Lecture Notes on Electrodynamics

Electricity, magnetism and Electrodynamics

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Undergraduate Lecture Notes

Last updated on January 18, 2026

IEHE, Bhopal

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1 Vector Calculus

1.1 Scalars and Vectors

A **scalar** is a physical quantity completely specified by a magnitude alone. Examples include temperature, mass, electric potential, and energy.

A **vector** is a physical quantity specified by both magnitude and direction. Examples include displacement, velocity, force, and the electric and magnetic fields.

Definition 1.1. A vector $\mathbf{A}$ in three-dimensional Cartesian coordinates can be written as

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}, \tag{1.1}$$

where A_x, A_y, and A_z are the components of $\mathbf{A}$ along the unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ respectively.

The magnitude of $\mathbf{A}$ is given by

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{1.2}$$

1.2 Vector Algebra

1.2.1 Addition and Subtraction

The sum of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined component-wise:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}. \tag{1.3}$$

Vector subtraction is defined similarly.

1.2.2 Scalar (Dot) Product

Definition 1.2. The scalar product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta, \tag{1.4}$$

where θ is the angle between the vectors.

In Cartesian coordinates,

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{1.5}$$

1.2.3 Vector (Cross) Product

Definition 1.3. The vector product of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \, \hat{\mathbf{n}}, \tag{1.6}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$, determined by the right-hand rule.

In component form,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.7}$$

1.3 Differential Vector Operators

1.3.1 Gradient

Definition 1.4. The gradient of a scalar field $\phi(x, y, z)$ is defined as

$$\nabla\phi = \hat{\mathbf{x}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\phi}{\partial z}. \tag{1.8}$$

The gradient points in the direction of maximum increase of the scalar field and its magnitude gives the rate of change in that direction.

1.3.2 Divergence

Definition 1.5. The divergence of a vector field $\mathbf{A}$ is defined as

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{1.9}$$

Divergence measures the net outward flux per unit volume from a point.

1.3.3 Curl

Definition 1.6. The curl of a vector field $\mathbf{A}$ is defined as

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{1.10}$$

The curl represents the local rotational tendency of the vector field.

1.4 Integral Theorems

1.4.1 Gauss's Divergence Theorem

Law 1.1 (Gauss's Divergence Theorem)**.** *Let $\mathbf{A}(\mathbf{r})$ be a continuously differentiable vector field defined throughout a volume V bounded by a closed surface S with outward-directed unit normal $\hat{\mathbf{n}}$. Then the total flux of $\mathbf{A}$ through the surface S is equal to the volume integral of the divergence of $\mathbf{A}$ over V, that is,*

$$\oint_S \mathbf{A} \cdot d\mathbf{a} = \int_V (\boldsymbol{\nabla} \cdot \mathbf{A})\, d\tau, \tag{1.11}$$

where $d\mathbf{a} = \hat{\mathbf{n}}\, da$ is the outward-directed surface area element and $d\tau$ is the volume element.

Gauss's divergence theorem converts a surface integral into a volume integral. Physically, it states that the net outward flux of a vector field through a closed surface equals the total strength of sources contained within the enclosed volume.

1.4.2 Stokes' Theorem

Law 1.2 (Stokes' Theorem)**.** *Let $\mathbf{A}(\mathbf{r})$ be a continuously differentiable vector field defined on an open surface S bounded by a closed curve C. If $\hat{\mathbf{n}}$ is the unit normal to the surface S and the direction of traversal of C is related to $\hat{\mathbf{n}}$ by the right-hand rule, then the line integral of $\mathbf{A}$ around C is equal to the surface integral of the curl of $\mathbf{A}$ over S, that is,*

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{a}. \tag{1.12}$$

Stokes' theorem relates the circulation of a vector field around a closed curve to the flux of its curl through any surface bounded by that curve. Physically, it expresses the idea that local rotational behavior of a field gives rise to macroscopic circulation.

1.5 Important Vector Identities

The following vector identities involving the del operator (∇) are extensively used in electrodynamics, particularly in the formulation and manipulation of Maxwell's equations and in solving boundary-value problems.

1.5.1 First-Order Del Operator Identities

Important Result

$$\boldsymbol{\nabla}(\phi + \psi) = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\psi, \tag{1.13}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{A} + \mathbf{B}) = \boldsymbol{\nabla} \cdot \mathbf{A} + \boldsymbol{\nabla} \cdot \mathbf{B}, \tag{1.14}$$

$$\boldsymbol{\nabla} \times (\mathbf{A} + \mathbf{B}) = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times \mathbf{B}, \tag{1.15}$$

Important Result

$$\boldsymbol{\nabla}(\phi\psi) = \phi\,\boldsymbol{\nabla}\psi + \psi\,\boldsymbol{\nabla}\phi, \tag{1.16}$$
$$\boldsymbol{\nabla}\cdot(\phi\mathbf{A}) = \phi\,\boldsymbol{\nabla}\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\nabla}\phi, \tag{1.17}$$
$$\boldsymbol{\nabla}\times(\phi\mathbf{A}) = \phi\,\boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\phi\times\mathbf{A}, \tag{1.18}$$

Important Result

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}), \tag{1.19}$$
$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}, \tag{1.20}$$

1.5.2 Identities Involving Curl and Divergence

Important Result

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0, \tag{1.21}$$
$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0, \tag{1.22}$$
$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}, \tag{1.23}$$

These identities impose strong constraints on the possible forms of vector fields and play a fundamental role in Maxwell's equations.

1.5.3 Second-Order Del Operator Identities

Important Result

$$\nabla^2\phi = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi), \tag{1.24}$$
$$\nabla^2\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}), \tag{1.25}$$

Important Result

$$\boldsymbol{\nabla}\left(\nabla^2\phi\right) = \nabla^2(\boldsymbol{\nabla}\phi), \tag{1.26}$$
$$\boldsymbol{\nabla}\times\left(\nabla^2\mathbf{A}\right) = \nabla^2(\boldsymbol{\nabla}\times\mathbf{A}), \tag{1.27}$$
$$\boldsymbol{\nabla}\cdot\left(\nabla^2\mathbf{A}\right) = \nabla^2(\boldsymbol{\nabla}\cdot\mathbf{A}), \tag{1.28}$$

Important Result

$$\nabla^2(\phi\psi) = \phi\,\nabla^2\psi + \psi\,\nabla^2\phi + 2\,\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi, \tag{1.29}$$

1.5.4 Identities Involving Directional Derivatives

Important Result

$$(\mathbf{A} \cdot \nabla)\phi = \mathbf{A} \cdot \boldsymbol{\nabla}\phi, \tag{1.30}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}), \tag{1.31}$$

1.5.5 Useful Special Identities

Important Result

$$\boldsymbol{\nabla}(r) = \hat{\mathbf{r}}, \tag{1.32}$$

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{\mathbf{r}}, \tag{1.33}$$

$$\nabla^2\left(\frac{1}{r}\right) = 0 \quad (r \neq 0), \tag{1.34}$$

$$\boldsymbol{\nabla} \times (\hat{\mathbf{r}}) = 0. \tag{1.35}$$

1.6 Spherical Coordinate System

1.6.1 Definition of Spherical Coordinates

In the spherical coordinate system, the position of a point in space is specified by the coordinates (r, θ, ϕ), where:

- r is the radial distance from the origin,
- θ is the polar angle measured from the positive z-axis $(0 \leq \theta \leq \pi)$,
- ϕ is the azimuthal angle measured in the xy-plane from the positive x-axis $(0 \leq \phi < 2\pi)$.

The transformation between Cartesian and spherical coordinates is given by

$$x = r\sin\theta\cos\phi, \tag{1.36}$$

$$y = r\sin\theta\sin\phi, \tag{1.37}$$

$$z = r\cos\theta. \tag{1.38}$$

Conversely,

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{1.39}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right), \tag{1.40}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{1.41}$$

1.6.2 Unit Vectors in Spherical Coordinates

The spherical coordinate system uses three mutually orthogonal unit vectors:

- $\hat{\mathbf{r}}$ in the radial direction,
- $\hat{\boldsymbol{\theta}}$ in the direction of increasing θ,
- $\hat{\boldsymbol{\phi}}$ in the direction of increasing ϕ.

These unit vectors vary with position and are related to the Cartesian unit vectors by

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \tag{1.42}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \tag{1.43}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}. \tag{1.44}$$

1.6.3 Differential Length, Area, and Volume Elements

In spherical coordinates, the differential displacement vector is

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}. \tag{1.45}$$

The corresponding differential area elements are

$$d\mathbf{a}_r = r^2\sin\theta\,d\theta\,d\phi\;\hat{\mathbf{r}}, \tag{1.46}$$

$$d\mathbf{a}_\theta = r\sin\theta\,dr\,d\phi\;\hat{\boldsymbol{\theta}}, \tag{1.47}$$

$$d\mathbf{a}_\phi = r\,dr\,d\theta\;\hat{\boldsymbol{\phi}}. \tag{1.48}$$

The differential volume element is

$$d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi. \tag{1.49}$$

1.6.4 Gradient, Divergence, and Curl in Spherical Coordinates

For a scalar field $\phi(r, \theta, \phi)$, the gradient is

$$\boldsymbol{\nabla}\phi = \hat{\mathbf{r}}\frac{\partial\phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\phi}. \tag{1.50}$$

For a vector field $\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$, the divergence is

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}. \tag{1.51}$$

The curl of $\mathbf{A}$ is given by

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}. \tag{1.52}$$

1.6.5 Laplacian in Spherical Coordinates

The Laplacian of a scalar field $\phi(r,\theta,\phi)$ in spherical coordinates is

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2}. \tag{1.53}$$

1.7 Cylindrical Coordinate System

1.7.1 Definition of Cylindrical Coordinates

In the cylindrical coordinate system, a point in space is specified by the coordinates (ρ,ϕ,z), where:

- ρ is the perpendicular distance from the z-axis,
- ϕ is the azimuthal angle measured in the xy-plane from the positive x-axis $(0 \le \phi < 2\pi)$,
- z is the same coordinate as in Cartesian coordinates.

The transformation between Cartesian and cylindrical coordinates is given by

$$x = \rho\cos\phi, \tag{1.54}$$
$$y = \rho\sin\phi, \tag{1.55}$$
$$z = z. \tag{1.56}$$

Conversely,

$$\rho = \sqrt{x^2+y^2}, \tag{1.57}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right), \tag{1.58}$$
$$z = z. \tag{1.59}$$

1.7.2 Unit Vectors in Cylindrical Coordinates

The cylindrical coordinate system employs three mutually orthogonal unit vectors:

- $\hat{\boldsymbol{\rho}}$ in the direction of increasing ρ,

- $\hat{\boldsymbol{\phi}}$ in the direction of increasing ϕ,
- $\hat{\mathbf{z}}$ along the positive z-axis.

These unit vectors are related to the Cartesian unit vectors by

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \tag{1.60}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \tag{1.61}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}. \tag{1.62}$$

1.7.3 Differential Length, Area, and Volume Elements

The differential displacement vector in cylindrical coordinates is

$$d\mathbf{l} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}. \tag{1.63}$$

The differential area elements are

$$d\mathbf{a}_\rho = \rho\,d\phi\,dz\;\hat{\boldsymbol{\rho}}, \tag{1.64}$$

$$d\mathbf{a}_\phi = d\rho\,dz\;\hat{\boldsymbol{\phi}}, \tag{1.65}$$

$$d\mathbf{a}_z = \rho\,d\rho\,d\phi\;\hat{\mathbf{z}}. \tag{1.66}$$

The differential volume element is

$$d\tau = \rho\,d\rho\,d\phi\,dz. \tag{1.67}$$

1.7.4 Gradient, Divergence, and Curl in Cylindrical Coordinates

For a scalar field $\phi(\rho, \phi, z)$, the gradient is

$$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\rho}}\frac{\partial\phi}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial\phi}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial\phi}{\partial z}. \tag{1.68}$$

For a vector field $\mathbf{A} = A_\rho\hat{\boldsymbol{\rho}} + A_\phi\hat{\boldsymbol{\phi}} + A_z\hat{\mathbf{z}}$, the divergence is

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}. \tag{1.69}$$

The curl of $\mathbf{A}$ is given by

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}. \tag{1.70}$$

1.7.5 Laplacian in Cylindrical Coordinates

The Laplacian of a scalar field $\phi(\rho, \phi, z)$ in cylindrical coordinates is

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\phi^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{1.71}$$

Problems

Problem 1.1. For the scalar field

$$\phi(x, y, z) = x^2y + yz^3,$$

find the gradient $\boldsymbol{\nabla}\phi$ at the point $(1, -1, 2)$. *Answer:* $\boldsymbol{\nabla}\phi = -2\hat{\mathbf{x}} + 9\hat{\mathbf{y}} - 12\hat{\mathbf{z}}$

Problem 1.2. Given the vector field

$$\mathbf{A} = x^2\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + z^2\hat{\mathbf{z}},$$

calculate $\boldsymbol{\nabla}\cdot\mathbf{A}$ and $\boldsymbol{\nabla}\times\mathbf{A}$. *Answer:* $\boldsymbol{\nabla}\cdot\mathbf{A} = 2x + z + 2z,\ \boldsymbol{\nabla}\times\mathbf{A} = y\hat{\mathbf{x}}$

Problem 1.3. Show that

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0$$

for the scalar field $\phi = r^2$.

Problem 1.4. Verify that

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0$$

for $\mathbf{A} = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$.

Problem 1.5. Find $\nabla^2\phi$ for

$$\phi(r) = \frac{1}{r}, \quad r \neq 0.$$

Answer: $\nabla^2\phi = 0$

Problem 1.6. Using spherical coordinates, find

$$\boldsymbol{\nabla}\cdot\left(\frac{1}{r^2}\hat{\mathbf{r}}\right).$$

Answer: 0 *for* $r \neq 0$

Problem 1.7. Evaluate the flux of

$$\mathbf{A} = r\hat{\mathbf{r}}$$

through a sphere of radius R. *Answer:* $4\pi R^3$

Problem 1.8. Using Gauss's theorem, evaluate

$$\oint_S (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})\cdot d\mathbf{a}$$

over a cube of side a centered at the origin. *Answer:* $3a^3$

Problem 1.9. Using Stokes' theorem, evaluate

$$\oint_C \mathbf{A} \cdot d\mathbf{l}$$

for $\mathbf{A} = y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$, where C is a circle of radius R in the xy-plane. *Answer:* $-2\pi R^2$

Problem 1.10. Find the divergence of

$$\mathbf{A} = \rho\hat{\boldsymbol{\rho}}$$

in cylindrical coordinates. *Answer:* $\boldsymbol{\nabla} \cdot \mathbf{A} = 2$

Problem 1.11. Find the curl of

$$\mathbf{A} = \rho^2\hat{\boldsymbol{\phi}}$$

in cylindrical coordinates. *Answer:* $2\rho\,\hat{\mathbf{z}}$

Problem 1.12. Show that if $\phi = \phi(z)$ only, then

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial z^2}.$$

Answer: True

Problem 1.13. For $\mathbf{A} = r^n\hat{\mathbf{r}}$, find $\boldsymbol{\nabla} \cdot \mathbf{A}$ and determine n such that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ for $r \neq 0$. *Answer:* $\boldsymbol{\nabla} \cdot \mathbf{A} = (n+2)r^{n-1}$, $n = -2$

Problem 1.14. Starting from

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0},$$

derive the integral form of Gauss's law. *Answer:* $\oint \mathbf{E} \cdot d\mathbf{a} = \dfrac{Q_{enc}}{\varepsilon_0}$

2 Electrostatics

2.1 Electric Charge

Electric charge is a fundamental property of matter responsible for electrical interactions.

Postulate 2.1. Electric charge is conserved; the total charge of an isolated system remains constant.

Charge is quantized and occurs in integral multiples of the elementary charge $e = 1.602 \times 10^{-19}\,\mathrm{C}$.

2.2 Coulomb's Law

The force between two stationary point charges was first quantified by Coulomb.

