Boise State Math 275 (Ultman)

Worksheet 1.3: Introduction to the Dot and Cross Products

From the Toolbox (what you need from previous classes)

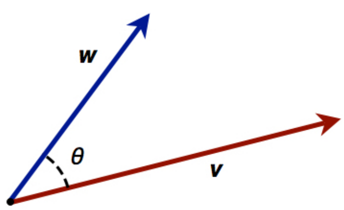
- ☐ Trigonometry: Sine and cosine functions.
- ☐ Vectors: Know what a vector is. Be able to compute the magnitude of a vector.

The dot and cross products are vector operations with many important applications. Today, you will focus on computation and geometric interpretations.

In this worksheet, you will:

- ☐ Compute the dot and cross product of two vectors, using both the algebraic and geometric definitions.

Definitions



$$\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle = v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}$$
$$\boldsymbol{w} = \langle w_1, w_2, w_3 \rangle = w_1\,\hat{\boldsymbol{\imath}} + w_2\,\hat{\boldsymbol{\jmath}} + w_3\,\hat{\boldsymbol{k}}$$

Dot Product

Algebraic Definition: $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

Geometric Definition: $\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}| \cos\theta$

Cross Product

Algebraic Definition:
$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Geometric Definition:

Magnitude: $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}| \sin\theta$ = area of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$

Direction: determined by the right-hand rule, and orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$

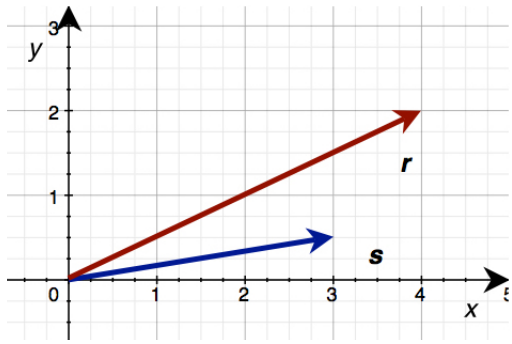
Model 1: The Dot Product — Algebraic and Geometric Definitions

The dot product can be computed in two different ways:

Algebraic Definition: $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2$

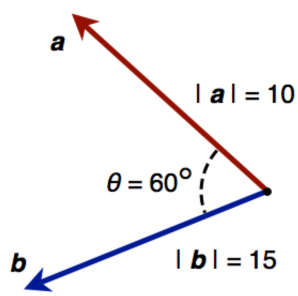
Geometric Definition: $\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}| \cos\theta$

Diagram 1A:



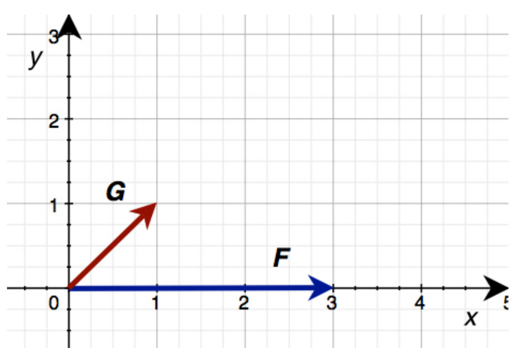
Vectors: $\boldsymbol{r}$, $\boldsymbol{s}$.

Diagram 1B:



Vectors $\boldsymbol{a}$, $\boldsymbol{b}$.

Diagram 1C:



Vectors $\boldsymbol{F}$, $\boldsymbol{G}$.

Critical Thinking Questions

In this section, you will practice computing the dot product, using both the algebraic and the geometric definitions.

(Q1) Only two out of the three diagrams 1A, 1B, 1C contain enough information for you to determine the component forms of their vectors. Which ones are they?

(Q2) One of the diagrams out of the three 1A, 1B, 1C gives explicit information about the magnitudes of the vectors and the angle between them. Which one is it?

(Q3) Compute the dot product $\boldsymbol{r} \cdot \boldsymbol{s}$ for the vectors in Diagram 1A. Which definition did you use, and why did you choose that definition?

