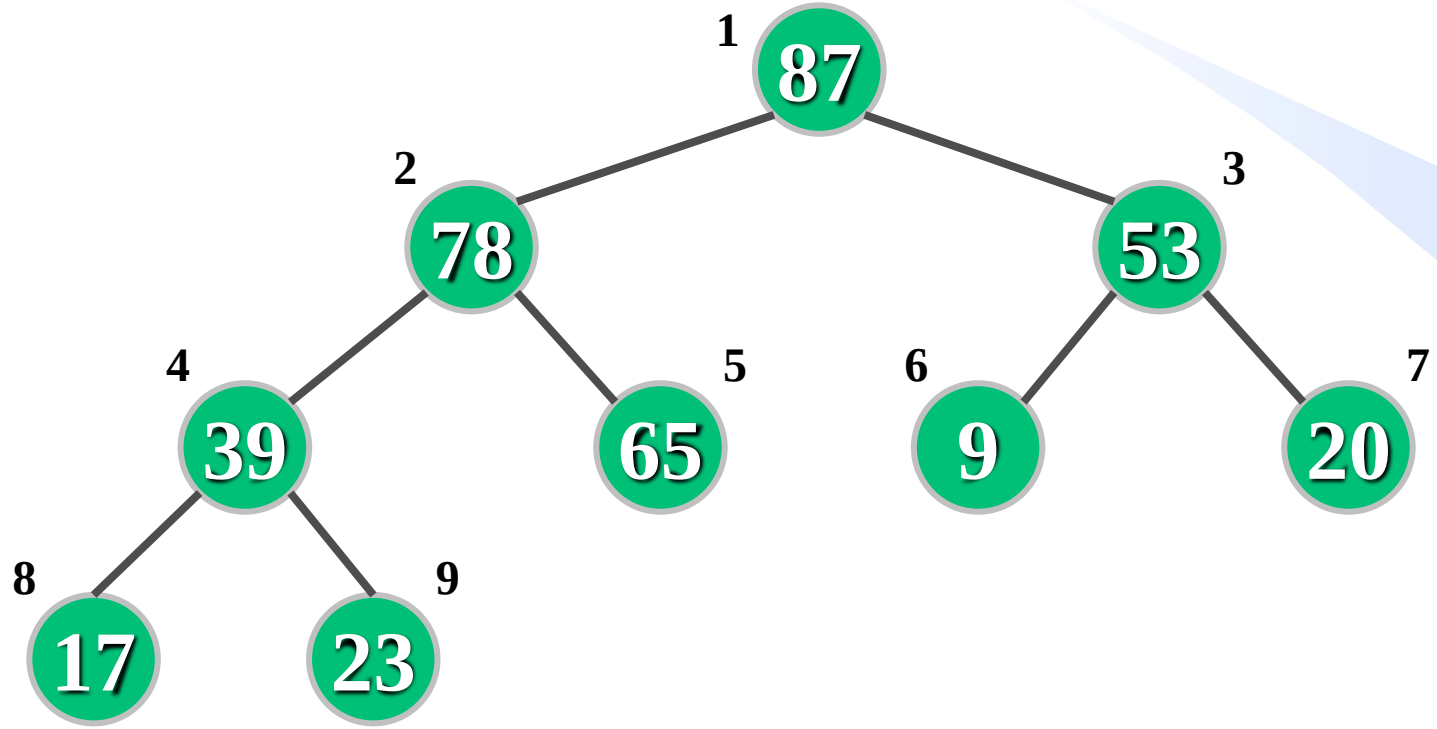
下沉操作



# 堆的上浮操作

A

当大根堆的元素值R[*i*]变大时，该结点可能会上浮；



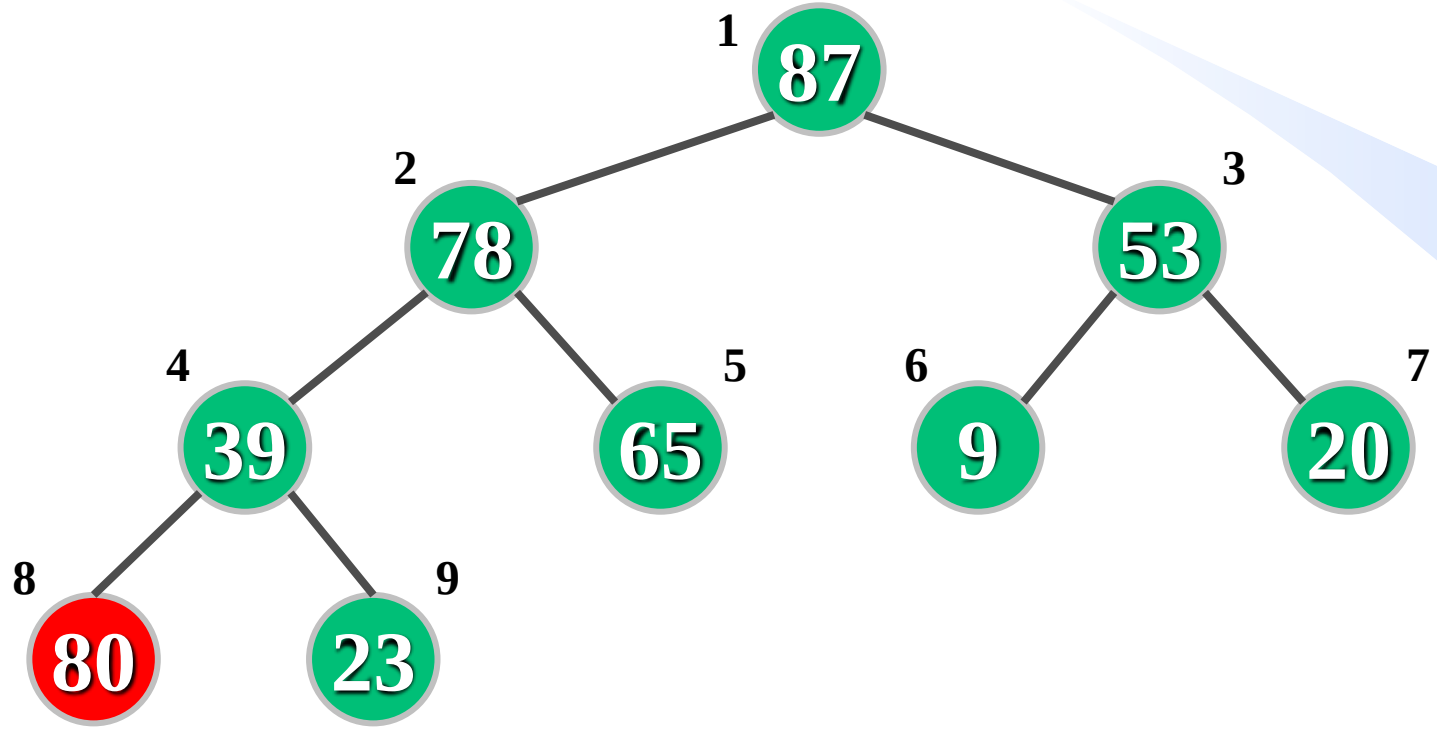

将R[8]修改为80

| 87 | 78 | 53 | 39 | 65 | 9 | 20 | 17 | 23 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

# 堆的上浮操作

A

当大根堆的元素值R[*i*]变大时，该结点可能会上浮；



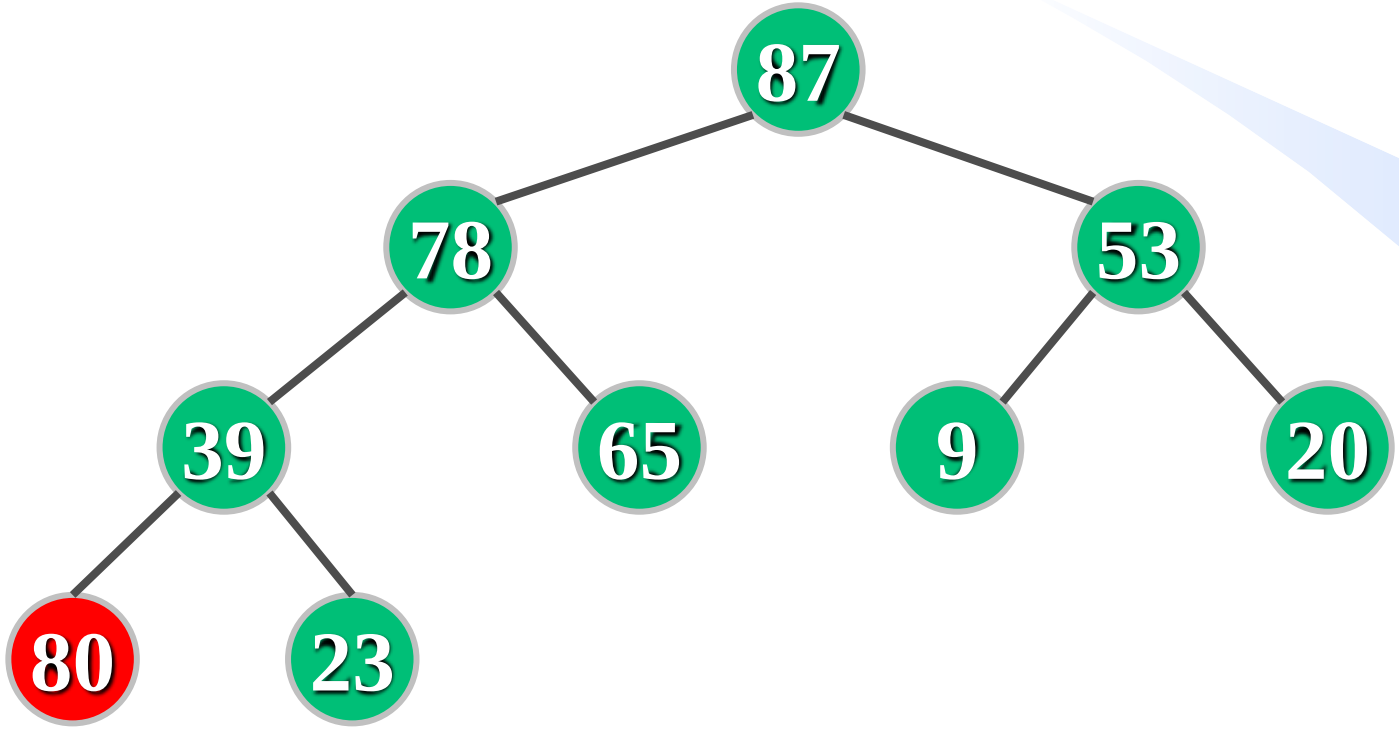

将R[8]修改为80

| 87 | 78 | 53 | 39 | 65 | 9 | 20 | 80 | 23 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

# 堆的上浮操作

A

当大根堆的元素值R[*i*]变大时，该结点可能会上浮；



# 堆的上浮操作

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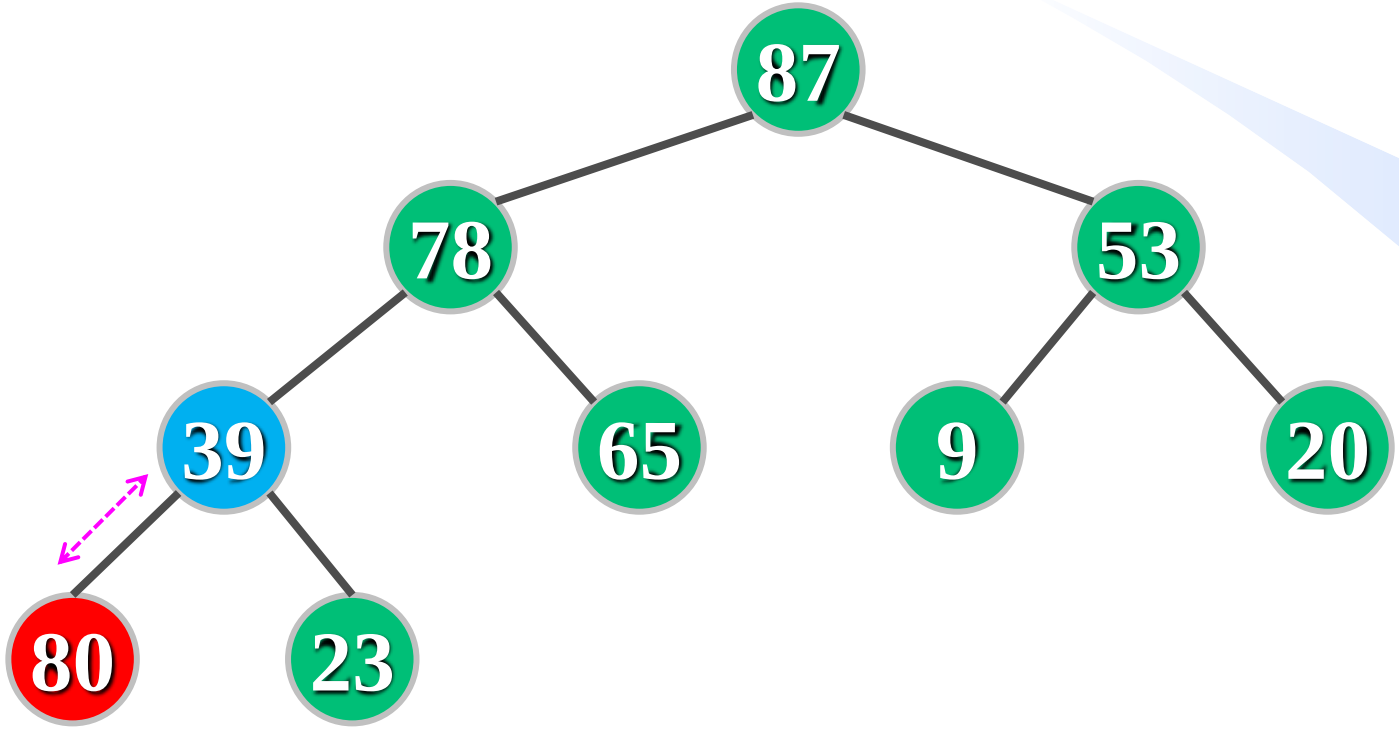
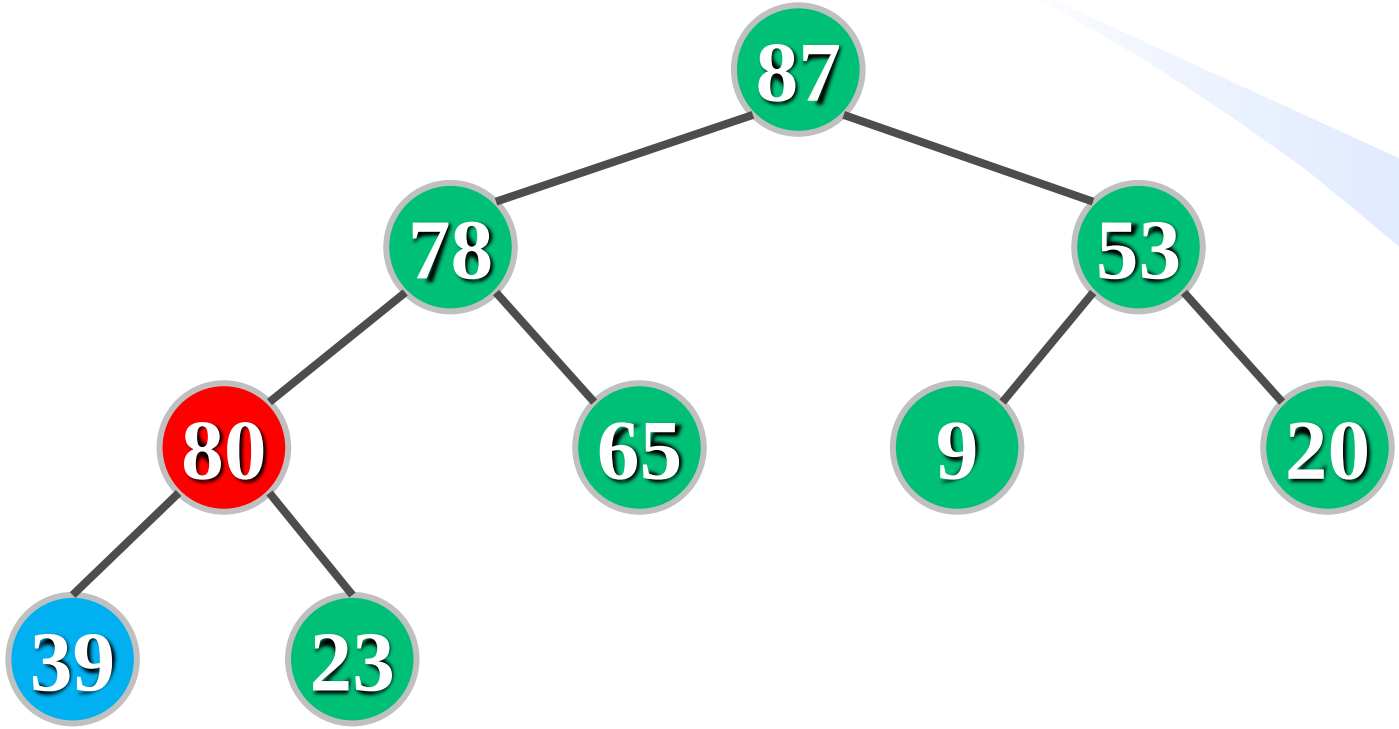
当大根堆的元素值R[*i*]变大时，该结点可能会上浮；