Law 2.1 (Coulomb's Law)**.** *The electrostatic force* $\mathbf{F}$ *between two point charges* q_1 *and* q_2 *separated by a distance* r *is given by*

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}, \tag{2.1}$$

where $\hat{\mathbf{r}}$ *is the unit vector from* q_1 *to* q_2 *and* ε_0 *is the permittivity of free space.*

The force acts along the line joining the charges and is repulsive for like charges and attractive for unlike charges.

2.3 Electric Field

2.3.1 Definition of Electric Field

The electric field is defined as the force per unit positive test charge.

Definition 2.1. The electric field $\mathbf{E}$ at a point in space is defined as

$$\mathbf{E} = \lim_{q\to 0}\frac{\mathbf{F}}{q}. \tag{2.2}$$

For a point charge q located at the origin,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{2.3}$$

2.3.2 Principle of Superposition

Since the electrostatic force obeys the principle of superposition, the electric field, being defined as force per unit charge, must also obey the same principle.

Postulate 2.2. The net electric field at any point due to a system of charges is equal to the vector sum of the electric fields produced by the individual charges acting independently.

Electric Field Due to Discrete Charges

For a system of N point charges q_i located at positions $\mathbf{r}_i$, the electric field at a field point $\mathbf{r}$ is given by

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}. \tag{2.4}$$

Each charge produces an electric field as if it were acting alone, and the resultant field is obtained by vector addition.

Electric Field Due to Continuous Charge Distributions

When the charge distribution is continuous, the discrete sum is replaced by an integral. For a volume charge density $\rho(\mathbf{r}')$, the electric field at point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d\tau'. \tag{2.5}$$

Similar expressions can be written for line and surface charge distributions by replacing $\rho \, d\tau$ with $\lambda \, dl$ or $\sigma \, da$, respectively.

Electric Field Due to Line and Surface Charge Distributions

For a *line charge distribution* characterized by a linear charge density $\lambda(\mathbf{r}')$, the electric field at point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dl'. \tag{2.6}$$

For a *surface charge distribution* described by a surface charge density $\sigma(\mathbf{r}')$, the electric field at point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, da'. \tag{2.7}$$

In each case, the vector $\mathbf{r} - \mathbf{r}'$ points from the source charge element to the field point, and the integration extends over the entire charge distribution.

2.4 Electric Flux

Definition 2.2. The electric flux through a surface S is defined as

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a}. \tag{2.8}$$

Electric flux provides a measure of the number of electric field lines passing through a given surface.

Remark 2.1. In electrostatics, the electric field may be viewed as a measure of the density of electric flux in space. Regions of stronger electric field correspond to regions of higher electric flux density.

2.5 Gauss's Law

2.5.1 Statement of Gauss's Law

Law 2.2. *Gauss's law states that the total electric flux emerging from any closed surface is directly proportional to the total electric charge enclosed within that surface. The proportionality constant is the permittivity of free space.*

In compact notation, Gauss's law is expressed as

$$\Phi_E = \frac{Q_{enc}}{\varepsilon_0}, \tag{2.9}$$

where Φ_E *is the total electric flux through the closed surface and* Q_{enc} *is the total charge enclosed.*

Remark 2.2. electric flux measures how much electric field passes through a closed surface, and this flux is determined solely by the charge enclosed inside the surface, independent of the shape or size of the surface.

2.5.2 Integral Form of Gauss's Law

The electric flux through a closed surface S is defined as

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{a}. \tag{2.10}$$

The total charge enclosed within the surface can be written in terms of the volume charge density ρ as

$$Q_{\text{enc}} = \int_V \rho \, d\tau, \tag{2.11}$$

where V is the volume enclosed by the surface S.

Substituting these expressions into the flux–charge relation, Gauss's law in integral form becomes

Law 2.3 (Integral Form of Gauss's law)**.**

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \int_V \rho \, d\tau. \tag{2.12}$$

This form of Gauss's law is particularly useful for evaluating electric fields in systems possessing spherical, cylindrical, or planar symmetry.

2.5.3 Differential Form of Gauss's Law

Gauss's divergence theorem relates the flux of a vector field through a closed surface to the volume integral of its divergence:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V (\boldsymbol{\nabla} \cdot \mathbf{E})\, d\tau. \tag{2.13}$$

Applying this theorem to the integral form of Gauss's law, we obtain

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{E})\, d\tau = \frac{1}{\varepsilon_0} \int_V \rho\, d\tau. \tag{2.14}$$

Since the above relation must hold for any arbitrary volume V, the integrands must be equal at every point in space. Hence, Gauss's law in differential form is

Law 2.4 (Differential Form of Gauss's Law)**.**

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{2.15}$$

2.6 Applications of Gauss's Law

Gauss's law is most useful when the charge distribution possesses a high degree of symmetry. In such cases, an appropriate Gaussian surface can be chosen so that the electric field is either constant or zero over the surface, allowing straightforward evaluation of the flux integral.

2.6.1 Point Charge (Coulomb's Law)

Consider a point charge q placed at the origin. Due to spherical symmetry, the electric field at any point depends only on the radial distance r and is directed radially outward.

Choose a spherical Gaussian surface of radius r centered at the charge. On this surface, $\mathbf{E}$ has constant magnitude and is normal to the surface everywhere.

The electric flux is

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = E \oint_S da = E(4\pi r^2). \tag{2.16}$$

The charge enclosed is $Q_{\text{enc}} = q$. Applying Gauss's law,

$$E(4\pi r^2) = \frac{q}{\varepsilon_0}. \tag{2.17}$$

Hence,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}, \tag{2.18}$$

which is Coulomb's law in field form.

2.6.2 Electric Field of an Electric Dipole

Consider two point charges $+q$ and $-q$ separated by a distance $2a$, forming an electric dipole.

The electric field at a point $\mathbf{r}$ is obtained by superposition:

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-. \tag{2.19}$$

Using Coulomb's law,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} - \frac{q(\mathbf{r}+\mathbf{a})}{|\mathbf{r}+\mathbf{a}|^3}\right]. \tag{2.20}$$

For points far from the dipole ($r \gg a$), the electric field reduces to

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right], \tag{2.21}$$

where $\mathbf{p} = 2qa\,\hat{\mathbf{z}}$ is the electric dipole moment.

2.6.3 Infinite Long Straight Wire

Consider an infinitely long straight wire carrying a uniform linear charge density λ. Due to cylindrical symmetry, the electric field is radial and depends only on the distance ρ from the wire.

Choose a cylindrical Gaussian surface of radius ρ and length L, coaxial with the wire.

The electric field is perpendicular to the curved surface and parallel to the end caps. Hence, flux through the end caps is zero.

The total flux is

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = E(2\pi\rho L). \tag{2.22}$$

The enclosed charge is

$$Q_{\text{enc}} = \lambda L. \tag{2.23}$$

Applying Gauss's law,

$$E(2\pi\rho L) = \frac{\lambda L}{\varepsilon_0}. \tag{2.24}$$

Therefore,

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0\rho}\hat{\boldsymbol{\rho}}. \tag{2.25}$$

2.6.4 Infinite Plane Sheet of Charge

Consider an infinite plane sheet with uniform surface charge density σ. The electric field must be perpendicular to the sheet and independent of distance from it.

Choose a cylindrical Gaussian surface (pillbox) of cross-sectional area A, with its flat faces parallel to the sheet.

The flux through the curved surface is zero. The flux through the two flat faces is

$$\Phi_E = EA + EA = 2EA. \tag{2.26}$$

The enclosed charge is

$$Q_{\text{enc}} = \sigma A. \tag{2.27}$$

Applying Gauss's law,

$$2EA = \frac{\sigma A}{\varepsilon_0}. \tag{2.28}$$

Thus,

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}, \tag{2.29}$$

where $\hat{\mathbf{n}}$ is the unit normal to the sheet.

2.6.5 Uniformly Charged Solid Sphere

Consider a solid sphere of radius R with constant volume charge density ρ_0.

Electric Field Outside the Sphere ($r > R$)

Choose a spherical Gaussian surface of radius $r > R$. The total enclosed charge is

$$Q_{\text{enc}} = \rho_0 \frac{4}{3}\pi R^3. \tag{2.30}$$

Applying Gauss's law,

$$E(4\pi r^2) = \frac{1}{\varepsilon_0}\rho_0 \frac{4}{3}\pi R^3. \tag{2.31}$$

Hence,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}, \tag{2.32}$$

where Q is the total charge of the sphere.

Electric Field Inside the Sphere ($r < R$)

For a Gaussian sphere of radius $r < R$, the enclosed charge is

$$Q_{\text{enc}} = \rho_0 \frac{4}{3}\pi r^3. \tag{2.33}$$

Applying Gauss's law,

$$E(4\pi r^2) = \frac{1}{\varepsilon_0}\rho_0 \frac{4}{3}\pi r^3. \tag{2.34}$$

Thus,

$$\mathbf{E} = \frac{\rho_0 r}{3\varepsilon_0}\hat{\mathbf{r}}. \tag{2.35}$$

2.6.6 Uniformly Charged Spherical Shell

Consider a thin spherical shell of radius R carrying uniform surface charge density σ.

Outside the Shell ($r > R$)

The enclosed charge is

$$Q = 4\pi R^2\sigma. \tag{2.36}$$

Applying Gauss's law,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}. \tag{2.37}$$

Inside the Shell ($r < R$)

No charge is enclosed within the Gaussian surface. Hence,

$$\mathbf{E} = 0. \tag{2.38}$$

2.7 Curl of the Electrostatic Field

2.7.1 Curl of the Electric Field Due to a Point Charge

Consider a point charge q placed at the origin. The electric field at a distance r is given by Coulomb's law,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{2.39}$$

This field depends only on the radial coordinate and is directed along $\hat{\mathbf{r}}$. To evaluate its curl, we use the expression for curl in spherical coordinates.

For a purely radial field of the form $\mathbf{E} = E_r(r)\hat{\mathbf{r}}$, the curl vanishes identically:

$$\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0} \qquad (r \neq 0). \tag{2.40}$$

Thus, everywhere in space except at the location of the charge, the electric field due to a point charge is curl-free.

Law 2.5. *In electrostatics, the curl of the electric field vanishes:*

$$\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}. \tag{2.41}$$

This result is independent of the specific charge configuration and holds for all static electric fields.

2.7.2 Conservative nature of Electrostatic field

Theorem 2.1 (Conservativeness of Electrostatic Field)**.** *The electrostatic field* $\mathbf{E}$ *in a region with stationary charges is conservative; that is, the line integral around any closed path is zero:*

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \tag{2.42}$$

Proof. Consider a set of stationary point charges $\{q_i\}$ at positions $\mathbf{r}_i$. The electric field at a point $\mathbf{r}$ is given by the principle of superposition:

$$\mathbf{E}(\mathbf{r}) = \sum_i \frac{1}{4\pi\varepsilon_0}\frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}. \tag{2.43}$$

Curl of a single point charge field.

For a single point charge at the origin:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{2.44}$$

This is a radial field. Using the vector identity for a radial function $f(r)\hat{\mathbf{r}}$, its curl is identically zero:

$$\nabla \times \mathbf{E} = 0 \quad \text{for } r \neq 0. \tag{2.45}$$

Superposition principle.

$$\nabla \times \mathbf{E} = \sum_i \nabla \times \mathbf{E}_i = 0. \tag{2.46}$$

Hence, the electrostatic field due to any configuration of stationary charges is irrotational.
Apply Stokes' theorem.
For any closed curve C bounding a surface S, Stokes' theorem gives

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{a}. \tag{2.47}$$

Since $\nabla \times \mathbf{E} = 0$, the right-hand side vanishes:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \tag{2.48}$$

□

Remark 2.3. The fact that $\mathbf{E}$ is conservative implies that the work done by the electrostatic field on a test charge moving between any two points is independent of the path.

2.8 Scalar Potential

2.8.1 Existence of Scalar Potential

Theorem 2.2. *If the curl of a vector field vanishes in a simply connected region, then the field can be expressed as the gradient of a scalar potential.*

Proof. Since $\nabla \times \mathbf{E} = 0$ in electrostatics, there exists a scalar function V such that

$$\mathbf{E} = -\nabla V. \tag{2.49}$$

The negative sign is chosen by convention so that the electric field points in the direction of decreasing potential. □

2.8.2 Potential Difference

Theorem 2.3 (Fundamental Theorem of Gradient)**.** *For a scalar field $V(\mathbf{r})$, the line integral of its gradient between two points A and B is equal to the difference of the scalar at the endpoints:*

$$\int_A^B \nabla V \cdot d\mathbf{l} = V(B) - V(A). \tag{2.50}$$

Definition 2.3 (Electric Potential Difference)**.** Using $\mathbf{E} = -\nabla V$ in 2.50,
The potential difference between two points A and B is

$$V(B) - V(A) = -\int_A^B \mathbf{E} \cdot d\mathbf{l}. \tag{2.51}$$

Remark 2.4. This derivation shows that the potential difference depends only on the endpoints and is path-independent, a direct consequence of the conservative nature of the electrostatic field.

2.8.3 Potential Due to a Point Charge

Choosing $V(\infty) = 0$, the potential at a distance r from a point charge q is

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}. \tag{2.52}$$

2.8.4 Potential Due to Multiple Charges

For N point charges q_i located at positions $\mathbf{r}_i$, the potential at $\mathbf{r}$ is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}. \tag{2.53}$$

For continuous charge distributions:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}d\tau', \tag{2.54}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}dl', \tag{2.55}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}da'. \tag{2.56}$$

2.8.5 Poisson's and Laplace's Equations

Law 2.6 (Poisson's Equation)**.** *Using* $\mathbf{E} = -\boldsymbol{\nabla} V$ *in Gauss's law, we obtain*

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \implies \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V) = -\nabla^2 V = \frac{\rho}{\varepsilon_0}, \tag{2.57}$$

so

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}. \tag{2.58}$$

Remark 2.5. Poisson's equation relates the potential to the charge density in space.

Law 2.7 (Laplace's Equation)**.** *In charge-free regions (*$\rho = 0$*), Poisson's equation reduces to Laplace's equation:*

$$\nabla^2 V = 0. \tag{2.59}$$

2.8.6 Laplace's Equation in Spherical Coordinates

In spherical coordinates (r, θ, ϕ), assuming azimuthal symmetry $(\partial/\partial\phi = 0)$, Laplace's equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0. \tag{2.60}$$

Separation of Variables

Assume $V(r,\theta) = R(r)\Theta(\theta)$. Then

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0. \tag{2.61}$$

Setting each part equal to a constant $l(l+1)$, we get:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1), \tag{2.62}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1). \tag{2.63}$$

Above equation is Legendre equation. whose solution is given by legendre polynomial:

$$\Theta(\cos\theta) = P_l(\cos\theta) \tag{2.64}$$

Solution for Radial Part

The radial equation is

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l+1)R = 0, \tag{2.65}$$

whose general solution is

$$R(r) = Ar^l + Br^{-(l+1)}, \tag{2.66}$$

where A and B are constants determined by boundary conditions.

Full Solution

The general solution for the potential is

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta), \tag{2.67}$$

where P_l are the Legendre polynomials.

2.9 Uniqueness Theorem in Electrostatics

Theorem 2.4 (Uniqueness Theorem)**.** *In a given volume V, the solution of Poisson's equation*

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \tag{2.68}$$

has unique solution if either

1. *the value of the potential V is specified on the boundary surface S (Dirichlet boundary condition), or*

2. *the normal derivative* $\frac{\partial V}{\partial n}$ *is specified on* S *(Neumann boundary condition), together with the total charge in* V*.*

2.9.1 Proof of the Uniqueness Theorem

Proof. Assume that there exist two solutions V_1 and V_2 satisfying Poisson's equation in the volume V and the same boundary conditions on the surface S.

