Representations of the Poincaré Group

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Chapter 1

The Poincaré group

1.1 Characterisation of the Poincaré group

In this section, we will study the characterisation and the properties of the Poincaré group. We start from $\eta_{\mu\nu}$, the Minkowski metric,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu} \tag{1.1}$$

and work with 'natural' units *i.e.* $c = \hbar = 1$
Any transformation Λ that satisfies the following condition is a Lorentz transformation

$$\boxed{\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}} \tag{1.2}$$

The Lorentz condition can also we written as

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\eta^{\rho\sigma} = \eta^{\mu\nu} \tag{1.3}$$

Proof: Multiply equation 1.2 by $\eta^{\sigma\alpha}\Lambda^{\beta}{}_{\alpha}$

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}(\Lambda^{\nu}{}_{\sigma}\eta^{\sigma\alpha}\Lambda^{\beta}{}_{\alpha}) = \eta_{\rho\sigma}\eta^{\sigma\alpha}\Lambda^{\beta}{}_{\alpha} = \delta_{\rho}{}^{\alpha}\Lambda^{\beta}{}_{\alpha} = \Lambda^{\beta}{}_{\rho} = \delta_{\mu}{}^{\beta}\Lambda^{\mu}{}_{\rho} = \eta_{\mu\nu}\eta^{\nu\beta}\Lambda^{\mu}{}_{\rho} \tag{1.4}$$

which one may relabel in order to get equation 1.3.

The aim here is to show that these transformations form a group, *i.e.* we need to prove the group properties; closure and the existence of the identity element and that of the inverse.

- Closure: First perform a coordinate transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$ where Λ satisfies the Lorentz condition 1.2 and then a second one as follows

$$x''^{\mu} = \bar{\Lambda}^{\mu}{}_{\rho} x'^{\rho} + \bar{a}^{\mu} = \bar{\Lambda}^{\mu}{}_{\rho}(\Lambda^{\rho}{}_{\nu} x^{\nu} + a^{\rho}) + \bar{a}^{\mu} = \underbrace{(\bar{\Lambda}^{\mu}{}_{\rho}\Lambda^{\rho}{}_{\nu})}_{\text{Lorentz transf}}\ x^{\nu} + \underbrace{(\bar{\Lambda}^{\mu}{}_{\rho} x^{\nu} + \bar{a}^{\mu})}_{\text{another translation}} \tag{1.5}$$

When $\Lambda^{\mu}{}_{\nu}$ and $\bar{\Lambda}^{\mu}{}_{\nu}$ both satisfy the Lorentz condition, so does their product and hence we have the composition rule for these transformations:

$$T(\bar{\Lambda}, \bar{a})T(\Lambda, a) = T(\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}) \tag{1.6}$$

- Identity: The identity transformation is $T(1, 0)$.
Proof: Using the composition rule 1.6 we have

$$T(1, 0)T(\Lambda, a) = T(\Lambda, a) \tag{1.7}$$

as required.

- Inverse: The inverse trasformation is $T(\Lambda^{-1}, -\Lambda^{-1}a)$
Proof: Again using equation 1.6

$$T(\Lambda^{-1}, -\Lambda^{-1}a)T(\Lambda, a) = T(1, \Lambda^{-1}a - \Lambda^{-1}a) = T(1, 0) \tag{1.8}$$

as required.

To every transformation T, there corresponds a unitary linear operator that acts on vectors in Hilbert space such that it takes $\Psi \to U(\Lambda, a)\Psi$ and obeys the composition rule

$$\boxed{U(\bar{\Lambda}, \bar{a})U(\Lambda, a) = U(\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a})} \tag{1.9}$$

With identity and inverse elements are

$$\boxed{U(1, 0)} \tag{1.10}$$

and

$$\boxed{U(\Lambda^{-1}, -\Lambda^{-1}a)} \tag{1.11}$$

respectively.
The whole group of such transformation is known as the *inhomogeneous Lorentz group* or ***Poincaré group*** . Now we will consider two different subgroups of this group.
The first one is the group of transformations with no translations present *i.e.* $a^{\mu} = 0$. This group is known as the *homogeneous Lorentz group* or simply the ***Lorentz group***.

Using the Lorentz condition 1.2, the property of the product of determinants and the fact that $\det(\eta) = 1$ we have

$$(\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1 \tag{1.12}$$

For $\Lambda^\mu{}_\nu$ and $\bar{\Lambda}^\mu{}_\nu$ we have

$$(\bar{\Lambda}\Lambda)^0{}_0 = \bar{\Lambda}^0{}_0\Lambda^0{}_0 + \bar{\Lambda}^0{}_1\Lambda^1{}_0 + \bar{\Lambda}^0{}_2\Lambda^2{}_0 + \bar{\Lambda}^0{}_3\Lambda^3{}_0 \tag{1.13}$$

According to equations 1.2 and 1.3

$$\eta_{\mu\nu}\Lambda^\mu{}_0\Lambda^\nu_0 = \eta_{00} = +1 \Longrightarrow (\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0\Lambda^i{}_0 = 1 + \Lambda^0{}_i\Lambda^0{}_i \tag{1.14}$$

So $(\Lambda^0{}_0)^2 \geq 1$ which in turn implies that either $\Lambda^0{}_0 \geq +1$ or $\Lambda^0{}_0 \leq -1$.
Also, note that the length of the 3-vector $(\Lambda^1{}_0, \Lambda^2{}_0, \Lambda^3{}_0)$ is $\sqrt{(\Lambda^0{}_0)^2 - 1}$ according to equation 1.14. Similarly $(\bar{\Lambda}^0{}_1, \bar{\Lambda}^0{}_2, \bar{\Lambda}^0{}_3)$ has length $\sqrt{(\bar{\Lambda}^0{}_0)^2 - 1}$ and so the scalar product of the two 3-vectors is bounded,

$$\begin{aligned}
&|\bar{\Lambda}^0{}_1\Lambda^1{}_0 + \bar{\Lambda}^0{}_2\Lambda^2{}_0 + \bar{\Lambda}^0{}_3\Lambda^3{}_0| \leq \sqrt{(\Lambda^0{}_0)^2 - 1}\sqrt{(\bar{\Lambda}^0{}_0)^2 - 1} \Rightarrow \\
&|(\bar{\Lambda}\Lambda)^0{}_0 - \bar{\Lambda}^0{}_0\Lambda^0{}_0| \leq \sqrt{(\Lambda^0{}_0)^2 - 1}\sqrt{(\bar{\Lambda}^0{}_0)^2 - 1} \Rightarrow \\
&-\sqrt{(\Lambda^0{}_0)^2 - 1}\sqrt{(\bar{\Lambda}^0{}_0)^2 - 1} \leq (\bar{\Lambda}\Lambda)^0{}_0 - \bar{\Lambda}^0{}_0\Lambda^0{}_0 \Rightarrow \\
&(\bar{\Lambda}\Lambda)^0{}_0 \geq \bar{\Lambda}^0{}_0\Lambda^0{}_0 - \sqrt{(\Lambda^0{}_0)^2 - 1}\sqrt{(\bar{\Lambda}^0{}_0)^2 - 1} \geq 1
\end{aligned} \tag{1.15}$$

We know the identity is where $\Lambda = 1$ and so in order to preserve continuity we need to take $(\Lambda^0{}_0)^2 \geq 1$ (see Figure 1.1) and $\det \Lambda = 1$. This is to ensure that they have the same sign as the identity since we want to be able to obtain any Lorentz transformation from the identity by a continuous change of parameters. Hence it forms a subgroup of Lorentz transformations know as the *proper orthochronous Lorentz group*.

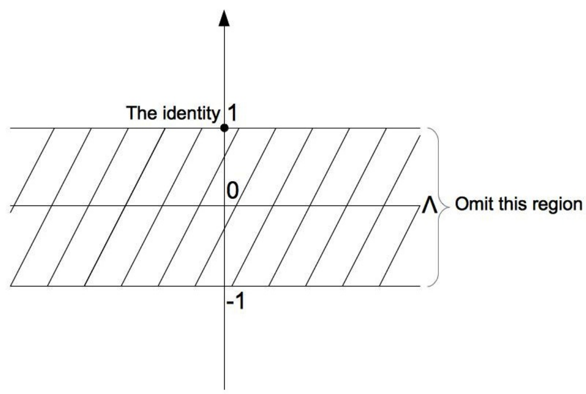

Figure 1.1: Allowed region

1.2 The Poincaré algebra

In this section we aim to find information about the symmetries of the Poincaré group using the properties of group elements near the identity. For this purpose, consider the infinitesimal Poincaré transformation

$$\Lambda^{\mu\nu} = \delta^{\mu\nu} + \omega^{\mu\nu} \ , \ a^{\mu} = \epsilon^{\mu} \tag{1.16}$$

Using the Lorentz condition 1.2 to first order in ω we must have

$$\eta_{\rho\sigma} = \eta_{\mu\nu}(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) = \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + O(\omega^2) \tag{1.17}$$

but since η is symmetric with respect to the change of its two indices

$$\omega_{\rho\sigma} + \omega_{\sigma\rho} = 0 \Rightarrow \omega_{\rho\sigma} = -\omega_{\sigma\rho} \tag{1.18}$$

i.e. $\omega_{\mu\nu}$ is a rank-2 antisymmetric tensor in four dimensions and so it has 6 independent components with zeros on the diagonal. On the other hand, ϵ^{μ} is a 4-vector so it has 4 components. Therefore in order to specify a Poincaré transformation we need 10 independent parameters.
For an infinitesimal Lorentz transformation one can make a Taylor expansion as follows

$$\boxed{U(1+\omega,\epsilon) = 1 - \frac{1}{2}i\omega_{\mu\nu}M^{\mu\nu} + i\epsilon_{\mu}P^{\mu} + \dots} \tag{1.19}$$

where $M^{\mu\nu}$ and P^{μ} are Hermitian operators so that U is a unitary operator.
Here is a good point to stop and explore these operators in more detail.

We can make an infinitesimal change in position and time coordinates of a quantum state and then Taylor expand as follows

$$\Psi(\underline{x}+\underline{\epsilon}, t+\tau) = \Psi(\underline{x}, t) + i\underline{\epsilon}\cdot\underbrace{(-i\underline{\nabla}\Psi)}_{\hat{\underline{P}}\Psi} - i\tau\underbrace{(i\frac{d}{dt}\Psi)}_{\hat{H}\Psi} \tag{1.20}$$

In other words, the momentum operator and the Hamiltonian correspond to space and time translations respectively. Putting them together, we form the momentum-energy four vector P^{μ} which corresponds to translations in space-time. By the same analogy, the angular momentum operator is the generator of space rotations (See later (1.77) for a more rigorous proof).
Now consider the general Lorentz transformations from one frame to another *i.e.* a boost. We will now prove that boosts are rotations in Minkowski space, *i.e.* hyperbolic rotations. Consider a boost in the x direction:

$$\begin{cases} t' = \gamma(t - \frac{vx}{c^2}) \\ x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases} \tag{1.21}$$

Since we are working in natural unit, $c = 1$ and so we can define $\beta = \frac{v}{c} = v$. Therefore the transformation can be written as follows:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{1.22}$$

Since y and z do not change in this transformation, for simplicity, we may only consider t and x from now on.
Define the rapidity variable ϕ such that $\tanh\phi = \beta$, so

$$\gamma = \frac{1}{\sqrt{1-\beta}} = \frac{1}{\sqrt{1-\tanh^2\phi}} = \cosh\phi \tag{1.23}$$

$$\tanh\phi = \frac{\sinh\phi}{\cosh\phi} \Rightarrow \sinh\phi = \beta\gamma \tag{1.24}$$

Hence

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{1.25}$$

which is a hyperbolic rotation, as required.

A nice trick: Consider two contracted tensors $\omega_{\mu\nu}M^{\mu\nu}$. It is possible to prove that when $\omega_{\mu\nu}$ is symmetric (antisymmetric), $M^{\mu\nu}$ has to be symmetric (antisymmetric) as well.
Let

$$M^{\mu\nu} = \frac{1}{2}\underbrace{(M^{\mu\nu} + M^{\nu\mu})}_{\text{Symm part } M^{\mu\nu}_{(s)}} + \frac{1}{2}\underbrace{(M^{\mu\nu} - M^{\nu\mu})}_{\text{AntiSymm part } M^{\mu\nu}_{(a)}} \tag{1.26}$$

Now contract this with $\omega_{\mu\nu}$,

- $\omega_{\mu\nu}$ is symmetric: The antisymmetric part of M vanishes

$$\omega_{\mu\nu}M^{\mu\nu} - \omega_{\mu\nu}M^{\nu\mu} = \omega_{\mu\nu}M^{\mu\nu} - \omega_{\nu\mu}M^{\mu\nu} = \omega_{\mu\nu}M^{\mu\nu} - \omega_{\mu\nu}M^{\mu\nu} = 0 \tag{1.27}$$

- $\omega_{\mu\nu}$ is antisymmetric: The symmetric part of M vanishes

$$\omega_{\mu\nu}M^{\mu\nu} + \omega_{\mu\nu}M^{\nu\mu} = \omega_{\mu\nu}M^{\mu\nu} + \omega_{\nu\mu}M^{\mu\nu} = \omega_{\mu\nu}M^{\mu\nu} - \omega_{\mu\nu}M^{\mu\nu} = 0 \tag{1.28}$$

Therefore, going back to equation 1.19 and knowing that ω is antisymmetric, it is immediately implied that M is antisymmetric with respect to the exchange its two indices and so it has 6 independent components.

$$M^{\mu\nu} = -M^{\nu\mu} \tag{1.29}$$

We now know that p^μ is the 4-momentum operator since it corresponds to translations. Also, $M^{\mu\nu}$ is a rotation operator which contains the usual angular momentum 3-vector as well is the boost 3-vector *i.e.* 6 independent components, as expected.