# 堆的上浮操作

A

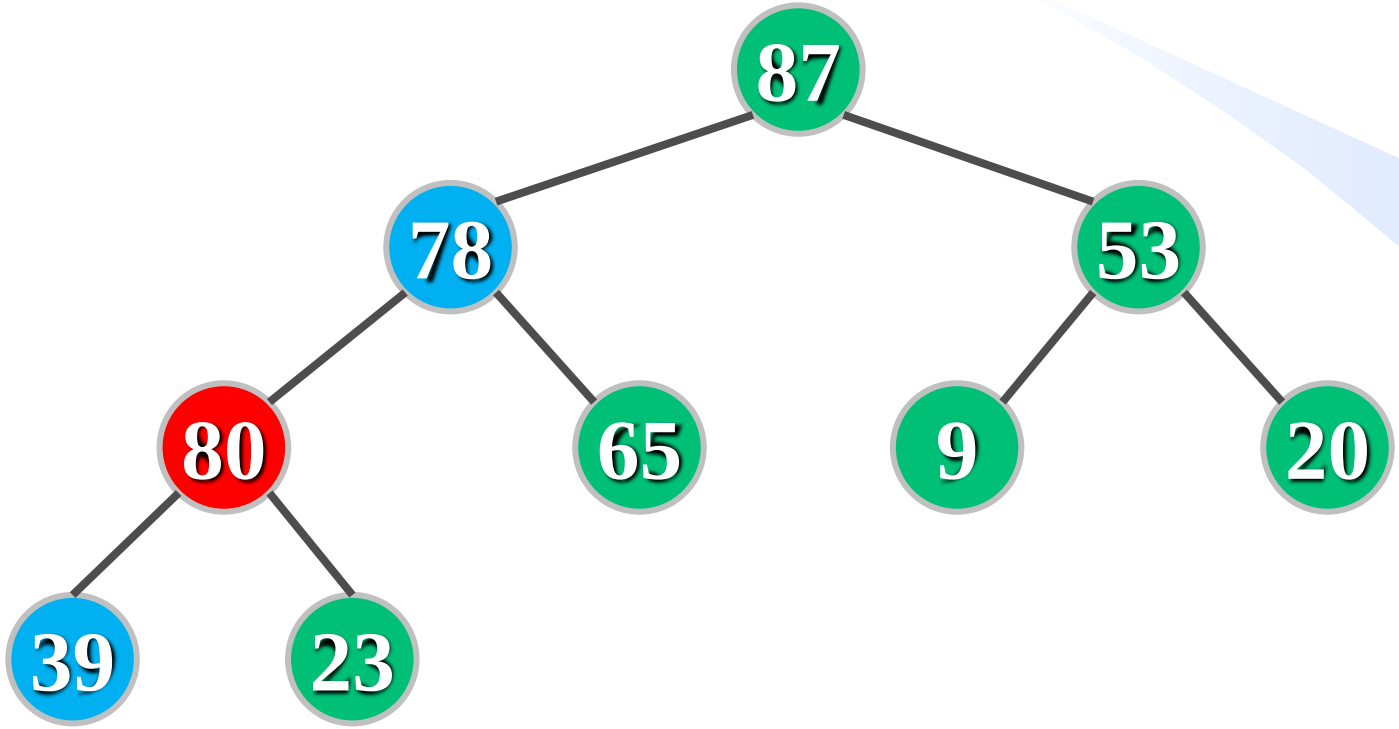
当大根堆的元素值R[*i*]变大时，该结点可能会上浮；



# 堆的上浮操作

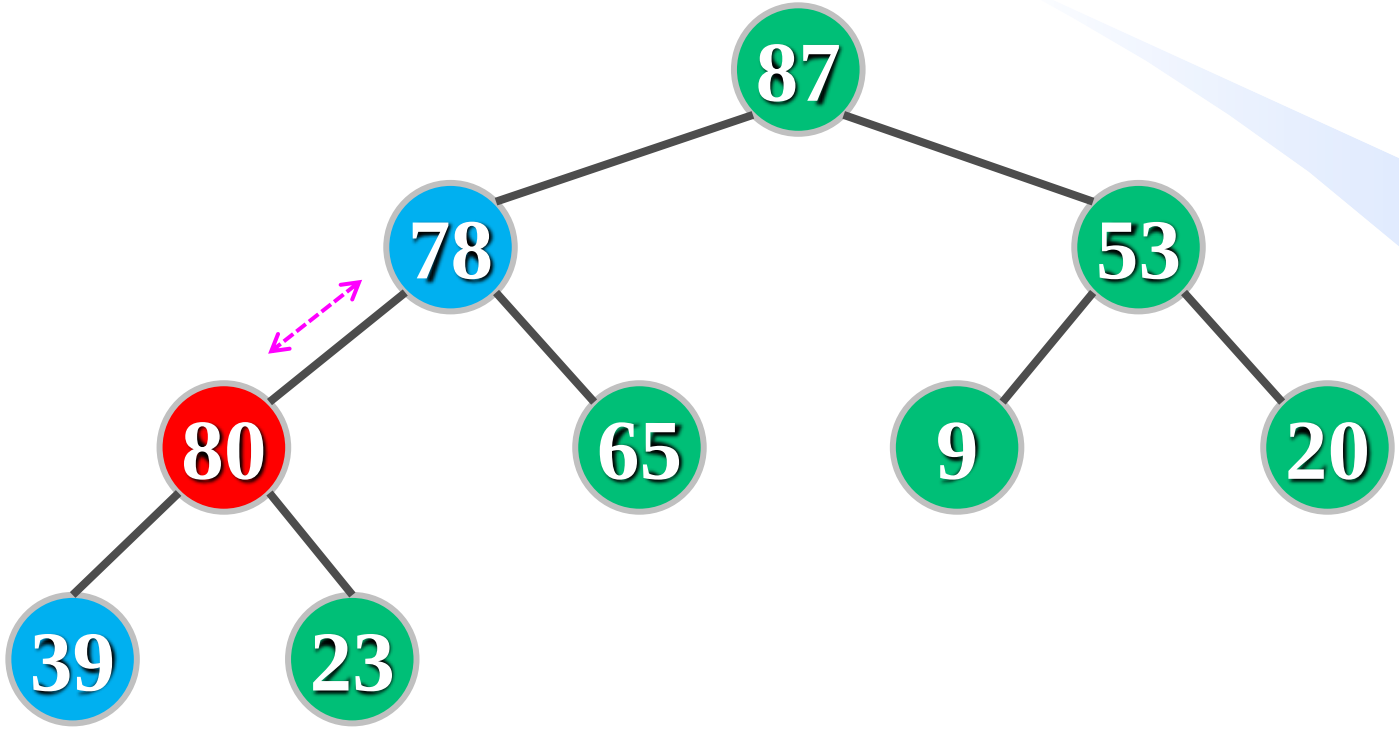
A

当大根堆的元素值R[*i*]变大时，该结点可能会上浮；



# 堆的上浮操作

当大根堆的元素值R[*i*]变大时，该结点可能会上浮；

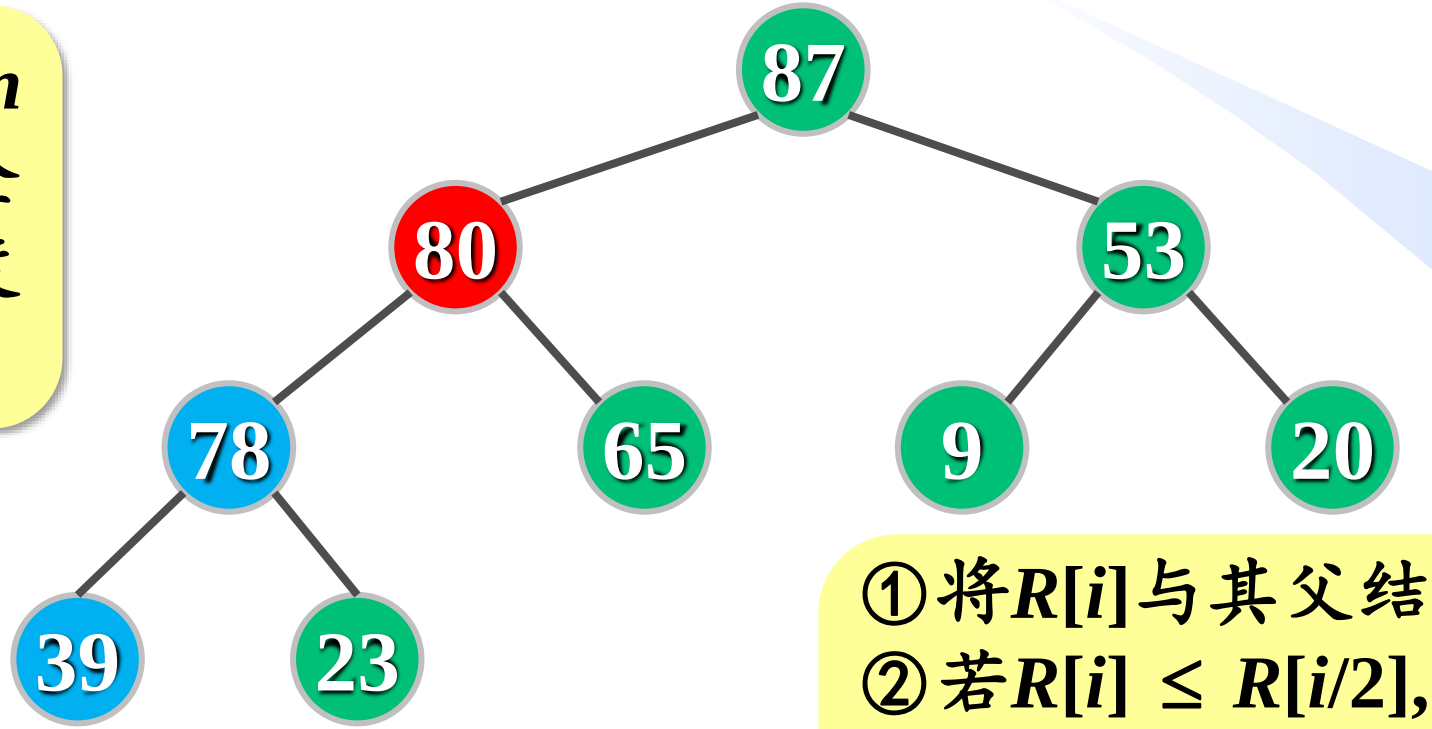


# 堆的上浮操作

A

当大根堆的元素值R[*i*]变大时，该结点可能会上浮；

回顾：具有*n*个结点的完全二叉树的高度是$\lfloor \log_2 n \rfloor$.




①将*R*[*i*]与其父结点*R*[*i*/2]比较；
②若*R*[*i*] ≤ *R*[*i*/2]，则已满足堆序性，算法结束；
③否则交换*R*[*i*]和*R*[*i*/2]，令*i*上行指向其父亲*i*←*i*/2，执行①。

时间复杂度
取决于树的高度
O(log*n*)

# 堆的上浮操作

A

```
void ShiftUp(int R[], int n, int i){
    //堆元素R[i]上浮,数组R[ ]存储堆,n为堆包含的元素个数
    while(i>1 && R[i]>R[i/2]){   //R[i]比父亲大且i不是根
        swap(R[i], R[i/2]);      //交换R[i]和父亲
        i/=2;                    //结点i继续上浮
    }
}
```

时间复杂度O(log*n*)

也称为Swim或PercolateUp

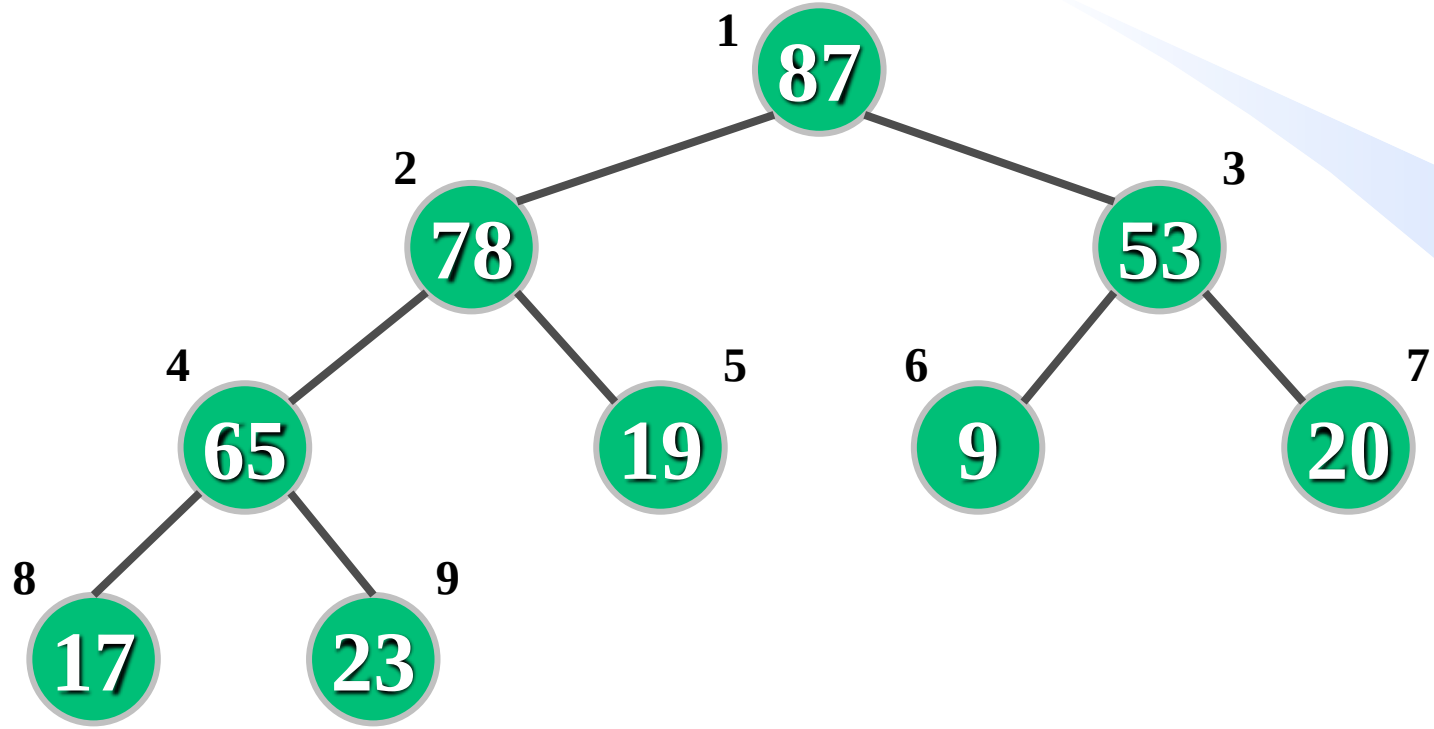
```
void swap(int &a, int &b){
    int temp = a;
    a = b;
    b = temp;
}
```

①将$R[i]$与其父结点$R[i/2]$比较；
②若$R[i] \le R[i/2]$，则已满足堆序性，算法结束；
③否则交换$R[i]$和$R[i/2]$，令$i$指向其父亲$i \leftarrow i/2$，执行①。

# 堆的下沉操作

A

当大根堆的元素值R[*i*]变小时，该结点可能会下沉；



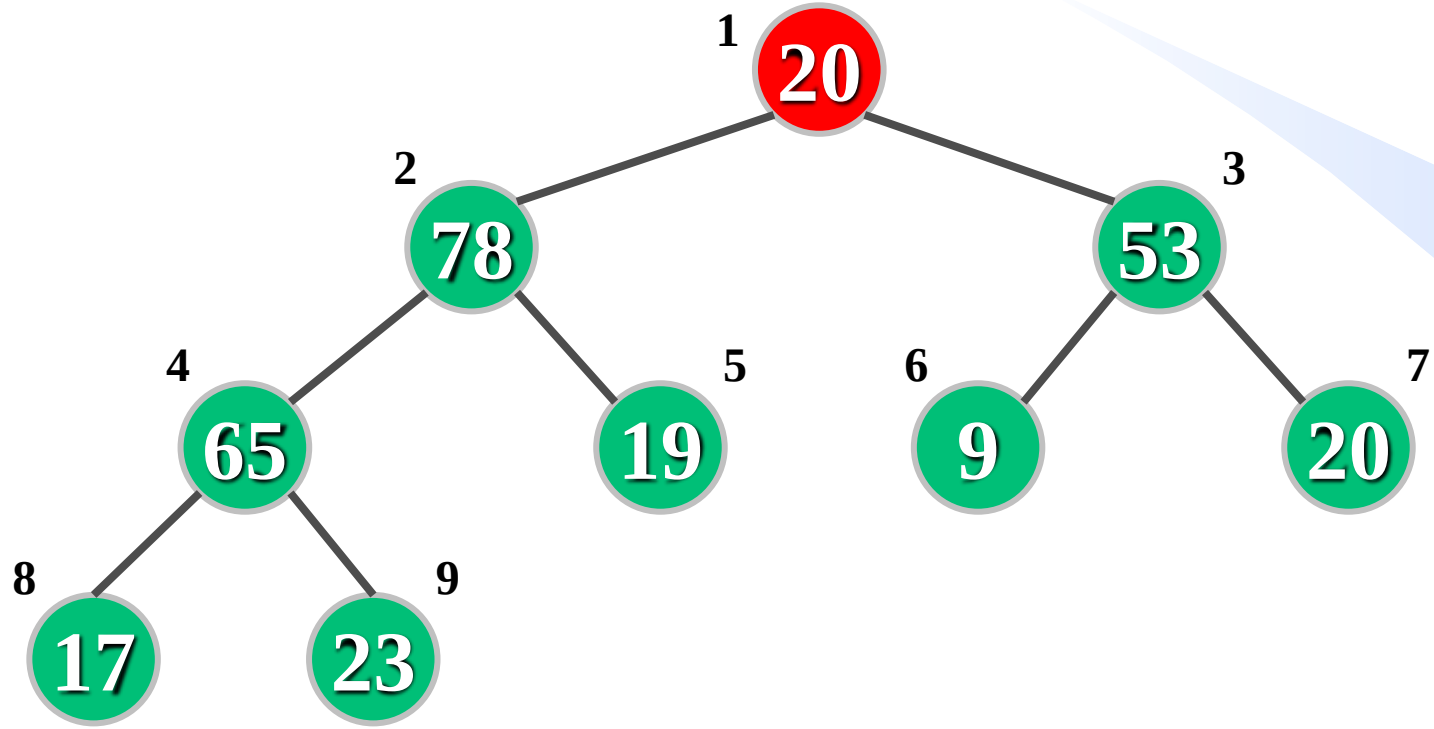

将R[1]修改为20

| 87 | 78 | 53 | 65 | 19 | 9 | 20 | 17 | 23 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

# 堆的下沉操作

A

当大根堆的元素值R[*i*]变小时，该结点可能会下沉；




将R[1]修改为20

| 20 | 78 | 53 | 65 | 19 | 9 | 20 | 17 | 23 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

# 堆的下沉操作

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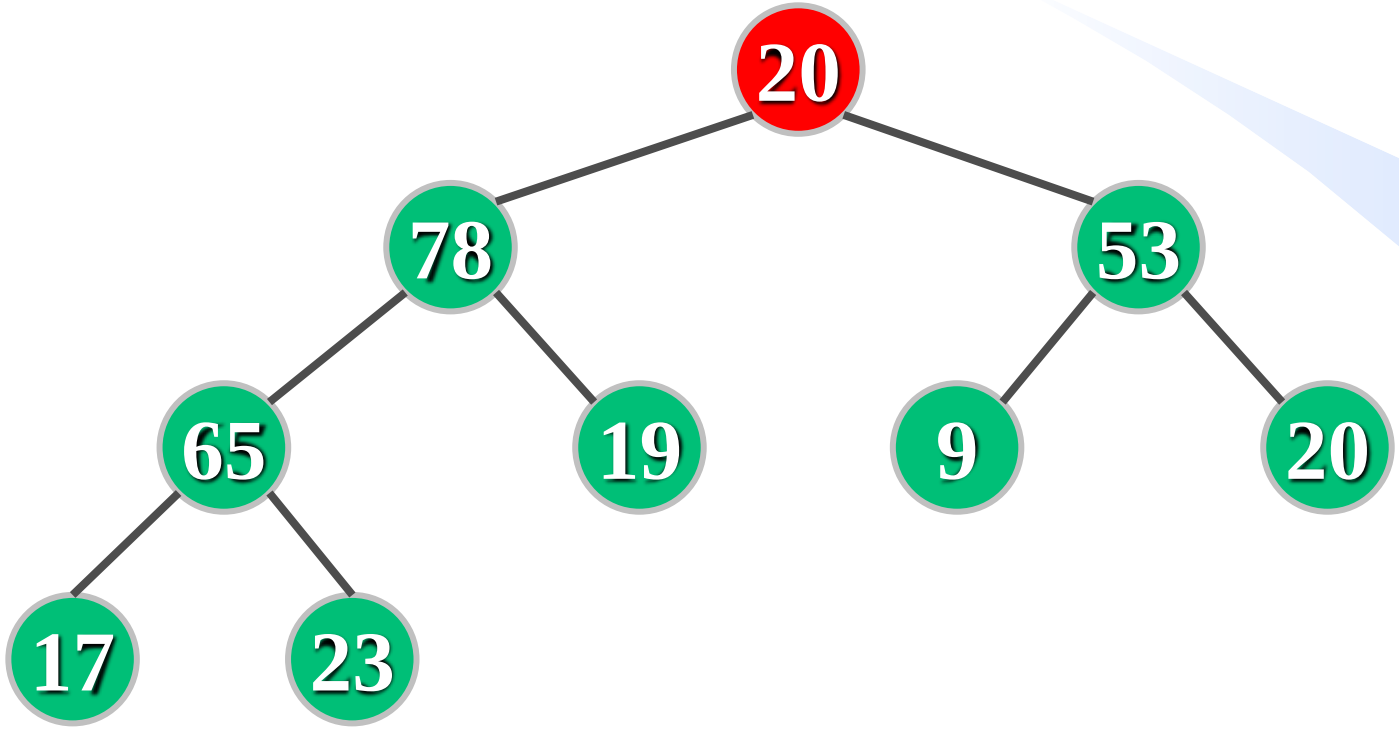
当大根堆的元素值R[*i*]变小时，该结点可能会下沉；



# 堆的下沉操作

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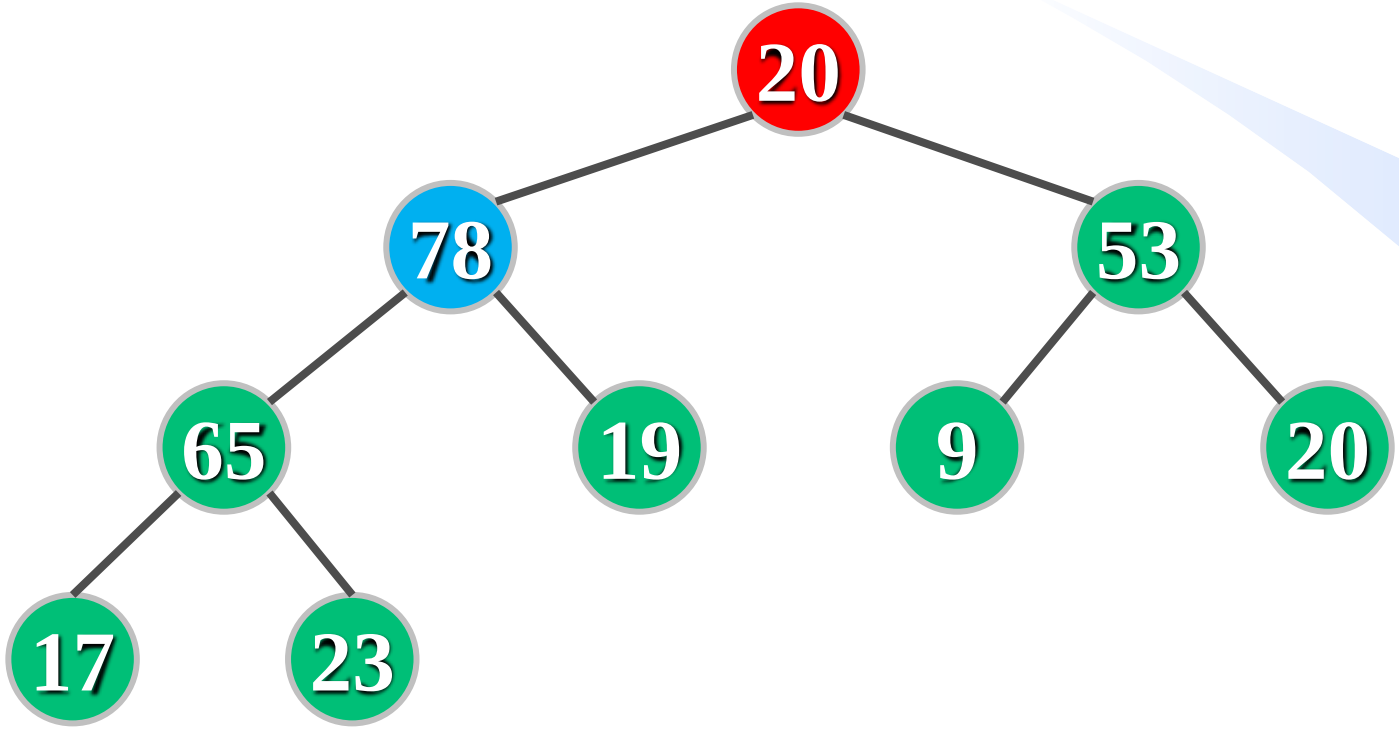
当大根堆的元素值R[*i*]变小时，该结点可能会下沉；