Define their difference

$$\phi = V_1 - V_2. \tag{2.69}$$

Since both V_1 and V_2 satisfy Poisson's equation with the same charge density ρ,

$$\nabla^2\phi = \nabla^2 V_1 - \nabla^2 V_2 = 0. \tag{2.70}$$

Thus, ϕ satisfies Laplace's equation in V.

Consider the volume integral

$$\int_V \phi\nabla^2\phi \, d\tau = 0. \tag{2.71}$$

Using the vector identity

$$\phi\nabla^2\phi = \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\phi) - (\boldsymbol{\nabla}\phi)^2,$$

we obtain

$$\int_V \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\phi)\, d\tau - \int_V (\boldsymbol{\nabla}\phi)^2 \, d\tau = 0. \tag{2.72}$$

Applying Gauss's divergence theorem,

$$\oint_S \phi\boldsymbol{\nabla}\phi\cdot d\mathbf{a} = \int_V (\boldsymbol{\nabla}\phi)^2 \, d\tau. \tag{2.73}$$

For Dirichlet boundary conditions, $V_1 = V_2$ on S, hence $\phi = 0$ on S. Therefore, the surface integral vanishes:

$$\oint_S \phi\boldsymbol{\nabla}\phi\cdot d\mathbf{a} = 0. \tag{2.74}$$

This implies

$$\int_V (\boldsymbol{\nabla}\phi)^2 \, d\tau = 0. \tag{2.75}$$

Since $(\boldsymbol{\nabla}\phi)^2 \geq 0$ everywhere, the only possibility is

$$\boldsymbol{\nabla}\phi = 0 \quad \text{throughout } V. \tag{2.76}$$

Hence ϕ is a constant. But since $\phi = 0$ on the boundary, this constant must be zero:

$$\phi = 0 \quad \Rightarrow \quad V_1 = V_2. \tag{2.77}$$

Thus, the solution is unique. □

Remark 2.6. The uniqueness theorem implies that once the charge distribution and appropriate boundary conditions are specified, the electrostatic potential and electric field are completely determined. Any method that produces a solution satisfying Poisson's equation and the boundary conditions must therefore give the correct physical solution.

3 Polarization

3.1 Induced Dipole

If an atom is placed in a Electric field, atomic nuclie gets shifted in the direction of electric field and center of surrounding electron cloud get shifted to oppoiste of magnetic field. and hence induced a dipole moment.

Induced dipole moment:

$$\vec{p} = \alpha \vec{E} \tag{3.1}$$

where α is called atomic polarizibility.

For molecules, induced dipole moment is diffent when we apply electric field in different directions of its orientation.

In general, Let say induced dipole moment in the molecule is p_x, if electric field is applied $\vec{E}$ in any random direction, than

$$p_x = \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \tag{3.2}$$

and so on... or Equivalently,

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xy} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yy} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{3.3}$$

Matric α_{ij} is known as polarizibility tensor.

Its always possible to choose the orientation (principle axes) of the molecule such that all off diagonal terms vanishes.

Definition 3.1. Macroscopically, In dielectric material, We define Polarization as the Induced dipole moment per unit volume.

$$\vec{P} = N\alpha\vec{E} \tag{3.4}$$

where N is Number of dipoles per unit Volume.

3.2 Polarization of Dielectrics

When a dielectric is placed in an external electric field, polarization occurs due to the slight displacement of bound charges. Each molecule acquires an induced electric dipole moment, and the collective effect is described by the polarization vector.

3.2.1 Electric Potential Due to an Induced Electric Dipole

Consider an induced electric dipole moment $\mathbf{p}$ located at the origin. The electric potential at a point $\mathbf{r}$ due to this dipole is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}. \tag{3.5}$$

3.2.2 Rewriting the Potential Using Vector Identities

Using the identity

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}, \tag{3.6}$$

the potential may be written as

$$V(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\int_V \mathbf{P}(\mathbf{r}')\cdot\nabla\left(\frac{1}{r}\right)\, d\tau'. \tag{3.7}$$

Applying the vector identity

$$\nabla\cdot\left(\frac{\mathbf{P}}{r}\right) = \frac{\nabla\cdot\mathbf{P}}{r} + \mathbf{P}\cdot\nabla\left(\frac{1}{r}\right), \tag{3.8}$$

we obtain

$$\mathbf{P}\cdot\nabla\left(\frac{1}{r}\right) = \nabla\cdot\left(\frac{\mathbf{P}}{r}\right) - \frac{\nabla\cdot\mathbf{P}}{r}. \tag{3.9}$$

Substituting into the expression for the potential and applying the divergence theorem, we find

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_V \frac{-\nabla\cdot\mathbf{P}}{r}\, d\tau + \oint_S \frac{\mathbf{P}\cdot\hat{\mathbf{n}}}{r}\, da\right]. \tag{3.10}$$

3.2.3 Bound Volume and Surface Charge Densities

Comparing with the general expression for the potential due to charge distributions, we identify:

$$\rho_b = -\nabla\cdot\mathbf{P} \quad \text{(bound volume charge density)}, \tag{3.11}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \quad \text{(bound surface charge density)}. \tag{3.12}$$

Hence, the electric potential due to polarization can be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_V \frac{\rho_b}{r}\, d\tau + \oint_S \frac{\sigma_b}{r}\, da\right]. \tag{3.13}$$

3.3 Gauss's Law in Matter

When electric fields exist inside material media, the presence of bound charges due to polarization requires a careful reformulation of Gauss's law.

3.3.1 Free and Bound Charges

In a dielectric medium, electric charges are classified into two types:

- **Free charges**: Charges that can move freely under the influence of an electric field (e.g. conduction electrons or externally placed charges). Their volume charge density is denoted by ρ_f.
- **Bound charges**: Charges that arise due to polarization of the material. These charges are tied to atoms or molecules and cannot move freely.

The bound charges are characterized by the polarization vector $\mathbf{P}$:

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P}, \tag{3.14}$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}. \tag{3.15}$$

The *total* charge density in matter is therefore

$$\rho = \rho_f + \rho_b. \tag{3.16}$$

3.3.2 Gauss's Law in Vacuum

In electrostatics, Gauss's law for the electric field $\mathbf{E}$ is

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{3.17}$$

This law involves the *total* charge density ρ, irrespective of whether the charges are free or bound. Substituting $\rho = \rho_f + \rho_b$,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\varepsilon_0}(\rho_f + \rho_b). \tag{3.18}$$

Using $\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P}$, we obtain

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \boldsymbol{\nabla} \cdot \mathbf{P}\right). \tag{3.19}$$

3.3.3 Derivation of Gauss's Law in Matter

Rearranging the above equation,

$$\boldsymbol{\nabla} \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f. \tag{3.20}$$

This motivates the definition of a new vector field, called the *electric displacement field.*

3.3.4 Electric Displacement Field

Definition 3.2 (Electric Displacement Field)**.** The electric displacement field $\mathbf{D}$ is defined as

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{3.21}$$

In terms of $\mathbf{D}$, Gauss's law in matter takes the simple form

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f.} \tag{3.22}$$

In integral form,

$$\boxed{\oint \mathbf{D} \cdot d\mathbf{a} = Q_{\text{free, enc}}.} \tag{3.23}$$

Thus, the flux of $\mathbf{D}$ through a closed surface depends *only* on the free charge enclosed. All effects of bound charges are absorbed into the definition of $\mathbf{D}$.

- The electric field $\mathbf{E}$ responds to both free and bound charges.
- The displacement field $\mathbf{D}$ responds only to free charges.
- The complexity of polarization is encoded in $\mathbf{P}$, allowing $\mathbf{D}$ to simplify electrostatic boundary-value problems in matter.

3.3.5 Example: Point Charge in a Linear Dielectric

Consider a point free charge q placed at the origin of an infinite, homogeneous, linear dielectric.

By spherical symmetry,

$$\mathbf{D} = D(r)\,\hat{\mathbf{r}}. \tag{3.24}$$

Applying Gauss's law for $\mathbf{D}$ over a spherical surface of radius r,

$$\oint \mathbf{D} \cdot d\mathbf{a} = D(r)(4\pi r^2) = q. \tag{3.25}$$

Hence,

$$\boxed{\mathbf{D}(r) = \frac{q}{4\pi r^2}\,\hat{\mathbf{r}}.} \tag{3.26}$$

If the dielectric is linear and isotropic, $\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$, so

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \varepsilon = \varepsilon_0(1 + \chi_e). \tag{3.27}$$

Therefore, the electric field is

$$\boxed{\mathbf{E}(r) = \frac{q}{4\pi\varepsilon r^2}\,\hat{\mathbf{r}}.} \tag{3.28}$$

This result shows that the effect of the dielectric is to reduce the electric field by a factor of $\varepsilon/\varepsilon_0$ compared to vacuum.

3.4 Electric Field at the Center of a Uniformly Polarized Sphere

Consider a sphere of radius R that is uniformly polarized with a constant polarization vector

$$\mathbf{P} = P\,\hat{\mathbf{z}}. \tag{3.29}$$

Our goal is to calculate explicitly the electric field at the center of the sphere due to the bound charges produced by this polarization.

Bound Charge Distribution

In a polarized dielectric, the bound volume and surface charge densities are

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P}, \tag{3.30}$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}. \tag{3.31}$$

Since the polarization is uniform,

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = 0. \tag{3.32}$$

However, bound surface charge exists on the spherical surface:

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P\cos\theta, \tag{3.33}$$

where θ is the polar angle measured from the direction of polarization.

Electric Field Due to a Surface Charge Element

Consider a surface element dA located at (R, θ, ϕ) on the sphere. The bound charge on this element is

$$dq = \sigma_b\,dA = P\cos\theta\,dA. \tag{3.34}$$

The distance from the surface element to the center of the sphere is R. The electric field at the center due to dq is

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}(-\hat{\mathbf{r}}), \tag{3.35}$$

where the minus sign indicates that the field points from the charge element toward the center.

Substituting for dq,

$$d\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{P\cos\theta}{R^2}\hat{\mathbf{r}}\,dA. \tag{3.36}$$

Symmetry Considerations

By spherical symmetry, all transverse components of the electric field cancel upon integration. Only the component along the polarization direction survives.

Using

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \cos\theta, \tag{3.37}$$

the z-component of the field is

$$dE_z = -\frac{1}{4\pi\varepsilon_0} \frac{P\cos^2\theta}{R^2}\, dA. \tag{3.38}$$

Surface Integration

The surface element on a sphere is

$$dA = R^2 \sin\theta \, d\theta \, d\phi. \tag{3.39}$$

Substituting,

$$dE_z = -\frac{P}{4\pi\varepsilon_0} \cos^2\theta \sin\theta \, d\theta \, d\phi. \tag{3.40}$$

Integrating over the entire surface,

$$E_z = -\frac{P}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta. \tag{3.41}$$

The angular integrals are

$$\int_0^{2\pi} d\phi = 2\pi, \tag{3.42}$$

and with the substitution $u = \cos\theta$,

$$\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = \int_{-1}^{1} u^2 \, du = \frac{2}{3}. \tag{3.43}$$

Final Result

Combining the results,

$$E_z = -\frac{P}{4\pi\varepsilon_0}(2\pi)\left(\frac{2}{3}\right) = -\frac{P}{3\varepsilon_0}. \tag{3.44}$$

In vector form, the electric field at the center of the uniformly polarized sphere is

$$\boxed{\mathbf{E}_{\text{center}} = -\frac{\mathbf{P}}{3\varepsilon_0}.} \tag{3.45}$$

Remarks

- The electric field inside the uniformly polarized sphere is uniform.
- The field arises entirely from bound surface charges.
- The field is directed opposite to the polarization vector.
- This result forms the basis of the Lorentz local field and the Clausius–Mossotti relation.

3.5 Clausius–Mossotti Relation

The macroscopic electric properties of a dielectric arise from the microscopic response of its constituent atoms or molecules to an applied electric field. The Clausius–Mossotti relation establishes a connection between the microscopic polarizability of individual molecules and the macroscopic dielectric constant of the material.

3.5.1 Polarization and Molecular Polarizability

Consider a dielectric composed of identical, non-interacting molecules. When an external electric field $\mathbf{E}$ is applied, each molecule acquires an induced electric dipole moment proportional to the *local electric field* $\mathbf{E}_{\text{loc}}$ acting on it:

$$\mathbf{p} = \alpha \mathbf{E}_{\text{loc}}, \tag{3.46}$$

where α is the molecular polarizability.

If the number density of molecules is N, the macroscopic polarization is

$$\mathbf{P} = N\mathbf{p} = N\alpha \mathbf{E}_{\text{loc}}. \tag{3.47}$$

Thus, the key problem reduces to determining the local electric field experienced by a molecule inside the dielectric.

3.5.2 Local Electric Field: Lorentz Cavity Method

The macroscopic electric field $\mathbf{E}$ appearing in Maxwell's equations is an average field and does not represent the actual field acting on an individual molecule. To compute the local field, we imagine a small spherical cavity (called the *Lorentz cavity*) centered on a given molecule.

The local electric field at the center of the cavity is the sum of three contributions:

$$\mathbf{E}_{\text{loc}} = \mathbf{E} + \mathbf{E}_{\text{near}} + \mathbf{E}_{\text{cav}}, \tag{3.48}$$

where

- $\mathbf{E}$ is the macroscopic field,
- $\mathbf{E}_{\text{near}}$ is the field due to dipoles inside the cavity,
- $\mathbf{E}_{\text{cav}}$ is the field due to polarization charges on the cavity surface.

For cubic or isotropic media, the contribution from nearby dipoles averages to zero:

$$\mathbf{E}_{\text{near}} = 0. \tag{3.49}$$

3.5.3 Field Due to the Polarized Cavity

The polarization $\mathbf{P}$ of the dielectric induces bound surface charge on the spherical cavity. The surface charge density is

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}, \tag{3.50}$$

where $\hat{\mathbf{n}}$ is the outward normal to the cavity surface.

The electric field at the center of a uniformly polarized sphere due to this surface charge is a standard result:

$$\mathbf{E}_{\text{cav}} = \frac{1}{3\varepsilon_0}\mathbf{P}. \tag{3.51}$$

Therefore, the local electric field becomes

$$\boxed{\mathbf{E}_{\text{loc}} = \mathbf{E} + \frac{1}{3\varepsilon_0}\mathbf{P}.} \tag{3.52}$$

3.5.4 Relation Between Polarization and Macroscopic Field

Substituting the expression for $\mathbf{E}_{\text{loc}}$ into the polarization equation,

$$\mathbf{P} = N\alpha\left(\mathbf{E} + \frac{1}{3\varepsilon_0}\mathbf{P}\right). \tag{3.53}$$

Rearranging,

$$\left(1 - \frac{N\alpha}{3\varepsilon_0}\right)\mathbf{P} = N\alpha\mathbf{E}. \tag{3.54}$$

Solving for $\mathbf{P}$,

$$\mathbf{P} = \frac{N\alpha}{1 - \dfrac{N\alpha}{3\varepsilon_0}}\mathbf{E}. \tag{3.55}$$

3.5.5 Derivation of the Clausius–Mossotti Relation

From macroscopic electromagnetism, polarization is related to the electric field by

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}, \tag{3.56}$$

where χ_e is the electric susceptibility.

Equating the two expressions for $\mathbf{P}$,

$$\varepsilon_0\chi_e = \frac{N\alpha}{1 - \dfrac{N\alpha}{3\varepsilon_0}}. \tag{3.57}$$

Rewriting,

$$\frac{\chi_e}{1 + \dfrac{\chi_e}{3}} = \frac{N\alpha}{3\varepsilon_0}. \tag{3.58}$$

Using $\varepsilon_r = 1 + \chi_e$, we finally obtain the *Clausius–Mossotti relation*:

$$\boxed{\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N\alpha}{3\varepsilon_0}.} \tag{3.59}$$

3.5.6 Physical Interpretation and Validity

The Clausius–Mossotti relation shows that the macroscopic dielectric constant is determined by the microscopic polarizability and number density of molecules.