(Q4) Compute the dot product $\boldsymbol{a} \cdot \boldsymbol{b}$ for the vectors in Diagram 1B. Which definition did you use, and why did you choose that definition?

(Q5) Compute the dot product $\boldsymbol{F} \cdot \boldsymbol{G}$ for the vectors in Diagram 1C. Which definition did you use, and why did you choose that definition?

(Q6) Now, compute the dot product $\boldsymbol{F} \cdot \boldsymbol{G}$ for the vectors in Diagram 1C using the *other* definition of the dot product. How does this compare to your answer to (Q5)?

($\oplus$ Q7) Suppose you know that two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular. What is the value of the dot product $\boldsymbol{v} \cdot \boldsymbol{w}$?

($\oplus$ Q8) What is the value of the dot product $\vec{0} \cdot \boldsymbol{w}$? *($\vec{0}$ is the* **zero vector***: the vector whose components all equal zero.)*

($\oplus$ Q9) Suppose you know that the angle between two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is $\theta = \pi/3$, and $\boldsymbol{v} \cdot \boldsymbol{w} = 4$. What (if anything) can you say about the magnitudes $|\boldsymbol{v}|$ and $|\boldsymbol{w}|$?

Model 2: The Cross Product — Algebraic Definition

DIAGRAM 2:

Computing 2×2 determinants: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Algebraic Definition of the Cross Product:

$$\boldsymbol{v} = v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}} \qquad \boldsymbol{w} = w_1\,\hat{\boldsymbol{\imath}} + w_2\,\hat{\boldsymbol{\jmath}} + w_3\,\hat{\boldsymbol{k}}$$

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{1}$$

$$= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \hat{\boldsymbol{\imath}} - \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \hat{\boldsymbol{\jmath}} + \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \hat{\boldsymbol{k}} \tag{2}$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \hat{\boldsymbol{\imath}} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \hat{\boldsymbol{\jmath}} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \hat{\boldsymbol{k}} \tag{3}$$

$$= \Big(v_2 w_3 - v_3 w_2\Big)\,\hat{\boldsymbol{\imath}} - \Big(\qquad\qquad\qquad \Big)\,\hat{\boldsymbol{\jmath}} + \Big(\qquad\qquad\qquad \Big)\,\hat{\boldsymbol{k}} \tag{4}$$

Critical Thinking Questions

In this section, you will practice computing the cross product using the algebraic definition.

(Q10) In Diagram 2, look at the 3×3 array in front of the $\hat{\boldsymbol{\imath}}$ basis vector in Equation (2). Draw a line through the **top row** and the **first (left) column**. Compare the remaining values to the 2×2 array in front of the $\hat{\boldsymbol{\imath}}$ basis vector in Equation (3) — they should be the same.

(Q11) Now, look at the 3×3 array in front of the $\hat{\boldsymbol{\jmath}}$ basis vector in Equation (2). Draw a line through the **top row** and the **second (middle) column**. Compare the remaining values to the 2×2 array in front of the $\hat{\boldsymbol{\jmath}}$ basis vector in Equation (3) — they should be the same.

(Q12) Finally, look at the 3×3 array in front of the $\hat{\boldsymbol{k}}$ basis vector in Equation (2). Draw a line through the _______ **row** and the ___________________ **column** so that the remaining values are the same as in the 2×2 array in front of the $\hat{\boldsymbol{k}}$ basis vector in Equation (3).

(Q13) Use the rule for computing 2×2 determinants at the top of Diagram 2 to complete Equation (4) in Diagram 2.

Equation (4) in Diagram 2 is the (algebraic) definition of the cross product $\boldsymbol{v} \times \boldsymbol{w}$, but it's difficulty to memorize. You may find it is easier to remember how to compute $\boldsymbol{v} \times \boldsymbol{w}$ if you begin with Equation (1), work through the process of crossing out rows and columns, then compute the three 2×2 determinants.