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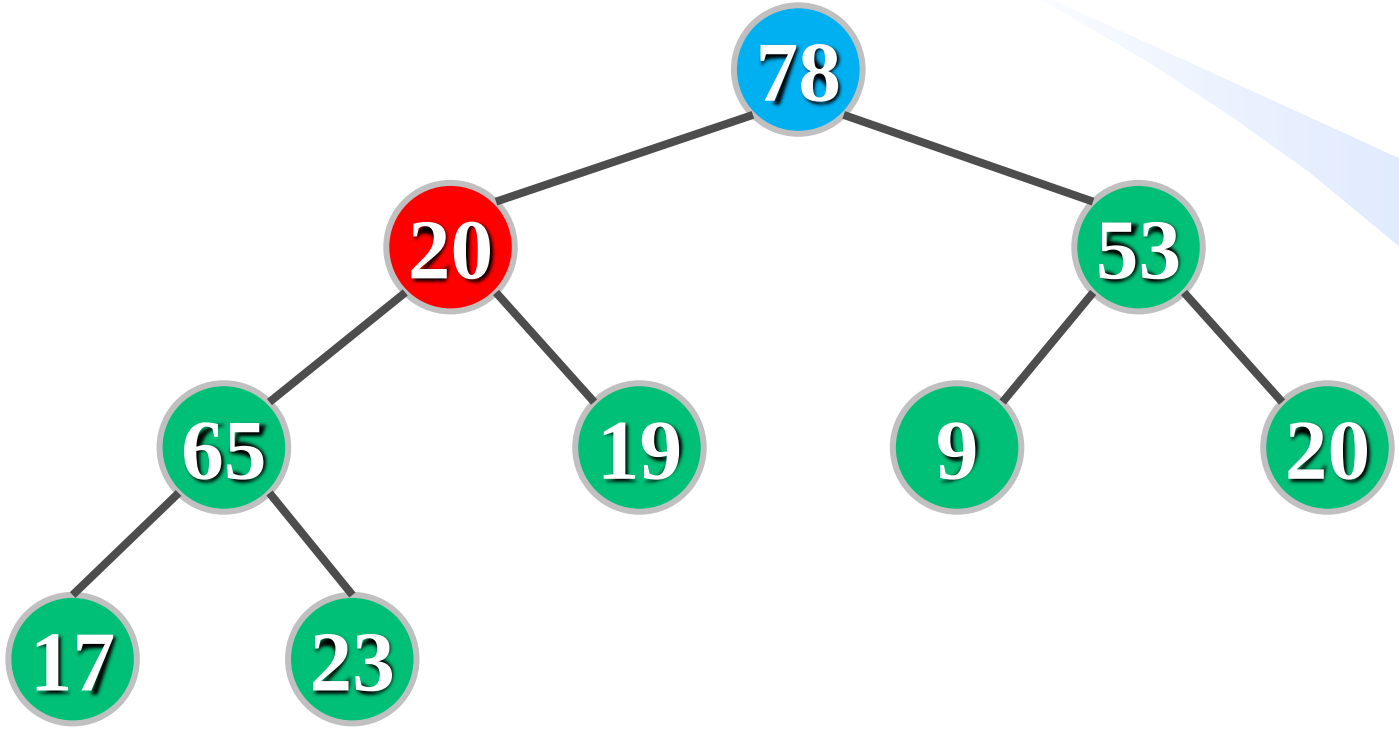
当大根堆的元素值R[*i*]变小时，该结点可能会下沉；



# 堆的下沉操作

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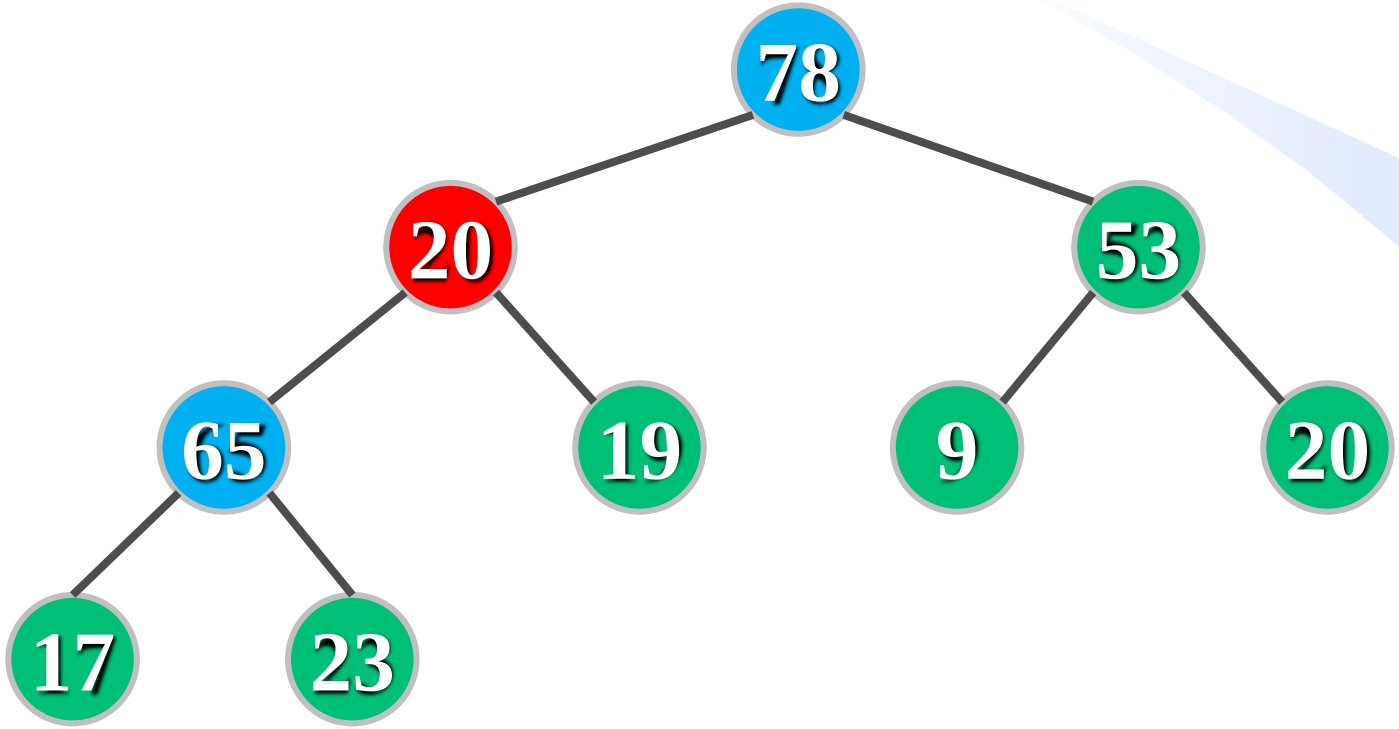
当大根堆的元素值R[*i*]变小时，该结点可能会下沉；



# 堆的下沉操作

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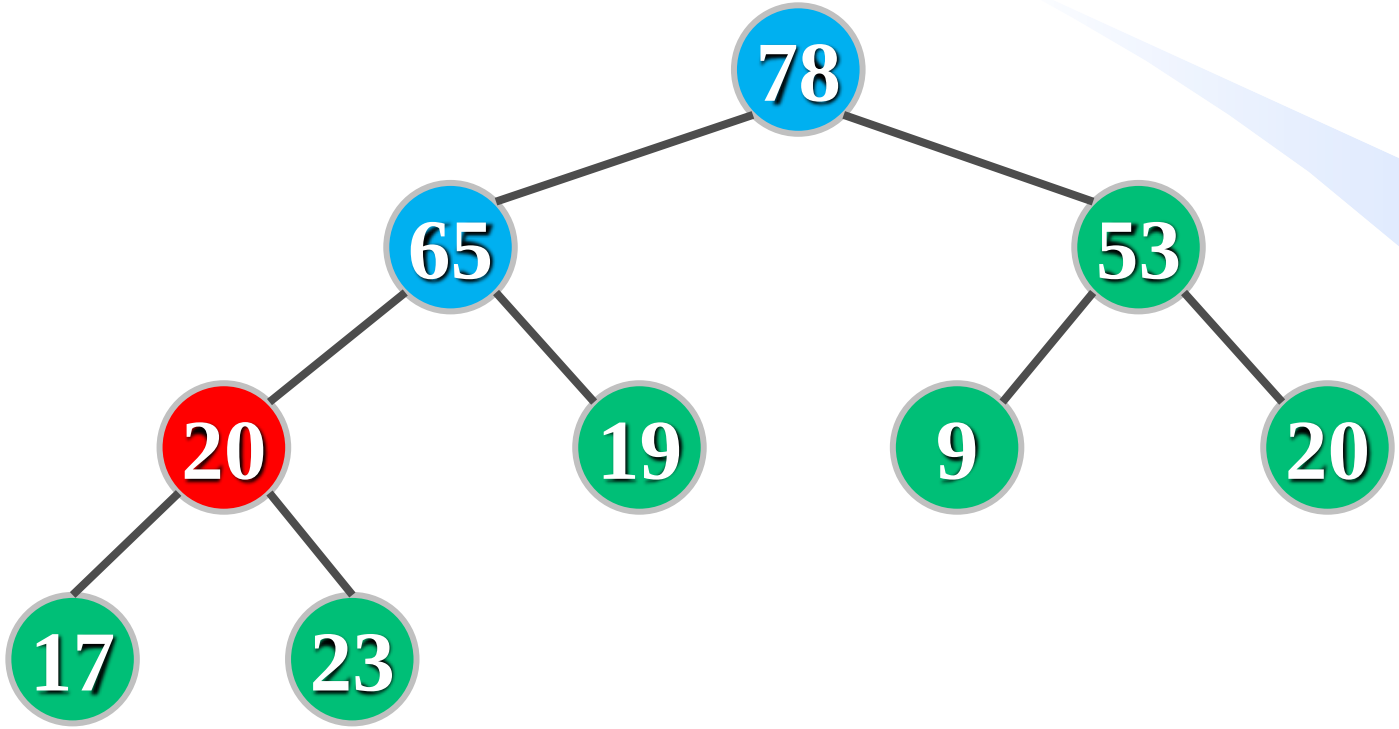
当大根堆的元素值R[*i*]变小时，该结点可能会下沉；



# 堆的下沉操作

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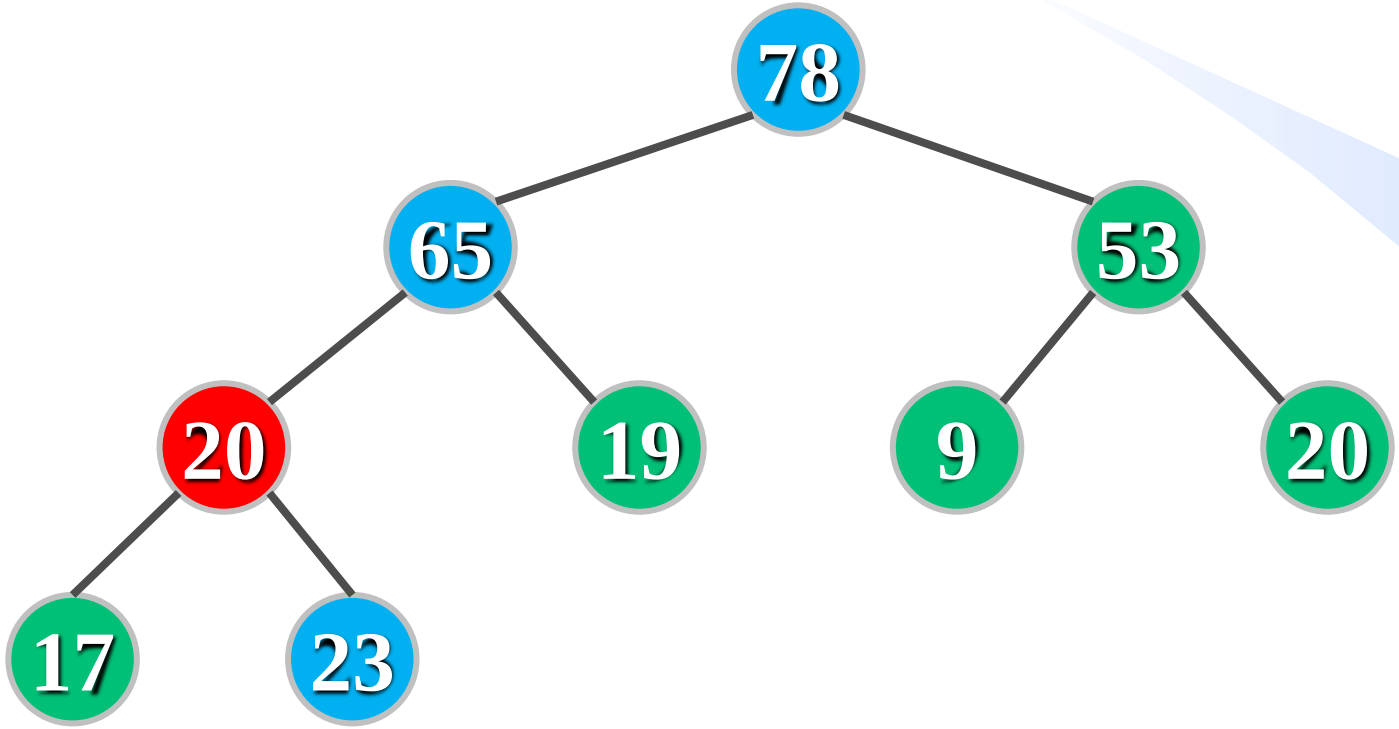
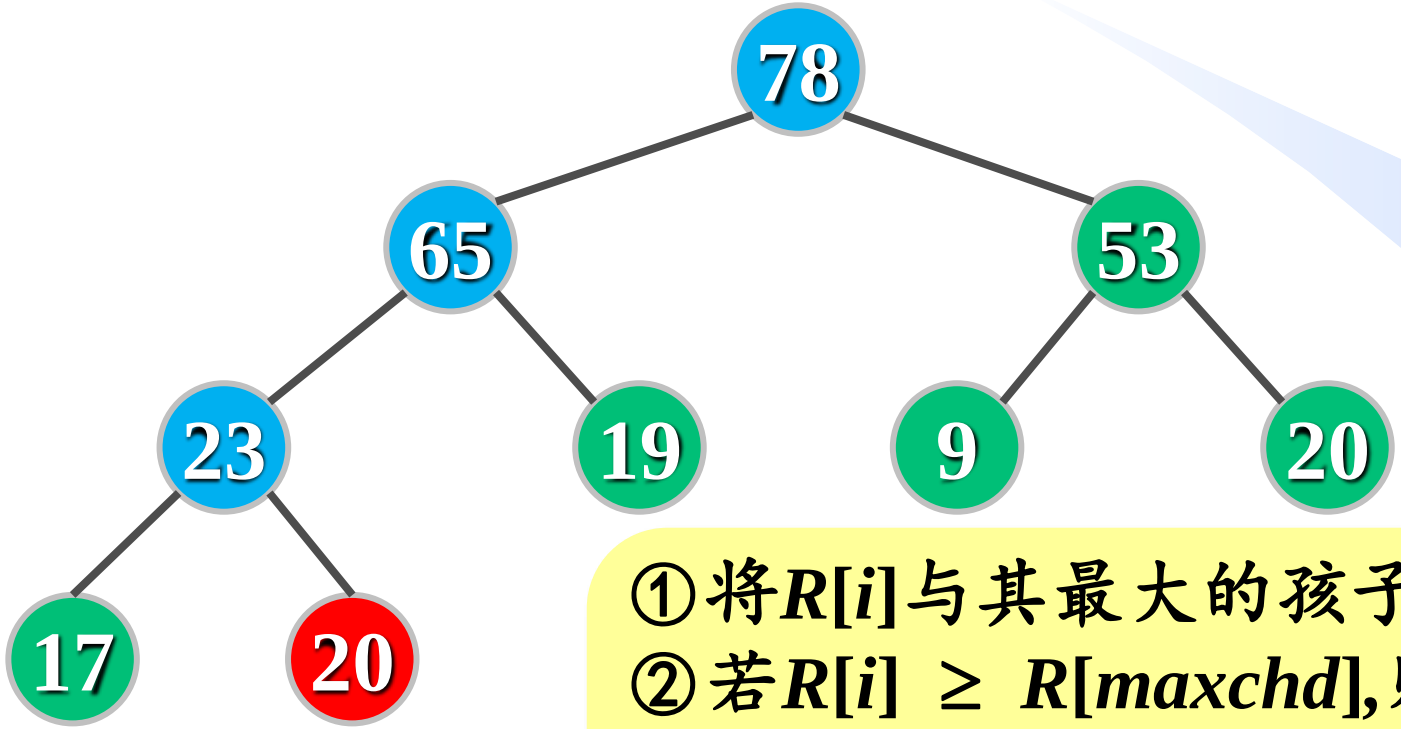
当大根堆的元素值R[*i*]变小时，该结点可能会下沉；



# 堆的下沉操作

A

当大根堆的元素值R[*i*]变小时，该结点可能会下沉；

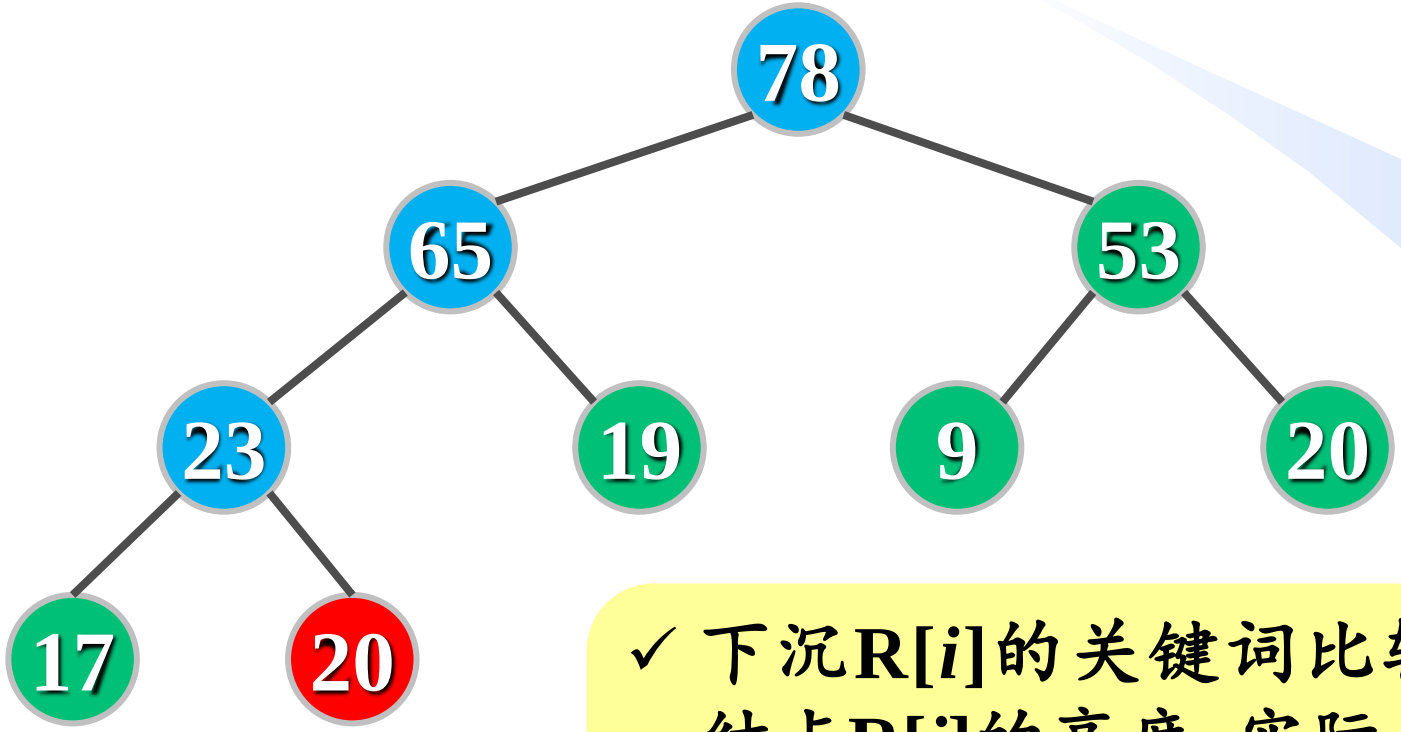



①将*R*[*i*]与其最大的孩子*R*[*maxchd*]比较；
②若*R*[*i*] ≥ *R*[*maxchd*],则已满足堆序性，算法结束；
③否则交换*R*[*i*]和*R*[*maxchd*]，令*i*下行*i*←*maxchd*，返回①

# 堆的下沉操作

A

当大根堆的元素值R[*i*]变小时，该结点可能会下沉；




- ✓ 下沉R[*i*]的关键词比较次数：取决于结点R[*i*]的高度，实际是高度×2；
- ✓ 每个非叶结点对应2次关键词比较；
- ✓ 时间复杂度O(log*n*)

# 堆的下沉操作

A

```
void ShiftDown(int R[], int n, int i) {
    //堆元素R[i]下沉, 数组R[ ]存储堆, n为堆包含的元素个数
    while(i <= n/2){          //i最多下行至最后一个非叶结点
        int maxchd = 2*i;   // 假定最大孩子为左孩子
        if(maxchd+1<=n && R[maxchd]<R[maxchd+1])
            maxchd++;         //i的右孩子是最大孩子
        if(R[i] >= R[maxchd]) return;
        swap(R[maxchd],R[i]); // R[i]的最大孩子比R[i]大
        i = maxchd;                  // 结点i继续下沉
    }
}
```

若i有右孩子

i的右孩子比左孩子大

也称为Sink或PercolateDown

时间复杂度 O(log*n*)