The relation is valid under the assumptions:

- The dielectric is isotropic and homogeneous,
- Molecular interactions are weak,
- Polarization is linear and induced,
- The local field is well described by the Lorentz cavity model.

It fails in strongly polar materials, dense media, and near resonance frequencies.

3.6 Langevin Theory of Polarization

The Langevin theory explains the polarization of polar dielectrics by considering the statistical orientation of permanent electric dipoles in an external electric field.

3.6.1 Polar Dielectrics and Permanent Dipoles

In polar dielectrics, molecules possess a *permanent electric dipole moment* $\mathbf{p}$ even in the absence of an external electric field. In thermal equilibrium and without an applied field, the dipoles are randomly oriented, resulting in zero net polarization.

When an external electric field $\mathbf{E}$ is applied, it tends to align the dipoles, producing a net polarization. This alignment, however, is opposed by thermal agitation.

3.6.2 Potential Energy of a Dipole in an Electric Field

For a dipole moment $\mathbf{p}$ placed in a uniform electric field $\mathbf{E}$, the potential energy is

$$\boxed{U = -\mathbf{p} \cdot \mathbf{E} = -pE\cos\theta,} \tag{3.60}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{E}$.

3.6.3 Statistical Distribution of Dipole Orientations

At temperature T, the probability that a dipole has orientation between θ and $\theta + d\theta$ is given by the Boltzmann factor:

$$dP \propto e^{-U/k_B T}\, d\Omega, \tag{3.61}$$

where

$$d\Omega = 2\pi \sin\theta \, d\theta$$

is the element of solid angle.

Thus,

$$dP = C\, e^{(pE\cos\theta)/(k_B T)}\, 2\pi \sin\theta \, d\theta, \tag{3.62}$$

where C is a normalization constant.

3.6.4 Average Dipole Moment

The component of the dipole moment along the field direction is $p\cos\theta$. The thermal average is

$$\langle p_z \rangle = \frac{\int_0^\pi p\cos\theta\, e^{(pE\cos\theta)/(k_B T)}\, 2\pi \sin\theta\, d\theta}{\int_0^\pi e^{(pE\cos\theta)/(k_B T)}\, 2\pi \sin\theta\, d\theta}. \tag{3.63}$$

Define the dimensionless parameter

$$\boxed{a = \frac{pE}{k_B T}.} \tag{3.64}$$

Then

$$\langle p_z \rangle = p \frac{\int_0^\pi \cos\theta\, e^{a\cos\theta} \sin\theta\, d\theta}{\int_0^\pi e^{a\cos\theta} \sin\theta\, d\theta}. \tag{3.65}$$

3.6.5 Evaluation of the Integrals

Make the substitution

$$x = \cos\theta, \qquad dx = -\sin\theta\, d\theta.$$

The denominator becomes

$$\int_{-1}^{1} e^{ax}\, dx = \frac{2\sinh a}{a}. \tag{3.66}$$

The numerator becomes

$$\int_{-1}^{1} x e^{ax}\, dx = \frac{2}{a}\left(\cosh a - \frac{\sinh a}{a}\right). \tag{3.67}$$

Hence,

$$\langle p_z \rangle = p\left(\coth a - \frac{1}{a}\right). \tag{3.68}$$

3.6.6 Langevin Function

Define the **Langevin function**

$$\boxed{L(a) = \coth a - \frac{1}{a}.} \tag{3.69}$$

Thus, the average dipole moment per molecule is

$$\boxed{\langle p_z \rangle = p\, L(a).} \tag{3.70}$$

3.6.7 Polarization of the Medium

If N is the number of molecules per unit volume, the polarization is

$$\boxed{\mathbf{P} = Np\,L(a)\,\hat{\mathbf{E}}.} \tag{3.71}$$

This is the **Langevin formula** for polarization.

3.6.8 Weak Field Limit

For weak fields ($a \ll 1$), expand the Langevin function:

$$L(a) \approx \frac{a}{3}. \tag{3.72}$$

Then

$$\boxed{\mathbf{P} = \frac{Np^2}{3k_BT}\mathbf{E}.} \tag{3.73}$$

Comparing with $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$, the electric susceptibility is

$$\boxed{\chi_e = \frac{Np^2}{3\epsilon_0 k_BT}.} \tag{3.74}$$

3.6.9 Remarks

- Polarization increases with electric field strength.
- Polarization decreases with temperature due to thermal disorder.
- The Langevin theory applies to polar gases and dilute polar dielectrics.

3.6.10 Langevin–Debye Formula

The Langevin theory accounts for the polarization arising from the *orientation of permanent dipoles*. In real dielectrics, however, molecules also acquire an *induced dipole moment* in the presence of an electric field. The total polarization is therefore the sum of orientational and induced contributions.

Induced Polarization

For a non-conducting molecule placed in an electric field $\mathbf{E}$, the induced dipole moment is

$$\mathbf{p}_{\text{ind}} = \alpha\mathbf{E}, \tag{3.75}$$

where α is the molecular polarizability.

If N is the number of molecules per unit volume, the induced polarization is

$$\mathbf{P}_{\text{ind}} = N\alpha\mathbf{E}. \tag{3.76}$$

Orientational Polarization

From Langevin theory, the orientational polarization of permanent dipoles is

$$\mathbf{P}_{\text{or}} = NpL(a)\,\hat{\mathbf{E}}, \qquad a = \frac{pE}{k_B T}. \tag{3.77}$$

In the weak-field limit ($a \ll 1$),

$$\mathbf{P}_{\text{or}} = \frac{Np^2}{3k_B T}\mathbf{E}. \tag{3.78}$$

Total Polarization

The total polarization is

$$\mathbf{P} = \mathbf{P}_{\text{ind}} + \mathbf{P}_{\text{or}} = N\alpha\mathbf{E} + \frac{Np^2}{3k_B T}\mathbf{E}. \tag{3.79}$$

Thus,

$$\boxed{\mathbf{P} = N\left(\alpha + \frac{p^2}{3k_B T}\right)\mathbf{E}.} \tag{3.80}$$

Langevin–Debye Formula for Susceptibility

Using the macroscopic relation

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \tag{3.81}$$

the electric susceptibility becomes

$$\boxed{\chi_e = \frac{N}{\epsilon_0}\left(\alpha + \frac{p^2}{3k_B T}\right).} \tag{3.82}$$

This expression is known as the **Langevin–Debye formula**.

Physical Interpretation

- The first term, α, represents **induced (electronic and ionic) polarization** and is independent of temperature.
- The second term represents **orientational polarization** and varies as $1/T$.
- At high temperatures, orientational polarization becomes weak.
- At low temperatures, orientational polarization dominates.

Experimental Significance

The Langevin–Debye formula explains the observed temperature dependence of the dielectric constant in polar gases and liquids. By measuring χ_e as a function of temperature, one can determine both the molecular polarizability α and the permanent dipole moment p.

3.7 Statistical Theory: The Langevin-Debye Formula

For polar molecules with a permanent dipole moment $\mathbf{p}_0$, polarization involves the alignment of dipoles against thermal agitation.[1, 10, 20] The potential energy is $U = -\mathbf{p}_0 \cdot \mathbf{E} = -p_0 E \cos\theta$. Following Boltzmann statistics, the average dipole moment is [5, 9, 15]:

$$\langle p_z \rangle = p_0 \left(\coth u - \frac{1}{u} \right) = p_0 L(u) \tag{3.83}$$

where $u = p_0 E / k_B T$ and $L(u)$ is the **Langevin function**.[15, 20, 21] In the high-temperature limit ($u \ll 1$), $L(u) \approx u/3$, yielding the orientational polarizability $\alpha_{\text{orient}} = p_0^2 / 3k_B T$.[20, 21, 22] The total polarizability α results in the **Langevin-Debye formula** [10, 20, 15]:

$$\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{N}{3\epsilon_0} \left(\alpha_{\text{induced}} + \frac{p_0^2}{3k_B T} \right) \tag{3.84}$$

3.8 Energy Stored in the Electric Field

In electrostatics, work must be done to assemble a system of charges against electric forces. This work is stored in the configuration of the system and is referred to as *electrostatic potential energy*. In field theory, this energy is understood as being stored in the electric field itself.

3.8.1 Work Done in Assembling a Charge Distribution

Consider bringing charges from infinity to form a given charge distribution. The work done in bringing an infinitesimal charge dq to a point with electric potential V is

$$dW = V\, dq. \tag{3.85}$$

For a continuous volume charge distribution with charge density $\rho(\mathbf{r})$, the total electrostatic energy is

$$W = \frac{1}{2} \int \rho(\mathbf{r}) V(\mathbf{r})\, dV. \tag{3.86}$$

The factor $\frac{1}{2}$ appears to avoid double counting the interaction energy between pairs of charges.

3.8.2 Energy in Terms of Electric Field

Using Gauss's law in differential form,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{3.87}$$

we may write

$$W = \frac{\varepsilon_0}{2} \int V (\boldsymbol{\nabla} \cdot \mathbf{E})\, dV. \tag{3.88}$$

Applying the vector identity

$$\boldsymbol{\nabla} \cdot (V\mathbf{E}) = V(\boldsymbol{\nabla} \cdot \mathbf{E}) + \mathbf{E} \cdot \boldsymbol{\nabla} V, \tag{3.89}$$

and using $\mathbf{E} = -\boldsymbol{\nabla} V$, we obtain

$$V(\boldsymbol{\nabla} \cdot \mathbf{E}) = \boldsymbol{\nabla} \cdot (V\mathbf{E}) + E^2. \tag{3.90}$$

Substituting into the expression for energy,

$$W = \frac{\varepsilon_0}{2} \int \left[\boldsymbol{\nabla} \cdot (V\mathbf{E}) + E^2 \right] dV. \tag{3.91}$$

Using the divergence theorem,

$$\int \boldsymbol{\nabla} \cdot (V\mathbf{E})\, dV = \oint V\mathbf{E} \cdot d\mathbf{a}. \tag{3.92}$$

If the charge distribution is localized, both V and $\mathbf{E}$ vanish sufficiently rapidly at infinity, so the surface integral is zero. Hence,

$$W = \frac{\varepsilon_0}{2} \int E^2\, dV. \tag{3.93}$$

3.8.3 Electric Field Energy Density

The integrand in the above expression represents the energy stored per unit volume in the electric field. Thus, the *energy density* of the electric field is

$$\boxed{u_E = \frac{1}{2}\varepsilon_0 E^2.} \tag{3.94}$$

This result shows that energy is not localized on charges alone but is distributed throughout space wherever an electric field exists.

The expression for electric field energy density implies that:

- Energy exists in empty space wherever an electric field is present.
- Work is required to establish an electric field configuration.
- The field itself acts as the carrier of energy, independent of the presence of charges.

Important Result

The electric field stores energy with density $u_E = \frac{1}{2}\varepsilon_0 E^2$, establishing the field—not the charges—as the fundamental repository of electrostatic energy.

3.9 Capacitance

When electric charges are placed on conductors, work must be done against electrostatic forces. The ability of a conductor or a system of conductors to store electric charge by storing electrostatic energy is quantified by a physical quantity called *capacitance*.

3.9.1 Concept of Capacitance

Consider an isolated conductor carrying a charge Q. The electric potential V of the conductor increases as charge is added to it. For a given geometry and surrounding medium, the potential is found to be directly proportional to the charge:

$$V \propto Q. \tag{3.95}$$

This linear relationship holds as long as the medium is linear and the geometry remains unchanged.

The constant of proportionality between charge and potential is called the *capacitance*.

Definition 3.3 (Capacitance)**.** The capacitance C of a conductor is defined as the ratio of the charge stored on it to the electric potential it acquires:

$$\boxed{C = \frac{Q}{V}.} \tag{3.96}$$

The SI unit of capacitance is the farad (F), where

$$1\ \mathrm{F} = 1\ \mathrm{C\ V^{-1}}.$$

Remark 3.1. Capacitance depends only on the geometry of the conductors and the permittivity of the medium, and not on the charge or potential individually.

3.9.2 Capacitors

In practice, capacitance is realized using a system of two conductors separated by an insulating medium (dielectric). Such a device is called a *capacitor*. If the conductors carry charges $+Q$ and $-Q$ and have a potential difference V, then the capacitance of the capacitor is

$$C = \frac{Q}{V}. \tag{3.97}$$

3.9.3 Parallel Plate Capacitor

Consider a parallel plate capacitor consisting of two large conducting plates of area A, separated by a distance d, with vacuum between them.

Electric Field

The surface charge density on the plates is

$$\sigma = \frac{Q}{A}. \tag{3.98}$$

The electric field between the plates is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}. \tag{3.99}$$

Potential Difference

The potential difference between the plates is

$$V = Ed = \frac{Qd}{\varepsilon_0 A}. \tag{3.100}$$

Capacitance

Thus, the capacitance is

$$\boxed{C = \frac{\varepsilon_0 A}{d}.} \tag{3.101}$$

If a dielectric of permittivity ε fills the space between the plates,

$$C = \frac{\varepsilon A}{d} = \kappa \frac{\varepsilon_0 A}{d}, \tag{3.102}$$

where $\kappa = \varepsilon/\varepsilon_0$ is the dielectric constant.

3.9.4 Spherical Capacitor

Consider two concentric conducting spherical shells of radii a (inner shell) and b (outer shell), carrying charges $+Q$ and $-Q$ respectively.

Electric Field

Using Gauss's law, the electric field in the region $a < r < b$ is

$$E(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}. \tag{3.103}$$

Potential Difference

The potential difference between the shells is

$$V = \int_a^b E(r)\, dr = \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right). \tag{3.104}$$

Capacitance

Therefore,

$$\boxed{C = 4\pi\varepsilon_0 \frac{ab}{b-a}.} \tag{3.105}$$

3.9.5 Cylindrical Capacitor

Consider a cylindrical capacitor formed by two long coaxial conducting cylinders of radii a and b ($b > a$) and length L, with $L \gg b$.

Electric Field

Using Gauss's law, the electric field in the region $a < r < b$ is

$$E(r) = \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{r}. \tag{3.106}$$

Potential Difference

The potential difference between the cylinders is

$$V = \int_a^b E(r)\,dr = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right). \tag{3.107}$$

Capacitance

Hence, the capacitance of the cylindrical capacitor is

$$\boxed{C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}.} \tag{3.108}$$

3.9.6 Factors Affecting Capacitance

From the above examples, we observe that capacitance depends on:

- Geometry and size of the conductors,
- Separation between conductors,
- Permittivity of the medium between them.

Important Result

Capacitance is a purely geometrical property of a conductor system and measures its ability to store electric charge by storing energy in the electric field.

3.9.7 Energy stored in Parallel Plate Capacitor

Consider a parallel plate capacitor of plate area A and separation d, carrying charge $\pm Q$.

The electric field between the plates is

$$E = \frac{Q}{\varepsilon_0 A}. \tag{3.109}$$

The energy stored in the capacitor is

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon_0 A}{d}(Ed)^2. \tag{3.110}$$

This simplifies to

$$U = \frac{\varepsilon_0}{2} E^2 (Ad), \tag{3.111}$$

which is precisely the field energy density multiplied by the volume between the plates.

4 Magnetostatics

Magnetostatics deals with magnetic fields produced by steady (time-independent) electric currents. In this regime, charge densities and currents do not vary with time, and electromagnetic radiation effects are absent. The theory of magnetostatics forms a crucial pillar of classical electrodynamics and provides the foundation for understanding magnetic materials, inductors, transformers, and electric motors.

4.1 Ohm's Law

4.1.1 Microscopic Origin of Electric Current

Consider a conductor containing mobile charge carriers of charge q and number density n. When an electric field $\mathbf{E}$ is applied, each charge experiences a force

$$\mathbf{F} = q\mathbf{E}. \tag{4.1}$$

Due to frequent collisions with lattice ions, the charges do not accelerate indefinitely. Instead, they acquire a small average velocity called the *drift velocity* $\mathbf{v}_d$, directed opposite to $\mathbf{E}$ for electrons.