Consider the product

$$U(\Lambda, a)U(1+\omega, \epsilon)U^{-1}(\Lambda, a) = U(\Lambda(1+\omega)\Lambda^{-1},\ \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a) \tag{1.30}$$

where we have used the composition rule 1.9 and 1.11.
Expanding to first order in ω and ϵ,

$$U(\Lambda, a)[-\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu} + \epsilon_\mu P^\mu]U^{-1}(\Lambda, a) = -\frac{1}{2}(\Lambda\omega\Lambda^{-1})_{\rho\sigma}M^{\rho\sigma} + (\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_\rho P^\rho \tag{1.31}$$

Therefore for ω on the RHS we have,

$$\begin{aligned}
&-\frac{1}{2}\Lambda_\rho{}^\mu\omega_{\mu\nu}(\Lambda^{-1})^\nu{}_\sigma M^{\rho\sigma} - \Lambda_\rho{}^\mu\omega_{\mu\nu}(\Lambda^{-1})^\nu{}_\sigma a^\sigma P^\rho = \\
&-\frac{1}{2}\Lambda_\rho{}^\mu\omega_{\mu\nu}(\Lambda^{-1})^\nu{}_\sigma M^{\rho\sigma} - \frac{1}{2}[\Lambda_\rho{}^\mu\omega_{\mu\nu}(\Lambda^{-1})^\nu{}_\sigma a^\sigma P^\rho - \Lambda_\rho{}^\mu\omega_{\nu\mu}(\Lambda^{-1})^\nu{}_\sigma a^\sigma P^\rho] = \\
&-\frac{1}{2}(\Lambda_\rho{}^\mu\omega_{\mu\nu}\Lambda_\sigma{}^\nu M^{\rho\sigma} + \Lambda_\rho{}^\mu\omega_{\mu\nu}\Lambda_\sigma{}^\nu a^\sigma P^\rho - \Lambda_\rho{}^\nu\omega_{\mu\nu}\Lambda_\sigma{}^\mu a^\sigma P^\rho) = \\
&-\frac{1}{2}(\Lambda_\rho{}^\mu\omega_{\mu\nu}\Lambda_\sigma{}^\nu M^{\rho\sigma} + \Lambda_\rho{}^\mu\omega_{\mu\nu}\Lambda_\sigma{}^\nu a^\sigma P^\rho - \Lambda_\sigma{}^\nu\omega_{\mu\nu}\Lambda_\rho{}^\mu a^\rho P^\sigma)
\end{aligned} \tag{1.32}$$

Where, on the second line we have used the fact that w is antisymmetric, on the third line we have relabelled μ and ν and ρ and σ on the fourth line.
Equating coefficients of ω and ϵ on both side of the equation we get,

$$U(\Lambda, a)M^{\mu\nu}U^{-1}(\Lambda, a) = \Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu(M^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho) \tag{1.33}$$

$$U(\Lambda, a)P^\mu U^{-1}(\Lambda, a) = \Lambda_\rho{}^\mu P^\rho \tag{1.34}$$

Note: The equation above is simply stating how the operator P transforms under U, writing it as a linear combination of operators on the right hand side.
using 1.11 to first order in ω and ϵ we have,

$$\begin{aligned}
U^{-1}(\Lambda, a) &= U(\Lambda^{-1}, -\Lambda^{-1}a) = U((1+\omega)^{-1}, -(1+\omega)^{-1}\epsilon) \\
&= U(1-\omega, -(1-\omega)\epsilon) = U(1-\omega, -\epsilon)
\end{aligned} \tag{1.35}$$

the LHS of equation 1.34 becomes

$$\begin{aligned}
&(1 - \frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta} + i\epsilon_\alpha P^\alpha)P^\mu(1 + \frac{i}{2}\omega_{\gamma\xi}M^{\gamma\xi} - i\epsilon_\gamma P^\gamma) \\
=&P^\mu - \frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}P^\mu + i\epsilon_\alpha P^\alpha P^\mu + \frac{i}{2}\omega_{\gamma\xi}P^\mu M^{\gamma\xi} - i\epsilon_\gamma P^\mu P^\gamma \\
=&P^\mu + i(-\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma}P^\mu + \epsilon_\rho P^\rho P^\mu + \frac{1}{2}\omega_{\rho\sigma}P^\mu M^{\rho\sigma} - \epsilon_\rho P^\mu P^\rho) \\
=&P^\mu + i[-\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma} + i\epsilon_\rho P^\rho, P^\mu]
\end{aligned} \tag{1.36}$$

and the RHS

$$\Lambda_\rho{}^\mu P^\rho = (\delta_\rho{}^\mu + \omega_\rho{}^\mu)P^\rho \tag{1.37}$$

Setting LHS = RHS,

$$i[-\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma} + i\epsilon_\rho P^\rho, P^\mu] = \omega_\rho{}^\mu P^\rho \tag{1.38}$$

Similarly for $M^{\mu\nu}$ the of equation 1.33 LHS is,

$$M^{\mu\nu} + i[-\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma} + i\epsilon_\rho P^\rho, M^{\mu\nu}] \tag{1.39}$$

and the RHS,

$$\begin{aligned}
\Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu M^{\rho\sigma} =& (\delta_\rho{}^\mu + \omega_\rho{}^\mu)(\delta_\sigma{}^\nu + \omega_\sigma{}^\nu)(M^{\rho\sigma} - \epsilon^\rho P^\sigma + \epsilon^\sigma P^\rho) \\
=& M^{\mu\nu} + \omega_\rho{}^\mu M^{\rho\nu} + \omega_\sigma{}^\nu M^{\mu\sigma} - \epsilon^\mu P^\nu + \epsilon^\nu P^\mu \\
=& M^{\mu\nu} + \eta^{\mu\sigma}\omega_{\rho\sigma}M^{\rho\nu} + \eta^{\nu\rho}\omega_{\sigma\rho}M^{\mu\sigma} - \epsilon^\mu P^\nu + \epsilon^\nu P^\mu \\
=& M^{\mu\nu} + \frac{1}{2}[\eta^{\mu\sigma}\omega_{\rho\sigma}M^{\rho\nu} + \eta^{\mu\rho}\omega_{\sigma\rho}M^{\sigma\nu} + \eta^{\nu\rho}\omega_{\sigma\rho}M^{\mu\sigma} + \eta^{\nu\sigma}\omega_{\rho\sigma}M^{\mu\rho}] - \epsilon^\mu P^\nu + \epsilon^\nu P^\mu \\
=& M^{\mu\nu} + \frac{1}{2}[\eta^{\mu\sigma}\omega_{\rho\sigma}M^{\rho\nu} - \eta^{\mu\rho}\omega_{\rho\sigma}M^{\sigma\nu} - \eta^{\nu\rho}\omega_{\rho\sigma}M^{\mu\sigma} + \eta^{\nu\sigma}\omega_{\rho\sigma}M^{\mu\rho}] - \epsilon^\mu P^\nu + \epsilon^\nu P^\mu \\
=& M^{\mu\nu} + \frac{1}{2}\omega_{\rho\sigma}[\eta^{\mu\sigma}M^{\rho\nu} - \eta^{\mu\rho}M^{\sigma\nu} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}] - \epsilon^\mu P^\nu + \epsilon^\nu P^\mu
\end{aligned} \tag{1.40}$$

Equating we get,

$$i[-\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma} + \epsilon_\rho P^\rho, M^{\mu\nu}] = \frac{1}{2}\omega_{\rho\sigma}[\eta^{\mu\sigma}M^{\rho\nu} - \eta^{\mu\rho}M^{\sigma\nu} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}] - \epsilon^\mu P^\nu + \epsilon^\nu P^\mu \tag{1.41}$$

Finally, we extract the coefficients of ω and ϵ and use the fact that $M^{\mu\nu}$ is antisymmetric,

$$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho})} \tag{1.42}$$

and

$$\boxed{[P^\rho, M^{\mu\nu}] = i(\eta^{\rho\mu}P^\nu - \eta^{\rho\nu}P^\mu)} \tag{1.43}$$

On the other hand, equation 1.38 implies the following,

$$\boxed{[P^\mu, P^\rho] = 0} \tag{1.44}$$

So we have the momentum, angular momentum and boost 3-vectors,

$$\mathbf{P} = \left\{P^1, P^2, P^3\right\} \tag{1.45}$$

$$\mathbf{J} = \left\{M^{23}, M^{31}, M^{12}\right\} \tag{1.46}$$

$$\mathbf{K} = \left\{M^{01}, M^{02}, M^{03}\right\} \tag{1.47}$$

The above commutation relations imply,

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{1.48}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \tag{1.49}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k \tag{1.50}$$

$$[J_i, P_j] = i\epsilon_{ijk} P_k \tag{1.51}$$

$$[K_i, P_j] = -iH\delta_{ij} \tag{1.52}$$

$$[J_i, H] = [P_i, H] = [H, H] = 0 \tag{1.53}$$

$$[K_i, H] = -iP_i \tag{1.54}$$

Now we define *Pauli-Lubanski pseudo-vector* by

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma} \tag{1.55}$$

which has the following properties,

$$\boxed{W_\mu P^\mu = 0} \tag{1.56}$$

Proof: Use the fact the P's commute

$$\begin{aligned} W_\mu P^\mu &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma P^\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\mu P^\sigma = \frac{1}{2}\epsilon_{\sigma\nu\rho\mu} M^{\nu\rho} P^\sigma P^\mu \\ &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma P^\mu = -W_\mu P^\mu \;\Rightarrow W_\mu P^\mu = 0 \end{aligned} \tag{1.57}$$

$$\boxed{[W_\mu, P_\nu] = 0} \tag{1.58}$$

Proof:

$$
\begin{aligned}
[W^\mu, P^\alpha] =& \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}([P_\nu, P^\alpha]M_{\rho\sigma} + P_\nu[M_{\rho\sigma}, P^\alpha]) = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\eta^{\alpha\beta}([P_\alpha, P_\beta]M_{\rho\sigma} + P_\nu[M_{\rho\sigma}, P_\beta]) \\
=& \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\eta^{\alpha\beta}P_\nu(\eta_{\beta\rho}P_\sigma - \eta_{\beta\sigma}P_\rho) = \frac{i}{2}(\epsilon^{\mu\nu\alpha\sigma}P_\nu P_\sigma - \epsilon^{\mu\nu\rho\alpha}P_\nu P_\rho) \\
=& \frac{i}{4}(\epsilon^{\mu\nu\alpha\sigma}P_\nu P_\sigma + \epsilon^{\mu\sigma\alpha\nu}P_\sigma P_\nu - \epsilon^{\mu\nu\rho\alpha}P_\nu P_\rho - \epsilon^{\mu\rho\nu\alpha}P_\rho P_\nu) \\
=& \frac{i}{4}(\epsilon^{\mu\nu\alpha\sigma}P_\nu P_\sigma + \epsilon^{\mu\nu\rho\alpha}P_\nu P_\rho - \epsilon^{\mu\nu\rho\alpha}P_\nu P_\rho - \epsilon^{\mu\nu\alpha\sigma}P_\nu P_\sigma) = 0
\end{aligned}
\tag{1.59}
$$

Where on the fourth line we have used the fact that P's commute and then relabelled the indices for the second and fourth terms. This commutation relation implies that the Pauli-Lubanski vector is invariant under translations.

Now define

$$
\tilde{M}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\rho\sigma} \tag{1.60}
$$

So that $W_\mu = P^\nu \tilde{M}_{\mu\nu}$. Multiplying both sides of equation 1.60 by $\epsilon^{\alpha\beta\mu\nu}$ one gets

$$
\epsilon^{\alpha\beta\mu\nu}\tilde{M}_{\mu\nu} = -\left(\delta^\alpha{}_\rho\delta^\beta{}_\sigma - \delta^\alpha{}_\sigma\delta^\beta{}_\rho\right)M^{\rho\sigma} = -2M^{\alpha\beta} \tag{1.61}
$$

where we have used the identity for the four dimensional Levi-Civita tensor. Therefore,

$$
M^{\alpha\beta} = \frac{-1}{2}\epsilon^{\alpha\beta\mu\nu}\tilde{M}_{\mu\nu} \tag{1.62}
$$

Let us determine how $\tilde{M}_{\mu\nu}$ transforms under a Lorentz transformation by considering

$$
\begin{aligned}
&[\tilde{M}_{\mu\nu}, M_{\alpha\beta}] = \frac{1}{2}\epsilon[M^{\rho\sigma}, M_{\alpha\beta}] = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}[M_{\rho\sigma}, M_{\alpha\beta}] \\
=& -\frac{i}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\left(\eta^{\rho\alpha}M^{\sigma\beta} - \eta^{\rho\beta}M^{\sigma\alpha} - \eta^{\sigma\alpha}M^{\rho\beta} + \eta^{\sigma\beta}M^{\rho\alpha}\right) \\
=& -\frac{i}{2}\left(\epsilon_{\mu\nu\alpha}{}^\sigma M_{\sigma\beta} - \epsilon_{\mu\nu\beta}{}^\sigma M_{\sigma\alpha} + \epsilon_{\mu\nu\alpha}{}^\rho M_{\rho\beta} - \epsilon_{\mu\nu\beta}{}^\rho M_{\rho\alpha}\right) = -i\left(\epsilon_{\mu\nu\beta}{}^\rho M_{\rho\beta} - \epsilon_{\mu\nu\beta}{}^\rho M_{\rho\alpha}\right) \\
=& -i\left(\epsilon_{\mu\nu\beta}{}^\rho(-\frac{1}{2}\epsilon_{\rho\beta\tau\kappa}\tilde{M}^{\tau\kappa}) - \epsilon_{\mu\nu\beta}{}^\rho(-\frac{1}{2}\epsilon_{\rho\alpha\tau\kappa}\tilde{M}^{\tau\kappa})\right) = -\frac{i}{2}\left(\epsilon_{\mu\nu\beta}{}^\rho\epsilon_{\beta\tau\kappa\rho} - \epsilon_{\mu\nu\beta}{}^\rho\epsilon_{\alpha\tau\kappa\rho}\right)\tilde{M}^{\tau\kappa} \\
=& -\frac{i}{2}[-\eta_{\mu\beta}(\eta_{\nu\tau}\eta_{\alpha\kappa} - \eta_{\nu\kappa}\eta_{\tau\alpha}) - \eta_{\mu\tau}(\eta_{\nu\kappa}\eta_{\alpha\beta} - \eta_{\nu\beta}\eta_{\kappa\alpha}) - \eta_{\mu\kappa}(\eta_{\nu\beta}\eta_{\alpha\tau} - \eta_{\nu\tau}\eta_{\beta\alpha}) - \alpha \leftrightarrow \beta]\tilde{M}^{\tau\kappa} \\
=& -\frac{i}{2}[-2\eta_{\mu\beta}\tilde{M}_{\nu\alpha} - \eta_{\alpha\beta}\tilde{M}_{\mu\nu} + \eta_{\nu\beta}\tilde{M}_{\mu\alpha} - \eta_{\nu\beta}\tilde{M}_{\alpha\mu} + \eta_{\alpha\beta}\tilde{M}_{\nu\mu} - \alpha \leftrightarrow \beta] \\
=& -i[\eta_{\mu\alpha}\tilde{M}_{\nu\beta} - \eta_{\mu\beta}\tilde{M}_{\nu\alpha} + \eta_{\nu\beta}\tilde{M}_{\mu\alpha} - \eta_{\nu\alpha}\tilde{M}_{\mu\beta}]
\end{aligned}
\tag{1.63}
$$

which is the same result as equation 1.42. In other words, $\tilde{M}_{\mu\nu}$ behaves like a second rank tensor under a Lorentz transformation. Note that on the fifth line we have used the identity for 4-dimensional Levi-Civita tensors:

$$\epsilon_{\mu\nu\beta}{}^{\rho}\epsilon_{\beta\tau\kappa\rho} = -\eta_{\mu\beta}(\eta_{\nu\tau}\eta_{\alpha\kappa} - \eta_{\nu\kappa}\eta_{\tau\alpha}) - \eta_{\mu\tau}(\eta_{\nu\kappa}\eta_{\alpha\beta} - \eta_{\nu\beta}\eta_{\kappa\alpha}) - \eta_{\mu\kappa}(\eta_{\nu\beta}\eta_{\alpha\tau} - \eta_{\nu\tau}\eta_{\beta\alpha}) \tag{1.64}$$