# 初始建堆

A

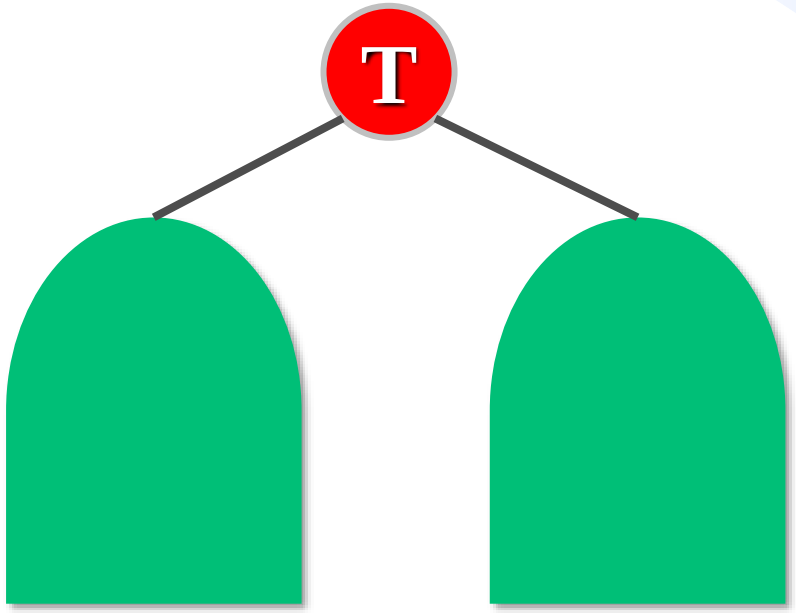
自顶向下

递归思路

① 建立T的左子堆

② 建立T的右子堆

③ 结点T下沉。




自底向上

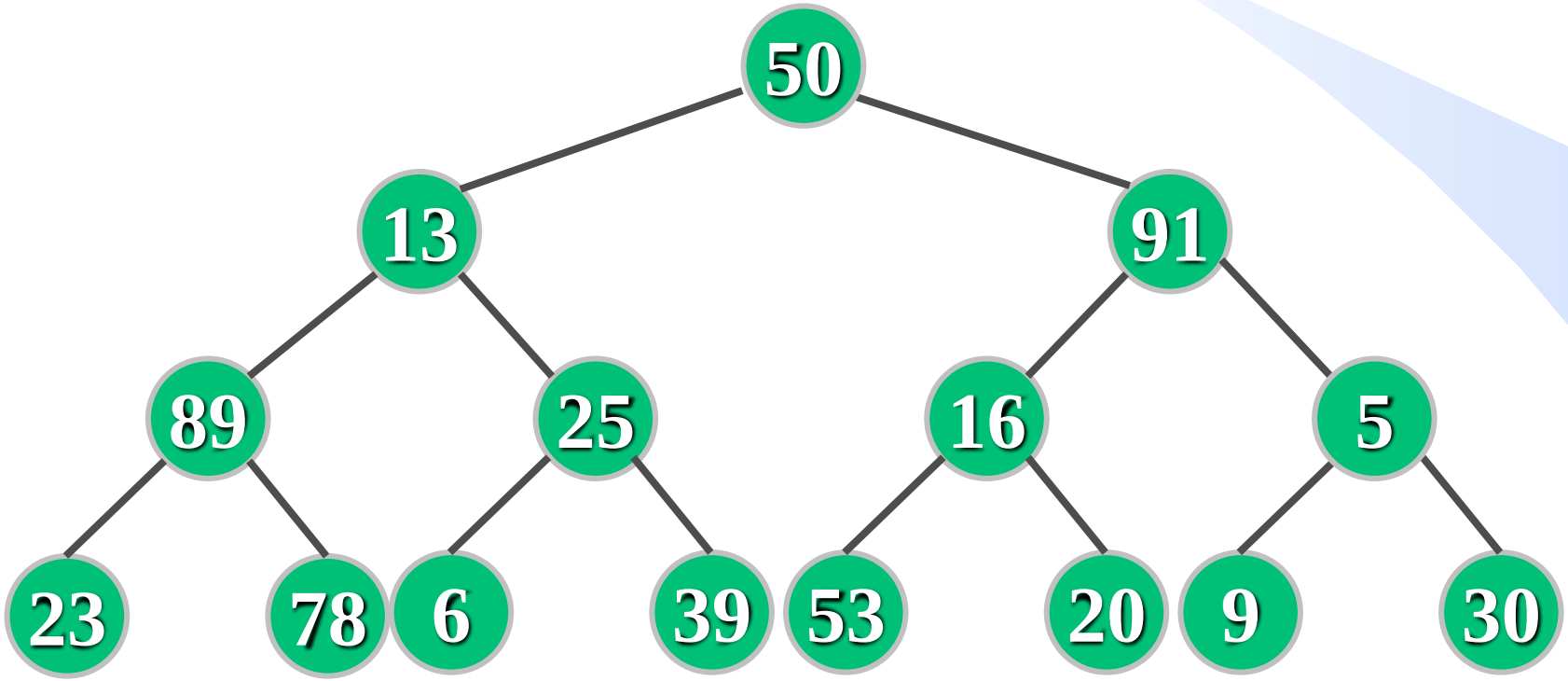
动态规划思路

从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。

# 初始建堆

A

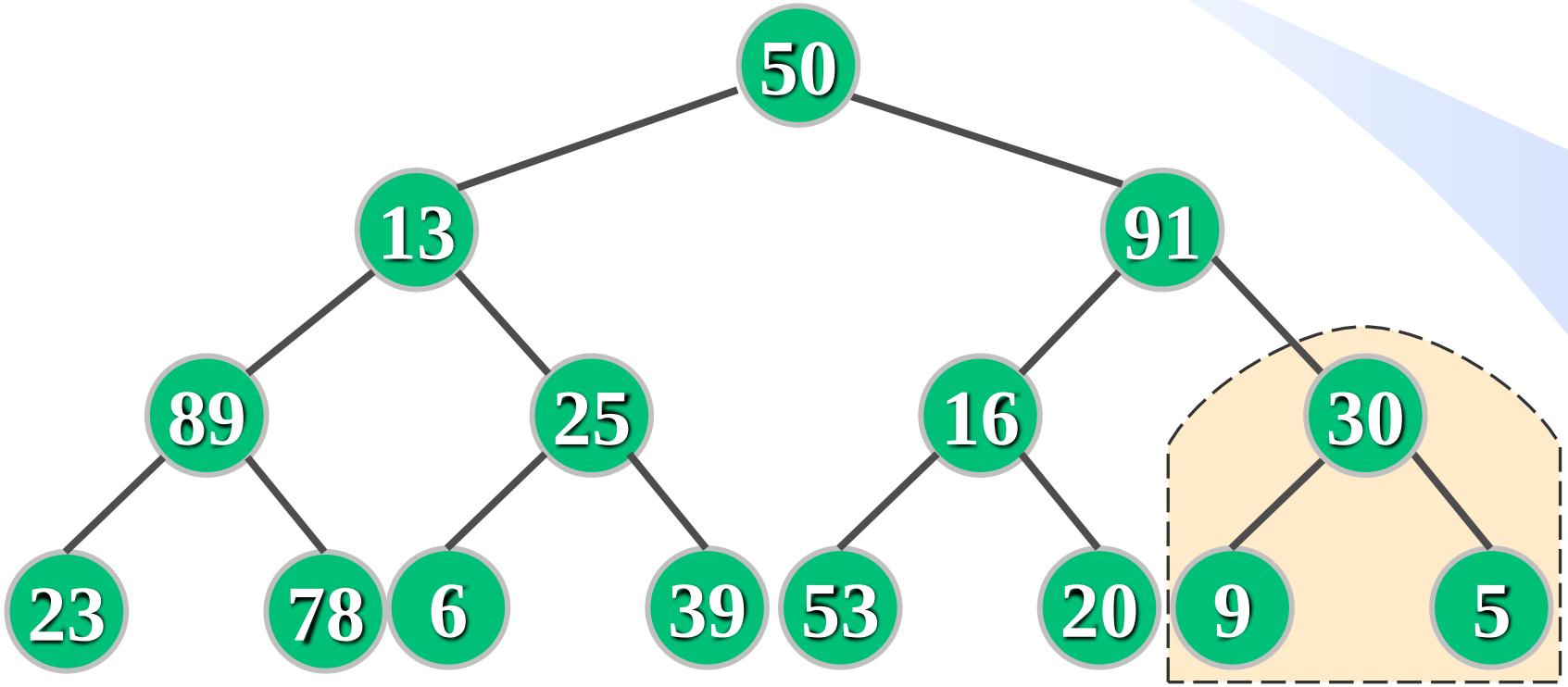
从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。



# 初始建堆

A

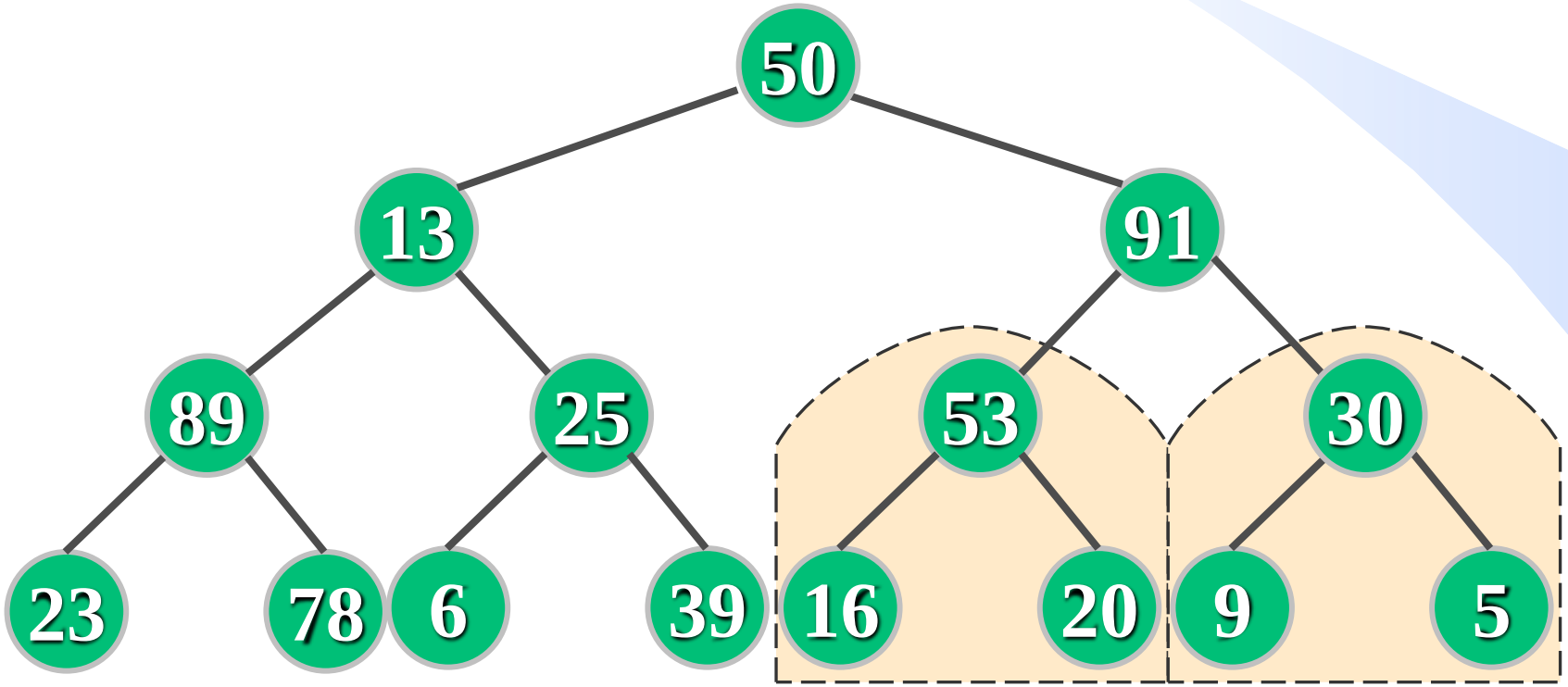
从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。



# 初始建堆

A

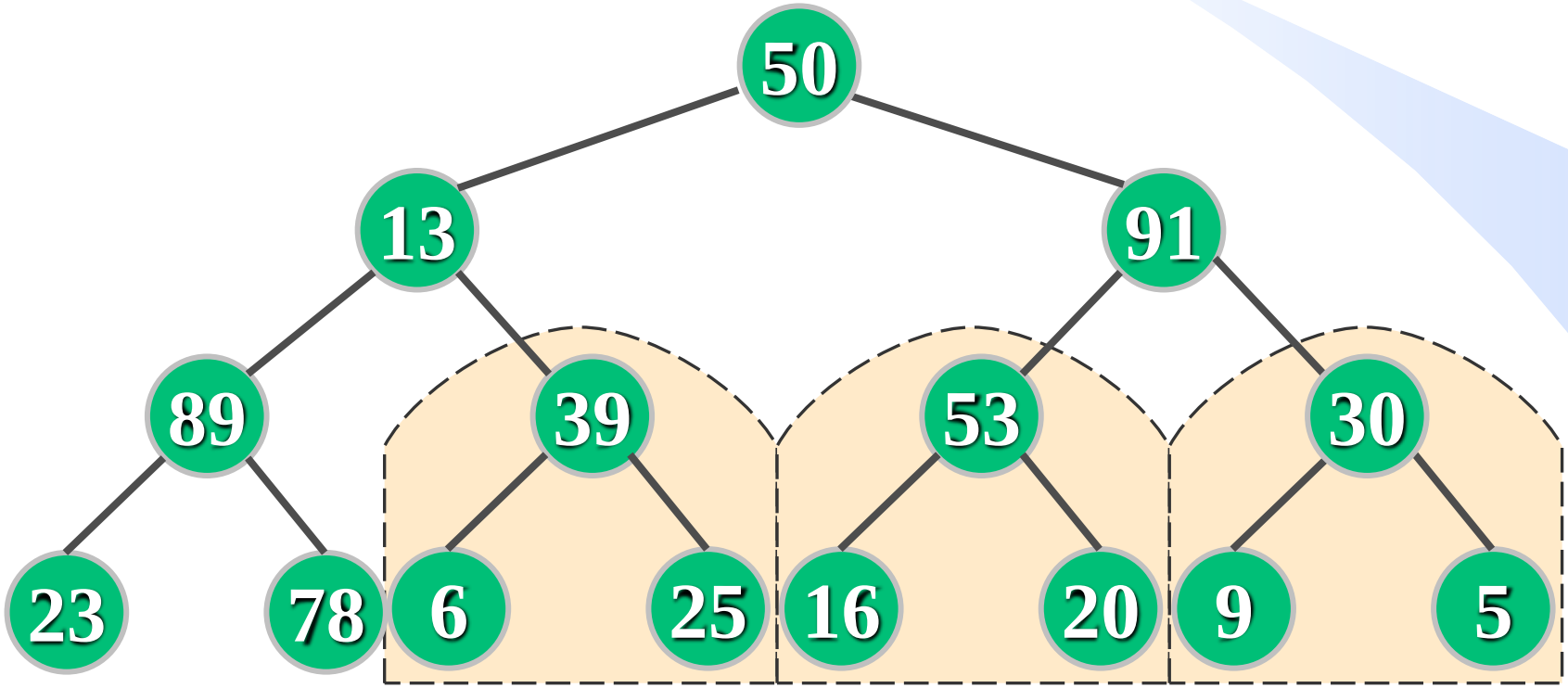
从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。



# 初始建堆

A

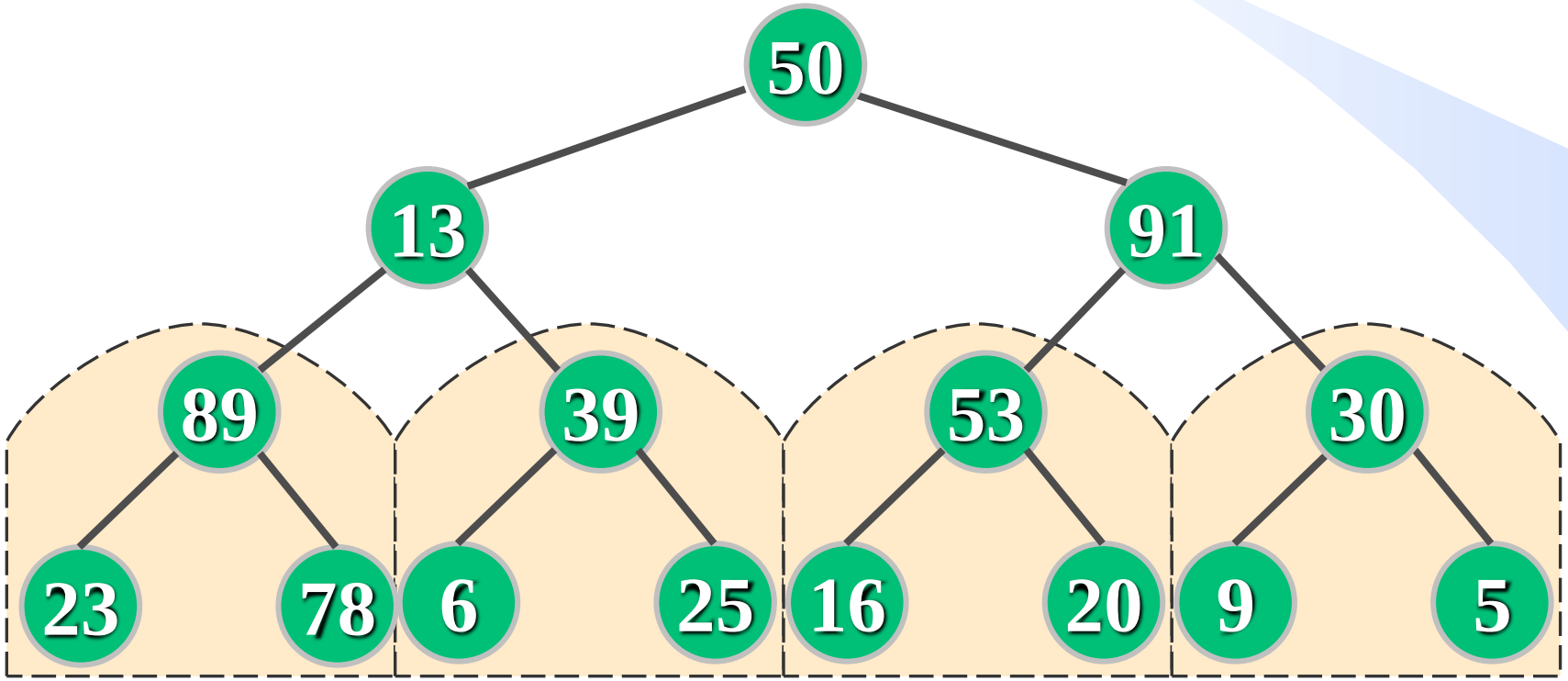
从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。



# 初始建堆

A

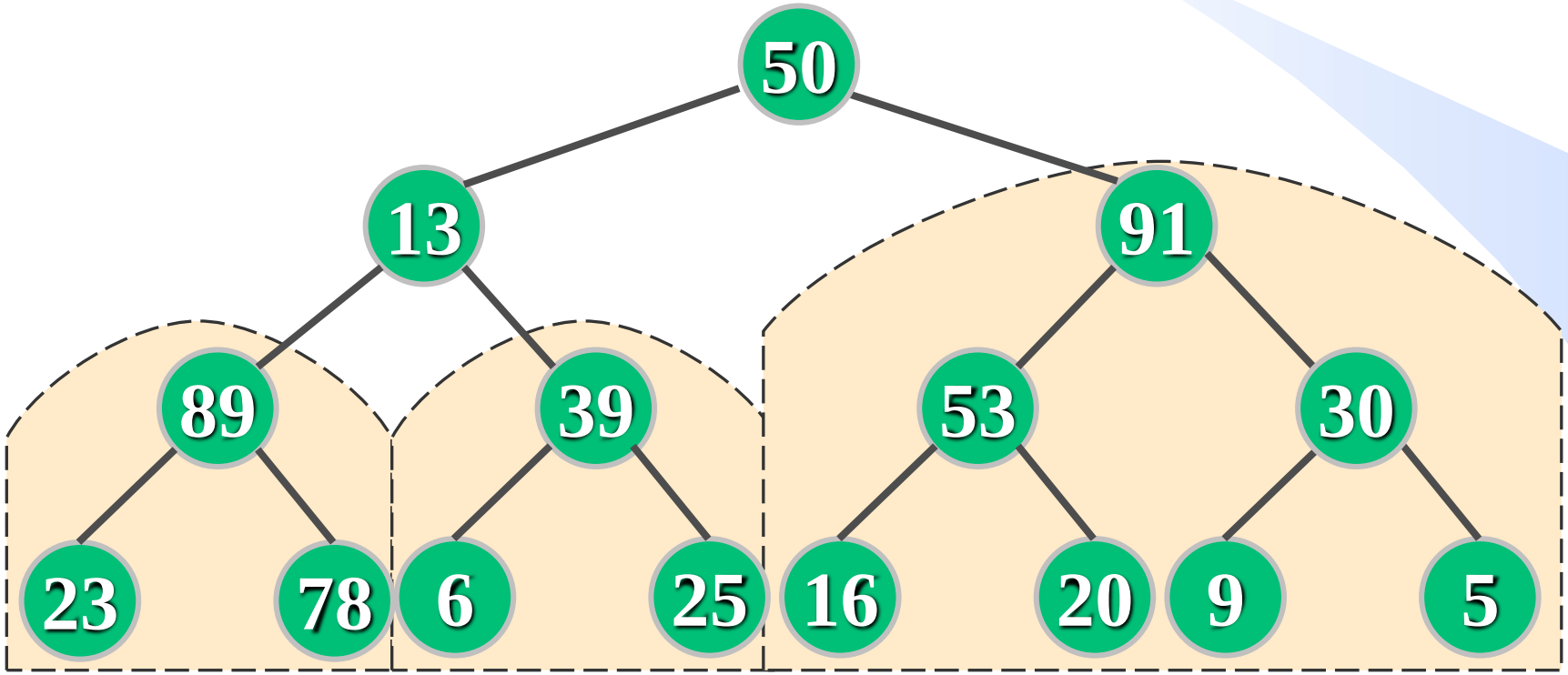
从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。