In steady state, the drift velocity is proportional to the applied electric field:

$$\mathbf{v}_d = \mu\mathbf{E}, \tag{4.2}$$

where μ is the mobility of the charge carriers.

4.1.2 Current Density

The electric current density $\mathbf{J}$ is defined as the charge flow per unit area per unit time. For charge carriers moving with drift velocity $\mathbf{v}_d$,

$$\mathbf{J} = nq\mathbf{v}_d. \tag{4.3}$$

Substituting for $\mathbf{v}_d$, we obtain

$$\mathbf{J} = nq\mu\mathbf{E}. \tag{4.4}$$

4.1.3 Local (Differential) Form of Ohm's Law

Defining the electrical conductivity σ as

$$\sigma = nq\mu, \tag{4.5}$$

we arrive at the local form of Ohm's law:

$$\boxed{\mathbf{J} = \sigma\mathbf{E}.} \tag{4.6}$$

Remark 4.1. Ohm's law in this form is a *constitutive relation*. It describes how a particular material responds to an applied electric field and is not a fundamental law of nature.

4.1.4 Macroscopic Form of Ohm's Law

Consider a uniform conductor of length L and cross-sectional area A. The current density is related to the total current I by

$$J = \frac{I}{A}. \tag{4.7}$$

The electric field inside the conductor is related to the potential difference V by

$$E = \frac{V}{L}. \tag{4.8}$$

Using the local form $\mathbf{J} = \sigma\mathbf{E}$, we obtain

$$\frac{I}{A} = \sigma\frac{V}{L}. \tag{4.9}$$

Rearranging,

$$\boxed{V = IR,} \tag{4.10}$$

where the resistance R of the conductor is

$$R = \frac{L}{\sigma A}. \tag{4.11}$$

This is the familiar macroscopic form of Ohm's law.

Important Result

Ohm's law is a material-dependent relation that connects the electric field to current density. It complements Maxwell's equations by specifying how currents arise in conducting media.

4.2 Lorentz Force and Magnetic Field

Definition 4.1 (Lorentz Force)**.** The force experienced by a charge q moving with velocity $\mathbf{v}$ in the presence of electric and magnetic fields is given by

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{4.12}$$

In magnetostatics, electric fields may be absent or irrelevant, and the magnetic force is

$$\mathbf{F}_m = q\,\mathbf{v} \times \mathbf{B}. \tag{4.13}$$

Remark 4.2. The magnetic force is always perpendicular to the velocity of the charge. Hence, it does no work and can only change the direction of motion, not the speed.

For a current-carrying conductor, the force on an element $d\ell$ carrying current I is

$$d\mathbf{F} = I\,d\ell \times \mathbf{B}. \tag{4.14}$$

4.3 Biot–Savart Law

The magnetic field produced by a steady current distribution is given by the Biot–Savart law.

Law 4.1 (Biot–Savart Law)**.** *The magnetic field at a point* $\mathbf{r}$ *due to a current element* $I\,d\ell$ *located at* $\mathbf{r}'$ *is*

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{I\,d\ell \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{4.15}$$

For a continuous current distribution,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\,d^3r'. \tag{4.16}$$

4.4 Applications of the Biot–Savart Law

4.4.1 Magnetic Field Due to an Infinitely Long Straight Wire

Consider a straight wire carrying a steady current I. According to the Biot–Savart law,

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\ell \times \mathbf{r}}{r^3}, \tag{4.17}$$

where $\mathbf{r}$ is the position vector from the current element to the observation point.

By symmetry, the magnetic field is tangential to a circle centered on the wire. Integration along the length of the wire yields

$$\boxed{\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\phi}}.} \tag{4.18}$$

This result agrees with that obtained using Ampère's circuital law.

4.4.2 Magnetic Field at the Center of a Circular Current Loop

Consider a circular loop of radius R carrying a steady current I.

Symmetry Considerations

At the center of the loop:

- All current elements are equidistant from the center.
- The magnetic field contributions perpendicular to the axis cancel.
- Only the axial components add constructively.

Calculation

For each current element,

$$dB = \frac{\mu_0}{4\pi} \frac{I\, d\ell \sin\theta}{R^2}, \tag{4.19}$$

where $\theta = 90°$.

Integrating over the entire loop,

$$B = \frac{\mu_0 I}{4\pi R^2} \oint d\ell = \frac{\mu_0 I}{4\pi R^2}(2\pi R). \tag{4.20}$$

Thus,

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{2R}\,\hat{\mathbf{z}}.} \tag{4.21}$$

For a coil of N turns,

$$\mathbf{B} = \frac{\mu_0 N I}{2R}\,\hat{\mathbf{z}}. \tag{4.22}$$

4.4.3 Magnetic Field on the Axis of a Circular Current Loop

Consider a circular loop of radius R carrying current I. We calculate the magnetic field at a point on the axis at a distance x from the center.

The axial component of the field due to an element is

$$dB_z = \frac{\mu_0}{4\pi} \frac{I\, d\ell \sin\theta}{r^2} \cos\theta, \tag{4.23}$$

where

$$r = \sqrt{R^2 + x^2}, \qquad \cos\theta = \frac{x}{r}.$$

Integrating over the loop,

$$\boxed{\mathbf{B}(x) = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}}\,\hat{\mathbf{z}}.} \tag{4.24}$$

For N turns,

$$\mathbf{B}(x) = \frac{\mu_0 N I R^2}{2(R^2 + x^2)^{3/2}}\,\hat{\mathbf{z}}. \tag{4.25}$$

4.4.4 Magnetic Field on the Axis of a Long Solenoid

A solenoid can be regarded as a superposition of closely spaced circular current loops.

Using the result for the field on the axis of a loop and integrating along the length of the solenoid, the magnetic field at a point on its axis is

$$\mathbf{B} = \frac{\mu_0 n I}{2}(\cos\theta_1 - \cos\theta_2)\,\hat{\mathbf{z}}, \tag{4.26}$$

where θ_1 and θ_2 are the angles subtended by the ends of the solenoid at the observation point.

For an infinitely long solenoid,

$$\boxed{\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}}.} \tag{4.27}$$

4.4.5 Magnetic Field Due to a Finite Straight Current Segment

Consider a straight conductor carrying current I. The magnetic field at a point at perpendicular distance r from the wire is

$$\boxed{B = \frac{\mu_0 I}{4\pi r}(\sin\theta_1 + \sin\theta_2),} \tag{4.28}$$

where θ_1 and θ_2 are the angles subtended by the ends of the wire at the observation point.

For an infinitely long wire, $\theta_1 = \theta_2 = 90^\circ$, recovering the standard result.

4.4.6 Helmholtz Coils

Two identical circular coils of radius R, each carrying current I and separated by a distance R, form a Helmholtz coil arrangement.

The magnetic field at the midpoint between the coils is

$$\boxed{B = \left(\frac{4}{5}\right)^{3/2}\frac{\mu_0 N I}{R}.} \tag{4.29}$$

This configuration produces a highly uniform magnetic field near the center.

Important Result

The Biot–Savart law is the fundamental law of magnetostatics. Ampère's law is a powerful consequence of it, applicable primarily in highly symmetric situations.

4.5 Ampère's Circuital Law

It plays a role in magnetostatics analogous to Gauss's law in electrostatics and is especially powerful in problems possessing a high degree of symmetry.

Law 4.2 (Ampère's Circuital Law)**.** *The line integral of the magnetic field around any closed path is proportional to the total current enclosed by that path:*

$$\oint \mathbf{B}\cdot d\ell = \mu_0 I_{enc}. \tag{4.30}$$

Here,

- $d\ell$ is an infinitesimal line element along the closed path,
- I_{enc} is the net steady current passing through any surface bounded by the path.

Remark 4.3. Only currents that pierce the surface bounded by the loop contribute to I_{enc}. Currents outside the loop do not affect the circulation integral.

4.5.1 Integral Form of Ampère's Law

Law 4.3 (Ampère's Circuital Law)**.** *The line integral of the magnetic field* **B** *around any closed path* C *is equal to* μ_0 *times the total steady current enclosed by the path:*

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 I_{enc}. \tag{4.31}$$

The enclosed current I_{enc} may be expressed in terms of the current density **J** as a surface integral:

$$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{S}, \tag{4.32}$$

where S is any open surface bounded by the closed curve C.

Therefore, Ampère's law in complete integral form may be written as

$$\boxed{\oint_C \mathbf{B} \cdot d\ell = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}.} \tag{4.33}$$

Remark 4.4. The result is independent of the choice of surface S, provided the current distribution is steady.

4.5.2 Differential Form of Ampère's Law

Using Stokes' theorem, the line integral of the magnetic field can be converted into a surface integral:

$$\oint_C \mathbf{B} \cdot d\ell = \int_S (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{S}. \tag{4.34}$$

Substituting into the integral form of Ampère's law, we obtain

$$\int_S (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}. \tag{4.35}$$

Since this equality holds for any arbitrary surface S, the integrands must be equal at every point in space. Hence, the differential form of Ampère's law is

$$\boxed{\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}.} \tag{4.36}$$

Important Result

The differential form of Ampère's law shows that magnetic fields are generated by electric currents and that the curl of **B** is nonzero only in regions where current density exists.

Important Result

The curl of the magnetic field is nonzero only in regions where current density exists.

4.6 Applications of Ampère's Circuital Law

Ampère's circuital law provides a powerful method for calculating magnetic fields produced by steady currents when the current distribution possesses a high degree of symmetry. In such cases, an appropriate closed path, called an *Amperian loop*, may be chosen so that the magnetic field is either constant in magnitude or has a fixed direction along the loop.

In this section, we apply Ampère's law to several physically important and highly symmetric current configurations.

4.6.1 Infinitely Long Straight Current-Carrying Wire

Consider an infinitely long straight wire carrying a steady current I.

Symmetry and Choice of Amperian Loop

The system exhibits cylindrical symmetry about the axis of the wire. Consequently,

- magnetic field lines form concentric circles centered on the wire,
- the magnitude of $\mathbf{B}$ depends only on the radial distance r from the wire,
- the direction of $\mathbf{B}$ is tangential to the circular field lines.

A circular Amperian loop of radius r, concentric with the wire, is therefore an appropriate choice.

Application of Ampère's Law

Along the chosen loop, $\mathbf{B}$ is everywhere parallel to $d\ell$ and has constant magnitude. Hence,

$$\oint \mathbf{B} \cdot d\ell = B \oint d\ell = B(2\pi r). \tag{4.37}$$

The current enclosed by the loop is $I_{\text{enc}} = I$. Ampère's law then gives

$$B(2\pi r) = \mu_0 I. \tag{4.38}$$

Solving for the magnetic field,

$$\boxed{\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r}\,\hat{\phi}.} \tag{4.39}$$

4.6.2 Long Cylindrical Conductor with Uniform Current Density

Consider a long cylindrical conductor of radius R carrying a total current I uniformly distributed over its cross-section.

Magnetic Field Outside the Conductor ($r \geq R$)

For an Amperian loop of radius $r \geq R$, the entire current I is enclosed. Hence,

$$\boxed{\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\phi}}, \qquad r \geq R.} \tag{4.40}$$

Magnetic Field Inside the Conductor ($r < R$)

The current density is uniform and given by

$$J = \frac{I}{\pi R^2}. \tag{4.41}$$

The current enclosed by an Amperian loop of radius $r < R$ is

$$I_{\text{enc}} = J\pi r^2 = I\frac{r^2}{R^2}. \tag{4.42}$$

Applying Ampère's law,

$$B(2\pi r) = \mu_0 I \frac{r^2}{R^2}, \tag{4.43}$$

which yields

$$\boxed{\mathbf{B}(r) = \frac{\mu_0 I}{2\pi R^2} r\,\hat{\boldsymbol{\phi}}, \qquad r < R.} \tag{4.44}$$

Thus, the magnetic field increases linearly with distance from the axis inside the conductor.

4.6.3 Infinite Plane Current Sheet

Consider an infinite plane carrying a uniform surface current density $\mathbf{K}$.

Symmetry Considerations

By translational symmetry in the plane and reflection symmetry about the sheet:

- the magnetic field has the same magnitude at all points equidistant from the sheet,
- the field is parallel to the plane of the sheet,
- the field directions on opposite sides of the sheet are opposite.

Application of Ampère's Law

Choose a rectangular Amperian loop that straddles the current sheet, with two sides of length ℓ parallel to the magnetic field.

The line integral becomes

$$\oint \mathbf{B} \cdot d\ell = 2B\ell. \tag{4.45}$$

The enclosed current is

$$I_{\text{enc}} = K\ell. \tag{4.46}$$

Ampère's law then gives

$$2B\ell = \mu_0 K\ell, \tag{4.47}$$

and hence

$$\boxed{B = \frac{\mu_0 K}{2}.} \tag{4.48}$$

4.6.4 Long Solenoid

Consider a long solenoid with n turns per unit length carrying a steady current I.

Symmetry Considerations

For a sufficiently long solenoid:

- the magnetic field inside is uniform and parallel to the axis,
- the magnetic field outside is negligibly small.

Application of Ampère's Law

Choose a rectangular Amperian loop with one side of length ℓ inside the solenoid and the opposite side outside.

Since the field outside is negligible, the line integral reduces to

$$\oint \mathbf{B} \cdot d\ell = B\ell. \tag{4.49}$$

The enclosed current is

$$I_{\text{enc}} = nI\ell. \tag{4.50}$$

Ampère's law then yields

$$\boxed{\mathbf{B} = \mu_0 nI\,\hat{\mathbf{z}}.} \tag{4.51}$$

4.6.5 Toroidal Solenoid

Consider a toroid consisting of N closely wound turns carrying a steady current I.

Symmetry Considerations

The toroidal geometry implies that:

- magnetic field lines are circular and lie entirely within the core,
- the magnetic field depends only on the radial distance r from the axis.

Application of Ampère's Law

Choose a circular Amperian loop of radius r within the core of the toroid. Along this loop, $\mathbf{B}$ is constant in magnitude and parallel to $d\ell$.

$$\oint \mathbf{B} \cdot d\ell = B(2\pi r). \tag{4.52}$$

The enclosed current is

$$I_{\text{enc}} = NI. \tag{4.53}$$

Therefore,

$$\boxed{\mathbf{B}(r) = \frac{\mu_0 NI}{2\pi r}\,\hat{\boldsymbol{\phi}}.} \tag{4.54}$$

For points outside the toroidal core, $I_{\text{enc}} = 0$, and hence $\mathbf{B} = 0$.

4.6.6 Coaxial Cable

A coaxial cable consists of an inner conductor carrying current I and a concentric outer conductor carrying current $-I$.

Applying Ampère's law, the magnetic field is found to be

$$\mathbf{B}(r) = \begin{cases} \dfrac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\phi}}, & \text{between the conductors,} \\ 0, & \text{outside the cable.} \end{cases} \tag{4.55}$$

Important Result

Ampère's circuital law is most effective in situations of high symmetry, where the magnetic field can be taken outside the line integral. In problems lacking such symmetry, the Biot–Savart law must be used instead.

4.7 Gauss's Law of Magnetostatics

One of the fundamental laws governing magnetic fields is *Gauss's law of magnetostatics*. Unlike electric fields, which originate from electric charges, magnetic fields exhibit a profoundly different behavior: there are no isolated magnetic charges (magnetic monopoles) in nature.

4.7.1 Statement of the Law

Gauss's law for magnetism states that the total magnetic flux through any closed surface is zero:

$$\oint_S \mathbf{B} \cdot d\mathbf{a} = 0. \tag{4.56}$$

Here, $d\mathbf{a}$ is the outward-directed area element of the closed surface S, and $\mathbf{B}$ is the magnetic field.

4.7.2 Physical Interpretation

The vanishing of magnetic flux through a closed surface implies that magnetic field lines have no beginning or end. Instead, they always form continuous closed loops. For every magnetic field line entering a closed surface, an equal amount of magnetic field exits the surface.

This is in sharp contrast to electrostatics, where Gauss's law relates the electric flux through a closed surface to the enclosed electric charge. The absence of magnetic monopoles means there is no magnetic analog of electric charge that could act as a source or sink of magnetic field lines.