We will use this result to prove the equation below

$$\boxed{[W_\mu, M_{\alpha\beta}] = -i(\eta_{\beta\mu}W_\alpha - \eta_{\alpha\mu}W_\beta)} \tag{1.65}$$

Proof:

$$\begin{aligned}
[W_\mu, M_{\alpha\beta}] =& [P^\nu \tilde{M}_{\mu\nu}, M_{\alpha\beta}] = P^\nu[\tilde{M}_{\mu\nu}, M_{\alpha\beta}] + \eta^{\gamma\nu}[P_\gamma, M_{\alpha\beta}]\tilde{M}_{\mu\nu} \\
=& -iP^\nu(\eta_{\mu\alpha}\tilde{M}_{\nu\beta} - \eta_{\mu\beta}\tilde{M}_{\nu\alpha} + \eta_{\nu\beta}\tilde{M}_{\mu\alpha} - \eta_{\nu\alpha}\tilde{M}_{\mu\beta}) \\
&+ i\eta^{\gamma\nu}\eta_{\gamma\alpha}P_\beta\tilde{M}_{\mu\nu} - i\eta^{\gamma\nu}\eta_{\gamma\beta}P_\alpha\tilde{M}_{\mu\nu} \\
=& i\eta_{\mu\alpha}W_\beta - i\eta_{\mu\beta}W_\alpha - iP_\beta\tilde{M}_{\mu\alpha} + iP_\alpha\tilde{M}_{\mu\beta} + iP_\beta\tilde{M}_{\mu\alpha} - iP_\alpha\tilde{M}_{\mu\beta} \\
=& i\eta_{\alpha\mu}W_\beta - i\eta_{\beta\mu}W_\alpha
\end{aligned} \tag{1.66}$$

In other words, W^μ transforms as a 4-vector under homogeneous Lorentz transformations.

Using the equation above,

$$\boxed{[W_\mu, W_\alpha] = -i\epsilon_{\mu\alpha\beta\tau}W^\beta P^\tau} \tag{1.67}$$

Proof:

$$\begin{aligned}
[W_\mu, W_\alpha] =& \frac{1}{2}\epsilon_{\alpha\beta\gamma\tau}[W_\mu, M^{\beta\gamma}P^\tau] = \frac{1}{2}\epsilon_{\alpha\beta\gamma\tau}\left([W_\mu, M^{\beta\gamma}]P^\tau + M^{\beta\gamma}[W_\mu, P^\tau]\right) \\
=& \frac{1}{2}\epsilon_{\alpha\beta\gamma\tau}\eta^{\beta\kappa}\eta^{\gamma\xi}[W_\mu, M_{\kappa\xi}]P^\tau = \frac{-i}{2}\epsilon_{\alpha\beta\gamma\tau}\eta^{\beta\kappa}\eta^{\gamma\xi}\left(\eta_{\xi\mu}W_k - \eta_{\kappa\mu}W_\xi\right)p^\tau \\
=& \frac{-i}{2}\left(\epsilon_{\alpha\beta\mu\tau}W^\beta - \epsilon_{\alpha\mu\gamma\tau}W^\gamma\right)P^\tau = \frac{-i}{2}\left(\epsilon_{\mu\alpha\beta\tau}W^\beta + \epsilon_{\mu\alpha\gamma\tau}W^\gamma\right)P^\tau \\
=& -i\epsilon_{\mu\alpha\beta\tau}W^\beta P^\tau
\end{aligned} \tag{1.68}$$

$$
\begin{aligned}
W^2 = W^\mu W_\mu =& \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma \epsilon_{\mu\alpha\beta\gamma} M^{\alpha\beta} P^\gamma = \frac{1}{4}(\epsilon^{\nu\rho\sigma\mu}\epsilon_{\alpha\beta\gamma\mu}) M_{\nu\rho} P_\sigma M^{\alpha\beta} P^\gamma \\
=& \frac{1}{4}[-\delta^\nu{}_\alpha \delta^\rho{}_\beta \delta^\sigma{}_\gamma - \delta^\nu{}_\beta \delta^\rho{}_\gamma \delta^\sigma{}_\alpha - \delta^\nu{}_\gamma \delta^\rho{}_\alpha \delta^\sigma{}_\beta \\
& + \delta^\nu{}_\beta \delta^\rho{}_\alpha \delta^\sigma{}_\gamma + \delta^\nu{}_\alpha \delta^\rho{}_\gamma \delta^\sigma{}_\beta + \delta^\nu{}_\gamma \delta^\rho{}_\beta \alpha^\sigma{}_\alpha]\ M_{\nu\rho} P_\sigma M^{\alpha\beta} P^\gamma \\
=& \frac{1}{4}[-M_{\alpha\beta}P_\gamma - M_{\beta\gamma}P_\alpha - M_{\gamma\alpha}P_\beta + M_{\beta\alpha}P_\gamma + M_{\alpha\gamma}P_\beta + M_{\gamma\beta}P_\alpha] M^{\alpha\beta} P^\gamma \\
=& -\frac{1}{2} M_{\alpha\beta} P_\gamma M^{\alpha\beta} P^\gamma + \frac{1}{2} M_{\gamma\beta} P_\alpha M^{\alpha\beta} P^\gamma + \frac{1}{2} M_{\alpha\gamma} P_\beta M^{\alpha\beta} P^\gamma \\
=& -\frac{1}{2} M_{\alpha\beta} P_\gamma M^{\alpha\beta} P^\gamma + M_{\gamma\beta} P_\alpha M^{\alpha\beta} P^\gamma \\
=& -\frac{1}{2} M_{\alpha\beta} \eta_{\tau\gamma} P^\tau M^{\alpha\beta} P^\gamma + M_{\gamma\beta} \eta_{\tau\alpha} P^\tau M^{\alpha\beta} P^\gamma \\
=& \left[-\frac{1}{2} M_{\alpha\beta}\eta_{\tau\gamma} + M_{\gamma\beta}\eta_{\tau\alpha}\right] \left[M^{\alpha\beta} P^\tau + i(\eta^{\tau\alpha} P^\beta + \eta^{\tau\beta} P^\alpha)\right] P^\gamma \\
=& -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} P^2 - \frac{i}{2}(M_{\gamma\beta} P^\beta P^\gamma + M_{\alpha\gamma} P^\alpha P^\gamma) \\
& + M_{\gamma\beta} M^{\alpha\beta} P_\alpha P^\gamma + i(4 M_{\gamma\beta} P^\beta P^\gamma + M_{\gamma\alpha} P^\alpha P^\gamma) \\
=& -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} P^2 + M_{\gamma\beta} M^{\alpha\beta} P^\gamma P_\alpha
\end{aligned}
\tag{1.69}
$$

on the sixth and tenth lines we have made use of the antisymmetry of M and equation 1.43 on the eighth. W^2 is Lorentz invariant (since it is the length of a 4-vector) and hence commutes with all the other generators.
The other Lorentz invariant entity is the mass operator $P^2 = P^\mu P_\mu$ which again, commutes with all generators. Therefore, W^2 and P^2 are the Casimir operators for the Poincaré algebra.

For a pure translation we have,

$$
U(1, a')U(1, a) = U(1, a' + a) \tag{1.70}
$$

Therefore, a total translation can be made using many infinitesimal ones *i.e.*

$$
U(1, a) = U(1, N\delta a) = \underbrace{U(1, \delta a)U(1, \delta a)\ldots U(1, \delta a)}_{\text{N times}} = [U(1, \delta a)]^N \tag{1.71}
$$

Now, we let $N \to \infty$ and expand the right hand side of the equation to first order in a using equation 1.19 to get

$$
U(1, a) = \lim_{N\to\infty} [1 + \frac{i}{N} P^\mu a_\mu]^N \tag{1.72}
$$

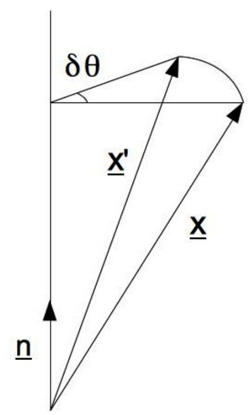

Figure 1.2: Rotation

Hence the operator for pure translations acting on the Hilbert space is

$$\boxed{U(1,a) = e^{iP^\mu a_\mu}} \tag{1.73}$$

We know that (see Figure 1.2)

$$\mathbf{x}' = \mathbf{x} + (\mathbf{n} \times \mathbf{x})\delta\theta \tag{1.74}$$

is the *active* rotation of vector $\mathbf{x}$ by an angle $\delta\theta$ about axis $\mathbf{n}$.
Now consider the case where the coordinate system is rotated by angle $\delta\theta$ rather than the vector itself. This is known as the *passive* transformation. This transformation is equivalent to an *active* rotation of the vector by $-\delta\theta$. So for the case of the rotation operator acting on the state vector we have:

$$\psi'(\underline{x}) = U(R(\theta))\psi(\underline{x}) = \psi(R^{-1}(\theta)\underline{x}) \tag{1.75}$$

and so for an infinitesimal rotation we do a Taylor expansion using the formula below for small a,

$$\psi(\underline{x} + \underline{a}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\underline{a} \cdot \underline{\nabla})^n \psi(\underline{x}) = \exp(\underline{a} \cdot \underline{\nabla})\psi(\underline{x}) \tag{1.76}$$

$$\begin{aligned}\psi(\underline{x} + (\underline{n} \times \underline{x})\delta\theta) &= \exp(\delta\theta\ (\underline{n} \times \underline{x}) \cdot \underline{\nabla})\psi(\underline{x}) = \exp(i\delta\theta\ (\underline{n} \times \underline{x}) \cdot (-i\underline{\nabla}))\psi(\underline{x}) \\ &= \exp(i\delta\theta\ (\underline{n} \times \underline{x}) \cdot \underline{p})\psi(\underline{x}) = \exp(i\delta\theta\ (\underline{x} \times \underline{p}) \cdot \underline{n})\psi(\underline{x}) \\ &= \exp(i\delta\theta\ \underline{n} \cdot \underline{J})\psi(\underline{x}) = \exp(i\underline{\delta\theta} \cdot \underline{J})\psi(\underline{x})\end{aligned} \tag{1.77}$$

where we have used $a = (\underline{n} \times \underline{x})\delta\theta$.

Knowing that any rotation is a combination of many infinitesimal ones, we have

$$\underbrace{U(\delta\theta)U(\delta\theta)\ldots U(\delta\theta)}_{\text{N times}} = U(N\delta\theta) = U(\theta) \tag{1.78}$$

Therefore,

$$\boxed{U(R(\theta),0) = e^{i\underline{J}\cdot\underline{\theta}}} \tag{1.79}$$

1.3 Connection with one-particle states

The task here is to classify the one-particle states by their transformation under the Poincaré group. First, we need to choose a label for the physical state vector. Since P^2 and W^2 are the Casimir operators for the algebra and so commute with all the other generator, it is possible to label the states by p and σ where the latter, as we will see later on, represents the spin of the particle. So for the energy-momentum operator

$$P^\mu \Psi_{p,\sigma} = p^\mu \Psi_{p,\sigma} \tag{1.80}$$

According to equation 1.73, the states transform under translations as

$$U(1,a)\Psi_{p,\sigma} = e^{i\underline{p}\cdot\underline{a}}\Psi_{p,\sigma} \tag{1.81}$$

Using equation 1.34, we have

$$\begin{aligned} P^\mu \underbrace{U(\Lambda)\Psi_{p,\sigma}} =& U(\Lambda)[U^{-1}(\Lambda)P^\mu U(\Lambda)]\Psi_{p,\sigma} = U(\Lambda)((\Lambda^{-1})_\rho{}^\mu P^\rho)\Psi_{p,\sigma} \\ =& \Lambda^\mu{}_\rho p^\rho \underbrace{U(\Lambda)\Psi_{\Lambda p,\sigma}} \end{aligned} \tag{1.82}$$

Therefore, by comparing it to equation 1.80, one can write $U(\Lambda)\Psi_{\rho,\sigma}$ as a linear combination

$$U(\Lambda)\Psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda,p)\Psi_{\Lambda p,\sigma'} \tag{1.83}$$

In general, it may be possible for C to be made block diagonal so that each block furnishes a representation of the Poincaré group.Therefore, our goal from now on is to find the irreducible representations of the Poincaré group.