# 初始建堆

A

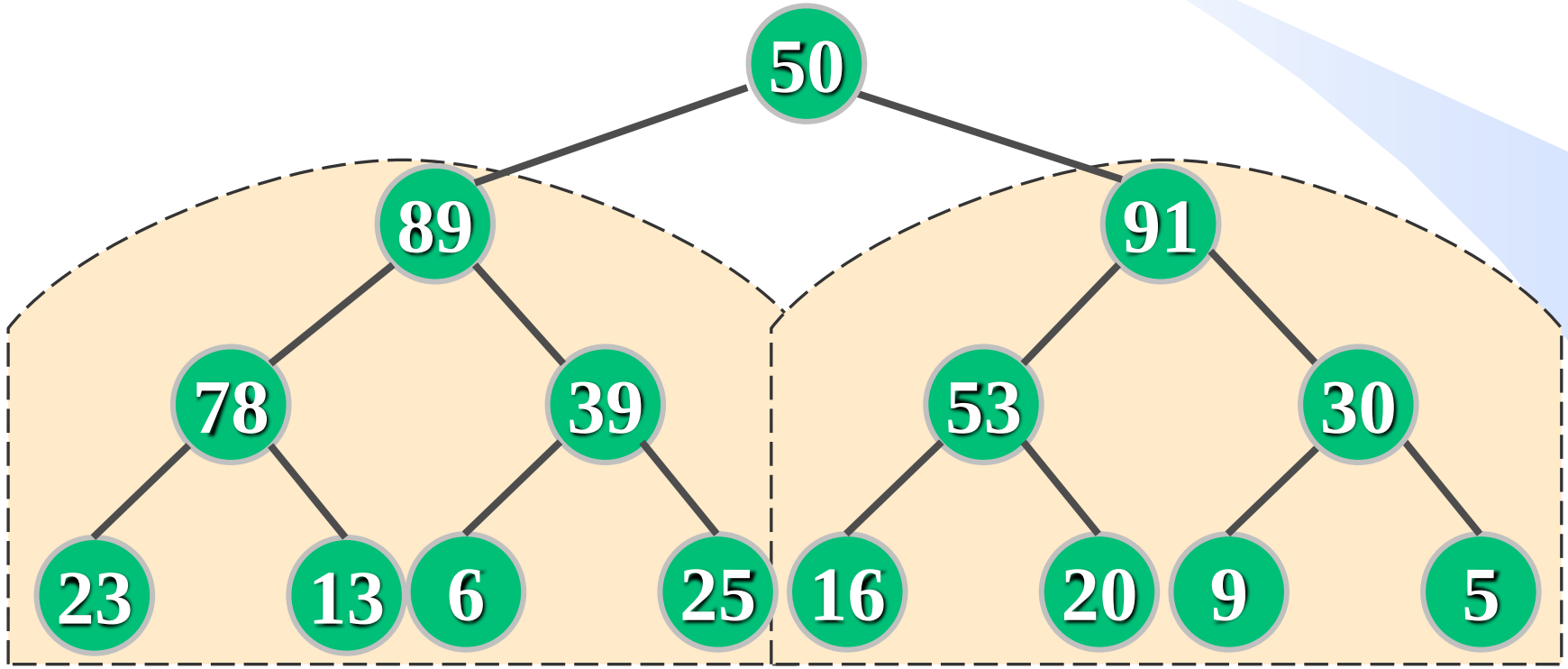
从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。



# 初始建堆

A

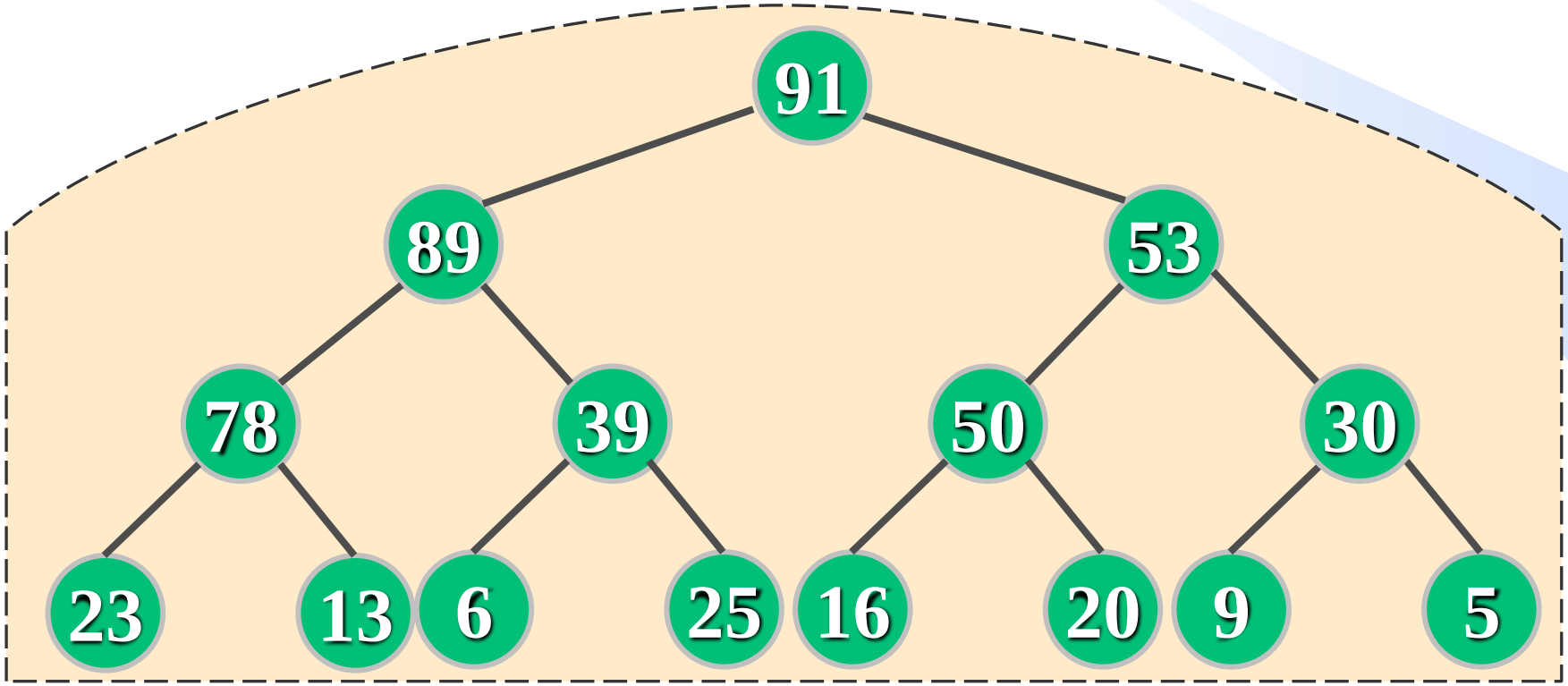
从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。



# 初始建堆

A

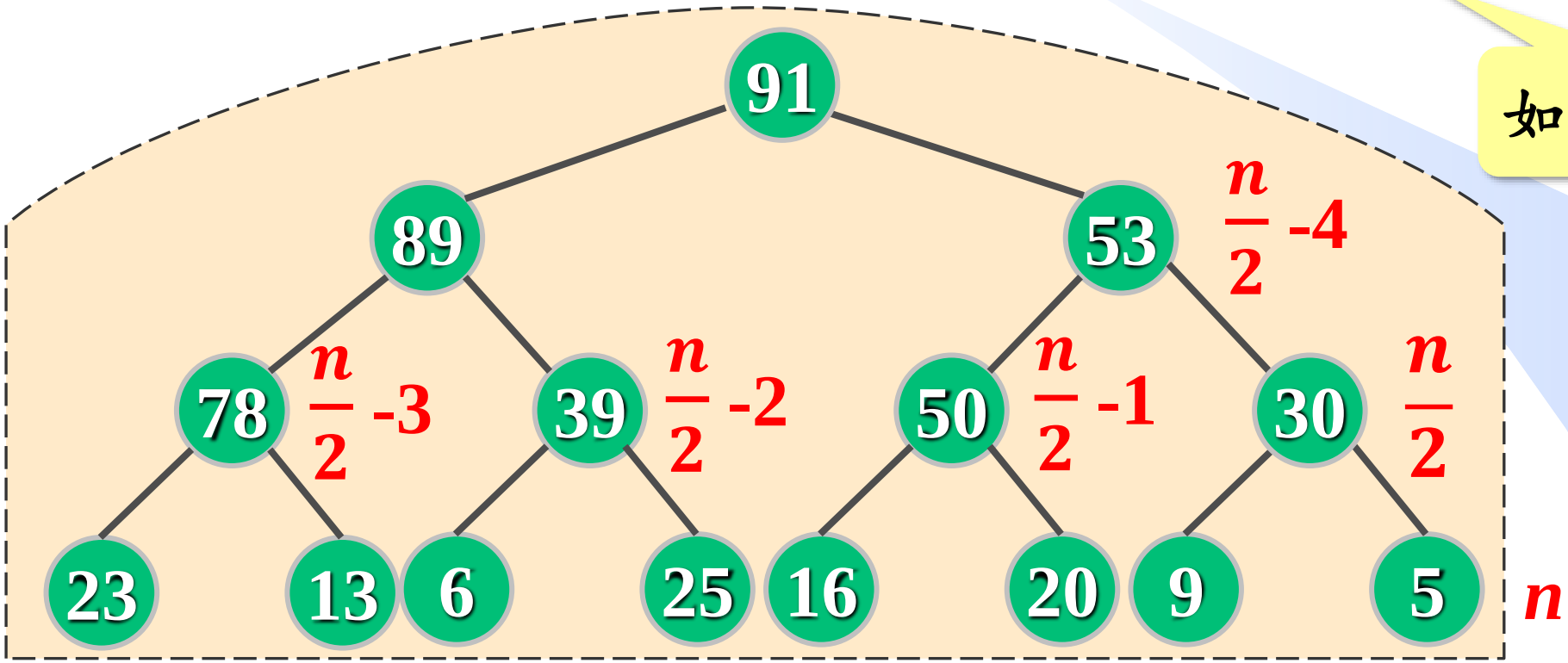
从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。



# 初始建堆

A

从最后一个非叶结点开始，依次建立以每个非叶结点为根的子堆。




如何实现?

从最后一个非叶结点开始，依次下沉$\lfloor n/2 \rfloor, \lfloor n/2 \rfloor - 1, \ldots, 1$。

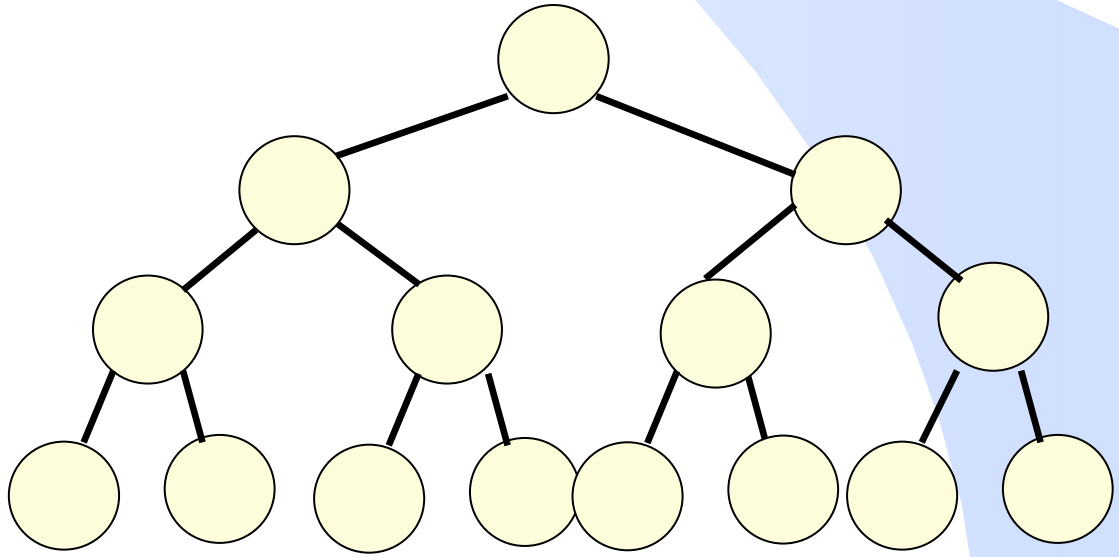
# 初始建堆算法

从最后一个非叶结点开始，依次下沉结点$\lfloor n/2 \rfloor, \lfloor n/2 \rfloor - 1, \ldots, 1$。

```
void BuildHeap(int R[],int n){
    for(int i=n/2; i>=1; i--)
        ShiftDown(R,n,i); //建立以i为根的堆，即下沉i
}
```

**Floyd建堆算法**
**自底向上建堆**

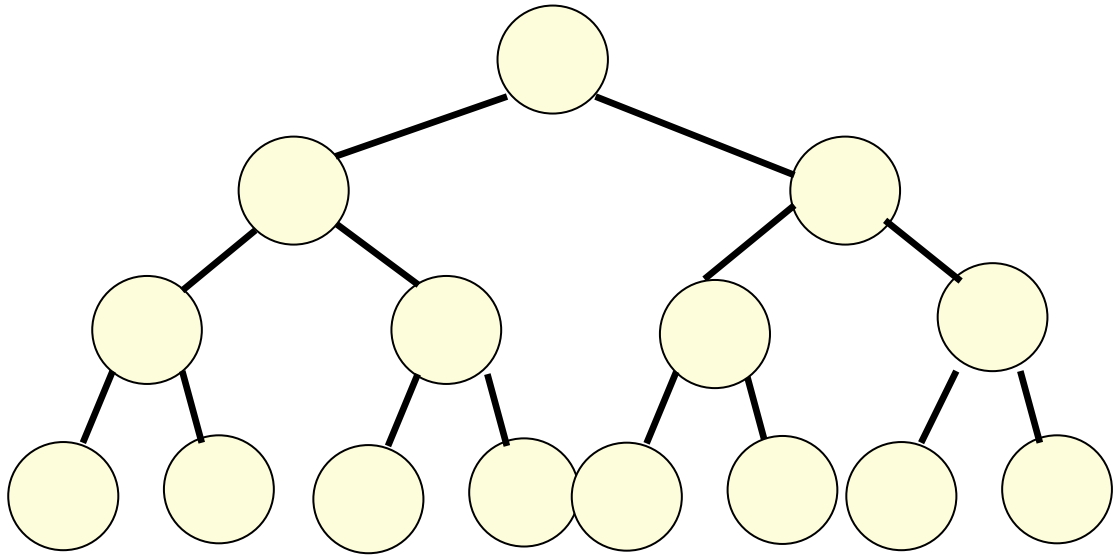
**最坏时间复杂度**
**O(*n*log*n*)?**



# 初始建堆算法的时间复杂度

B

- 下沉*R*[*i*]的关键词比较次数取决于*R*[*i*]的高度。
- 建堆算法是下沉了每个非叶结点，故总时间取决于各结点的高度之和。



# 初始建堆算法的时间复杂度

B

$$T = \frac{n}{4} + 2 \cdot \frac{n}{8} + 3 \cdot \frac{n}{16} + 4 \cdot \frac{n}{32} + \cdots$$

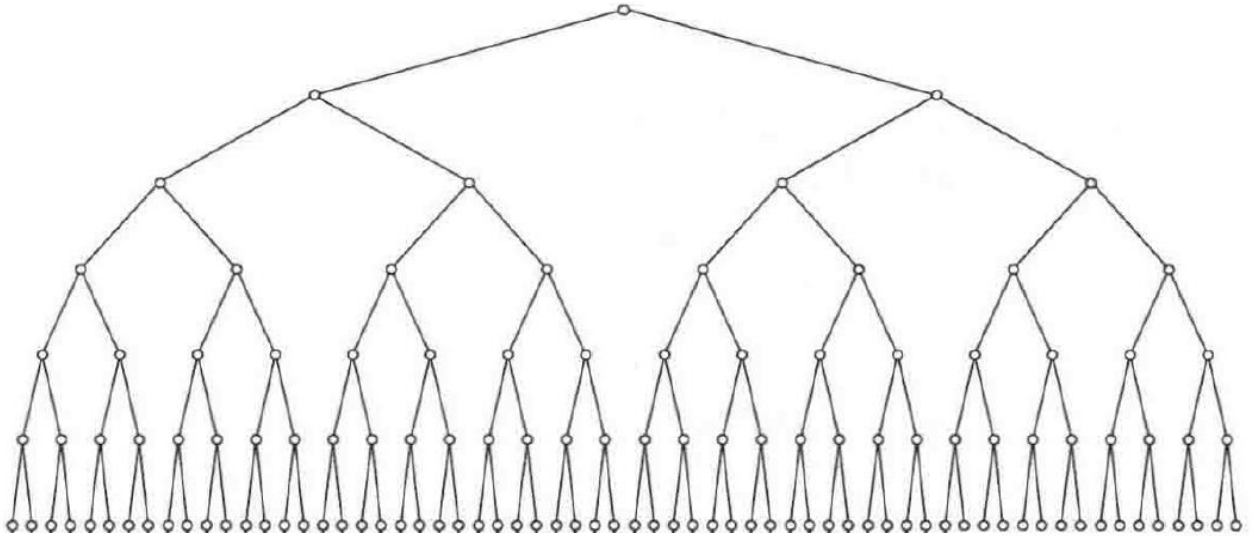
$$2T = \frac{n}{2} + 2 \cdot \frac{n}{4} + 3 \cdot \frac{n}{8} + 4 \cdot \frac{n}{16} + \cdots$$

$$2T - T = \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \frac{n}{16} + \frac{n}{32} + \cdots$$

$$T = n(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots)$$

$$= n\left(\frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^k\right)}{1 - \frac{1}{2}}\right) = n\left(1 - \left(\frac{1}{2}\right)^k\right) < n$$

**各结点的高度之和小于$n$**
**算法最坏情况时间复杂度O($n$)**



# 课下思考

B

若规定sum++为基本运算，以下选项中_____能最精确的反映下面算法的时间复杂度。【吉林大学21级期末考试题】

```
void f(int n){
    int t=0, sum=0, i, j;
    for(i=n; i>1; i/=2) {
        t++;
        for(j=0; j<t*i; j++)
            sum++;
    }
}
```

A. O($n^3$)　　B. O($n\log n$)　　C. O($n^2$)　　D. O($n$)

# 课下思考

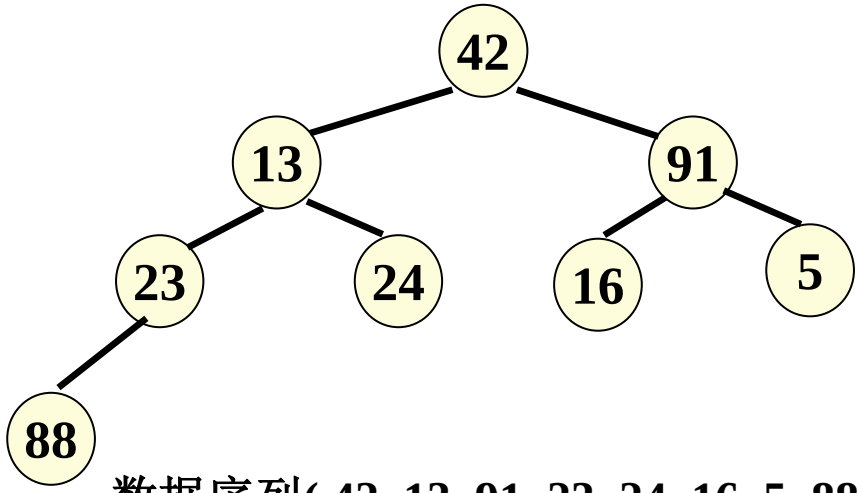
A

判断题：将$n$个元素建成一个堆，至少需要O($n$log$n$)时间。【清华大学考研题】

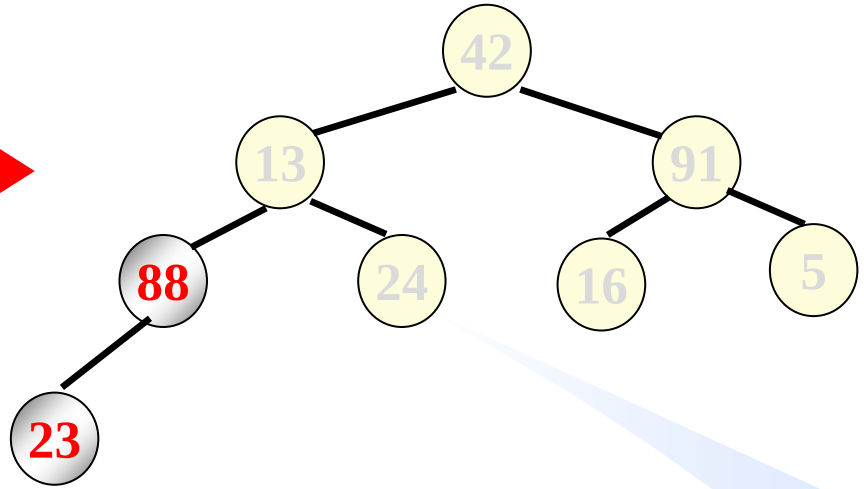
使用初始建堆算法，将数据序列( 42, 13, 91, 23, 24, 16, 5, 88 )建成的大根堆为_____。【吉林大学21级期末考试题】