(Q14) Compute $\boldsymbol{v} \times \boldsymbol{w}$ for $\boldsymbol{v} = 3\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 4\,\hat{\boldsymbol{k}}$ and $\boldsymbol{w} = 5\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{k}} = 5\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\boldsymbol{k}}$.

(Q15) Using the same vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ from (Q14), compute $\boldsymbol{w} \times \boldsymbol{v}$.

(Q16) Based on your answers to (Q14) and (Q15): Since $\boldsymbol{w} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{w}$, this means $\boldsymbol{v} \times \boldsymbol{w}$ and $\boldsymbol{w} \times \boldsymbol{v}$ have the ____________________ magnitude, but _________________________ directions.

(Q17) Circle and correct the errors in the following equation (compare to Diagram 2, Equation (4)):

$$\boldsymbol{v} \times \boldsymbol{w} = \Big(v_2 w_3 - v_3 w_2\Big)\,\hat{\boldsymbol{\imath}} + \Big(v_1 w_3 - v_3 w_1\Big)\,\hat{\boldsymbol{\jmath}} + \Big(v_1 w_2 - v_2 w_1\Big)\,\hat{\boldsymbol{k}}$$

(Q18) Circle and correct the errors in the following equation :

$$\boldsymbol{v} \times \boldsymbol{w} = \Big(v_2 w_3 + v_3 w_2\Big)\,\hat{\boldsymbol{\imath}} - \Big(v_1 w_3 + v_3 w_1\Big)\,\hat{\boldsymbol{\jmath}} + \Big(v_1 w_2 + v_2 w_1\Big)\,\hat{\boldsymbol{k}}$$

($\oplus$ Q19) What is the value of the cross product $\vec{0} \times \boldsymbol{w}$? *($\vec{0}$ is the 3-d zero vector, $\vec{0} = \langle 0, 0, 0\rangle$)*

($\oplus$ Q20) **Scalar multiples can be factored out of cross products: if $\boldsymbol{v} = c\boldsymbol{u}$, then $\boldsymbol{v} \times \boldsymbol{w} = c(\boldsymbol{u} \times \boldsymbol{w})$.**
Show that $\langle 2, 4, 6\rangle \times \langle 3, 1, 5\rangle = 2(\langle 1, 2, 3\rangle \times \langle 3, 1, 5\rangle)$.

Interlude: The Right-Hand Rule

We will go over this as a class: If you get to it early and want to begin working on it, here's a link to a video (by BraunVideos) demonstrating the right-hand rule. (Video may auto-play.)

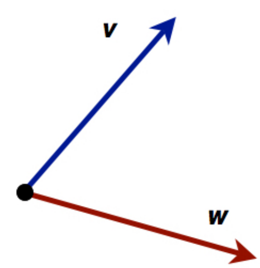
The **right-hand rule** is a method for determining how three vectors are positioned in space relative to each other.

If a set of three vectors satisfies the right-hand rule, we will say that they are **positively oriented**. If not, we will say they are **negatively oriented**.

There are several ways to remember the right-hand rule. Here is one:

- Start with three vectors $\{\boldsymbol{v}, \boldsymbol{w}.\boldsymbol{u}\}$ *(the order of the three vectors matters!)*.
- Hold out your right hand so that your wrist is at the base point, your hand and fingers point in a straight line along $\boldsymbol{v}$, and your palm faces $\boldsymbol{w}$.
- Make a fist with your thumb sticking up (like you are hitch-hiking). If the third vector $\boldsymbol{u}$ points in the direction of your thumb, then $\boldsymbol{u}$ satisfies the right-had rule and $\{\boldsymbol{v}, \boldsymbol{w}.\boldsymbol{u}\}$ are positively oriented.

The cross product of two vectors satisfies the right-hand rule.



Question: Looking at two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ pictured above, use the right-hand rule to determine the directions of the cross products $\boldsymbol{v} \times \boldsymbol{w}$ and $\boldsymbol{w} \times \boldsymbol{v}$. Which cross product points towards you (the viewer), and which points away from you?