Consider the *forward light cone*, for any 4-vector q^μ

$$\bar{V}_+ = \{q^\mu : q^0 \geq 0, q^2 \geq 0\} \tag{1.84}$$

Then

$$q^\nu \in \bar{V}_+ \ \Rightarrow \ q'^\mu = \Lambda^\mu{}_\nu q^\nu \in \bar{V}_+ \ , \forall \ \Lambda^\mu{}_\nu \in SO(3,1) \tag{1.85}$$

Proof:

$$q'^\mu q'_\mu = \Lambda^\mu{}_\nu q^\nu \Lambda_\mu{}^\tau q_\tau = \left(\Lambda^{-1}\right)^\tau{}_\mu \Lambda^\mu{}_\nu q^\nu q_\tau = \delta^\tau{}_\nu q^\nu q_\tau = q^\tau q_\tau = q^2 \geq 0 \tag{1.86}$$

and so the second condition is satisfied. As for the first one,

$$q'^0 = \Lambda^0{}_\nu q^\mu = \Lambda^0{}_0 q^0 + \Lambda^0{}_1 q^1 + \Lambda^0{}_2 q^2 + \Lambda^0{}_3 q^3 \tag{1.87}$$

We have the 3-vectors, $(\Lambda^0{}_1, \Lambda^0{}_2, \Lambda^0{}_3)$ which has length $\sqrt{(\Lambda^0{}_0)^2 - 1}$ by equation 1.14, and (q^1, q^2, q^3) which has length $|\underline{q}|$. Taking the scalar product

$$\begin{aligned}
&|\Lambda^0{}_1 q^1 + \Lambda^0{}_2 q^2 + \Lambda^0{}_3 q^3| \leq |\underline{q}|\sqrt{(\Lambda^0{}_0)^2 - 1} \Rightarrow \\
&|q'^0 - \Lambda^0{}_0 q^0| \leq |\underline{q}|\sqrt{(\Lambda^0{}_0)^2 - 1} \Rightarrow \\
&-|\underline{q}|\sqrt{(\Lambda^0{}_0)^2 - 1} \leq q'^0 - \Lambda^0{}_0 q^0 \Rightarrow \\
&\Lambda^0{}_0 q^0 - |\underline{q}|\sqrt{(\Lambda^0{}_0)^2 - 1} \leq q'^0
\end{aligned} \tag{1.88}$$

but since we are dealing with proper orthochronous subgroup we must have $\Lambda^0{}_0 \geq 0$ so $\Lambda^0{}_0 > \sqrt{(\Lambda^0{}_0)^2 - 1}$. Also, for light-like 4-vector invariant length $q^2 = (q^0)^2 - |\underline{q}|^2$ and given $q^0 \geq 0$ by equation 1.84, it is implied that $q^0 \geq |\underline{q}|$. Therefore, $0 \leq q'^0$ and so the first condition is also satisfied. **Q.E.D**

Note that p^2 and the sign of p^0 (for $p^2 \geq 0$) are left invariant by all proper orthochronous Lorentz transformations. So one can choose a *standard 4-momentum* k^μ and apply a Lorentz boost such that

$$p^\mu = L^\mu{}_\nu(p) k^\mu \tag{1.89}$$

and *define* states with momentum p

$$\Psi_{p,\sigma} \equiv N(p) U(L(p)) \Psi_{k,\sigma} \tag{1.90}$$

where $N(p)$ is a normalisation factor, *i.e.* a number.
When we act on $\Psi_{p,\sigma}$ with a homogeneous Lorentz transformation $U(\Lambda)$ and use the composition rule for the RHS, we get

$$\begin{aligned}
U(\Lambda)\Psi_{p,\sigma} =& N(p) U(\Lambda L(p)) \Psi_{k,\sigma} \\
=& N(p) U(L(\Lambda p)) U(L^{-1}(\Lambda p) \Lambda L(p)) \Psi_{k,\sigma}
\end{aligned} \tag{1.91}$$

According to equation 1.89, the transformation $L^{-1}(\Lambda p)\Lambda L(p)$, acting on k, takes

$$k \rightarrow p \rightarrow \Lambda p \rightarrow k \tag{1.92}$$

i.e. this transformation is a member of the subgroup of the homogeneous Lorentz group consisting of Lorentz transformations $W^\mu{}_\nu$ that leave the standard 4-momentum invariant:

$$\boxed{W^\mu{}_\nu k^\nu = k^\mu} \tag{1.93}$$

This subgroup is known as the *little group.*

1.3.1 Little group

For any W satisfying equation 1.93, we can write

$$U(W)\Psi_{k,\sigma} = \sum_{\sigma'} D_{\sigma'\sigma}(W)\Psi_{k,\sigma'} \tag{1.94}$$

Since W leaves k invariant, the only degree of freedom that can change is σ. Also, for $\bar{W}, W \in$ little group, we have

$$\begin{aligned}\sum_{\sigma'} D_{\sigma'\sigma}(\bar{W}W)\Psi_{k,\sigma'} = U(\bar{W}W)\Psi_{k,\sigma} = U(\bar{W})U(W)\Psi_{k,\sigma}\\ =U(\bar{W})\sum_{\sigma''} D_{\sigma''\sigma}(W)\Psi_{k,\sigma''} = \sum_{\sigma''} D_{\sigma''\sigma}(W)[U(\bar{W})\Psi_{k,\sigma''}]\\ =\sum_{\sigma'}\sum_{\sigma''} D_{\sigma''\sigma}(W)D_{\sigma'\sigma''}(\bar{W})\Psi_{k,\sigma'}\end{aligned} \tag{1.95}$$

Comparing the LHS with the RHS

$$D_{\sigma'\sigma}(\bar{W}W) = \sum_{\sigma''} D_{\sigma'\sigma''}(\bar{W})D_{\sigma''\sigma}(W) \tag{1.96}$$

This is precisely the definition of the representation of a group. So it can be said that $D(W)$ furnish a representation of the little group.
So for a particular element of the little group

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p)\Lambda L(p) \tag{1.97}$$

we have, using 1.91

$$\begin{aligned}U(\Lambda)\Psi_{p,\sigma} =& N(p)U(L(\Lambda p))U(W(\Lambda,p))\Psi_{k,\sigma}\\ =& N(p)\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))U(L(\Lambda p))\Psi_{k,\sigma'}\\ =& \left(\frac{N(p)}{N(\Lambda p)}\right)\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\Lambda p,\sigma'}\end{aligned} \tag{1.98}$$

where on the third line, we have used equation 1.90.
Remember that the aim is to find the coefficients $C_{\sigma\sigma'}$. So we need to find the representation of the little group *i.e.* $D(W)$ and the normalisation factors. In other words, the representations of the Poincaré group are being derived using the representations of the little group. This method is known as the method of *induced representations*.

Table 1.1 summarises different classes of 4-momenta and the corresponding little group element.

Table 1.1: Momenta and Little Group

Particle Type	Condition	Standard k^μ	Little Group
(a)Massive	$p^2 = M^2 > 0, p^0 > 0$	$(M, 0, 0, 0)$	$SO(3)$
(b)Massive	$p^2 = M^2 > 0, p^0 < 0$	$(-M, 0, 0, 0)$	$SO(3)$
(c)Massless	$p^2 = 0, p^0 > 0$	$(\kappa, 0, 0, \kappa)$	$ISO(2)$
(d)Massless	$p^2 = 0, p^0 < 0$	$(-\kappa, 0, 0, \kappa)$	$ISO(2)$
(e)Tachyonic	$p^2 = -N^2 < 0$	$(0, 0, 0, N)$	$SO(1, 2)$
(f)Vacuum	$p^\mu = 0$	$(0, 0, 0, 0)$	$SO(1, 3)$

We will now briefly discuss each case. The connection between the little group and the Pauli-Lubanski vector defined in section 1.55 and a more detailed explanation of cases (a) and (c) will be presented in the next two sections.

- Case (a): Since we are in the rest frame of a massive particle, all ordinary three dimensional rotations leave k^μ invariant. Therefore the little group is all rotations in the three spacial dimensions *i.e.* $SO(3)$.

- Case (b): This is not a physical state because the energy is negative. Since state of lowest energy is always more favourable, we keep decreasing the energy down to $-\infty$ which is not physical. So we define the vacuum state to be at zero. The rotation group is again, $SO(3)$ for the same reason as above.

- Case (c): Covers particles of mass zero, since the *mass operator* $p^2 = 0$. The energy p^0 is more than zero. Therefore it represents a physical state. As well as that, the little group is the group of Euclidean Geometry consisting of rotations and translations in two dimensions, *i.e.* $ISO(2)$.

- Case (d): Again, it does not represent a physical state since the energy $p^0 < 0$.
- Case (e): This is the space-like case and so it does not represent a physical state. The little group is the group of rotations that that keep the third direction fixed *i.e.* 2+1 dimensional rotations keeping the third axis fixed $SO(1,2)$
- Case (f): This is a physical state and it corresponds to the vacuum. The standard 4-vector is at the origin and so the all rotations in space-time leave k^μ invariant. Hence the little group is $SO(1,3)$.

Before we proceed to building the representations of the Poincaré group for the massive (a) and massless (c) cases, let us consider the normalisation of the states.
We choose the states with standard momentum k^μ to be orthonormal. Form quantum mechanics, we know that the scalar product of the eigenstates of a Hermitian operator with eigenvalues $\mathbf{k}'$ and $\mathbf{k}$ is

$$\langle\Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle = \delta^3(\mathbf{k}'-\mathbf{k})\delta_{\sigma'\sigma} \tag{1.99}$$

Therefore, using 1.94 and 1.98 , it can be said that the representations of the little group are unitary:

$$D^\dagger(W) = D^{-1}(W) \tag{1.100}$$

However, for arbitrary momenta the scalar product is

$$\begin{aligned}
\langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle &\overset{1.90}{=}\langle\Psi_{p',\sigma'}|N(p)U(L(p))\Psi_{k,\sigma}\rangle = N(p)\langle U^\dagger(L(p))\Psi_{p',\sigma'}|\Psi_{k,\sigma}\rangle \\
&=N(p)\langle U^{-1}(L(p))\Psi_{p',\sigma'}|\Psi_{k,\sigma}\rangle = N(p)\langle U(L^{-1}(p))\Psi_{p',\sigma'}|\Psi_{k,\sigma}\rangle \\
&\overset{1.98}{=}N(p)\frac{N^*(p')}{N^*(L^{-1}(p)p')}\langle\sum_{\sigma''}D_{\sigma''\sigma'}(W(L^{-1}(p),p'))\Psi_{L^{-1}(p)p',\sigma''}|\Psi_{k,\sigma}\rangle \\
&=N(p)\frac{N^*(p')}{N^*(k')}\sum_{\sigma''}D^*_{\sigma''\sigma'}(W(L^{-1}(p),p'))\langle\Psi_{k',\sigma''}|\Psi_{k,\sigma}\rangle \\
&\overset{1.99}{=}N(p)N^*(p')\sum_{\sigma''}D^*_{\sigma''\sigma'}(W(L^{-1}(p),p'))\delta_{\sigma\sigma''}\delta(\mathbf{k}'-\mathbf{k}) \\
&=N(p)N^*(p')D^*_{\sigma\sigma'}(W(L^{-1}(p),p'))\delta(\mathbf{k}'-\mathbf{k})
\end{aligned} \tag{1.101}$$

where on the fifth line we take $N^*(k) = 1$ and on the sixth line the fact that the delta function only picks up the term $\sigma = \sigma''$ from the sum.
For $p = p'$ the transformation $W(L^{-1}(p), p) = 1$ and the scalar product of the two states is

$$\langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle = |N(p)|^2\delta_{\sigma,\sigma'}\delta(\mathbf{k}'-\mathbf{k}) \tag{1.102}$$

The remaining task now is to find the proportionality factor relating $\delta(\mathbf{k}'-\mathbf{k})$ and $\delta(\mathbf{p}'-\mathbf{p})$.

Consider the integral of an arbitrary scalar function $f(p)$ over 4-momenta with $p^2 = M^2 \geq 0$ and $p^0 > 0$ *i.e.* (a) and (c),

$$\begin{aligned}
&\int d^4p \ \delta(p^2 - M^2)\theta(p^0)f(p) = \int d^3\underline{p} \ dp^0 \ \delta((p^0)^2 - \mathbf{p}^2 - M^2)\theta(p^0)f(p^0, \mathbf{p}) \\
&= \int d^3\mathbf{p} \ dp^0 \ \delta((p^0)^2 - (\sqrt{\mathbf{p}^2 + M^2})^2)\theta(p^0)f(p^0, \mathbf{p}) \\
&= \int d^3\mathbf{p} \ dp^0 \ \frac{1}{2\sqrt{\mathbf{p}^2 + M^2}}[\delta(p^0 - \sqrt{\mathbf{p}^2 + M^2}) + \delta(p^0 + \sqrt{\mathbf{p}^2 + M^2})]\theta(p^0)f(p^0, \mathbf{p}) \quad (1.103) \\
&= \int d^3\mathbf{p} \ dp^0 \ \frac{1}{2\sqrt{\mathbf{p}^2 + M^2}} \ \delta(p^0 - \sqrt{\mathbf{p}^2 + M^2})f(p^0, \mathbf{p}) \\
&= \int d^3\mathbf{p} \ \frac{1}{2\sqrt{\mathbf{p}^2 + M^2}}f(\sqrt{\mathbf{p}^2 + M^2}, \mathbf{p})
\end{aligned}$$

where on the second line we have used the identity

$$\delta(x^2 - x_0^2) = \frac{1}{2|x|}(\delta(x - x_0) + \delta(x + x_0)) \tag{1.104}$$

the appearance of the delta function $\delta(p^2 - M^2)$ is due to the condition $p^2 = M^2 \geq 0$. As well as that, the step function $\theta(p^0)$ is required to impose the condition $p^0 > 0$ since $\theta(x) = 1$ for $x \geq 0$, $\theta(x) = 0$ for $x \leq 0$. This integral is invariant under Lorentz transformations *i.e.* it is a *Lorentz scalar*. The reason for this is that $f(p)$ is a scalar and so Lorentz invariant. Also,

$$d^4p' = \underbrace{\det|\frac{\partial p'}{\partial p}|}_{1} d^4p \tag{1.105}$$

is a Lorentz invariant. Finally, the step function $\theta(p^0)$ also remains invariant under Lorentz transformations since the sign of $p^0 > 0$ does not change.
$p^2 - M^2$ is known as the *mass shell* and the invariant volume element is

$$\frac{d^3\mathbf{p}}{\sqrt{\mathbf{p}^2 + M^2}} \tag{1.106}$$

By the definition of the delta function

$$\begin{aligned}
F(\mathbf{p}) &= \int F(\mathbf{p}')\delta^3(\mathbf{p} - \mathbf{p}')d^3\mathbf{p}' \\
&= \int F(\mathbf{p}')[\sqrt{\mathbf{p'^2} + M^2} \ \delta^3(\mathbf{p}' - \mathbf{p})]\frac{d^3\mathbf{p}'}{\sqrt{\mathbf{p}'^2 + M^2}}
\end{aligned} \tag{1.107}$$

We know that the LHS is invariant and so must be the RHS. We also know that the volume element is invariant, and so the term in the brackets *i.e.*

$$\sqrt{\mathbf{p'^2}+M^2}\ \delta^3(\mathbf{p'}-\mathbf{p}) = p^0\ \delta^3(\mathbf{p'}-\mathbf{p}) \tag{1.108}$$

does not change from going from one frame to another, then

$$p^0\ \delta^3(\mathbf{p'}-\mathbf{p}) = k^0\ \delta^3(\mathbf{k'}-\mathbf{k}) \tag{1.109}$$

Hence, substituting into equation 1.99

$$\langle \Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle = |N(p)|^2 \delta_{\sigma'\sigma}\left(\frac{p^0}{k^0}\right)\delta^3(\mathbf{p'}-\mathbf{p}) \tag{1.110}$$

So in order to normalise, we take $N(p)$ to be

$$N(p) = \sqrt{\frac{k^0}{p^0}} \tag{1.111}$$

In the two sections that follow, we will derive the representations of Poincaré group for massive and massless cases (a,c) and find the connection between the corresponding little group and the Pauli-Lubanski vector.