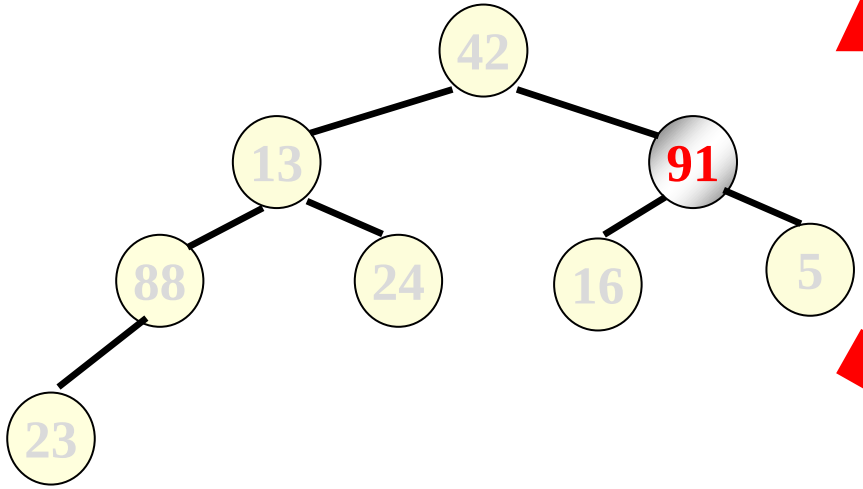
A. 91, 42, 88, 16, 5, 23, 24, 13　　B. 91, 88, 42, 24, 23, 16, 5, 13

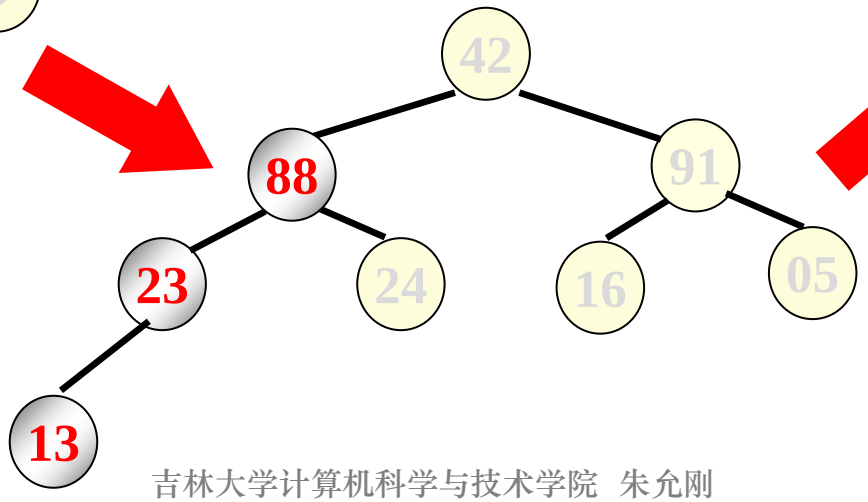
C. 91, 88, 42, 23, 24, 5, 16, 13　　D. 91, 88, 42, 23, 24, 16, 5, 13

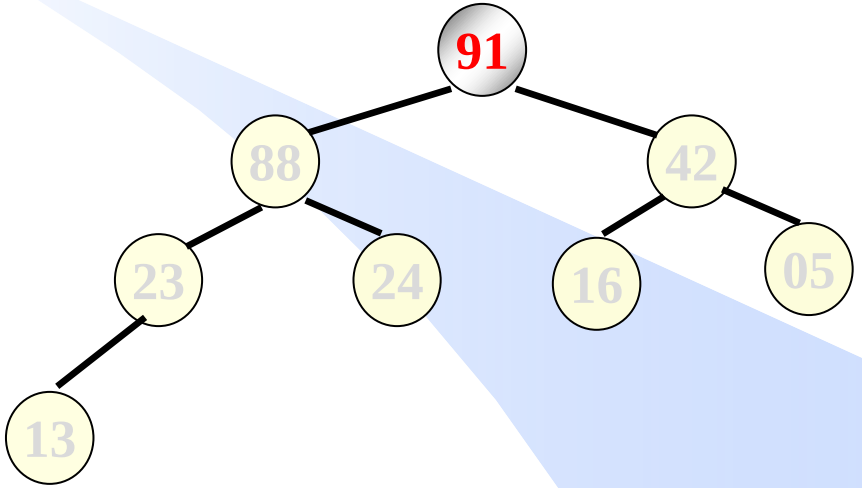



数据序列( 42, 13, 91, 23, 24, 16, 5, 88 )




A. 91, 42, 88, 16, 5, 23, 24, 13
B. B. 91, 88, 42, 24, 23, 16, 5, 13
C. 91, 88, 42, 23, 24, 5, 16, 13
**D. 91, 88, 42, 23, 24, 16, 5, 13**

# 课下思考

A

使用初始建堆算法，将数据序列(6, 4, 3, 5, 8, 9, 12, 10)建成的大根堆为为_____。【2022级吉林大学期末考试题】

A. 12, 10, 6, 4, 8, 9, 3, 5

B. 12, 10, 9, 5, 8, 6, 3, 4

C. 12, 9, 10, 6, 8, 3, 4, 5

D. 6, 4, 8, 3, 5, 9, 12, 10

# 课下思考

A

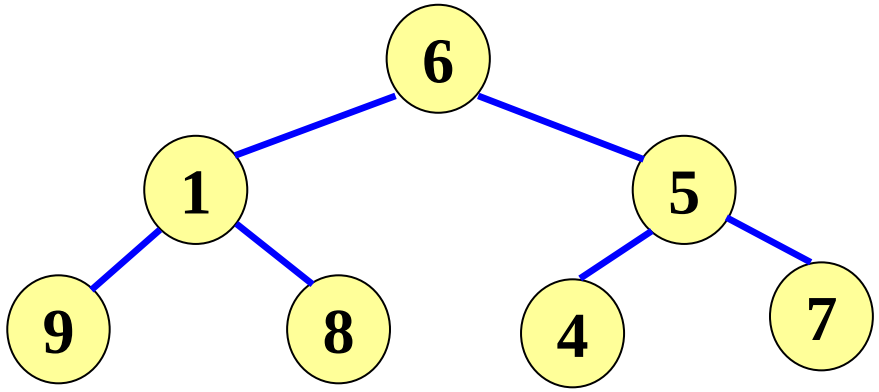
将数据序列 (6, 1, 5, 9, 8, 4, 7)建成大根堆，序列变化过程为_____。【2018年考研题全国卷】

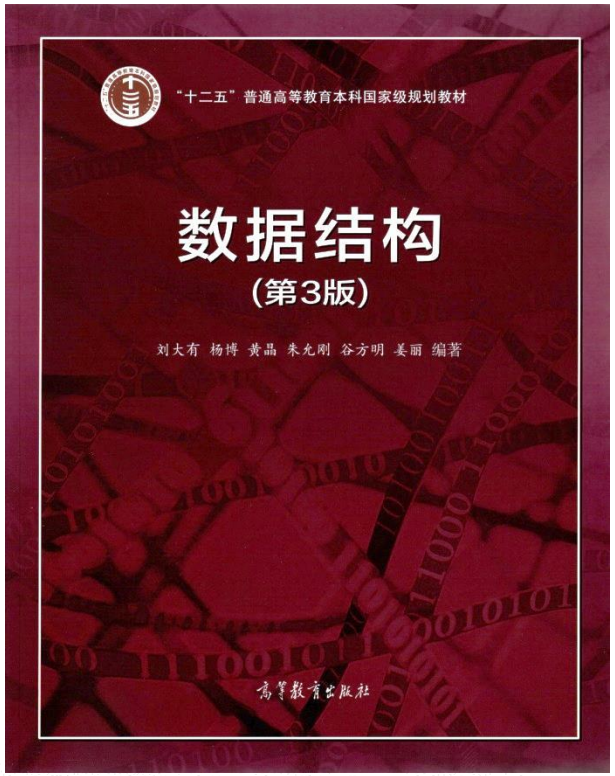
A. 6 1 7 9 8 4 5 → 6 9 7 1 8 4 5 → 9 6 7 1 8 4 5 → 9 8 7 1 6 4 5