4.7.3 Differential Form

Using the divergence theorem,

$$\oint_S \mathbf{B} \cdot d\mathbf{a} = \int_V (\boldsymbol{\nabla} \cdot \mathbf{B})\, dV, \tag{4.57}$$

where V is the volume enclosed by the surface S.

Since the surface integral vanishes for *any* closed surface, the integrand itself must be zero everywhere. Hence, Gauss's law of magnetostatics in differential form is

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{B} = 0.} \tag{4.58}$$

4.7.4 Consequences

- The condition $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ expresses the nonexistence of magnetic monopoles.
- Magnetic field lines are continuous and closed, never starting or ending within space.
- This property ensures the consistency of Ampère's law and the existence of a magnetic vector potential $\mathbf{A}$ such that
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{4.59}$$

Important Result

Gauss's law of magnetostatics encapsulates a basic empirical fact of nature: magnetic monopoles do not exist. As a result, the net magnetic flux through any closed surface is always zero.

4.8 Magnetic Vector Potential

Gauss's law of magnetostatics,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{4.60}$$

implies a fundamental mathematical property of the magnetic field: it is *solenoidal*. A solenoidal vector field can always be expressed as the curl of another vector field. This observation motivates the introduction of the *magnetic vector potential*.

4.8.1 Definition of Vector Potential

The magnetic vector potential $\mathbf{A}$ is defined such that

$$\boxed{\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}.} \tag{4.61}$$

This definition automatically satisfies Gauss's law of magnetostatics, since the divergence of a curl is identically zero:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0. \tag{4.62}$$

Thus, the vector potential provides a natural and consistent description of magnetic fields in magnetostatics.

4.8.2 Vector Potential Due to a Steady Current

For steady currents, the vector potential at a point $\mathbf{r}$ due to a current density $\mathbf{J}(\mathbf{r}')$ is given by

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'.} \tag{4.63}$$

For a thin wire carrying current I, this reduces to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\ell'}{|\mathbf{r} - \mathbf{r}'|}. \tag{4.64}$$

Taking the curl of $\mathbf{A}$ reproduces the Biot–Savart law for the magnetic field.

4.8.3 Gauge Freedom

The vector potential is not uniquely defined. If $\mathbf{A}$ is a valid vector potential, then

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi \tag{4.65}$$

also produces the same magnetic field for any scalar function $\chi(\mathbf{r})$, since

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\chi) = 0. \tag{4.66}$$

This freedom in choosing $\mathbf{A}$ is known as *gauge freedom*.

4.8.4 Choice of Gauge

In magnetostatics, a particularly convenient choice is the *Coulomb gauge*, defined by

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{4.67}$$

Under this gauge condition, the vector potential satisfies the Poisson equation

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.} \tag{4.68}$$

This equation closely parallels the Poisson equation for the scalar potential in electrostatics.

Important Result

The magnetic vector potential exists because $\nabla \cdot \mathbf{B} = 0$. Although not uniquely defined, it provides a powerful and indispensable framework for describing magnetic fields and their sources.

5 Magnetization

5.1 Bound Currents

When matter is magnetized, these dipole moments acquire a net alignment, giving rise to macroscopic magnetic effects. The effective currents produced by this alignment are called *bound currents*.

5.1.1 Magnetic Dipole as a Current Loop

A magnetic dipole moment $\mathbf{m}$ may be represented by a small planar current loop carrying current I and enclosing area A, such that

$$\mathbf{m} = IA\,\hat{\mathbf{n}}, \tag{5.1}$$

where $\hat{\mathbf{n}}$ is the unit normal to the plane of the loop.

The magnetic field produced by such a dipole is entirely due to circulating electric currents. Therefore, magnetization in matter can be understood as arising from a distribution of microscopic current loops.

5.1.2 Magnetization as a Continuous Distribution of Dipoles

For a magnetized material, the magnetization vector is defined as

$$\mathbf{M}(\mathbf{r}) = \frac{d\mathbf{m}}{dV}, \tag{5.2}$$

representing the magnetic dipole moment per unit volume.

Consider a small volume element dV of magnetized material. The total magnetic dipole moment contained in this volume is

$$d\mathbf{m} = \mathbf{M}\,dV. \tag{5.3}$$

Each dipole corresponds to a tiny current loop. When many such loops fill the material, their currents combine to produce macroscopic current densities.

5.1.3 Origin of Bound Volume Current Density

To determine the effective current density arising from magnetization, consider a magnetized medium with spatially varying $\mathbf{M}(\mathbf{r})$.

Within the bulk of the material, adjacent microscopic current loops partially cancel each other. However, if $\mathbf{M}$ varies from point to point, this cancellation is incomplete, leading to a net current flow inside the material.

A careful summation of the microscopic loop currents over a small volume leads to the bound volume current density

$$\boxed{\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M}.} \tag{5.4}$$

Remark 5.1. The bound volume current exists only in regions where magnetization is non-uniform. For a uniformly magnetized medium, $\boldsymbol{\nabla} \times \mathbf{M} = 0$, and hence $\mathbf{J}_b = 0$ in the bulk.

5.1.4 Origin of Bound Surface Current Density

At the surface of a magnetized material, the microscopic current loops are no longer fully canceled, because the loops at the boundary are truncated.

As a result, there is a net surface current flowing along the boundary of the material. This current is called the *bound surface current*.

To calculate this current, consider a surface element with outward unit normal $\hat{\mathbf{n}}$. The effective surface current density is found to be

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}.} \tag{5.5}$$

This current flows tangentially along the surface and is perpendicular to both the magnetization vector and the surface normal.

5.1.5 Physical Picture of Bound Currents

The bound currents may be summarized as follows:

- **Volume bound current** $\mathbf{J}_b$ arises due to spatial variation of magnetization within the material.
- **Surface bound current** $\mathbf{K}_b$ arises due to the termination of microscopic current loops at the boundary.

Together, these currents account completely for the magnetic field produced by magnetized matter.

5.1.6 Comparison with Polarization in Electrostatics

There is a close analogy between magnetization in magnetostatics and polarization in electrostatics:

Electrostatics	**Magnetostatics**
Electric dipole moment $\mathbf{p}$	Magnetic dipole moment $\mathbf{m}$
Polarization $\mathbf{P}$	Magnetization $\mathbf{M}$
Bound charge density $\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P}$	Bound current density $\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M}$
Bound surface charge $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$	Bound surface current $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$

Important Result

Magnetization produces magnetic fields not by creating magnetic charges, but by generating effective bound currents. These bound currents, together with free currents, fully determine the magnetic field in matter.

Important Result

Magnetization does not introduce magnetic charges; instead, it produces effective bound currents.

5.2 Ampère's Law in a Medium

In vacuum, Ampère's law relates the magnetic field $\mathbf{B}$ to the total current $\mathbf{J}$ flowing through a surface via

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{\text{total}}, \tag{5.6}$$

or equivalently in integral form,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{total, enc}}. \tag{5.7}$$

5.2.1 Currents in Matter: Free and Bound Currents

In materials, the total current density $\mathbf{J}_{\text{total}}$ is naturally split into:

- **Free currents:** $\mathbf{J}_f$, which are externally driven (e.g., by wires, batteries).
- **Bound currents:** $\mathbf{J}_b$, which arise from the microscopic motion of charges inside atoms and molecules due to magnetization $\mathbf{M}$ of the medium.

  The bound current density can be expressed as

$$\mathbf{J}_b = \nabla \times \mathbf{M}. \tag{5.8}$$

  Thus, the total current is

$$\mathbf{J}_{\text{total}} = \mathbf{J}_f + \mathbf{J}_b. \tag{5.9}$$

5.2.2 Ampère's Law in Terms of Magnetization

Substituting $\mathbf{J}_{\text{total}} = \mathbf{J}_f + \nabla \times \mathbf{M}$ into Ampère's law gives

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M}). \tag{5.10}$$

Rearranging, we can write

$$\nabla \times (\mathbf{B} - \mu_0 \mathbf{M}) = \mu_0 \mathbf{J}_f. \tag{5.11}$$

5.2.3 Introduction of the Magnetic Field H

This motivates the definition of the **magnetic field intensity H**:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}. \tag{5.12}$$

In terms of **H**, Ampère's law takes the simpler form

$$\nabla \times \mathbf{H} = \mathbf{J}_f, \tag{5.13}$$

or in integral form,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{f, enc}}. \tag{5.14}$$

This is analogous to Gauss's law in a medium, where the displacement field $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ allows the field equation to be expressed in terms of *free charges* only:

$$\nabla \cdot \mathbf{D} = \rho_f. \tag{5.15}$$

Similarly, Ampère's law in a medium allows the magnetic field **H** to be determined solely by *free currents*, while the magnetization effects are absorbed into the definition of **H**.

6 Electromagnetic Induction

6.1 Magnetic Flux

The fundamental quantity underlying electromagnetic induction is the *magnetic flux*.

Definition 6.1 (Magnetic Flux)**.** The magnetic flux Φ_B through a surface S is defined as

$$\boxed{\Phi_B = \int_S \mathbf{B} \cdot d\mathbf{a},} \tag{6.1}$$

where $d\mathbf{a}$ is the vector area element normal to the surface.

For a uniform magnetic field $\mathbf{B}$ passing through a flat surface of area A making an angle θ with the normal,

$$\Phi_B = BA\cos\theta. \tag{6.2}$$

Physical Interpretation

Magnetic flux measures the total number of magnetic field lines passing through a surface. A change in flux may occur due to:

- Variation of the magnetic field strength,
- Change in the area of the loop,
- Change in orientation of the loop with respect to the field.

6.2 Faraday's Law of Electromagnetic Induction

Faraday discovered that an electromotive force (emf) is induced in a closed circuit whenever the magnetic flux linked with the circuit changes with time.

6.2.1 Integral Form of Faraday's Law

The induced emf $\mathcal{E}$ in a closed loop is given by

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}.} \tag{6.3}$$

Since emf is the line integral of the electric field,

$$\mathcal{E} = \oint \mathbf{E} \cdot d\boldsymbol{\ell}, \tag{6.4}$$

Faraday's law may be written as

$$\boxed{\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a}.} \tag{6.5}$$

Lenz's Law

The negative sign in Faraday's law expresses *Lenz's law*: *The induced emf always acts so as to oppose the change in magnetic flux that produces it.*

This is a direct consequence of energy conservation.

6.3 Differential Form of Faraday's Law

Applying Stokes' theorem to the integral form,

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = \int_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot d\mathbf{a}, \tag{6.6}$$

we obtain

$$\int_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot d\mathbf{a} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}. \tag{6.7}$$

Since the surface is arbitrary, the integrands must be equal:

$$\boxed{\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.} \tag{6.8}$$

Important Result

Unlike electrostatics, the electric field produced by a time-varying magnetic field is *non-conservative*. Hence, a scalar electric potential cannot be defined globally.

6.4 Motional emf

When a conductor moves in a magnetic field, an emf is induced even if the magnetic field is time-independent.

Consider a straight conductor of length ℓ moving with velocity $\mathbf{v}$ perpendicular to a uniform magnetic field $\mathbf{B}$.

The magnetic force on charge carriers is

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}), \tag{6.9}$$

leading to charge separation and an induced emf

$$\boxed{\mathcal{E} = B\ell v.} \tag{6.10}$$

This motional emf is consistent with Faraday's law since the magnetic flux through the circuit changes due to motion.

6.5 Self-Induction

A current-carrying circuit produces a magnetic field that links with the circuit itself. Any change in current therefore induces an emf in the same circuit. This phenomenon is called **self-induction**.

6.5.1 Self-Inductance

The magnetic flux Φ_B linked with a circuit is proportional to the current I:

$$\Phi_B = LI, \tag{6.11}$$

where L is the *self-inductance* of the circuit, measured in henrys (H).

The induced emf due to a changing current is given by Faraday's law:

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}.} \tag{6.12}$$

6.5.2 Self-Inductance of a Circular Loop

Consider a single circular loop of radius R, carrying current I.

Magnetic Field at the Center of the Loop

The magnetic field at the center of a single loop is

$$B = \frac{\mu_0 I}{2R}. \tag{6.13}$$

Magnetic Flux Through the Loop

The area of the loop is $A = \pi R^2$, so the flux through one turn is

$$\Phi_B = B \cdot A = \frac{\mu_0 I}{2R} \cdot \pi R^2 = \frac{\mu_0 \pi R I}{2}. \tag{6.14}$$

Self-Inductance

By definition, $\Phi_B = LI$, so

$$\boxed{L = \frac{\mu_0 \pi R}{2}}. \tag{6.15}$$

Remark: - For a coil of N turns, flux through all turns adds up:

$$\Phi_B = N \cdot \frac{\mu_0 \pi R I}{2} \quad \Rightarrow \quad L = \frac{\mu_0 N^2 \pi R}{2}. \tag{6.16}$$

6.5.3 Self-Inductance of a Solenoid

Consider a long solenoid of: - N turns - Length l - Cross-sectional area A - Carrying current I

Magnetic Field Inside the Solenoid

For a long solenoid, the magnetic field is uniform:

$$B = \mu_0 \frac{N}{l} I. \tag{6.17}$$

Magnetic Flux Through Each Turn

The flux through one turn is

$$\Phi_{\text{one turn}} = B \cdot A = \mu_0 \frac{N}{l} I \cdot A. \tag{6.18}$$

Total Flux Through All Turns

The solenoid has N turns, so total flux linking the solenoid is

$$\Phi_B = N\Phi_{\text{one turn}} = \mu_0 \frac{N^2 A}{l} I. \tag{6.19}$$

Self-Inductance

By definition, $\Phi_B = LI$, so the self-inductance of the solenoid is

$$\boxed{L = \mu_0 \frac{N^2 A}{l}}. \tag{6.20}$$

Remark: - The self-inductance increases with the square of the number of turns N. - Larger cross-sectional area A or longer solenoids with tightly wound turns also increase L.

6.5.4 Energy Stored in a Self-Inductor

A current I in an inductor stores magnetic energy in the field:

$$U = \frac{1}{2} L I^2. \tag{6.21}$$

This energy is released when the current decreases, producing a back-emf that opposes the change in current, consistent with Lenz's law.

6.6 Mutual Induction

Mutual induction occurs when a changing current in one circuit induces an emf in a nearby second circuit. This is the basis of transformers and many electrical devices.

6.6.1 Mutual Inductance

Consider two circuits, labeled 1 and 2. If a current I_1 in circuit 1 produces a magnetic flux Φ_{21} linking circuit 2, then the flux is proportional to I_1:

$$\Phi_{21} = M I_1, \tag{6.22}$$

where M is the *mutual inductance* between the two circuits, measured in henrys (H).

The emf induced in circuit 2 due to a changing current in circuit 1 is

$$\boxed{\mathcal{E}_2 = -\frac{d\Phi_{21}}{dt} = -M\frac{dI_1}{dt}.} \tag{6.23}$$

Similarly, a changing current in circuit 2 induces an emf in circuit 1:

$$\mathcal{E}_1 = -M\frac{dI_2}{dt}. \tag{6.24}$$

6.6.2 Mutual Inductance of Two Circular Coils

Consider two coaxial circular coils: - Coil 1: N_1 turns, radius R_1, carrying current I_1 - Coil 2: N_2 turns, radius R_2, placed coaxially at distance d from coil 1

Magnetic Field of Coil 1 at the Location of Coil 2

The magnetic field along the axis of a circular loop of radius R_1 carrying current I_1 at a distance x from its center is

$$B(x) = \frac{\mu_0 I_1 R_1^2}{2(R_1^2 + x^2)^{3/2}}. \tag{6.25}$$

For N_1 turns:

$$B_1(x) = N_1\frac{\mu_0 I_1 R_1^2}{2(R_1^2 + x^2)^{3/2}}. \tag{6.26}$$

Flux Through Coil 2

The flux through one turn of coil 2 is

$$\Phi_{\text{one turn}} = B_1(d) \cdot \pi R_2^2, \tag{6.27}$$

so total flux through N_2 turns of coil 2 is

$$\Phi_{21} = N_2 B_1(d) \cdot \pi R_2^2 = N_2\frac{\mu_0 N_1 I_1 R_1^2}{2(R_1^2 + d^2)^{3/2}}\pi R_2^2. \tag{6.28}$$

Mutual Inductance

By definition, $\Phi_{21} = MI_1$, so

$$\boxed{M = \frac{\mu_0 \pi N_1 N_2 R_1^2 R_2^2}{2(R_1^2 + d^2)^{3/2}}}. \tag{6.29}$$

This is the mutual inductance of two coaxial circular coils.