1.3.2 Massive case

Going back to the definition of the Pauli-Lubanksi vector 1.55, in the rest frame of the particle we have the standard momentum $k^\mu = (M, 0, 0, 0)$ and so

$$W^0 = 0\ ;\ W^i = -MJ^i \tag{1.112}$$

i.e. the spacial components of W are proportional to J^i which are in turn the generators of the little group SO(3).

In general, J = orbital angular momentum + intrinsic angular momentum (spin). However, since we are now in the rest frame of the particle *i.e.* $\underline{p} = 0$, J is in fact the total spin of the particle.

$$W^2 = W^\mu W_\mu = -M\underline{J}^2 = -Ms(s+1) \tag{1.113}$$

when acting on a state with total spin s and mass M. The representation is therefore labelled by (M, s). We will write $J \equiv S$ and $s = 0, \frac{1}{2}, 1, ...$ and $\sigma = -s, ..., s$.
There are $2s+1$ possible values for σ. The set of $2s+1$ states $\{|s,\sigma\rangle\}$ is called a *multiplet*.

Acting with S_i (or more generally J_i) on this orthonormal set of vectors, results in a linear combination of $\{|s,\sigma'\rangle\}$ vectors, i.e what we have here is an *irreducible representation* which is $2s+1$ dimensional.
In other words, the unitary representation of the little group $SO(3)$ can be broken up into a direct sum of the irreducible unitary representations $D^{(j)}_{\sigma'\sigma}(R)$.
These representations are built up using matrices for infinitesimal rotations $R_{ik} = \delta_{ik} + \Theta_{ik}$ where Θ_{ik} is antisymmetric with respect to the exchange of its two indices.

$$D^{(s)}_{\sigma'\sigma}(1+\Theta) = \delta_{\sigma'\sigma} - \frac{i}{2}\Theta_{ik}(S^{(s)}_{ik})_{\sigma'\sigma} \tag{1.114}$$

From Quantum Mechanics,

$$S^2|s,\sigma\rangle = s(s+1) \tag{1.115}$$

and

$$S_3|s,\sigma\rangle = \sigma|s,\sigma\rangle \tag{1.116}$$

We also know that

$$S_\pm|s,\sigma\rangle = N_\pm|s,\sigma\rangle \tag{1.117}$$

and the factors $N_\pm$ can be found to be

$$N_\pm = \sqrt{(s\mp\sigma)(s\pm\sigma\pm 1)} \tag{1.118}$$

Hence,

$$\left(S^{(s)}_{12}\right)_{\sigma'\sigma} = \left(S^{(s)}_{3}\right)_{\sigma'\sigma} = \langle s,\sigma|S_3|s,\sigma'\rangle = \sigma\delta_{\sigma\sigma'} \tag{1.119}$$

$$\left(S^{(s)}_{\pm}\right)_{\sigma'\sigma} = \left(S^{(s)}_{23} \pm S^{(s)}_{31}\right)_{\sigma'\sigma} = \langle s,\sigma|S_1 \pm S_2|s,\sigma'\rangle = \delta_{\sigma',\sigma\pm 1}\sqrt{(s\mp\sigma)(s\pm\sigma+1)} \tag{1.120}$$

Hence the representation of the group elements obtained by taking the exponential of the generators is

$$D^{(s)}(\theta)_{\sigma\sigma'} = \langle s,\sigma|e^{i\theta\cdot\mathbf{S}}|s,\sigma'\rangle \tag{1.121}$$

For a particle of mass $M>0$ and spin s, equation 1.98 now reads

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\Lambda p,\sigma'} \tag{1.122}$$

where $W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p)$ is known as the *Wigner rotation.* In order to calculate this rotation, one can apply a 'standard boost' $L(p)$

$$L(P) = \begin{pmatrix} \gamma & \hat{p}_1\sqrt{\gamma^2-1} & \hat{p}_2\sqrt{\gamma^2-1} & \hat{p}_3\sqrt{\gamma^2-1} \\ \hat{p}_1\sqrt{\gamma^2-1} & 1+(\gamma-1)\hat{p}_1^2 & (\gamma-1)\hat{p}_1\hat{p}_2 & (\gamma-1)\hat{p}_1\hat{p}_3 \\ \hat{p}_2\sqrt{\gamma^2-1} & (\gamma-1)\hat{p}_2\hat{p}_1 & 1+(\gamma-1)\hat{p}_2^2 & (\gamma-1)\hat{p}_2\hat{p}_3 \\ \hat{p}_3\sqrt{\gamma^2-1} & (\gamma-1)\hat{p}_3\hat{p}_1 & (\gamma-1)\hat{p}_3\hat{p}_2 & 1+(\gamma-1)\hat{p}_3^2 \end{pmatrix} \tag{1.123}$$

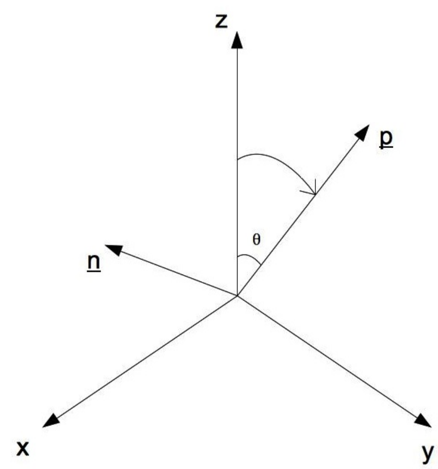

Figure 1.3: direction

such that $\hat{p}_i = \frac{p_i}{|\mathbf{p}|}$ with $\gamma = \sqrt{\frac{\mathbf{p}^2+M^2}{M}}$. (For a more detailed discussion of how $L(p)$ is actually derived, see the proof below.)

Note that when $\Lambda^\mu{}_\nu$ is an arbitrary 3-dimensional rotation $\mathscr{R}$, the Wigner rotation $W(\Lambda, p) = \mathscr{R},\ \forall p$.

Proof: We would like to go from a frame where the particle has momentum k, *i.e.* the rest frame, to a frame where the particle is viewed to have momentum p. We first make a boost, say in the third direction:

$$B(|\mathbf{p}|) = \begin{pmatrix} \gamma & 0 & 0 & \sqrt{\gamma^2-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\gamma^2-1} & 0 & 0 & \gamma \end{pmatrix} \tag{1.124}$$

where we have used the fact that $\gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \sqrt{\gamma^2-1} = -\beta\gamma$.

However, in order to get to the more general form of L we need to apply a 3-dimensional spacial rotation that takes us from the third direction into the direction of $\hat{p}$ according to Figure 1.3,

$$L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|)\mathbf{R^{-1}}(\hat{\mathbf{p}}) \tag{1.125}$$

with $R(\hat{\mathbf{p}})$ of the form

$$R(\hat{\mathbf{p}}) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_{3\times 3}(\hat{p}_i) \end{array}\right) \tag{1.126}$$

i.e. a rotation by some angle θ about axis $\underline{n}$, normal to the plane of z and $\hat{p}$ with $\underline{n} = \frac{\hat{z}\times \underline{p}}{|\underline{p}|}$ and $\cos\theta = \frac{\hat{z}\cdot \underline{p}}{|\underline{p}|}$.

Then for an arbitrary rotation $\Lambda = \mathscr{R}$, one can use equation 1.97

$$\begin{aligned} W(\mathscr{R}, p) =& L^{-1}(\mathscr{R}p)\mathscr{R}L(p) \\ =& \left[R(\mathscr{R}\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\mathscr{R}\hat{\mathbf{p}})\right]^{-1}\mathscr{R}R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) \\ =& R(\mathscr{R}\hat{\mathbf{p}})B^{-1}(|\mathbf{p}|)\underbrace{R^{-1}(\mathscr{R}\hat{\mathbf{p}})\mathscr{R}R(\hat{\mathbf{p}})}_{R(\theta)}B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) \end{aligned} \tag{1.127}$$

Note that $R^{-1}(\mathscr{R}\hat{\mathbf{p}})\mathscr{R}R(\hat{\mathbf{p}})$ takes the 3-axis into the direction of $\hat{\mathbf{p}}$ then to $\mathscr{R}\hat{\mathbf{p}}$ and finally back to the 3-axis. Since the 3-axis remains fixed under this transformation, it must be equivalent to a rotation by some angle θ about the 3-axis *i.e.*

$$R^{-1}(\mathscr{R}\hat{\mathbf{p}})\mathscr{R}R(\hat{\mathbf{p}}) = R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.128}$$

It can easily be verified that $R(\theta)$ commutes with $B(|\mathbf{p}|)$ which implies that

$$\begin{aligned} W(\mathscr{R}, p) =& R(\mathscr{R}\hat{\mathbf{p}})B^{-1}(|\mathbf{p}|)R(\theta)B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) \\ =& R(\mathscr{R}\hat{\mathbf{p}})R(\theta)R^{-1}(\hat{\mathbf{p}}) \overset{1.128}{=} \mathscr{R} \end{aligned} \tag{1.129}$$

Hence

$$\boxed{W(\mathscr{R}, p) = \mathscr{R}} \tag{1.130}$$

Q.E.D

Therefore, it can be concluded that under rotations, the states of a moving massive particle have the same transformations as that of the non-relativistic case. This can also be extended to multi-particle states.

1.3.3 Massless case

Consider the standard 4-momentum $k^\mu = (1,0,0,1)$ satisfying equation 1.93. When such a Lorentz transformation acts on a time-like 4-vector $t^\mu = (1,0,0,0)$, it results in a 4-vector Wt such that its

- Length: $(Wt)^\mu (Wt)_\mu = t^\mu t_\mu = 1$
- Scalar product with k: $(Wt)^\mu k_\mu = t^\mu k_\mu = 1$

The second condition is satisfied by any 4-vector of the form

$$(Wt)^\mu = (1+\xi, \alpha, \beta, \xi) \tag{1.131}$$

The first condition then implies that

$$\xi = \frac{\alpha^2 + \beta^2}{2} \tag{1.132}$$

Therefore, it can be said that $W^\mu{}_\nu$ acting on t has the same effect as the Lorentz transformation $S^\mu{}_\nu(\alpha,\beta)$, where S is

$$S^\mu{}_\nu(\alpha,\beta) = \begin{pmatrix} 1+\xi & \alpha & \beta & -\xi \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \xi & \alpha & \beta & 1-\xi \end{pmatrix} \tag{1.133}$$

But the question is how this matrix is actually derived. From equation 1.131, it can easily be observed that the first column of such a 4×4 matrix must be as above. Also when S acts on k it must leave it invariant and hence the last column. Now it remains to find the other two columns. We will use the fact that S is a Lorentz transformation

$$\eta_{\mu\nu} S^\mu{}_\rho S^\nu{}_\sigma = \eta_{\sigma\rho} \tag{1.134}$$

and as a result we may impose two conditions. The first one is that the 'length' of each column must be invariant and equal to 1. The second one is that the columns must be orthonormal to each other. We may also choose the block in the middle to be the identity and calculate the rest of the entires based on that. So are task is to find x, y, u, v

$$S^{\mu}{}_{\nu}(\alpha,\beta) = \begin{pmatrix} 1+\xi & x? & u? & -\xi \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \xi & y? & v? & 1-\xi \end{pmatrix} \tag{1.135}$$

For example, one can try $\rho = \sigma = 2$ Then, using the first condition

$$\begin{aligned} -1 =& S^0{}_2 S^0{}_2 - S^1{}_2 S^1{}_2 - S^2{}_2 S^2{}_2 - S^3{}_2 S^3{}_2 \\ =& u^2 - 0^2 - 1^2 - v^2 \Rightarrow u = \pm v \end{aligned} \tag{1.136}$$

Similarly by taking $\rho = \sigma = 1$ one gets $x = \pm y$. Imposing the second condition and setting $\rho = 2, \sigma = 3$ and then $\rho = 1, \sigma = 3$ we get $x = y = \alpha$ and $u = v = \beta$. Hence the result.