B. 6 9 5 1 8 4 7 → 6 9 7 1 8 4 5 → 9 6 7 1 8 4 5 → 9 8 7 1 6 4 5

C. 6 9 5 1 8 4 7 → 9 6 5 1 8 4 7 → 9 6 7 1 8 4 5 → 9 8 7 1 6 4 5

D. 6 1 7 9 8 4 5 → 7 1 6 9 8 4 5 → 7 9 6 1 8 4 5 → 9 7 6 1 8 4 5






# 堆排序

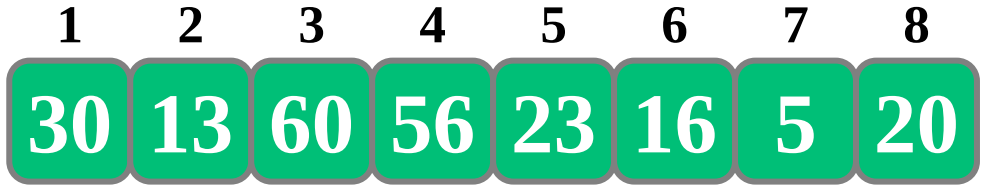
- 锦标赛排序
- 堆的概念及基本操作
- **堆排序算法**

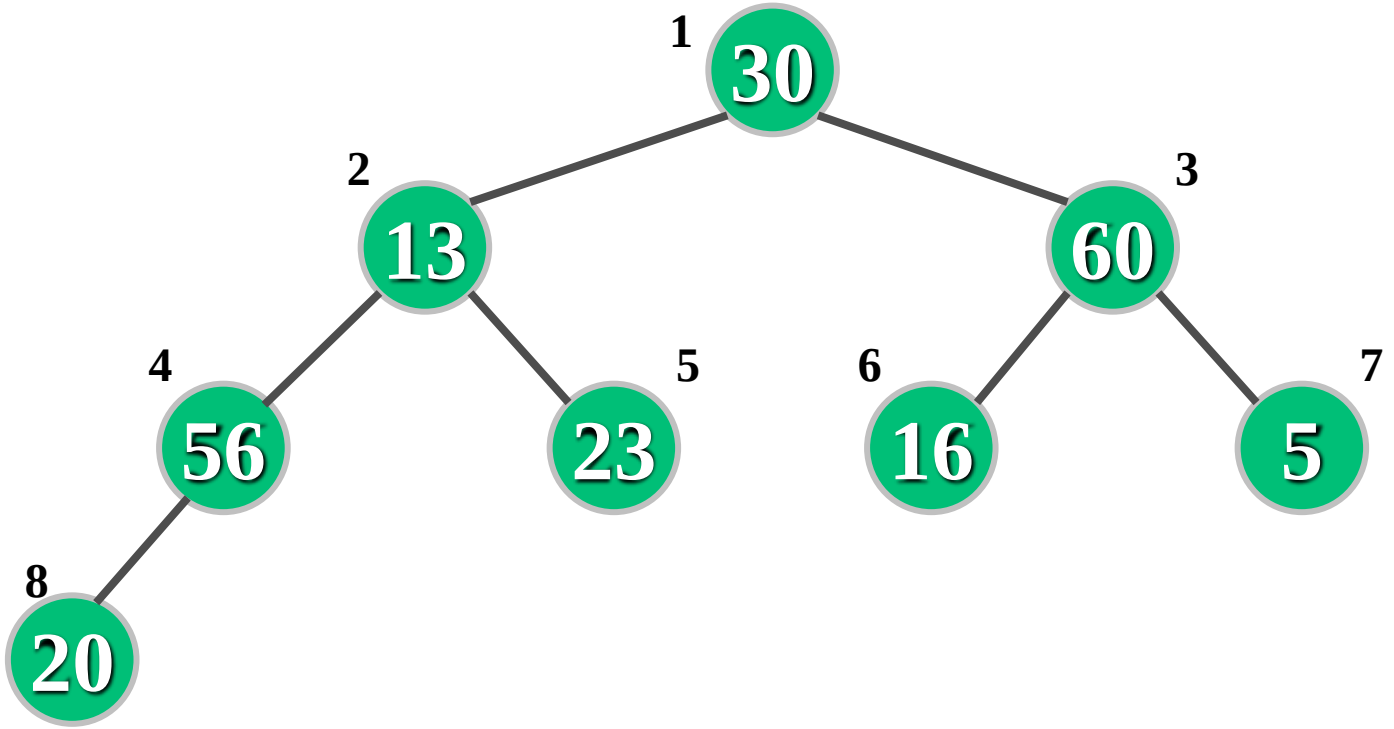
数据之法 结构之美 算法之道

zhuyungang@jlu.edu.cn

Last updated on 2025.5

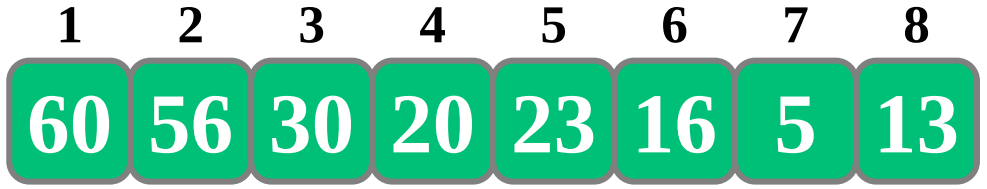
# 堆排序算法的思想

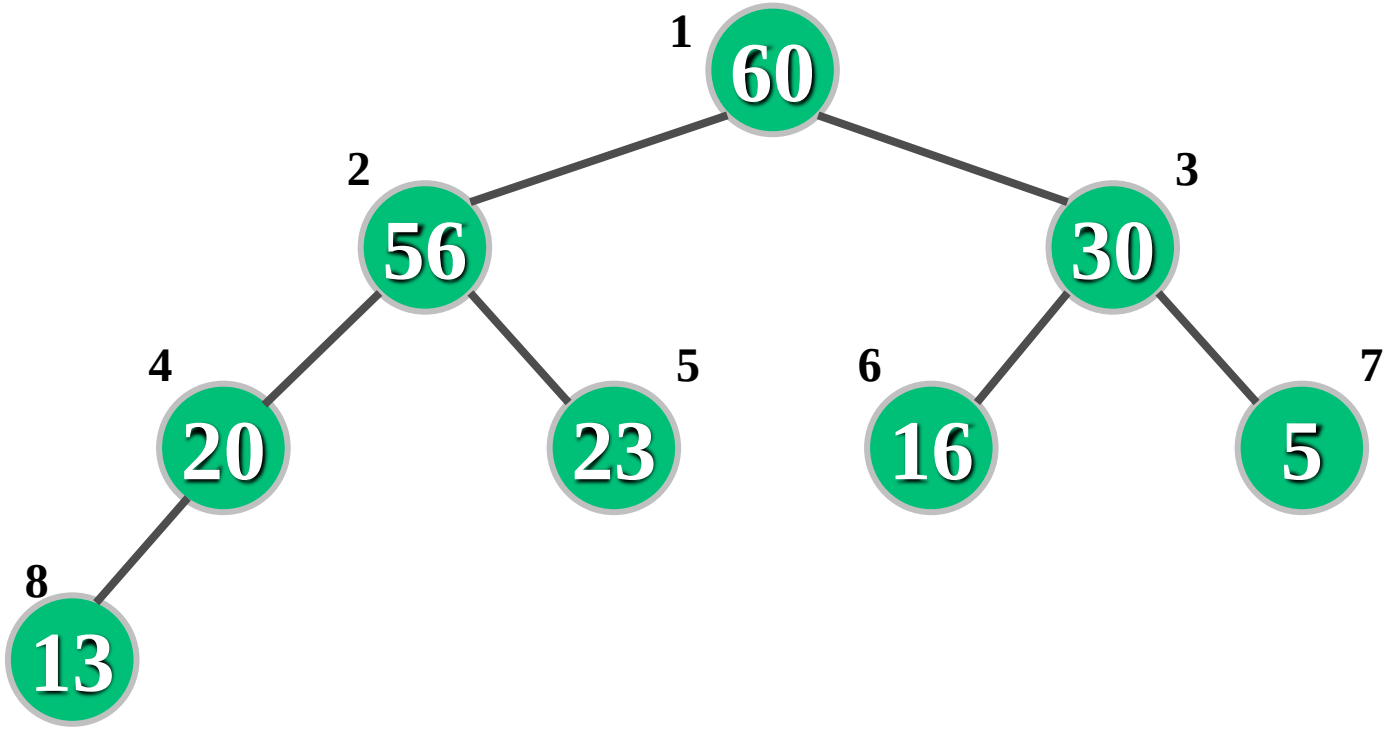



调用BuildHeap
将数组*R*建为堆

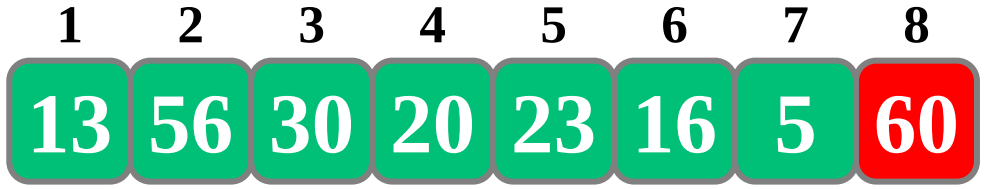
# 堆排序算法的思想

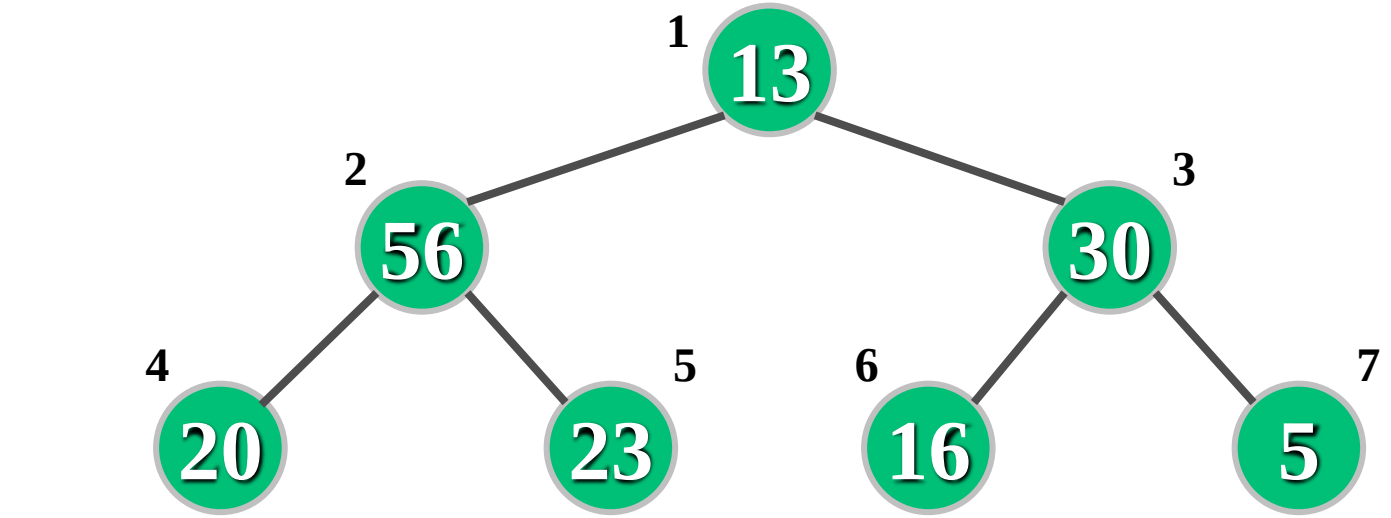



在前8个元素里找最大元素（即堆顶R[1]），与*R*[8]交换，使*R*[8]就位

# 堆排序算法的思想

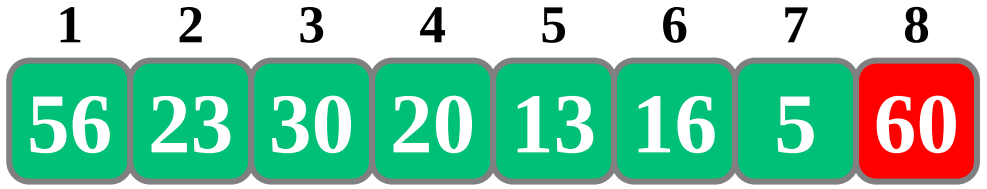


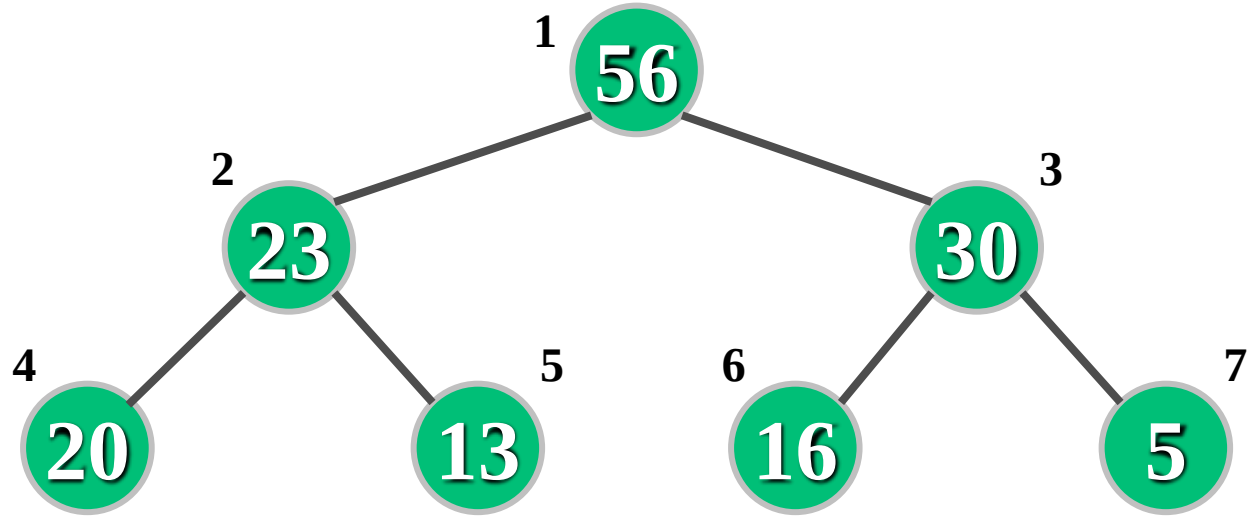





在前8个元素里找最大元素（即堆顶R[1]），与*R*[8]交换，使*R*[8]就位

# 堆排序算法的思想

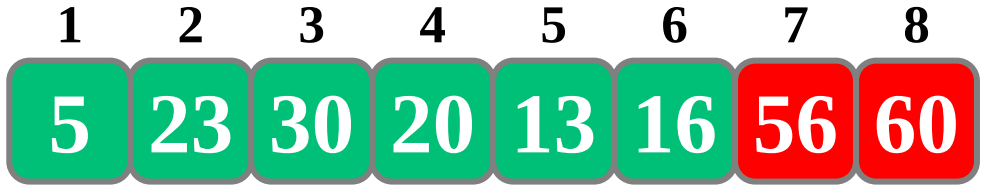


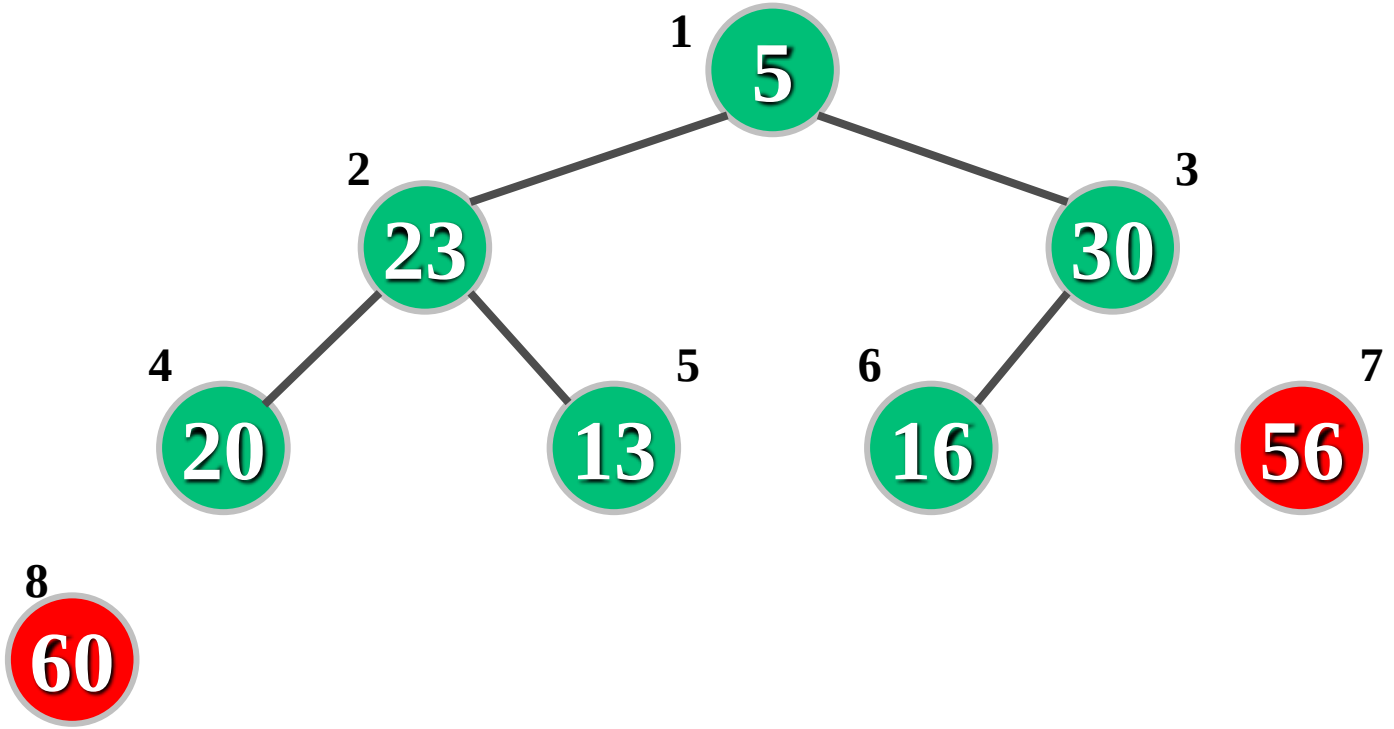




8
60

下沉根结点（堆顶R[1]）使前$R[1]...R[7]$重建为堆

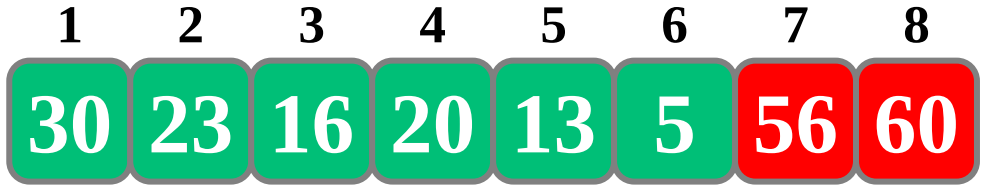
# 堆排序算法的思想

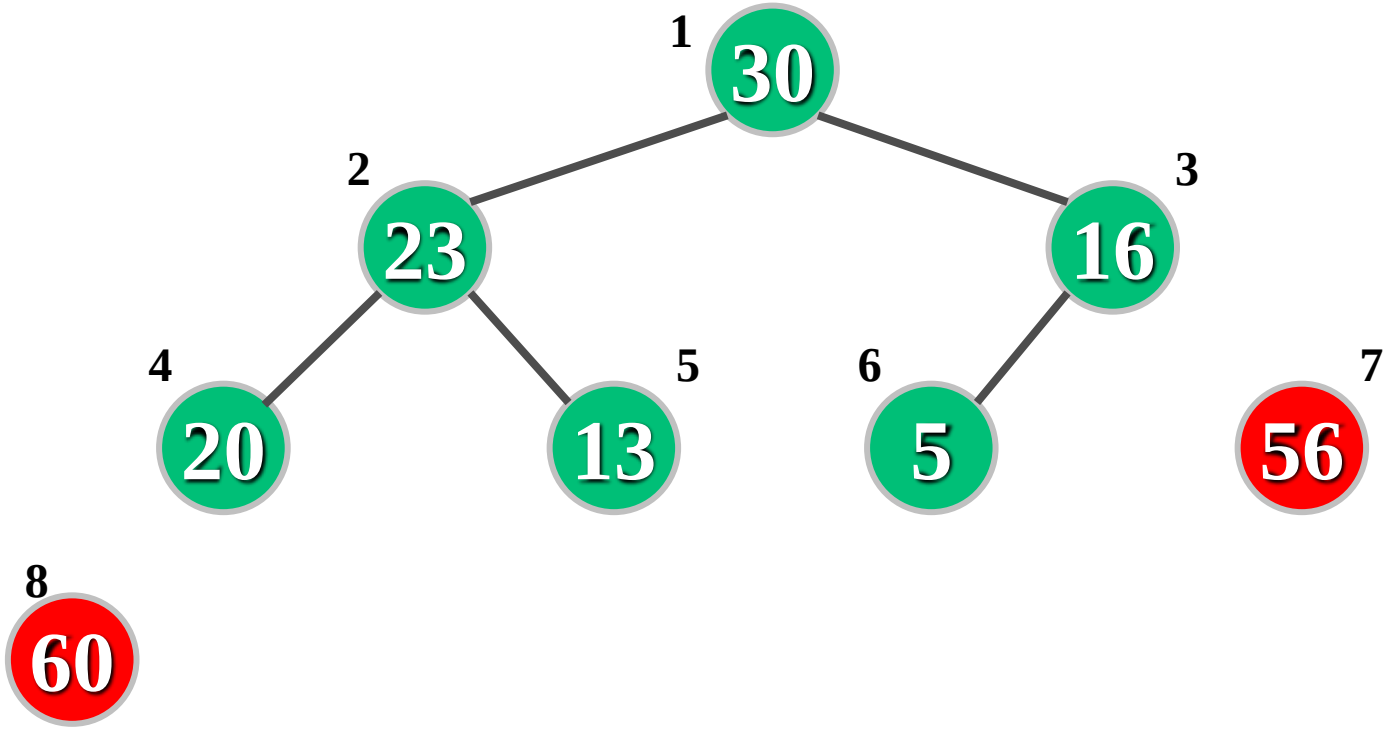



在前7个元素里找最大元素（即堆顶R[1]），与*R*[7]交换，使*R*[7]就位

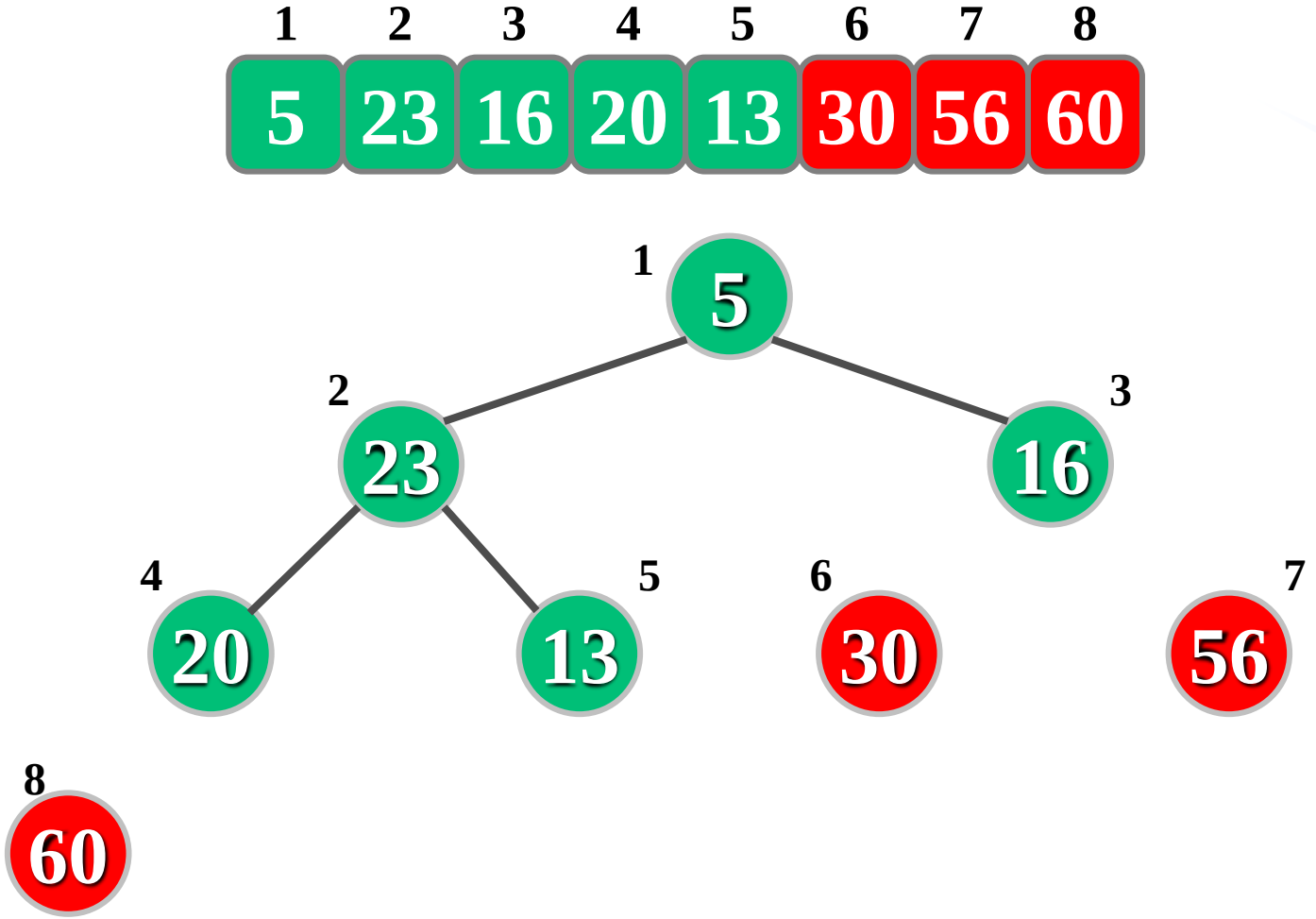
# 堆排序算法的思想




下沉根结点（堆顶$R[1]$）使前$R[1]...R[6]$重建为堆

# 堆排序算法的思想




在前6个元素里找最大元素（即堆顶R[1]），与R[6]交换，使R[6]就位

# 堆排序算法的思想

A

① 将待排序数组$R$建成一个大根堆。

② 在前$n$个元素的堆里选最大元素$R[1]$，与$R[n]$交换使$R[n]$就位，下沉根结点$R[1]$使$R[1]...R[n-1]$重建为堆。

③ 在前$n$-1个元素的堆里选最大元素$R[1]$，与$R[n-1]$交换使$R[n-1]$就位，下沉根结点$R[1]$使$R[1]...R[n-2]$重建为堆。

④ 在前$n$-2个元素的堆里选最大元素$R[1]$，与$R[n-2]$交换使$R[n-2]$就位，下沉根结点$R[1]$使$R[1]...R[n-3]$重建为堆。

⑤ .....

上述操作反复进行，直到调整范围只剩下一个元素$R[1]$为止。

此时，$R[1]$是$n$个元素中最小的，且数组$R$已按递增排列。

# 堆排序算法的粗略描述

A

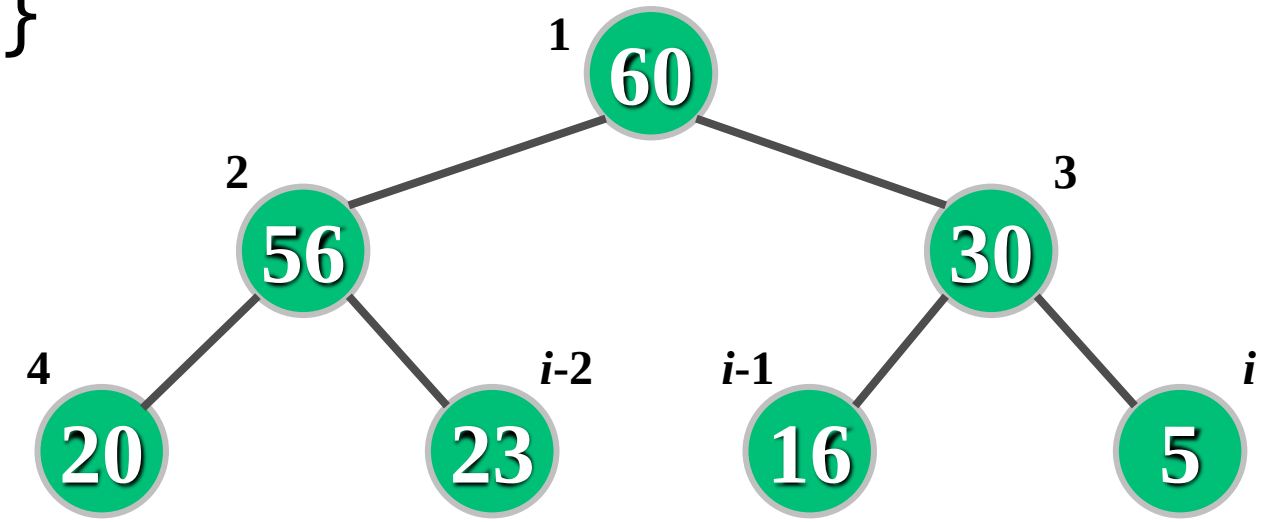
① 建立包含$R[1], R[2], \ldots, R[n]$的堆；

② for(int i=n; i>1; i--){ //$i$标识当前处理的堆尾

$R[1] \leftrightarrow R[i]$; //根$R[1]$和堆尾交换

下沉$R[1]$使$R[1]\ldots R[i-1]$重建为堆

}




虽然操作的是数组，但背后隐藏的灵魂是二叉树

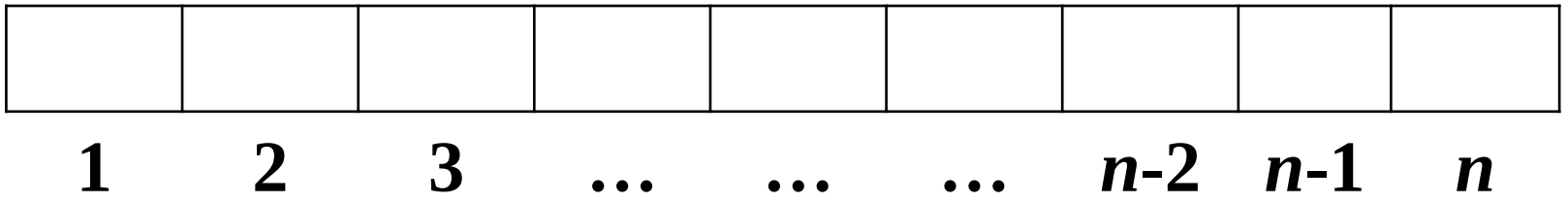
# 堆排序算法

A

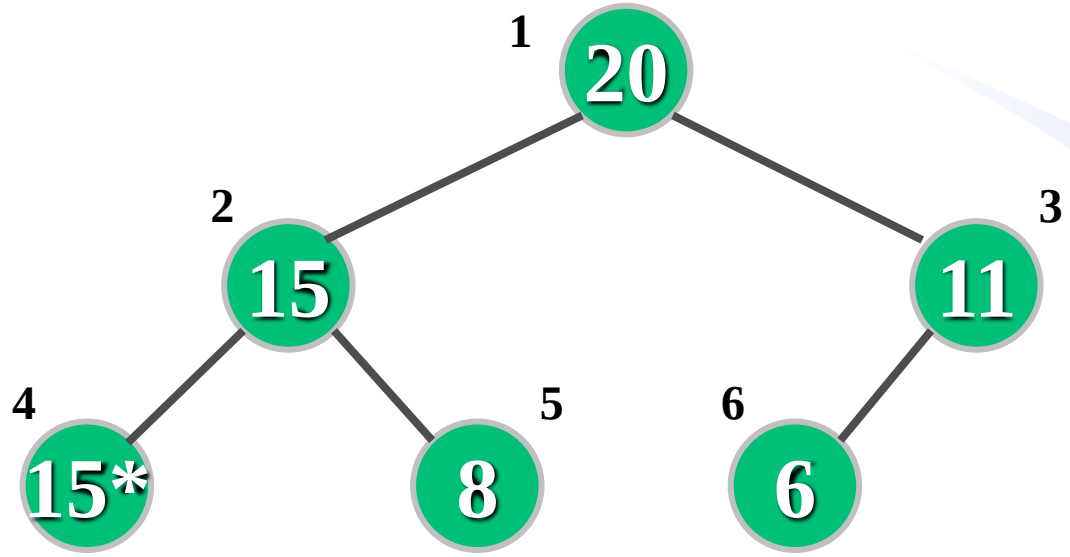
```
void HeapSort(int R[],int n){  //堆排序R[1]...R[n]
    BuildHeap(R, n);           //将R建为堆
    for(int i=n; i>1; i--){    //i为当前堆的堆尾
        swap(R[1], R[i]);      //前i个元素的最大者R[1]与R[i]交换
        ShiftDown(R,i-1,1);    //下沉R[1]使R[1]...R[i-1]重建为堆
    }
}
```

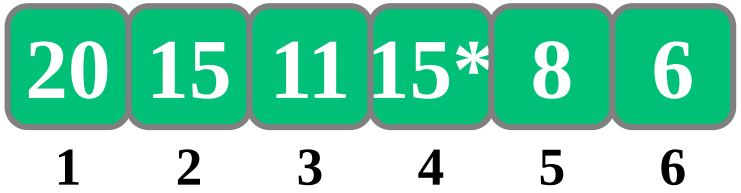
时间复杂度
$O(n\log n)$

```
void swap(int &a, int &b){
    int temp = a;
    a = b;
    b = temp;
}
```