6.6.3 Mutual Inductance of Two Solenoids

Consider: - Solenoid 1: N_1 turns, length l_1, cross-sectional area A_1, carrying current I_1 - Solenoid 2: N_2 turns, length l_2, cross-sectional area A_2, coaxial and overlapping completely

Magnetic Field of Solenoid 1

Inside a long solenoid, the magnetic field is uniform:

$$B_1 = \mu_0 \frac{N_1}{l_1} I_1. \tag{6.30}$$

Flux Through Solenoid 2

Assuming solenoid 2 is completely inside solenoid 1, the flux through each turn of solenoid 2 is

$$\Phi_{\text{one turn}} = B_1 A_2 = \mu_0 \frac{N_1 I_1}{l_1} A_2. \tag{6.31}$$

Total flux through N_2 turns:

$$\Phi_{21} = N_2 \Phi_{\text{one turn}} = \mu_0 \frac{N_1 N_2 A_2}{l_1} I_1. \tag{6.32}$$

Mutual Inductance

By definition, $\Phi_{21} = M I_1$, so

$$\boxed{M = \mu_0 \frac{N_1 N_2 A_2}{l_1}}. \tag{6.33}$$

Remark: - If the solenoids are not fully overlapping, only the overlapping length contributes to flux. - For coaxial solenoids of equal cross-section A, M simplifies to $\mu_0 \frac{N_1 N_2 A}{l_1}$.

6.6.4 Energy in Mutually Coupled Inductors

For two inductors with currents I_1 and I_2, the total magnetic energy stored is

$$U = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2, \tag{6.34}$$

where L_1 and L_2 are the self-inductances of the two circuits. The last term represents the energy due to mutual coupling.

7 Maxwell's Equations

Electrodynamics deals with time-dependent electric and magnetic fields and their mutual coupling. The central result of this theory is a unified set of equations—*Maxwell's equations*—which describe the behavior of electromagnetic fields and predict the existence of electromagnetic waves.

7.1 Continuity Equation

The **continuity equation** expresses the principle of **conservation of charge** (or mass, in fluid mechanics) in a differential form. It states that the rate at which charge density decreases in a volume is equal to the net current flowing out of that volume.

7.1.1 Derivation of the Continuity Equation

Consider a fixed volume V in space bounded by a closed surface S. Let $\rho(\mathbf{r}, t)$ be the charge density, and $\mathbf{J}(\mathbf{r}, t)$ be the current density at position $\mathbf{r}$ and time t.

The total charge inside the volume is

$$Q(t) = \int_V \rho(\mathbf{r}, t)\, d\tau, \tag{7.1}$$

where $d\tau$ is the volume element.

The rate of change of charge inside the volume is

$$\frac{dQ}{dt} = \int_V \frac{\partial \rho}{\partial t}\, d\tau. \tag{7.2}$$

According to the principle of conservation of charge, the rate of decrease of charge inside the volume is equal to the net outward current through the surface S:

$$\text{Net outward current} = \oint_S \mathbf{J} \cdot d\mathbf{a}, \tag{7.3}$$

where $d\mathbf{a}$ is the outward normal surface element.

Thus,

$$\int_V \frac{\partial \rho}{\partial t}\, d\tau = -\oint_S \mathbf{J} \cdot d\mathbf{a}. \tag{7.4}$$

7.1.2 Differential Form

Using the **divergence theorem**, which states

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{J})\, d\tau, \tag{7.5}$$

we can write

$$\int_V \frac{\partial \rho}{\partial t}\, d\tau = -\int_V (\nabla \cdot \mathbf{J})\, d\tau. \tag{7.6}$$

Since this must hold for any arbitrary volume V, the integrands are equal everywhere in space. Therefore, the **continuity equation in differential form** is

$$\boxed{\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0}. \tag{7.7}$$

This is the general proof of the continuity equation.

1. **Conservation of Charge:** The continuity equation ensures that charge can neither be created nor destroyed; it only flows from one region to another.

2. **Link Between Current and Charge Density:** It connects the local current density $\mathbf{J}$ to the change in charge density ρ, allowing us to analyze time-dependent charge distributions.

7.1.3 Steady and Unsteady Currents

- **Steady Current:** $\partial\rho/\partial t = 0$, so $\nabla \cdot \mathbf{J} = 0$. The current density is divergence-free; charge is locally constant.

- **Time-Varying Current:** $\partial\rho/\partial t \neq 0$, so $\nabla \cdot \mathbf{J} \neq 0$. Charge density changes in time according to the net flow of current.

7.2 Ampère's Law for Non-Steady Currents

7.2.1 Ampère's Law for Steady Currents

In magnetostatics (steady currents), Ampère's law relates the curl of the magnetic field $\mathbf{B}$ to the current density $\mathbf{J}$:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}} \quad \text{or in integral form} \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}. \tag{7.8}$$

Here, I_{enc} is the total current passing through a surface bounded by the loop.

7.2.2 Inconsistency with the Continuity Equation

Consider the continuity equation for charge conservation:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{7.9}$$

Taking the divergence of Ampère's law gives

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} \quad \Rightarrow \quad 0 = \mu_0 \nabla \cdot \mathbf{J}. \tag{7.10}$$

This implies

$$\nabla \cdot \mathbf{J} = 0, \tag{7.11}$$

which is only true if the current is steady ($\partial \rho / \partial t = 0$).

Conclusion: The original Ampère's law is valid only for **steady currents**. For time-varying currents ($\partial \rho / \partial t \neq 0$), it violates charge conservation.

7.2.3 Introducing an Additional Source: Displacement Current

To generalize Ampère's law, we assume that in addition to the conduction current $\mathbf{J}$, there exists another source of magnetic field $\mathbf{J}_d$, called the **displacement current**, such that

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \mathbf{J}_d). \tag{7.12}$$

7.2.4 Determining the Displacement Current $\mathbf{J}_d$

Take divergence on both sides:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot (\mathbf{J} + \mathbf{J}_d) \quad \Rightarrow \quad 0 = \mu_0 \nabla \cdot (\mathbf{J} + \mathbf{J}_d). \tag{7.13}$$

Use continuity equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{7.14}$$

Thus,

$$\nabla \cdot \mathbf{J}_d = \frac{\partial \rho}{\partial t}. \tag{7.15}$$

Use Gauss's law: From Gauss's law,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = \epsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}). \tag{7.16}$$

Identify displacement current:

$$\nabla \cdot \mathbf{J}_d = \epsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) \quad \Rightarrow \quad \mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{7.17}$$

7.2.5 Displacement Current in Linear Dielectric Media

If the medium is a linear dielectric with permittivity ϵ, the electric displacement field is

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}. \tag{7.18}$$

Then, the displacement current in the medium becomes

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}.. \tag{7.19}$$

7.2.6 Modified Ampère-Maxwell Law

Finally, the generalized Ampère's law, valid for time-varying currents and in media, is

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}}. \tag{7.20}$$

Significance: - Resolves the inconsistency with the continuity equation. - Explains how changing electric fields generate magnetic fields even in the absence of conduction currents. - Fundamental to the propagation of electromagnetic waves.

7.2.7 Integral Form of Ampère-Maxwell Law

To obtain the integral form, we use Stokes' theorem, which relates the line integral of a vector field around a closed loop to the surface integral of its curl over the surface bounded by the loop:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{a}. \tag{7.21}$$

Substituting the differential form into the surface integral:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_S \mu_0 \mathbf{J} \cdot d\mathbf{a} + \int_S \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a}. \tag{7.22}$$

Conduction current contribution: $\int_S \mathbf{J} \cdot d\mathbf{a} = I_{\text{enc}}$, the total conduction current passing through the surface.

Displacement current contribution: $\int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} = \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{a}$, which represents the rate of change of electric flux through the surface.

Integral form:

$$\boxed{\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{a}}. \tag{7.23}$$

This is the integral form of the generalized Ampère-Maxwell law, valid for time-varying fields.

7.2.8 Ampere-Maxwell Equation in Media

In a linear, isotropic, homogeneous medium, the electric and magnetic fields are modified by the material properties:

- Electric displacement field: $\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$
- Magnetic field intensity: $\mathbf{H} = \frac{\mathbf{B}}{\mu} - \mathbf{M}$

Differential form in media:

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \tag{7.24}$$

where $\mathbf{J}_f$ is the free (conduction) current.

Integral form in media:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{f, enc}} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{a}. \tag{7.25}$$

Remarks:

1. The displacement current in a medium is given by $\frac{\partial \mathbf{D}}{\partial t}$, which includes the polarization of the material.

2. The law now explicitly separates *free currents* from *bound currents* inside the medium.

3. This generalization is essential for electromagnetic wave propagation in materials and ensures consistency with charge conservation.

7.3 Maxwell's Equations

Maxwell's equations form a complete and self-consistent set of fundamental laws governing electric and magnetic fields. They unify electrostatics, magnetostatics, and electromagnetic induction, and predict the existence of electromagnetic waves.

7.3.1 Maxwell's Equations in Free Space

In free space (vacuum), electric and magnetic fields are described directly in terms of **E** and **B**. The four Maxwell's equations in differential form are:

1. **Gauss's Law for Electricity**

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \tag{7.26}$$

   This relates the electric field to the charge density ρ.

2. **Gauss's Law for Magnetism**

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{7.27}$$

   This expresses the non-existence of magnetic monopoles.

3. **Faraday's Law of Electromagnetic Induction**

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{7.28}$$

   A time-varying magnetic field produces an electric field.

4. **Ampère-Maxwell Law**

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}} \tag{7.29}$$

   This shows that magnetic fields are produced by both conduction currents and time-varying electric fields.

Together, these equations fully describe classical electromagnetism in vacuum.

7.3.2 Maxwell's Equations in Matter

In material media, charges and currents are classified as *free* and *bound*. To separate material response from external sources, the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$ are introduced:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \tag{7.30}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}. \tag{7.31}$$

Maxwell's equations in matter take the form:

1. **Gauss's Law in Matter**

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f} \tag{7.32}$$

where ρ_f is the free charge density.

2. **Gauss's Law for Magnetism**

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{7.33}$$

3. **Faraday's Law**

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{7.34}$$

4. **Ampère-Maxwell Law in Matter**

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}} \tag{7.35}$$

where $\mathbf{J}_f$ is the free current density.

These equations are particularly useful in linear, isotropic media where $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$.

7.3.3 Boundary Conditions at Interfaces

Maxwell's equations impose constraints on the behavior of fields at the boundary between two media.

Electric Field Boundary Conditions

- **Normal component of D:**

$$\boxed{(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{n} = \sigma_f} \tag{7.36}$$

where σ_f is the free surface charge density.

- **Tangential component of E:**

$$\boxed{(\mathbf{E}_2 - \mathbf{E}_1) \times \hat{n} = 0} \tag{7.37}$$

Magnetic Field Boundary Conditions

- **Normal component of B:**

$$\boxed{(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{n} = 0} \tag{7.38}$$

- **Tangential component of H:**

$$\boxed{(\mathbf{H}_2 - \mathbf{H}_1) \times \hat{n} = \mathbf{K}_f} \tag{7.39}$$

  where $\mathbf{K}_f$ is the free surface current density.

Significance: These boundary conditions are essential for solving electromagnetic problems involving interfaces, such as capacitors, waveguides, dielectric boundaries, and magnetic materials.

7.4 Tensor Formulation of Maxwell's Equations

7.4.1 Four-Vector and Field Tensor Notation

To write Maxwell's equations in relativistically invariant form, we introduce four-vectors.

Four-Position and Four-Current

The space-time four-vector is

$$x^\mu = (ct,\, x,\, y,\, z), \tag{7.40}$$

and the four-current is

$$\boxed{J^\mu = (c\rho,\, \mathbf{J})} \tag{7.41}$$

The space-time derivative operator is

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t},\, \nabla\right). \tag{7.42}$$

The continuity equation becomes

$$\boxed{\partial_\mu J^\mu = 0,} \tag{7.43}$$

which is manifestly Lorentz invariant.

Electromagnetic Field Tensor

The electric and magnetic fields are combined into the antisymmetric **electromagnetic field tensor**

$$\boxed{F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}} \tag{7.44}$$

This tensor fully contains the six independent components of $\mathbf{E}$ and $\mathbf{B}$.

7.4.2 Maxwell's Equations in Matrix (Tensor) Form

Using the field tensor $F^{\mu\nu}$ and the four-current J^μ, Maxwell's equations reduce to two compact tensor equations.

Inhomogeneous Maxwell Equations

The Gauss law for electricity and the Ampère-Maxwell law combine into

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu} \tag{7.45}$$

Expanding this equation reproduces:

- $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
- $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$

Homogeneous Maxwell Equations

The Gauss law for magnetism and Faraday's law combine into

$$\boxed{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0} \tag{7.46}$$

This identity follows from the antisymmetry of $F_{\mu\nu}$ and expands to:

- $\nabla \cdot \mathbf{B} = 0$
- $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

7.4.3 Interpretation

The tensor equations

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu, \tag{7.47}$$

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0, \tag{7.48}$$

are **nothing but Maxwell's equations written in compact matrix form**.

They demonstrate that:

- Electric and magnetic fields are components of a single geometric object.
- Maxwell's equations are Lorentz invariant.
- Charge conservation follows naturally from the tensor formulation.

This formulation provides the natural bridge between classical electrodynamics and relativistic field theory.

7.5 Poynting Theorem

Poynting's theorem expresses the **local conservation of energy** in electromagnetic systems. It relates the flow of electromagnetic energy to the work done on charges and the energy stored in electric and magnetic fields.

7.5.1 Energy Density of Electromagnetic Fields

The energy stored per unit volume in the electric and magnetic fields is given by:

- **Electric field energy density**

$$u_E = \frac{1}{2}\epsilon_0 E^2 \tag{7.49}$$

- **Magnetic field energy density**

$$u_B = \frac{1}{2\mu_0} B^2 \tag{7.50}$$

The total electromagnetic energy density is therefore

$$\boxed{u = u_E + u_B = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2} \tag{7.51}$$

7.5.2 Mechanical Work Done on Charges

A charge q moving with velocity $\mathbf{v}$ in electromagnetic fields experiences the **Lorentz force**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{7.52}$$

The rate of mechanical work done on the charge is

$$\frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}. \tag{7.53}$$

Since

$$(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0,$$

only the electric field does work. Thus,

$$\frac{dW}{dt} = q\mathbf{E} \cdot \mathbf{v}. \tag{7.54}$$

For a continuous charge distribution with current density $\mathbf{J} = \rho\mathbf{v}$, the mechanical power per unit volume delivered to matter is

$$\boxed{p_{\text{mech}} = \mathbf{J} \cdot \mathbf{E}.} \tag{7.55}$$

7.5.3 Derivation of Poynting's Theorem

We now derive the energy conservation equation using Maxwell's equations.