We mentioned that acting on t, this matrix has the same 'effect' as W. This does not mean that they are equal but it does imply that $S^{-1}(\alpha,\beta)W$ is a Lorentz transformation that leaves the vector $(1,0,0,0)$ invariant. Therefore, it can only be a pure rotation. As shown above, $S^{\mu}{}_{\nu}$ also leaves $(1,0,0,1)$ invariant, *i.e.* the 3-axis remains fixed and so $S^{-1}(\alpha,\beta)W$ must be a rotation by an arbitrary angle θ about the 3-axis

$$S^{-1}(\alpha,\beta)W = R(\theta) \tag{1.137}$$

where

$$R^{\mu}{}_{\nu}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.138}$$

From equation 1.137, it can be concluded that the most general element of the little group has the form

$$W(\theta,\alpha,\beta) = S(\alpha,\beta)R(\theta) \tag{1.139}$$

The aim now is to identify this group. It can be checked that for those transformations with $\theta = 0$ we have $R = \mathbb{1}$ and

$$S(\bar{\alpha},\bar{\beta})S(\alpha,\beta) = S(\bar{\alpha}+\alpha,\bar{\beta}+\beta) \tag{1.140}$$

and for those with $\alpha = \beta = 0 \Rightarrow S = \mathbb{1}$,

$$R(\bar{\theta})R(\theta) = R(\bar{\theta}+\theta) \tag{1.141}$$

which is also additive. These transformations forming the two subgroups above are Abelian *i.e.* their elements commute with each other. As well as that, this first subgroup mentioned is invariant such that its elements are transformed into other elements of the same subgroup by any element of the group

$$R(\theta)S(\alpha,\beta)R^{-1}(\theta) = S(\alpha\cos\theta + \beta\sin\theta, -\alpha\sin\theta + \beta\cos\theta) \tag{1.142}$$

which can easily be verified by simple matrix multiplication.
Therefore the product of any group elements can be calculated using the above equations. These multiplication rules can be identified as those of the little group $ISO(2)$ *i.e.* the group of Euclidean Geometry consisting of rotations (by angle θ) and translations (by vector (α,β)) in two dimensions.

Here is good point to stop and turn our attention to the Pauli-Lubanksi vector.
Starting from the standard 4-momentum $k^\mu = (\kappa, 0, 0, \kappa)$ and the Pauli-Lubanski vector we have,

$$W_0 = \frac{1}{2}\epsilon_{0\mu\nu 3}M^{\mu\nu}k^3 = \kappa M^{12} = \kappa J_3 = -W_3 \tag{1.143}$$

$$W_1 = -\kappa(M^{23} + M^{20}) = -\kappa(J_1 + K_2) \tag{1.144}$$

$$W_2 = \kappa(M^{31} + M^{01}) = -\kappa(J_2 - K_1) \tag{1.145}$$

from which the following can be verified:

$$[W_1, W_2] = 0 \tag{1.146}$$

$$[J_3, W_1] = \kappa[J_3, -J_1 - K_2] = i\kappa(-J_2 + K_1) = +iW_2 \tag{1.147}$$

$$[J_3, W_2] = \kappa[J_3, -J_2 + K_1] = i\kappa(J_1 + K_2) = -iW_1 \tag{1.148}$$

which is in fact the algebra of the little group $ISO(2)$ for the massless case.

Groups with no invariant Abelian subgroups are called *semi-simple* groups. They have certain simple properties, hence the name. $ISO(2)$, like the Poincaré group, is not semi-simple. We will reproduce the Lie Algebra 1.146 to 1.148 using the general little group element W and infinitesimal transformations.

$$W(\theta,\alpha,\beta)^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \tag{1.149}$$

On the other hand, $W(\theta,\alpha,\beta) = S(\alpha,\beta)R(\theta)$ so using 1.133 and 1.138 and making the approximation $\cos(\theta) \approx 1$, $\sin(\theta) \approx \theta$ and $\xi = 0$ for infinitesimal θ, α, β, we get

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \alpha & \beta & 0 \\ -\alpha & 0 & -\theta & \alpha \\ -\beta & \theta & 0 & \beta \\ 0 & -\alpha & -\beta & 0 \end{pmatrix} \tag{1.150}$$

which is antisymmetric and has 6 independent components. and so the corresponding operator

$$U(W(\theta,\alpha,\beta)) = 1 - i\alpha A - i\beta B - i\theta J_3 \tag{1.151}$$

where

$$A = -J^{13} + J^{10} = J_2 + K_1 \tag{1.152}$$

$$B = -J^{23} + J^{20} = -J_1 + K_2 \tag{1.153}$$

are Hermitian operators. Then using equations 1.48 to 1.50

$$[J_3, A] = +iB \tag{1.154}$$

$$[J_3, B] = -iA \tag{1.155}$$

$$[A, B] = 0 \tag{1.156}$$

which is precisely the same algebra expected using the Pauli-Lubanksi vector mentioned above, *i.e.* the algebra of $ISO(2)$. Because operators A and B commute, they can be simultaneously diagonalised.

$$\begin{aligned} A\Psi_{k,a,b} &= a\Psi_{k,a,b} \\ B\Psi_{k,a,b} &= b\Psi_{k,a,b} \end{aligned} \tag{1.157}$$

Consider the transformation of A and B, by observing equation 1.142 and remembering that the coefficients of A and B in equation 1.151 are α and β respectively, we can find the transformation of A and B to be as follows

$$\begin{aligned} U[R(\theta)]AU^{-1}[R(\theta)] &= A\cos\theta - B\sin\theta \\ U[R(\theta)]BU^{-1}[R(\theta)] &= A\sin\theta + B\cos\theta \end{aligned} \tag{1.158}$$

This can also be derived directly by using the matrix form of J_3, A and B, *i.e.*

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.159}$$

$$A = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}}_{J_2} + \underbrace{\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{K_1} \tag{1.160}$$

Similarly

$$B = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}}_{-J_1} + \underbrace{\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{K_2} \tag{1.161}$$

and simply going through the multiplication $U[R(\theta)]AU^{-1}[R(\theta)]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = A\cos\theta - B\sin\theta \tag{1.162}$$

as expected. The calculation is similar for B.

Define $\Psi^{\theta}_{k,a,b} \equiv U^{-1}(R(\theta))\Psi_{k,a,b}$. Then for A, we have

$$\begin{aligned} &U[R(\theta)]AU^{-1}[R(\theta)]\Psi_{k,a,b} = (A\cos\theta - B\sin\theta)\Psi_{k,a,b} \;\Rightarrow \\ &U[R(\theta)]A\Psi^{\theta}_{k,a,b} = (a\cos\theta - b\sin\theta)\Psi_{k,a,b} \;\Rightarrow \\ &A\Psi^{\theta}_{k,a,b} = (a\cos\theta - b\sin\theta)\Psi^{\theta}_{k,a,b} \end{aligned} \tag{1.163}$$

Similarly

$$B\Psi^{\theta}_{k,a,b} = (a\sin\theta + b\cos\theta)\Psi^{\theta}_{k,a,b} \tag{1.164}$$

The problem is that if one finds one set of non-zero eigenvalues of A and B, one finds a whole continuum. However, this is in contrast with the observations that suggest massless particles have no continuous degree of freedom. As a result other restrictions need to be imposed.
We must require the physical states to be eigenvectors of A and B with eigenvalues a=b=0.

$$A\Psi_{k,\sigma} = B\Psi_{k,\sigma} = 0 \tag{1.165}$$

Note that here we have labeled the states by k, σ since a and b are now fixed.
Now, in order to distinguish these states from each other, we need to consider the eigenvalues of the remaining generator J_3.

$$J_3\Psi_{k,\sigma} = \sigma\Psi_{k,\sigma} \tag{1.166}$$

Since the standard 4-momentum $k = (1, 0, 0, 1)$, the 3-momentum vector $\underline{k} = (0, 0, 1)$ is in the the 3-direction which means that σ is the component of angular momentum in the direction of motion and it is called *helicity*.

Therefore, it can be said that the states are identified by the momentum in the 3-direction and the angular momentum in the direction of motion.

Since $U(W(\theta,\alpha,\beta)) = 1 - i\alpha A - i\beta B - i\theta J_3$ for infinitesimal α,β,θ, for finite α and β we have

$$U(S(\alpha,\beta)) = e^{-i(\alpha A+\beta B)} \tag{1.167}$$

and for finite θ

$$U(R(\theta)) = e^{-iJ_3\theta} \tag{1.168}$$

The little group element W can then be written as follows using equation 1.139

$$U(W)\Psi_{k,\sigma} = e^{-i(\alpha A+\beta B)}e^{-i\theta J_3}\Psi_{k,\sigma} = e^{-i\theta\sigma}\Psi_{k,\sigma} \tag{1.169}$$

where we have used the fact that the A and B have zero eigenvalues when acting on Ψ. Hence, according to equation 1.94

$$D_{\sigma'\sigma}(W) = e^{-i\theta\sigma}\delta_{\sigma'\sigma} \tag{1.170}$$

Then, using equations 1.98 and 1.111 the Lorentz transformation rule for a massless particle with helicity σ is given by

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}}\; e^{i\sigma\theta(\Lambda,p)}\Psi_{\Lambda p,\sigma} \tag{1.171}$$

where $\theta(\Lambda,p)$ is defined by

$$W(\Lambda,p) \equiv L^{-1}\Lambda p\Lambda L(p) \equiv S(\alpha(\Lambda,p),\beta(\Lambda,p))R(\theta(\Lambda,p)) \tag{1.172}$$

The allowed values of σ are restricted to integers and half integers due to topological considerations. In order to calculate the little group element W we need to set a standard Lorentz transformation that takes $k^\mu = (\kappa,0,0,\kappa) \;\rightarrow\; p^\mu$. We choose it such that it has the following form

$$L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|/\kappa)R^{-1}(\hat{\mathbf{p}}) \tag{1.173}$$

with $B(u)$ being a pure boost in the 3-direction:

$$B(u) \equiv \begin{pmatrix} (u^2+1)/2u & 0 & 0 & (u^2-1)/2u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (u^2-1)/2u & 0 & 0 & (u^2+1)/2u \end{pmatrix} \tag{1.174}$$

One can check that this is matrix a a Lorentz transformation (boost) by comparing it with the general boost along the 3-axis

$$\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{1.175}$$

where we must have $\gamma^2 - \beta^2\gamma^2 = 1$. In our case, we may set $\frac{u^2+1}{2u} \equiv \gamma$ and $\frac{u^2-1}{2u} \equiv \beta\gamma$, then it is easy to verify that the required relation $(\frac{u^2+1}{2u})^2 - (\frac{u^2-1}{2u})^2 = 1$ holds as required by a Lorentz transformation.

On the other hand $R(\hat{\mathbf{p}})$ is a pure rotation that takes the 3-axis into the direction of motion $\hat{\mathbf{p}}$. For example in polar coordinates

$$\hat{\mathbf{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{1.176}$$

then taking $R(\hat{\mathbf{p}})$ to be a rotation by angle θ about the 2-axis, taking $(0,0,1) \to (\sin\theta, 0, \cos\theta)$, followed by a rotation about the 3-axis by angle ϕ:

$$U(R(\hat{\mathbf{p}})) = e^{-i\phi J_3} e^{-i\theta J_2} \tag{1.177}$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Note that the reason we give $U(R(\hat{\mathbf{p}}))$ rather than $R(\hat{\mathbf{p}})$ is that shifting θ or ϕ by 2π, leads to the same result for $R(\hat{\mathbf{p}})$ but picks up a minus sign for $U(R(\hat{\mathbf{p}}))$ when it acts on states with half-integer spins. Since 1.177 is a rotation that takes the 3-axis into direction $\hat{\mathbf{p}}$, any other choice of $R(\hat{\mathbf{p}})$ will be different from this one by at most an initial rotation about the 3-axis (because it keeps this axis fixed) which means that one-particle states merely differ by a phase factor.

It is very important to note that helicity is Lorentz invariant *i.e.* a massless particle of helicity, say, σ looks the same (aside from its momentum) in all inertial reference frames. The reason lies in equation 1.171. It can be observed that σ remains invariant. This is in contrast with the massive case 1.122 were σ's mix under the Lorentz transformation and hence it is not invariant and it transforms among the $2s + 1$ possible values.

Chapter 2

Fields with spin 0, 1/2, 1

The aim of the chapter is to construct Lorentz-invariant field theories using fields of spin 0, 1/2 and 1 in connection with the representation of the Lorentz group. We will then discuss spinor fields in more detail and learn how to manipulate spinor indices.

2.1 Representations of the Lorentz group using $SU(2) \times SU(2)$

Let $U(\Lambda)$, just as before, a be unitary operator corresponding to a Lorentz transformation. A scalar field operator transforms under a Lorentz transformation as follows

$$U(\Lambda)\varphi(x)U^{-1}(\Lambda) = \varphi(\Lambda^{-1}x) \tag{2.1}$$

The derivative of this field transforms as

$$U(\Lambda)\partial^\mu\varphi(x)U^{-1}(\Lambda) = \Lambda^\mu{}_\rho\bar{\partial}^\rho\varphi(\Lambda^{-1}x) \tag{2.2}$$

where we put a bar at the top of ∂ to indicated that it is the derivative with respect the argument $\bar{x} = \Lambda^{-1}x$
This implies that a *vector field* A^μ and a *tensor field* $B^{\mu\nu}$ transform as

$$U(\Lambda)A^\rho(x)U^{-1}(\Lambda) = \Lambda^\mu{}_\rho A^\rho(\Lambda^{-1}x) \tag{2.3}$$

$$U(\Lambda)B^{\mu\nu}(x)U^{-1}(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma B^{\rho\sigma}(\Lambda^{-1}x) \tag{2.4}$$

respectively.
Note that the symmetry and antisymmetry properties of $B^{\mu\nu}$ are preserved under Lorentz

transformations.
Proof: Let $B^{\mu\nu} = B^{\nu\mu}$ then

$$B^{\mu\nu} \longrightarrow B'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma B^{\rho\sigma} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma B^{\sigma\rho} = \Lambda^\nu{}_\sigma \Lambda^\mu{}_\rho B^{\sigma\rho} = B'^{\nu\mu} \tag{2.5}$$

and so the symmetry is preserved. The antisymmetric case is similar. **Q.E.D**

We define the *trace* of B to be $T(x) \equiv \eta_{\mu\nu} B^{\mu\nu}$. This entity transforms like a scalar field under Lorentz:

$$U(\Lambda)T(x)U^{-1}(\Lambda) = T(\Lambda^{-1}x) \tag{2.6}$$

Proof: Using the Lorentz condition, $\eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}$,

$$T(x) = \eta_{\mu\nu} B^{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu B^{\mu\nu} = \eta_{\rho\sigma} B'^{\rho\sigma} \tag{2.7}$$

Therefore, one can write an arbitrary tensor $B^{\mu\nu}$ as

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4}\eta^{\mu\nu}T(x) \tag{2.8}$$

where $A^{\mu\nu}$ is an antisymmetric tensor and hence traceless, $S^{\mu\nu}$ is symmetric and traceless *i.e.* taking $\eta_{\mu\nu}S^{\mu\nu} = 0$. It is important to notice that the fields $A^{\mu\nu}, S^{\mu\nu}$ and T do not mix with one another under Lorentz transformations. This is because we just proved that the symmetry properties are preserved under Lorentz transformations and T is a scalar while the other two are tensors.
The main question now is that whether this can be further decomposed into smaller irreducible representations. We will come back to this question later on in this chapter.