Model 3: The Cross Product — Geometric Definition

The cross product $\boldsymbol{v} \times \boldsymbol{w}$ is a vector, so it has both **magnitude** and **direction**. If $\boldsymbol{v} = \vec{0}$ or $\boldsymbol{w} = \vec{0}$, then $\boldsymbol{v} \times \boldsymbol{w} = \vec{0}$. Otherwise:

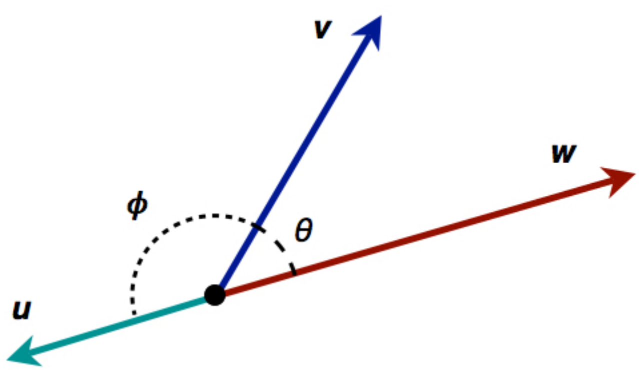
Geometric Definition of the Magnitude of $\boldsymbol{v} \times \boldsymbol{w}$ ($\boldsymbol{v}, \boldsymbol{w} \neq \vec{0}$):

$$|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}| \sin\theta$$

Geometric Definition of the Direction of $\boldsymbol{v} \times \boldsymbol{w}$ ($\boldsymbol{v}, \boldsymbol{w} \neq \vec{0}$):

- ∘ $\boldsymbol{v} \times \boldsymbol{w}$ is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$
- ∘ $\boldsymbol{v} \times \boldsymbol{w}$ points in the direction determined by the **right-hand rule**.

DIAGRAM 3:



$|\boldsymbol{u}| = 2$
$|\boldsymbol{v}| = 3$
$|\boldsymbol{w}| = 4$

$\theta = 45^\circ = \pi/4$
$\phi = 135^\circ = 3\pi/4$

Critical Thinking Questions

In this section, you will practice computing the magnitude and finding the direction of the cross product of two vectors using the geometric definitions.

(Q21) On Diagram 3, sketch the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$, and the parallelogram spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

Fact: Since $|\boldsymbol{v}||\boldsymbol{w}| \sin\theta$ is the area of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$, then the magnitude of the cross product $|\boldsymbol{v} \times \boldsymbol{w}|$ is the area of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$.

(Q22) In Diagram 3: Find the area of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$.

(Q23) Complete the table, using the information from Diagram 3. *When determining the direction of a cross product, assume that the paper is the plane containing $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, and the cross product points either towards you when looking at the diagram, or away from you when looking at the diagram.*

Cross Product	**Magnitude (Area of Parallelogram)**	**Direction (RHR)**
$\boldsymbol{v} \times \boldsymbol{w}$	$\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = 6\sqrt{2}$	Towards / $\boxed{\text{Away from}}$ the viewer
$\boldsymbol{w} \times \boldsymbol{v}$	$\lvert\boldsymbol{w} \times \boldsymbol{v}\rvert =$ ______________	Towards / Away from the viewer
$\boldsymbol{v} \times \boldsymbol{u}$	$\lvert\boldsymbol{v} \times \boldsymbol{u}\rvert =$ ______________	Towards / Away from the viewer
$\boldsymbol{u} \times \boldsymbol{v}$	$\lvert\boldsymbol{u} \times \boldsymbol{v}\rvert =$ ______________	Towards / Away from the viewer

(Q24) In Diagram 3: Use the given information to find the angle α between $\boldsymbol{u}$ and $\boldsymbol{w}$, and use this angle to compute $|\boldsymbol{u} \times \boldsymbol{w}| = |\boldsymbol{u}||\boldsymbol{w}| \sin \alpha$.

