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Homework # 1 - ejlouie , auyen

Q1

Part 1

Initialize array A, int i = 0, int j = 0, int length = size of A
for all i from i to length - 1
for all j from j to length - i - 1
if A[j] > A[j+1] // swap
initialize int temp
temp = A[j]
A[j] = A[j+1]
A[j+1] = temp

Part 2

The outer for loop makes n-i-1 steps. The inner for loop makes n-j-i-1 steps. In the worst case, the number of comparisons made with each iteration would be n-1, n-2, n-3, and so on.
The number of comparisons is

$$\sum_{j=0}^{n} n-j-1 = \frac{n(n+1)}{2} = O(n^2)$$

Part 3

Base Case: An array of size 1 is sorted.
Induction:
Loop Invariants:
At the end of each inner loop, A[j] is the largest
Everything to the right of A[i] is sorted.

The largest number always ends up at the end because it gets swapped due to being larger than everything else on the list. The same happens for the next largest until it reaches the largest value. This happens until the entire list is sorted

Q2

Inductive Hypothesis: The number of odd number subsets in $\{1, 2, ..., n\}$ is 2^{n-1}

Base Case: $n = 1$

$2^{1-1} = 2^0 = 1$

Inductive Step:

Set A_k has k numbers and 2^{k-1} subsets which contain an odd number of numbers.

Prove that A_{k+1}, which has $k+1$ numbers, has 2^k odd number sized subsets.

If you have a set A_{k+1}, which is $\{1, 2, ..., k, k+1\}$ i.e. A_k with another number added to it, you still have the 2^{k-1} subsets that have an odd number of numbers. If you add $k+1$ to each of the even number subsets, you now have an additional 2^{k+1} subsets with an odd number of numbers. Then, you add the 2^{k-1} initial odd number size subsets to the 2^{k-1} newly made odd number size subsets i.e. $2(2^{k-1})$, resulting in 2^k subsets with an odd number size.

3. $f(n) = a_0 + a_1 n + a_2 n^2 + \dots + a_k n^k$

show $F(n) \in \Theta(n^k)$ $\quad c_1 \cdot n^k \le f(n) \le c_2 \cdot n^k$

$f(n) = \Theta(n^k) \Rightarrow c_1 \le \lim_{n\to\infty} \frac{f(n)}{n^k} \le c_2$

$$c_1 \le \lim_{n\to\infty} \frac{a_0 + a_1 n + a_2 n^2 + \dots + a_k n^k}{n^k} \le c2$$

$$c_1 \le \lim_{n\to\infty} \frac{a_0}{n^k} + \lim_{n\to\infty} \frac{a_1 n}{n^k} + \lim_{n\to\infty} \frac{a_2 n^2}{n^k} + \dots + \lim_{n\to\infty} \frac{a_k n^k}{n^k} \le c_2$$

$$c_1 \le 0 + \lim_{n\to\infty} \frac{a_1}{n^{k-1}} + \lim_{n\to\infty} \frac{a_2}{n^{k-2}} + \dots + \lim_{n\to\infty} \frac{a_k}{n^{k-k}} \le c_2$$

$$c_1 \le 0 + 0 + 0 \dots + \lim_{n\to\infty} a_k \le c_2$$

$$c_1 \le a_k \le c_2$$

This shows that there exists a c_1 and c_2. Since a_k is a constant greater than 0, there is always a constant less than and greater than a_k. This shows that $f(n) \in \Theta(n^k)$

show $f(n) \notin O(n^{k'})$, for all $k' < k$ - proof by contradiction

suppose c and n exists

$F(n) = O(n^{k'}) \Rightarrow \lim_{n\to\infty} \frac{F(n)}{n^{k'}} \le c$

$$\lim_{n\to\infty} \frac{a_0 + a_1 n + a_2 n^2 + \dots + a_k n^k}{n^{k'}} \le c$$

$$\lim_{n\to\infty} \frac{a_0}{n^{k'}} + \lim_{n\to\infty} \frac{a_1 n}{n^{k'}} + \lim_{n\to\infty} \frac{a_2 n^2}{n^{k'}} + \dots + \lim_{n\to\infty} \frac{a_k n^k}{n^{k'}} \le c$$

$$0 + \lim_{n\to\infty} \frac{a_1}{n^{k'-1}} + \lim_{n\to\infty} \frac{a_2}{n^{k'-2}} + \dots + \lim_{n\to\infty} \frac{a_k}{n^{k'-k}} \le c$$

$$0 + 0 + 0 + \dots + \infty \le c$$

Since $k' < k$ as k gets bigger, $n^{k'-k}$ will get smaller and $f(n)$ will tend towards infinity. This is a contradiction because c cannot be greater than ∞, thus $f(n) \notin O(n^{k'})$

4. $\log_2 n = O(\sqrt{n})$ - show there exists c and n_0 such that

$0 \le \log_2 n \le c \cdot \sqrt{n}$ for all $n \ge n_0$

let $n_0 = 4$ $c = 1$

$0 \le \log_2 4 \le 1 \cdot \sqrt{4}$

$0 \le 2 \le 1 \cdot 2$

$0 \le 2 \le 2$

$\log_2 n \notin \Omega(\sqrt{n})$ - proof by contadiction

Suppose c and n_0 existed

$0 \le c\sqrt{n} \le \log_2 n$ for all $n \ge n_0$

$0 \le c \le \dfrac{\log_2 n}{\sqrt{n}}$ divide by $\sqrt{n}$

$\dfrac{\log_2 n}{\sqrt{n}} \ge c$ rearrange

$\lim_{n \to \infty} \dfrac{\log_2 n}{\sqrt{n}} \ge c$ take limit

$\lim_{n \to \infty} \dfrac{1/n\ln(2)}{1/2\sqrt{n}} \ge c$ take derivative by L'Hôpital's Rule

$\lim_{n \to \infty} \dfrac{2\sqrt{n}}{n\ln(2)} \ge c$ simplify

$\lim_{n \to \infty} \dfrac{2n^{1/2}}{n\ln(2)} \ge c$ change $\sqrt{n}$ to $n^{1/2}$

$\lim_{n \to \infty} \dfrac{2}{\sqrt{n}\ln(2)} \ge c$ simplify n

But as $n \to \infty$, $\dfrac{2}{\sqrt{n}\ln(2)} \to 0$. This prove to be a contradiction. c cannot be less than 0 since c must be positive, thus by contradiction $\log_2 n \notin \Omega(\sqrt{n})$

Q5.

The outer loop of insertion sort runs n times regardless of if the array is sorted or not. The inner loop only runs if an element is out of place. Normally the average time complexity of insertion sort is $O(n^2)$. Because only the last k elements are unsorted, the inner loop is only run k times. Therefore, the time complexity is $O(nk)$.