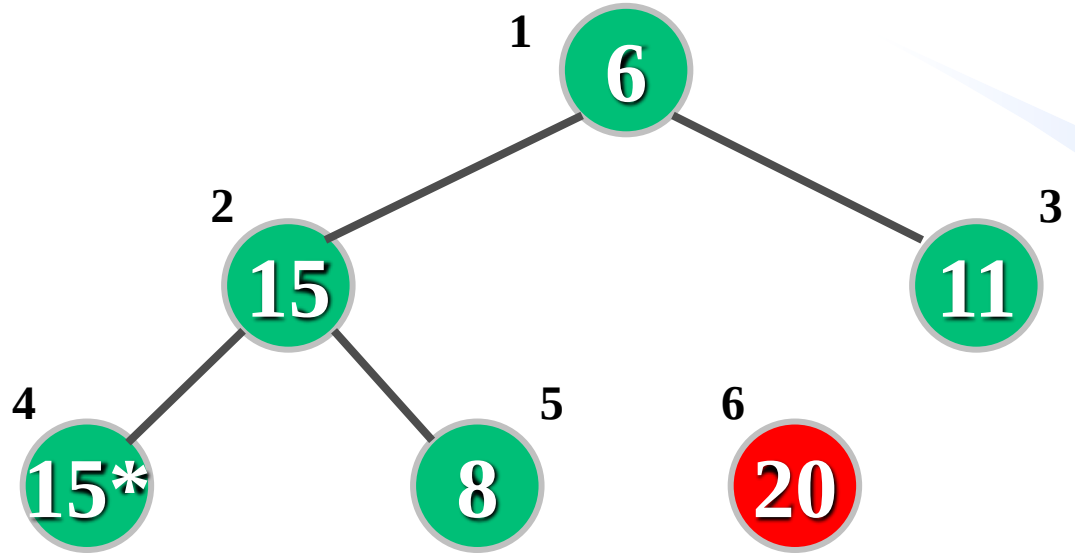
# 堆排序算法示例

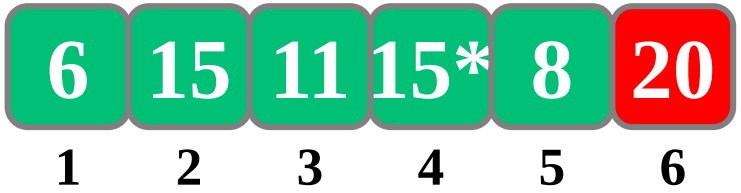



初始堆

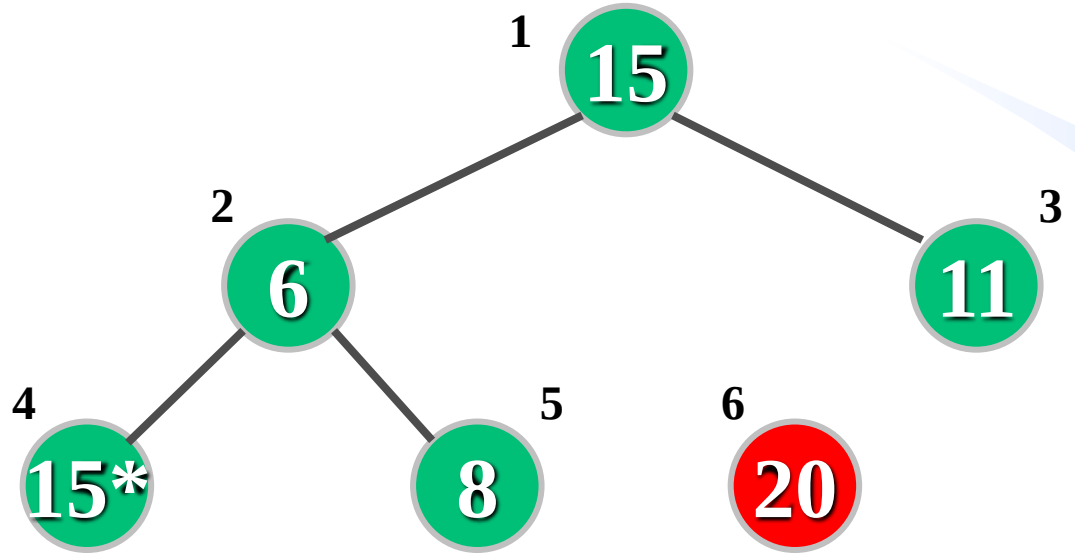
# 堆排序算法示例

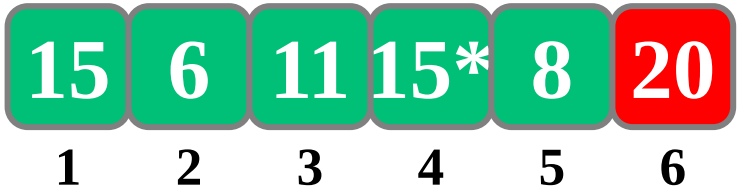



交换R[1]与R[6],
使R[6]就位

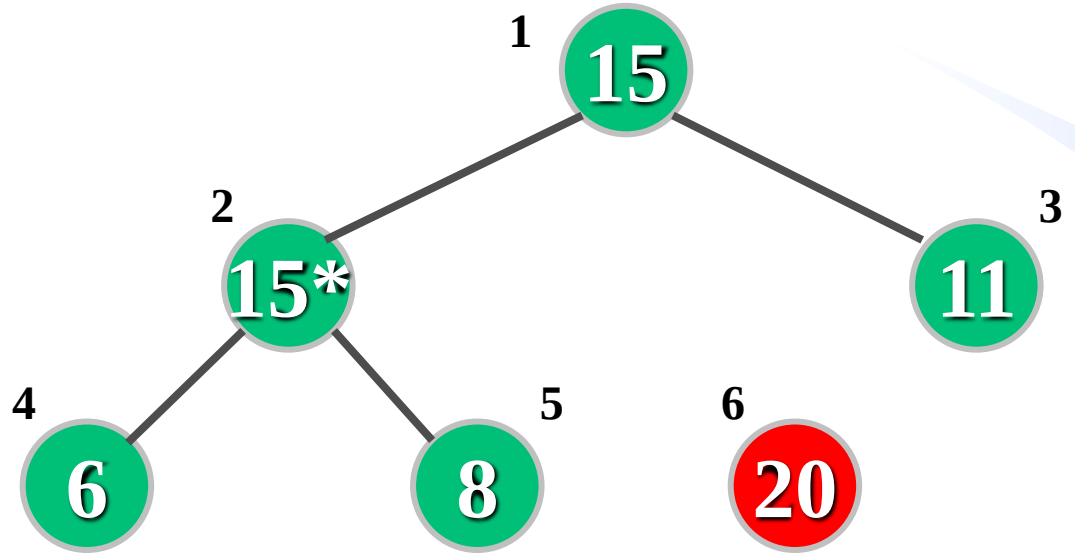
# 堆排序算法示例

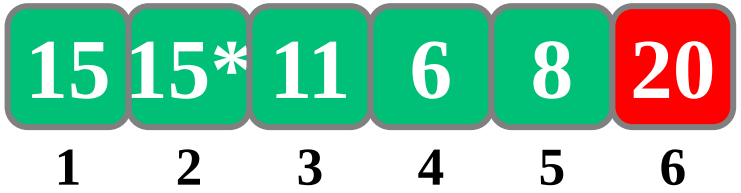



下沉R[1],使R[1]…R[5]重建为堆

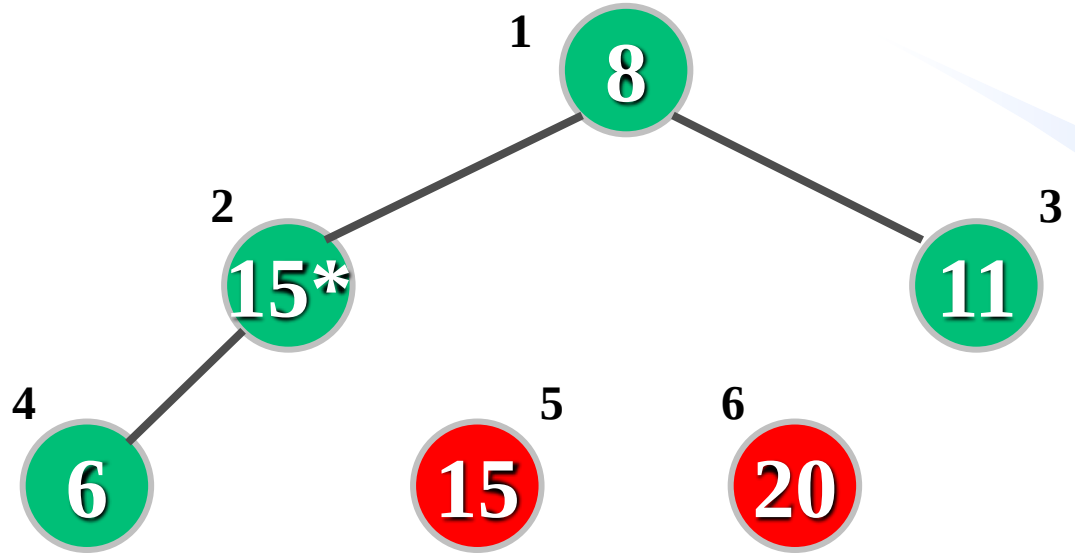
# 堆排序算法示例

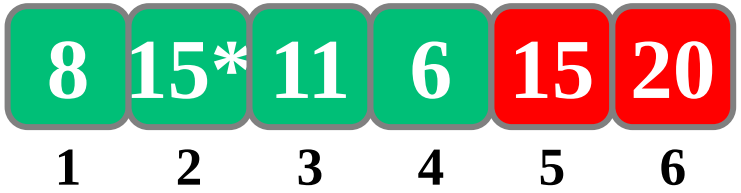



下沉R[1],使R[1]…R[5]重建为堆

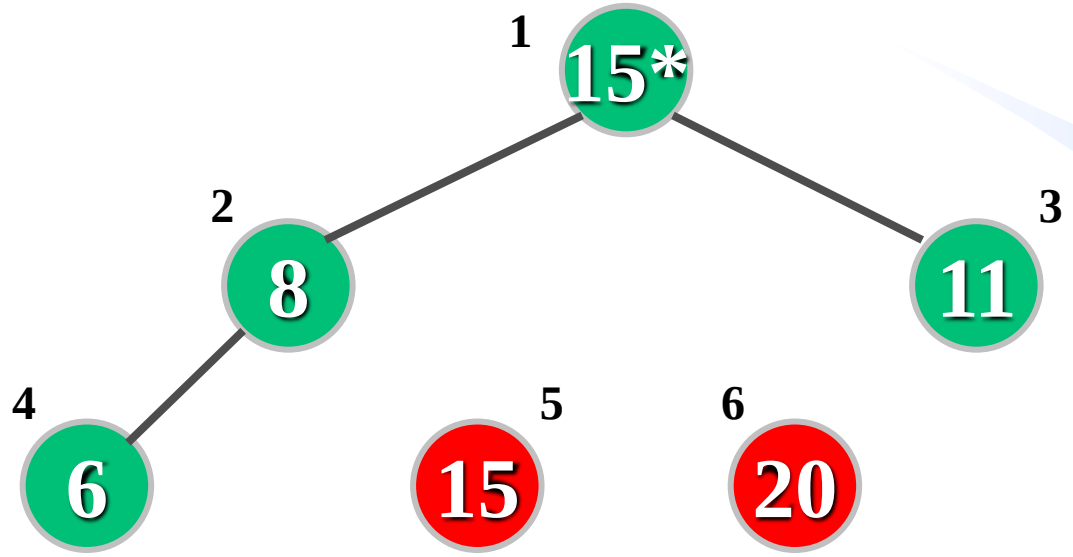
# 堆排序算法示例

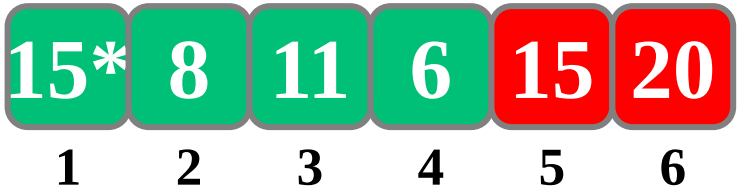



交换R[1]与R[5],使R[5]就位

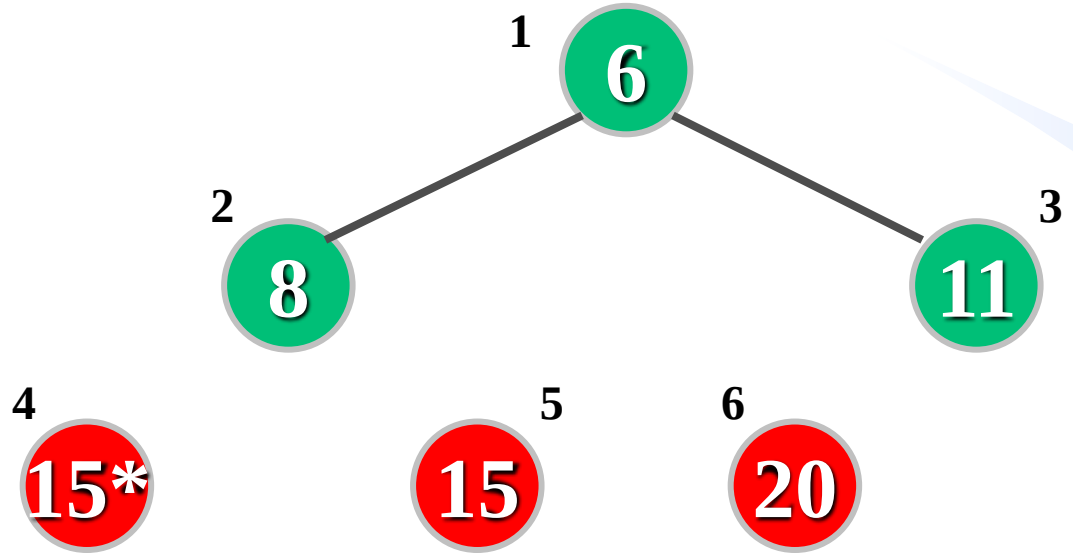
# 堆排序算法示例



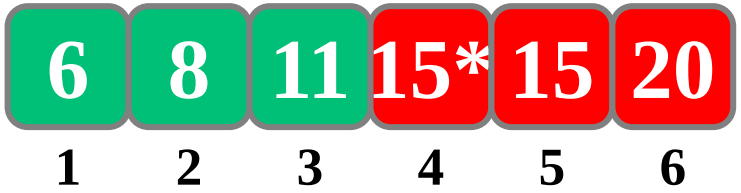




下沉R[1],使R[1]…R[4]重建为堆

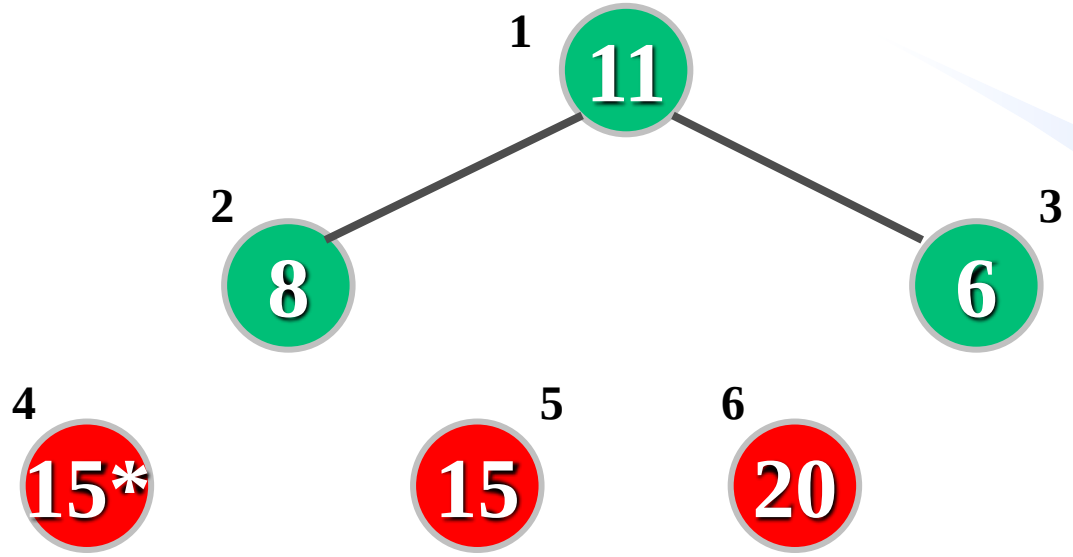
# 堆排序算法示例

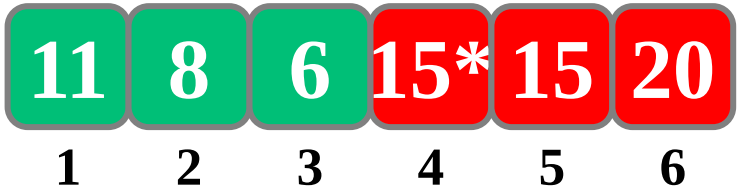



交换R[1]与R[4],
使R[4]就位



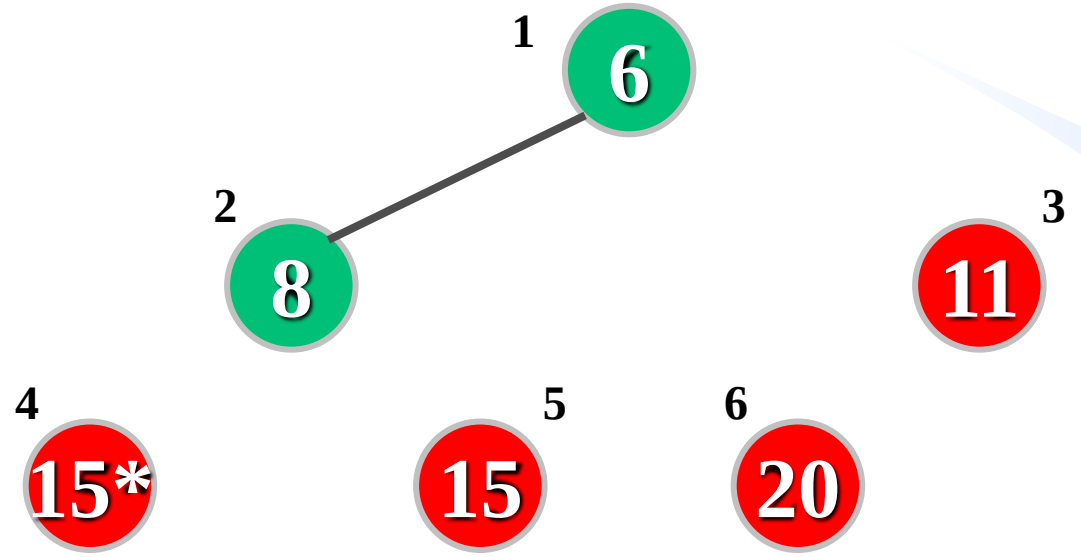
# 堆排序算法示例



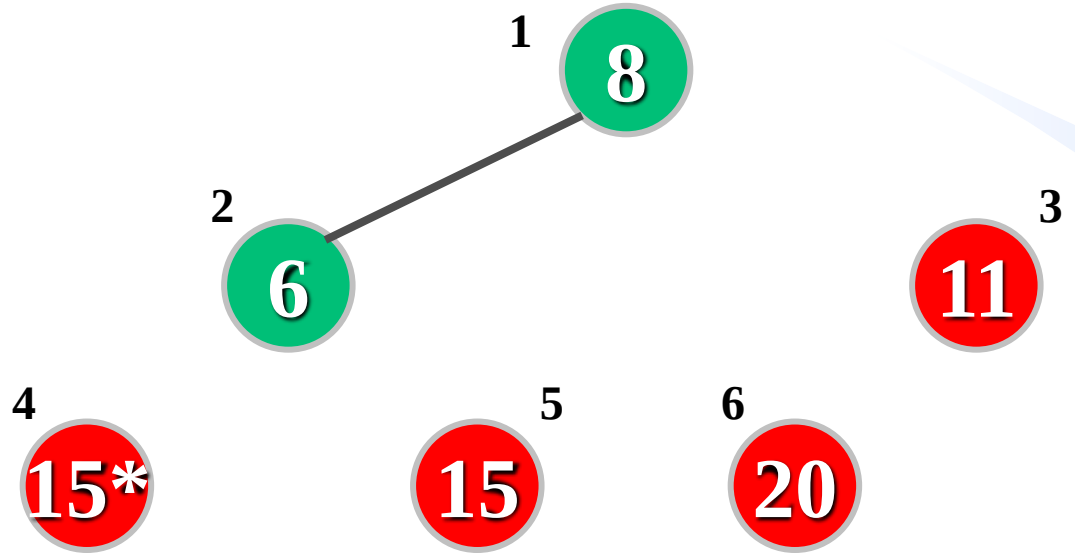

下沉R[1],使R[1]…R[3]重建为堆

# 堆排序算法示例




交换R[1]与R[3],
使R[3]就位

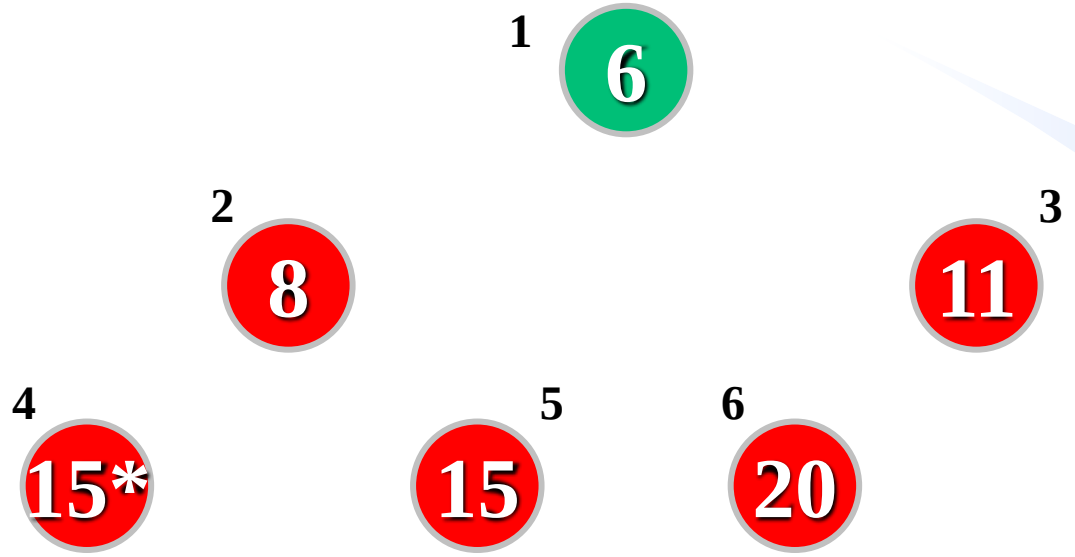
# 堆排序算法示例





下沉R[1],使R[1]…R[2]重建为堆

# 堆排序算法示例




交换R[1]与R[2],使R[2]就位


不稳定

# 堆排序算法总结

| 排序算法 | 时间复杂度 | | | 空间复杂度 | 稳定性 |
|---|---|---|---|---|---|
| | 最好 | 平均 | 最坏 | | |
| 堆排序 | O(*n*log*n*) | O(*n*log*n*) | O(*n*log*n*) | O(1) | 不稳定 |

对*n*个互异的随机关键词进行堆排序，关键词平均比较次数为2*n*log*n*-O(*n*loglog*n*)。

R. Schaffer, R. Sedgewick. The Analysis of Heapsort. *Journal of Algorithms*. 1993,15: 76-100.