(Q25) **Cross Product of Parallel Vectors:** If two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then the angle between them is either $\theta =$ _____ or $\theta =$ _____. So the magnitude of the cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is either:

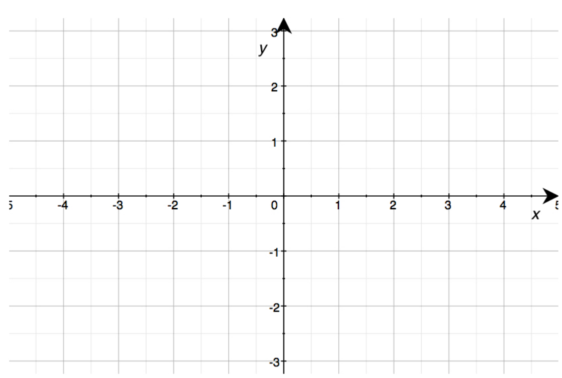
$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \_\_\_\_\_ = \_\_\_\_\_ \quad \text{or} \quad |\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \_\_\_\_\_ = \_\_\_\_\_.$$

The only vector with magnitude equal to zero is the zero vector. So, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then:

$$\boldsymbol{a} \times \boldsymbol{b} = \_\_\_\_\_\_\_.$$

($\oplus$ Q26) Return to Diagram 3. Suppose the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ lie in the xy-plane. Find the component forms of the vectors $\boldsymbol{v} \times \boldsymbol{w}$ and $\boldsymbol{w} \times \boldsymbol{v}$.

($\oplus$ Q27) Sketch vectors $\boldsymbol{F}$ and $\boldsymbol{G}$ so that $|\boldsymbol{F} \times \boldsymbol{G}| = 4$ and $\boldsymbol{F} \times \boldsymbol{G}$ points up out of the paper at the viewer.



Summary

	Dot Product	Cross Product
Output is a:	Vector / Scalar	Vector / Scalar
Order of vectors matters:	Yes / No	Yes / No
Trig function in the geometric definition is:	$\sin\theta$ / $\cos\theta$	$\sin\theta$ / $\cos\theta$

The algebraic definition of the dot product of two vectors $\boldsymbol{v} = \langle v_1, v_2, v_3\rangle$ and $\boldsymbol{w} = \langle w_1, w_2, w_3\rangle$ is $\boldsymbol{v} \cdot \boldsymbol{w} =$ __________________________________.

The algebraic definition of the cross product of two vectors $\boldsymbol{v} = v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}$ and $\boldsymbol{w} = w_1\,\hat{\boldsymbol{\imath}} + w_2\,\hat{\boldsymbol{\jmath}} + w_3\,\hat{\boldsymbol{k}}$ is:

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ & & \\ & & \end{vmatrix}$$

The geometric definition of the dot product of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is:

$\boldsymbol{v} \cdot \boldsymbol{w} =$ _________________________, where _____ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

The geometric definition of the **magnitude** of the cross product of two 3-d vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is:

$|\boldsymbol{v} \times \boldsymbol{w}| =$ _________________________, where _____ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

The **direction** of the cross product of two vectors $\boldsymbol{v} \times \boldsymbol{w}$ is orthogonal to $\boldsymbol{v}$ and $\boldsymbol{w}$, in the direction determined by the _______________-hand rule.

Computing the dot product using the algebraic definition will give [the same / a different] answer as using the geometric definition.

Computing the cross product using the algebraic definition will give [the same / a different] answer as using the geometric definition.

$\boldsymbol{v} \times \boldsymbol{w} =$ _____$\boldsymbol{w} \times \boldsymbol{v}$, so changing the order of the vectors ____________________ the direction of the cross product.

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, than $\boldsymbol{v} \cdot \boldsymbol{w} =$ _____. If $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel, then $\boldsymbol{v} \times \boldsymbol{w} =$ _____.

$\boldsymbol{v} \cdot \vec{0} =$ _____, and $\boldsymbol{v} \times \vec{0} =$ _____.