Consider a field $\varphi_A(x)$ where A is a Lorentz index. It transforms under Lorentz according to

$$U(\Lambda)\varphi_A(x)U^{-1}(\Lambda) = L_A{}^B(\Lambda)\varphi_B(\Lambda^{-1}x) \tag{2.9}$$

where $L_A{}^B(\Lambda)$ is a finite dimensional matrix that depends on Λ so that it results in a linear combination on the RHS of the equation. The group composition rule is obeyed

$$L_A{}^B(\Lambda')L_B{}^C(\Lambda) = L_A{}^C(\Lambda'\Lambda) \tag{2.10}$$

Therefore, $L_A{}^B(\Lambda)$ furnish the representation of the homogeneous Lorentz group, as seen before. Again, for and infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

$$U(1+\omega) = 1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} \tag{2.11}$$

with $M^{\mu\nu}$ being the generators of the Lorentz group and $\omega_{\mu\nu}$ being antisymmetric. As shown in the first chapter, the Lie algebra of the Lorentz group is specified by

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}) \tag{2.12}$$

From which the relations (1.48 to 1.50) between the angular momenta $J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk}$ and boosts $K_i \equiv M^{0i}$ are derived.
For an infinitesimal trasformation,

$$L_A{}^B(1+\omega) = \delta_A{}^B - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_A{}^B \tag{2.13}$$

Then, equation 2.9 implies

$$[M^{\mu\nu}, \varphi_A(x)] = \mathcal{L}^{\mu\nu}\varphi_A(x) + (S^{\mu\nu})_A{}^B\varphi_B(x) \tag{2.14}$$

where $\mathcal{L}^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$. Note that when μ and ν are the spacial coordinates, $\mathcal{L}^{\mu\nu}$ corresponds to orbital angular momentum.

Proof: The LHS of 2.9 to first order in ω reads

$$(1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})\varphi_A(x)(1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}) = \varphi_A(x) + \frac{i}{2}\omega_{\mu\nu}[\varphi_A(x), M^{\mu\nu}] \tag{2.15}$$

On the RHS of 2.9, we need make a taylor expansion: $\varphi_B(x - \omega x) = \varphi_B(x) - \omega_{\mu\nu}x^\nu\partial^\mu\varphi_B(x)$, then

$$\begin{aligned}&[\delta_A{}^B - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_A{}^B](\varphi_B(x) - \omega_{\mu\nu}x^\nu\partial^\mu\varphi_B(x))\\ =&\varphi_A(x) - \omega_{\mu\nu}x^\nu\partial^\mu\varphi_A(x) - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_A{}^B\varphi_B(x)\end{aligned} \tag{2.16}$$

We now make use of the antisymmetry of $\omega_{\mu\nu}$

$$-\omega_{\mu\nu}x^\nu\partial^\mu = \frac{1}{2}\left[-\omega_{\mu\nu}x^\nu\partial^\mu - \omega_{\nu\mu}x^\mu\partial^\nu\right] = \frac{1}{2}\left[-x^\nu\partial^\mu + x^\mu\partial^\nu\right]\omega_{\mu\nu} \tag{2.17}$$

Therefore, comparing the coefficients of ω on the left and the right hand sides of the equation, we have

$$[\varphi_A(x), M^{\mu\nu}] = \frac{1}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi_A(x) - (S^{\mu\nu})_A{}^B\varphi_B(x) \tag{2.18}$$

Substituting for $\mathcal{L}^{\mu\nu}$ gives the result. **Q.E.D**

It is not hard to show that both the differential operator and matrices $S^{\mu\nu}$ satisfy the same

commutation relations as generators $M^{\mu\nu}$, *i.e.* equation 2.12.
Proof: We first find the commutator

$$\begin{aligned}[x^\mu\partial^\nu, x^\rho\partial^\sigma] =&(x^\mu\partial^\nu)(x^\rho\partial^\sigma) - (x^\rho\partial^\sigma)(x^\mu\partial^\nu)\\ =&\eta^{\nu\rho}x^\mu\partial^\sigma + x^\mu x^\rho\partial^\nu\partial^\sigma - \eta^{\sigma\mu}x^\rho\partial^\nu - x^\rho x^\mu\partial^\sigma\partial^\nu\\ =&\eta^{\nu\rho}x^\mu\partial^\sigma - \eta^{\sigma\mu}x^\rho\partial^\nu\end{aligned} \tag{2.19}$$

then

$$\begin{aligned}&[\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}] = -[x^\mu\partial^\nu - x^\nu\partial^\mu, x^\rho\partial^\sigma - x^\sigma\partial^\rho]\\ =&-(\eta^{\nu\rho}x^\mu\partial^\sigma - \eta^{\sigma\mu}x^\rho\partial^\nu - \eta^{\mu\rho}x^\nu\partial^\sigma + \eta^{\sigma\nu}x^\rho\partial^\mu - \eta^{\nu\sigma}x^\mu\partial^\rho + \eta^{\rho\mu}x^\sigma\partial^\nu + \eta^{\mu\sigma}x^\nu\partial^\rho - \eta^{\rho\nu}x^\sigma\partial^\mu)\\ =&-\{\eta^{\nu\rho}(x^\mu\partial^\sigma - x^\sigma\partial^\mu) - \eta^{\sigma\mu}(x^\rho\partial^\nu - x^\nu\partial^\rho) - \eta^{\mu\rho}(x^\nu\partial^\sigma - x^\sigma\partial^\nu) + \eta^{\sigma\nu}(x^\rho\partial^\mu - x^\mu\partial^\rho)\}\\ =&\ i\{\eta^{\nu\rho}\mathcal{L}^{\mu\sigma} - \eta^{\sigma\mu}\mathcal{L}^{\rho\nu} - \eta^{\mu\rho}\mathcal{L}^{\nu\sigma} + \eta^{\sigma\nu}\mathcal{L}^{\rho\mu}\} = -i(\eta^{\mu\rho}\mathcal{L}^{\nu\sigma} - \eta^{\mu\sigma}\mathcal{L}^{\nu\rho} + \eta^{\nu\sigma}\mathcal{L}^{\mu\rho} - \eta^{\nu\rho}\mathcal{L}^{\mu\sigma})\end{aligned} \tag{2.20}$$

We know that the algebra 1.48 to 1.50 form the Lie algebra of $SO(3)$. Consider equation 1.48, from quantum mechanics we know that we can find three $(2j+1)\times(2j+1)$ hermitian matrices corresponding the the angular momenta in each direction. For example the matrix of J_3 is a diagonal matrix corresponding to an infinitesimal rotation by some angle θ about the third axis. The eigenvalues of J_3 are $m = -j, -j+1, ..., j$. The allowed values of j are 0, 1/2, 1, ... For a finite rotation, and some j (as seen in the previous chapter) we have,

$$d^{(j)}(\theta)_{m'm} = e^{im\theta}\delta_{m'm} \implies d^{(j)}(2\pi) = e^{i(2\pi m)} = \cos(2\pi m) = (-1)^{2m} = (-1)^{2j} \tag{2.21}$$

for $m = j$. Then, if j is an integer, rotation by 2π is equivalent to no rotation. However, when j is a half-integer, rotation by 2π picks up a minus sign. Therefore, the representations of the Lie algebra of $SO(3)$ is different with that of the group $SO(3)$. So we require a group that satisfies the Lie algebra given, but also solves the problem with the minus sign mentioned above. This group is known as the *Special Unitary* group or $SU(2)$. The properties of the two groups are summarised below:
We now define the operators N and $N^\dagger$ as the generators of $SU(2)$ group:

$$N_i \equiv \frac{1}{2}(J_i - iK_i) \tag{2.22}$$

$$N_i^\dagger \equiv \frac{1}{2}(J_i + iK_i) \tag{2.23}$$

and so In other words,

$$J_i = N_i + N_i^\dagger \tag{2.24}$$

and

$$K_i = i(N_i - N_i^\dagger) \tag{2.25}$$

Table 2.1: $SO(3)$ and $SU(2)$ properties

$SO(3)$	$SU(2)$
group of 3×3 orthogonal matrices	group of 2×2 unitary matrices
det = 1	det = 1
Real elements	Complex elements
$OO^T = \mathbb{1}$	$UU^\dagger = \mathbb{1}$

Equations 1.48 to 1.50 then imply that the two operators satisfy the following

$$[N_i, N_j] = i\epsilon_{ijk} N_k \tag{2.26}$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk} N_k^\dagger \tag{2.27}$$

$$[N_i, N_j^\dagger] = 0 \tag{2.28}$$

Note that we have two different $SU(2)$ algebras, one being the hermitian conjugate of the other. Hence the representation of the Lorentz group the 4 space-time dimensions is specified by **two** integers or half-integers, n and n'. These representations are labelled by (n, n'). The dimension of the representation is then determined by its number of components *i.e.* $(2n+1)(2n'+1)$. Since $J_i = N_i + N_i^\dagger$, given an n, one can determine the allowed values of j using the rule for addition of angular momenta: $j = |n - n'|, |n - n'| + 1, ..., n - n'$. The most important of these representations are given below:

$$(0, 0) \equiv \textit{scalar} \text{ or } \textit{singlet} \tag{2.29}$$

$$(\frac{1}{2}, 0) \equiv \textit{left-handed spinor} \tag{2.30}$$

$$(0, \frac{1}{2}) \equiv \textit{right-handed spinor} \tag{2.31}$$

$$(\frac{1}{2}, \frac{1}{2}) \equiv \textit{vector} \tag{2.32}$$

It is interesting to know why $(\frac{1}{2}, \frac{1}{2})$ is in fact the vector representation. Firstly, we know under a Lorentz transformation, the components of a 4-vector mix with one another and so the representation is irreducible. Secondly, a 4-vector is, of course, four dimensional so $(2n+1)(2n'+1) = 4$. The only possibilities are $(\frac{3}{2}, 0)$, $(0, \frac{3}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. However, we also know that rotating a 3-vector by 2π is equivalent to no rotation and hence it correspond to integer values of j. This implies that the first two candidates are not allowed for a 4-vector whose space components are a 3-vector. The $(\frac{1}{2}, \frac{1}{2})$ representation contains $j = 0, 1$ and so it is the only correct 4-vector representation.

2.2 Left and right handed spinor (Weyl) fields

The $(\frac{1}{2}, 0)$ representation of the Lie algebra of the Lorentz group contains the *left-handed spinor field* $\psi_a(x)$. This is also called the *left-handed Weyl field.* The index a runs from 1 to 2 and is known as the *left-handed spinor index.* ψ_a transforms under a Lorentz transformation according to

$$U(\Lambda)\psi_a(x)U^{-1}(\Lambda) = L_a{}^b(\Lambda)\psi_b(x)(\Lambda^{-1}x) \tag{2.33}$$

Note that $L_a{}^b(\Lambda)$ is a matrix in the $(\frac{1}{2}, 0)$ representation and satisfies the group composition rule

$$L_a{}^b(\Lambda')L_b{}^c(\Lambda) = L_a{}^c(\Lambda'\Lambda) \tag{2.34}$$

Again, for an infinitesimal rotation we have $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and so

$$L_a{}^b(1+\omega) = \delta_a{}^b - \frac{i}{2}\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b \tag{2.35}$$

Since $\omega_{\mu\nu}$ is antisymmetric, $(S_L^{\mu\nu})_a{}^b = -(S_L^{\nu\mu})_a{}^b$ also forms a set of 2×2 antisymmetric matrices. By the same methods seen before, it can be shown that they satisfy the same commutation relations as the generators $M^{\mu\nu}$:

$$[S_L^{\mu\nu}, S_L^{\rho\sigma}] = -i(\eta^{\mu\rho}S_L^{\nu\sigma} - \eta^{\mu\sigma}S_L^{\nu\rho} - \eta^{\nu\rho}S_L^{\mu\sigma} + \eta^{\nu\sigma}S_L^{\mu\rho}) \tag{2.36}$$

Using

$$U(1+\omega) = 1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} \tag{2.37}$$

and equation 2.33 we have

$$[M^{\mu\nu}, \psi_a(x)] = \mathcal{L}^{\mu\nu}\psi_a(x) + (S_L^{\mu\nu})_a{}^b\psi_b(x) \tag{2.38}$$

where $\mathcal{L}^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$, as before.
We will evaluate the fields at the origin of the space-time, $x^\mu = 0$, so that $\mathcal{L}^{\mu\nu} = 0$.
Recall that for the spacial components of $M^{\mu\nu}$ we have, $M^{ij} = \epsilon^{ijk}J_k$ with J_k being the angular momentum operator. So

$$\epsilon^{ijk}[J_k, \psi_a(0)] = (S_L^{ij})_a{}^b\psi_b(0) \tag{2.39}$$

In the $(\frac{1}{2}, 0)$ representation, the only allowed value of angular momentum is $j = \frac{1}{2}$. For a spin-$\frac{1}{2}$ operator, we have

$$(S_L^{ij})_a{}^b = \frac{1}{2}\epsilon_{ijk}\sigma_k \tag{2.40}$$

where σ_k, $k = 1,2,3$ are the familiar Pauli matrices, which are in fact the infinitesimal generators of $SU(2)$:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.41}$$