Start with Ampère-Maxwell law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{7.56}$$

Take the dot product with $\mathbf{E}$:

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mu_0 \mathbf{E} \cdot \mathbf{J} + \mu_0 \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}. \tag{7.57}$$

Use Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{7.58}$$

Take the dot product with $\mathbf{B}$:

$$\mathbf{B} \cdot (\nabla \times \mathbf{E}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{7.59}$$

Subtract the two equations
Using the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}), \tag{7.60}$$

we obtain

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = -\mu_0 \mathbf{E} \cdot \mathbf{J} - \mu_0 \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{7.61}$$

Rearrangement
Rewriting the time derivative terms,

$$\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2} \frac{\partial E^2}{\partial t}, \tag{7.62}$$

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2} \frac{\partial B^2}{\partial t}. \tag{7.63}$$

Thus,

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right) + \nabla \cdot \left(\frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right) + \mathbf{J} \cdot \mathbf{E} = 0.} \tag{7.64}$$

This is the **Poynting theorem**.

7.5.4 Poynting Vector and Physical Interpretation

The vector

$$\boxed{\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}} \tag{7.65}$$

is called the **Poynting vector**. It represents:

- Direction of electromagnetic energy flow
- Energy flux (power per unit area)

The theorem can be written compactly as

$$\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E} = 0,} \tag{7.66}$$

where u is the electromagnetic energy density.

7.5.5 Analogy with the Continuity Equation

Poynting's theorem has the same structure as the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{7.67}$$

Charge Conservation	**Energy Conservation**
ρ	u
$\mathbf{J}$	$\mathbf{S}$

Thus, electromagnetic energy behaves like a conserved "fluid" flowing through space.

Conclusion: Poynting's theorem shows that energy is stored in electromagnetic fields, transported through space by the Poynting vector, and transferred to matter via the work term $\mathbf{J} \cdot \mathbf{E}$.

8 Electromagnetic Waves

8.1 Wave Equation for Electromagnetic Fields in Vacuum

In free space (vacuum),

$$\rho = 0, \qquad \mathbf{J} = 0.$$

Maxwell's equations reduce to

$$\nabla \cdot \mathbf{E} = 0, \tag{8.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{8.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{8.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{8.4}$$

8.1.1 Wave Equation for the Electric Field

Take the curl of Faraday's law:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}). \tag{8.5}$$

Using the vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E},$$

and $\nabla \cdot \mathbf{E} = 0$, we get

$$-\nabla^2 \mathbf{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{8.6}$$

Hence,

$$\boxed{\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.} \tag{8.7}$$

8.1.2 Wave Equation for the Magnetic Field

Similarly, taking the curl of Ampère–Maxwell law,

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}), \tag{8.8}$$

and using $\nabla \cdot \mathbf{B} = 0$, we obtain

$$\boxed{\nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0.} \tag{8.9}$$

8.1.3 Speed of Electromagnetic Waves

Comparing with the standard wave equation,

$$\nabla^2 \mathbf{F} - \frac{1}{c^2} \frac{\partial^2 \mathbf{F}}{\partial t^2} = 0,$$

we identify

$$\boxed{c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.} \tag{8.10}$$

Thus electromagnetic waves propagate in vacuum with the speed of light.

8.2 Electromagnetic Waves in Material Media

In a linear, homogeneous, isotropic medium,

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}. \tag{8.11}$$

Assuming no free charges and currents,

$$\rho_f = 0, \qquad \mathbf{J}_f = 0.$$

Maxwell's equations become

$$\nabla \cdot \mathbf{E} = 0, \tag{8.12}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{8.13}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{8.14}$$
$$\nabla \times \mathbf{B} = \mu \epsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{8.15}$$

Following the same procedure as in vacuum, we obtain

$$\nabla^2 \mathbf{E} - \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{8.16}$$
$$\nabla^2 \mathbf{B} - \mu \epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0. \tag{8.17}$$

The wave speed in the medium is

$$\boxed{v = \frac{1}{\sqrt{\mu \epsilon}}.} \tag{8.18}$$

For a linear medium,

$$\epsilon = \epsilon_0 \epsilon_r, \qquad \mu = \mu_0 \mu_r. \tag{8.19}$$

Hence,

$$v = \frac{c}{\sqrt{\epsilon_r \mu_r}}. \tag{8.20}$$

The refractive index of the medium is defined as

$$\boxed{n = \frac{c}{v} = \sqrt{\epsilon_r \mu_r}.} \tag{8.21}$$

For most dielectrics,

$$\mu_r \approx 1, \quad \Rightarrow \quad n \approx \sqrt{\epsilon_r}.$$

8.3 Reflection and Refraction at a Plane Interface

Consider a plane interface between two homogeneous, isotropic media. The boundary surface is the plane

$$z = 0,$$

with medium 1 occupying $z < 0$ and medium 2 occupying $z > 0$.

Let

- $\mathbf{k}_i$, $\mathbf{k}_r$, $\mathbf{k}_t$ be the wave vectors of the incident, reflected, and transmitted waves respectively,
- v_1, v_2 be the wave velocities in medium 1 and medium 2,
- ω be the angular frequency, which is the same for all waves.

The magnitudes of the wave vectors are

$$k_i = k_r = \frac{\omega}{v_1}, \qquad k_t = \frac{\omega}{v_2}. \tag{8.22}$$

8.3.1 Phase Matching at the Boundary

At the boundary $z = 0$, the phases of the incident, reflected, and transmitted waves must match at all points on the interface. Hence,

$$\boxed{\mathbf{k}_i \cdot \mathbf{r} = \mathbf{k}_r \cdot \mathbf{r} = \mathbf{k}_t \cdot \mathbf{r} \qquad \text{at } z = 0.} \tag{8.23}$$

Since the boundary lies in the xy-plane, this condition implies equality of the components of the wave vectors parallel to the interface:

$$\boxed{k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t.} \tag{8.24}$$

From this, we immediately obtain the following fundamental results:

1. The incident, reflected, and transmitted waves all lie in the same plane (the *plane of incidence*).

2. Since $k_i = k_r$, it follows that

$$\boxed{\theta_i = \theta_r,}$$

which is the law of reflection.

3. Using $k = \omega/v$, we obtain Snell's law:

$$\boxed{\frac{\sin\theta_i}{\sin\theta_t} = \frac{v_1}{v_2} = \frac{n_2}{n_1},} \tag{8.25}$$

or equivalently,

$$\boxed{\frac{n_1}{n_2} = \frac{\sin\theta_t}{\sin\theta_i}.}$$

8.3.2 Field Geometry and Boundary Conditions

The plane of incidence is taken to be the xz-plane. The angles θ_i, θ_r, and θ_t are measured with respect to the normal (the z-axis).

In the absence of free surface charges and free surface currents, Maxwell's equations imply the boundary conditions:

$$\mathbf{E}_{\parallel}^{(1)} = \mathbf{E}_{\parallel}^{(2)}, \tag{8.26}$$

$$\mathbf{H}_{\parallel}^{(1)} = \mathbf{H}_{\parallel}^{(2)}. \tag{8.27}$$

These conditions determine the amplitudes of the reflected and transmitted waves.

8.3.3 Fresnel Equations

Define the parameters

$$\boxed{\alpha = \frac{\cos\theta_t}{\cos\theta_i}, \qquad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}.} \tag{8.28}$$

Electric Field Perpendicular to the Plane of Incidence

For polarization perpendicular to the plane of incidence, the electric field is parallel to the y-axis.

Applying the boundary conditions at $z = 0$:

$$E_i + E_r = E_t, \tag{8.29}$$

$$\frac{1}{\mu_1 v_1}(E_i - E_r)\cos\theta_i = \frac{1}{\mu_2 v_2} E_t \cos\theta_t. \tag{8.30}$$

Solving these equations, we obtain the Fresnel reflection coefficient:

$$\boxed{\frac{E_r}{E_i} = \frac{\alpha - \beta}{\alpha + \beta}.} \tag{8.31}$$

The transmission coefficient is

$$\boxed{\frac{E_t}{E_i} = \frac{2}{\alpha + \beta}.} \tag{8.32}$$

8.3.4 Brewster's Angle

8.4 Electromagnetic Waves in Conductors

When an electromagnetic wave propagates through a conducting medium, free charges respond to the electric field, leading to energy dissipation and attenuation of the wave. This behavior is fundamentally different from propagation in dielectrics.

8.4.1 Ohm's Law and Charge Density in Conductors

In a conductor, the current density is related to the electric field by Ohm's law:

$$\boxed{\mathbf{J} = \sigma \mathbf{E},} \tag{8.33}$$

where σ is the electrical conductivity of the medium.

The continuity equation is

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{8.34}$$

Using Ohm's law,

$$\nabla \cdot \mathbf{J} = \sigma \nabla \cdot \mathbf{E}. \tag{8.35}$$

From Gauss's law in a homogeneous medium,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}. \tag{8.36}$$

Hence,

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = 0. \tag{8.37}$$

The solution is

$$\rho(t) = \rho_0 e^{-(\sigma/\epsilon)t}. \tag{8.38}$$

Thus any excess charge density decays exponentially with time. After a very short transient interval, the conductor remains electrically neutral, and we may take

$$\boxed{\rho = 0} \tag{8.39}$$

for electromagnetic wave propagation in conductors.

8.4.2 Maxwell's Equations in a Conductor

For a linear, homogeneous conductor with $\rho = 0$ and $\mathbf{J} = \sigma\mathbf{E}$, Maxwell's equations become:

$$\nabla \cdot \mathbf{E} = 0, \tag{8.40}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{8.41}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{8.42}$$

$$\nabla \times \mathbf{B} = \mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}. \tag{8.43}$$

8.4.3 Wave Equation in a Conductor

Taking the curl of Faraday's law,

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}). \tag{8.44}$$

Using the vector identity and $\nabla \cdot \mathbf{E} = 0$,

$$\nabla^2\mathbf{E} = \mu\sigma\frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{8.45}$$

Thus the wave equation for the electric field in a conductor is

$$\boxed{\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu\sigma\frac{\partial \mathbf{E}}{\partial t} = 0.} \tag{8.46}$$

An identical equation holds for $\mathbf{B}$.

8.4.4 Plane Wave Solution and Complex Wave Vector

Assume a plane wave solution propagating in the $+z$ direction:

$$\mathbf{E}(z,t) = \mathbf{E}_0 e^{i(kz-\omega t)}. \tag{8.47}$$

Substitution into the wave equation gives the dispersion relation

$$k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega. \tag{8.48}$$

Hence the wave number is complex:

$$\boxed{k = \alpha + i\beta,} \tag{8.49}$$

where

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \tag{8.50}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}. \tag{8.51}$$

8.4.5 Attenuation and Skin Depth

The electric field becomes

$$\boxed{\mathbf{E}(z,t) = \mathbf{E}_0 e^{-\beta z}\cos(\alpha z - \omega t).} \tag{8.52}$$

Thus electromagnetic waves are attenuated exponentially inside a conductor.

The characteristic penetration depth (skin depth) is

$$\boxed{\delta = \frac{1}{\beta}.} \tag{8.53}$$

8.4.6 Phase Relation Between E and B

From Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{8.54}$$

we obtain

$$\mathbf{B} = \frac{1}{\omega}(\mathbf{k} \times \mathbf{E}). \tag{8.55}$$

Since k is complex, **B** is not in phase with **E**. The magnetic field lags the electric field by a phase angle ϕ, given by

$$\boxed{\tan\phi = \frac{\beta}{\alpha}.} \tag{8.56}$$

Thus, in a conducting medium:

- The wave amplitude decays exponentially.
- The phase velocity differs from that in free space.
- The electric and magnetic fields are not in phase.

Electromagnetic energy entering a conductor is dissipated as Joule heat due to induced currents. The attenuation of waves and the phase shift between fields are direct consequences of finite conductivity.

8.5 Frequency Dependence of Permittivity (Dispersion)

When the speed of propagation of an electromagnetic wave in a medium depends on its frequency, the medium is said to be **dispersive**. The phenomenon responsible for this frequency dependence is called **dispersion**.

Dispersion arises because the electric field of the wave interacts with bound charges in the medium, whose response depends on the frequency of the applied field.

8.5.1 Physical Origin of Dispersion

In non-conducting (dielectric) media, electrons are not free but are bound to atoms or molecules. When an external electric field is applied, these bound electrons are displaced slightly from their equilibrium positions, creating electric dipoles.

The motion of a bound electron can be modeled as that of a damped, driven harmonic oscillator.

8.5.2 Forces Acting on a Bound Electron

Consider an electron of charge $-q$ and mass m, displaced by a small distance x from its equilibrium position.

Binding Force

The restoring force due to the binding of the electron to the atom is taken to be linear:

$$\boxed{F_{\text{binding}} = -kx = -m\omega_0^2 x,} \tag{8.57}$$

where ω_0 is the natural angular frequency of the bound electron.

Damping Force

Energy can be lost due to various mechanisms, such as radiation of electromagnetic waves by the accelerating charge or collisions within the medium. This is modeled phenomenologically by a damping force:

$$\boxed{F_{\text{damping}} = -m\gamma\frac{dx}{dt},} \tag{8.58}$$

where γ is the damping constant.

Driving Force

In the presence of an electromagnetic wave, the electron experiences an oscillating electric field:

$$\mathbf{E}(t) = \mathbf{E}_0 e^{-i\omega t}. \tag{8.59}$$

The driving force is

$$\boxed{F_{\text{driving}} = qE_0 e^{-i\omega t}.} \tag{8.60}$$

8.5.3 Equation of Motion

The total force on the electron is the sum of the above contributions. Hence, the equation of motion is

$$\boxed{m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0 e^{-i\omega t}.} \tag{8.61}$$

8.5.4 Solution of the Equation of Motion

We seek a steady-state solution of the form

$$x(t) = x_0 e^{-i\omega t}. \tag{8.62}$$

Substituting into the equation of motion gives

$$\left(-m\omega^2 - im\gamma\omega + m\omega_0^2\right) x_0 = qE_0. \tag{8.63}$$

Solving for x_0,

$$\boxed{x_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0.} \tag{8.64}$$

8.5.5 Polarization and Electric Susceptibility

The electric dipole moment of a single electron is

$$p = qx. \tag{8.65}$$

If N is the number of such oscillators per unit volume, the polarization of the medium is

$$\mathbf{P} = Nqx = Nqx_0 e^{-i\omega t}. \tag{8.66}$$

Using the expression for x_0, we obtain

$$\boxed{\mathbf{P} = \epsilon_0 \chi(\omega) \mathbf{E},} \tag{8.67}$$

where the (complex) electric susceptibility is

$$\boxed{\chi(\omega) = \frac{Nq^2}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}.} \tag{8.68}$$

8.5.6 Complex Permittivity

The dielectric function is related to the susceptibility by

$$\boxed{\epsilon(\omega) = \epsilon_0 \left[1 + \chi(\omega)\right].} \tag{8.69}$$

Since $\chi(\omega)$ is complex, the permittivity can be written as

$$\epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega). \tag{8.70}$$

- $\epsilon'(\omega)$ determines the **dispersion** (variation of phase velocity with frequency),
- $\epsilon''(\omega)$ determines the **absorption** of the wave in the medium.

8.5.7 Absorption and Dispersion

Near the resonance frequency $\omega \approx \omega_0$:

- Absorption is maximum, corresponding to a peak in $\epsilon''(\omega)$.
- The real part $\epsilon'(\omega)$ varies rapidly with frequency.

This leads to two important regimes:

1. **Normal dispersion:**
$$\frac{dn}{d\omega} > 0$$
occurs away from resonance.

2. **Anomalous dispersion:**
$$\frac{dn}{d\omega} < 0$$
occurs in the vicinity of absorption bands.

8.5.8 Refractive Index

The (complex) refractive index $n(\omega)$ is defined by

$$\boxed{n^2(\omega) = \frac{\epsilon(\omega)}{\epsilon_0}.} \tag{8.71}$$

Writing
$$n(\omega) = n'(\omega) + in''(\omega),$$

- $n'(\omega)$ governs phase velocity and refraction,
- $n''(\omega)$ governs attenuation of the wave.

8.5.9 Cauchy's Formula

Far from resonance frequencies and in regions of weak absorption, the refractive index can be expanded as a power series:

$$\boxed{n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \cdots} \tag{8.72}$$

This empirical relation is known as **Cauchy's formula** and is widely used to describe normal dispersion in transparent media.