Note that the indices a, b run form 1 to 2 and identify each element of the 2×2 matrix, whereas i, j indices in S^{ij} are merely there to label the type of rotation we are considering. For example,(S_L^{12}) means rotation in the 1-2 plane, *i.e.* about the 3-axis. Then $(S_L^{12})_a{}^b = \frac{1}{2}\epsilon^{12k}\sigma_k = \frac{1}{2}\sigma_3$. Then, *e.g.* $(S_L^{12})_1{}^1 = +\frac{1}{2}$ and so on.
Now we need to identify the matrices for the boost operator $K_k = M^{0k}$ in the $(\frac{1}{2}, 0)$ representation. We also know that in this representation, $N_k^\dagger$ is zero since $n' = 0$. Equations 2.24 and 2.25 then imply that $K_k = iJ_k$ and so

$$(S_L^{0k})_a{}^b = \frac{1}{2} i\sigma_k \tag{2.42}$$

As we have already mentioned, hermitian conjugation swaps the two $SU(2)$ Lie algebras. Hence if one takes the hermitian conjugate of a field $\psi_a(x)$ in the $(\frac{1}{2}, 0)$ representation, it will result in a field in the $(0, \frac{1}{2})$ representation. This field is called the *right-handed spinor (Weyl) field*. We will adopt the notation

$$[\psi_a(x)]^\dagger = \psi_{\dot a}^\dagger(x) \tag{2.43}$$

for such a field. It transforms under Lorentz according to

$$U(\Lambda)\psi_{\dot a}^\dagger(x)U^{-1}(\Lambda) = R_{\dot a}{}^{\dot b}(\Lambda)\psi_{\dot b}^\dagger(x)(\Lambda^{-1}x) \tag{2.44}$$

with $R_{\dot a}{}^{\dot b}$ being a matrix in the $(0, \frac{1}{2})$ representation. The group composition rule satisfied by these matrices is

$$R_{\dot a}{}^{\dot b}(\Lambda')R_{\dot b}{}^{\dot c}(\Lambda) = R_{\dot a}{}^{\dot c}(\Lambda'\Lambda) \tag{2.45}$$

For an infinites rents transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ we have, just as before,

$$R_{\dot a}{}^{\dot b}(1 + \delta\omega) = \delta_{\dot a}{}^{\dot b} - \frac{i}{2}\delta\omega_{\mu\nu}(S_R^{\mu\nu})_{\dot a}{}^{\dot b} \tag{2.46}$$

where $(S_R^{\mu\nu})_{\dot a}{}^{\dot b} = -(S_R^{\nu\mu})_{\dot a}{}^{\dot b}$ are 2×2 matrices that obey the same commutation relation as the generators $M^{\mu\nu}$. Then

$$[M^{\mu\nu}, \psi_{\dot a}^\dagger(0)] = (S_R^{\mu\nu})_{\dot a}{}^{\dot b}\psi_{\dot b}^\dagger(0) \tag{2.47}$$

taking the hermitian conjugate, knowing that $M^{\mu\nu}$ is a hermitian operator, we get

$$[\psi_a(0), M^{\mu\nu}] = [(S_R^{\mu\nu})_{\dot a}{}^{\dot b}]^*\psi_b(0) \tag{2.48}$$

Comparing with $[M^{\mu\nu}, \psi_a(0)] = (S_L^{\mu\nu})_a{}^b \psi_b(0)$ derived before, implies that

$$(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} = -[(S_L^{\mu\nu})_a{}^b]^* \tag{2.49}$$

Let us now examine the properties of spinor indices in more detail. For this purpose, consider a field $C_{ab}(x)$ which carries two spinor indices. Under a Lorentz transformation

$$U(\Lambda) C_{ab} U^{-1}(\Lambda) = L_a{}^c L_b{}^d C_{cd}(\Lambda^{-1}x) \tag{2.50}$$

The question is whether the four components of C_{ab} can be broken into smaller part that do no mix under the Lorentz transformations. To address this question we need to refer back to quantum mechanics. We know that two particles, each with spin $\frac{1}{2}$, can be in a state of total spin zero or one. Then

- Total spin j=0: The antisymmetric combination

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{2.51}$$

- Total spin j=1: The symmetric combination

$$\begin{cases} |1,-1\rangle = |\downarrow\downarrow\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1,1\rangle = |\uparrow\uparrow\rangle \end{cases} \tag{2.52}$$

In other words, $\frac{1}{2} \otimes \frac{1}{2} = 0_A \oplus 1_S$. Note that the dimension of the left hand side is equal to $2 \times 2 = 4$ which is the same as that on the right hand side *i.e.* $1 + 3 = 4$. Therefore, for the Lorentz group we have

$$\underbrace{(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)}_{\text{dim} = 2\times 2 = 4} = \underbrace{(0,0)_A \oplus (1,0)_S}_{\text{dim} = 1+3=4} \tag{2.53}$$

Hence $C_{ab}(x)$ can be written as

$$C_{ab}(x) = \epsilon_{ab} D(x) + G_{ab}(x) \tag{2.54}$$

where $G_{ab} = G_{ba}$, $D(x)$ is a scalar field (corresponding to spin=0) and $\epsilon_{ab} = -\epsilon_{ba}$ is an antisymmetric constant defined such that $\epsilon_{21} = -\epsilon_{12} = +1$.
Under a Lorentz transformation, the RHS of the above equation becomes

$$U(\Lambda) \left[\epsilon_{ab} D(x) + G_{ab}(x)\right] U^{-1}(\Lambda) = \epsilon_{ab} D(\Lambda^{-1}x) + L_a{}^c L_b{}^d G_{cd}(\Lambda^{-1}x) \tag{2.55}$$

where we have used fact that $D(x)$ transforms as a scalar and ϵ is a constant.
On the other hand, using equation 2.50 and then 2.54 itself, we have

$$L_a{}^c L_b{}^d C_{cd}(\Lambda^{-1}x) = L_a{}^c L_b{}^d \epsilon_{cd} D(\Lambda^{-1}x) + L_a{}^c L_b{}^d G_{cd}(\Lambda^{-1}x) \tag{2.56}$$

Comparing the two, we get

$$L_a{}^c L_b{}^d \epsilon_{cd} = \epsilon_{ab} \tag{2.57}$$

which implies that ϵ_{ab} is an *invariant* under Lorentz transformations. In a sense, it is similar to the metric $\eta_{\mu\nu}$ which is also a Lorentz invariant:

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu} \tag{2.58}$$

In precisely the same way that we use $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ to raise and lower vector indices, one can use ϵ_{ab} and ϵ^{ab} to raise and lower left-handed spinor indices. We define

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{ab} \tag{2.59}$$

Then

$$\epsilon_{ab}\epsilon^{bc} = \delta_a{}^c \tag{2.60}$$

$$\epsilon^{ab}\epsilon_{bc} = \delta^a{}_c \tag{2.61}$$

One can easily check these by simple matrix multiplication. So we may define

$$\psi^a(x) \equiv \epsilon^{ab}\psi_b(x) \tag{2.62}$$

As well as that,

$$\psi_a = \epsilon_{ab}\psi^b = \epsilon_{ab}\epsilon^{bc}\psi_c = \delta_a{}^c\psi_c \tag{2.63}$$

Equation 2.62 may also be written as

$$\psi^a = \epsilon^{ab}\psi_b = -\epsilon^{ba}\psi_b = -\psi_b\epsilon^{ba} = \psi_b\epsilon^{ab} \tag{2.64}$$

It is important to be careful with the signs when contracting indices, *e.g.*

$$\psi^a\chi_a = \epsilon^{ab}\psi_b\chi_a = -\epsilon^{ba}\psi_b\chi_a = -\psi_b\chi^b \tag{2.65}$$

Using the same method, one can find an invariant symbol $\epsilon_{\dot{a}\dot{b}} = -\epsilon_{\dot{b}\dot{a}}$ for the second SU(2) factor, *i.e.* for a right-handed spinor field

$$(0, \frac{1}{2}) \otimes (0, \frac{1}{2}) = (0,0)_A \oplus (0,1)_S \tag{2.66}$$

We now have

$$\epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\dot{a}\dot{b}} \tag{2.67}$$

The same relations presented above hold if we replace all the undotted indices with the dotted ones.

Let $A_{a\dot{a}}(x)$ be a field carrying one undotted and one dotted index and so it belongs to the $(\frac{1}{2}, \frac{1}{2})$ representation. But we also said that $(\frac{1}{2}, \frac{1}{2})$ representation corresponds to the vector representation. The aim is to now write the field in the 'more natural' vector representation A^μ. For this purpose, we need to find a mapping which can be written as

$$A_{a\dot{a}}(x) = \sigma^\mu_{a\dot{a}} A_\mu(x) \tag{2.68}$$

where $\sigma^\mu_{a\dot{a}}$ is an invariant symbol. To understand why such a symbol must exist consider the an object $T^\mu_{a\dot{a}}$ that carries a Lorentz index μ as well as two left and right handed indices $a, \dot{a}$. It transforms under Lorentz according to

$$T^\mu_{a\dot{a}} = \Lambda^\mu{}_\nu L(\Lambda)_a{}^b R(\Lambda)_{\dot{a}}{}^{\dot{b}} T^\nu_{b\dot{b}} \tag{2.69}$$

so that

$$(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (0,0) \oplus ... \tag{2.70}$$

and so on the RHS we have the $(n, n') = (0,0)$ which corresponds to a singlet and so it is invariant *i.e.* when acting with N and $N^\dagger$ on such a state, we get zero. We take $\sigma^\mu_{a\dot{a}}$ to be the invariant part of $T^\mu_{a\dot{a}}$. It turns out to be possible to choose $\sigma^\mu_{a\dot{a}}$ to be

$$\sigma^\mu_{a\dot{a}} = (I, \vec{\sigma}) \tag{2.71}$$

but we need to check that this is indeed a viable choice and consistent with the conventions for $(S_L^{\mu\nu})_a{}^b$ and $(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}}$. Consider a combination of invariant symbols *e.g.* $\eta_{\mu\nu}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}}$. This whole object must also be invariant and because it carries two dotted and undotted indices, we would expect it to be proportional to $\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}$.

$$\eta_{\mu\nu}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \tag{2.72}$$

On the other hand

$$\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.73}$$

which implies

$$\sigma^\mu_{a\dot{a}}\sigma_{\mu b\dot{b}} = 2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} \tag{2.74}$$

as required. If this turned out not to be correct, 2.71 would not a a suitable choice for such a symbol. In a similar fashion, we must require $\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}}$ to be proportional to $\eta^{\mu\nu}$:

$$\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}} = 2\eta^{\mu\nu} \tag{2.75}$$

The proof of this equation and a more detailed discussion on the properties of ϵ_{ab} and its relation with $S^{\mu\nu}$ will be presented in the next section.
We can generalise the existence of such invariant symbols *i.e.* whenever the direction product of a representation includes a scalar (singlet), there exists an invariant symbol. For example from

$$(\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) = (0,0)_S \oplus (0,1)_A \oplus (1,0)_A \oplus (1,1)_S \tag{2.76}$$

we can conclude the existence of $\eta_{\mu\nu}$ since we know it is invariant under Lorentz transformations and it carries two vector indices. Another example is the four dimensional Levi-Civita symbol which is deduced using

$$(\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) = (0,0)_A \oplus ... \tag{2.77}$$

the symbol $\epsilon^{\mu\nu\rho\sigma}$ carries four vector indices and is antisymmetric on exchange of any pair of indices. It is normalised in such a way that $\epsilon^{0123} = +1$ and is invariant. This is because ${\Lambda^{\mu}}_{\alpha}{\Lambda^{\nu}}_{\beta}{\Lambda^{\rho}}_{\gamma}{\Lambda^{\sigma}}_{\tau}\epsilon^{\alpha\beta\gamma\tau}$ is antisymmetric with respect to exchange of any pair of indices that are not contracted hence it must be proportional to $\epsilon^{\mu\nu\rho\sigma}$.

This is a good point to refer back to the question raised in the previous section on whether or not a field carrying two vector indices $B^{\mu\nu}$ could be further decomposed into yet smaller irreducible representations, where we found $B^{\mu\nu}$ to be equal to

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4}\eta^{\mu\nu}T(x) \tag{2.78}$$

We will now compare this with equation 2.76. $T(x)$ is a scalar, so it correspond to the $(0,0)$ representation. $S^{\mu\nu}$ corresponds to $(1,1)_S$ since it is symmetric. Also, because it is traceless we conclude that it has $10-1=9$ independent components which is the same as the dimensionality of the $(1,1)$ representation. The antisymmetric field $A^{\mu\nu}(x)$ must therefore correspond to $(0,1)_A \oplus (1,0)_A$. However, a field in the $(0,1)_A$ representation carries two symmetric undotted indices. On the other hand, a field in the $(1,0)_A$ representation carries two symmetric dotted indices and is the hermitian conjugate of the previous field.Our aim is now to determine a mapping similar to 2.68 for $A^{\mu\nu}(x)$ in terms of a field G_{ab} and its hermitian conjugate $G^{\dagger}_{\dot{a}\dot{b}}$.
The generators $S^{\mu\nu}_L$ and $S^{\mu\nu}_R$ provide such a mapping. Note that since Pauli matrices are traceless, equations 2.40 and 2.42 imply that ${(S^{\mu\nu}_L)_a}^a = 0$. We also have

$${(S^{\mu\nu}_L)_a}^a = \epsilon^{ab}(S^{\mu\nu}_L)_{ab} = 0 \tag{2.79}$$

Then relabelling a and b and using the antisymmetry of ϵ

$$\epsilon^{ba}(S_L^{\mu\nu})_{ba} = -\epsilon^{ab}(S_L^{\mu\nu})_{ba} = 0 \tag{2.80}$$

Therefore,

$$(S_L^{\mu\nu})_{ab} = (S_L^{\mu\nu})_{ba} \tag{2.81}$$

is symmetric on the exchange of its two spinor indices. A similar method can be used to show that $(S_R^{\mu\nu})_{\dot{a}\dot{b}} = (S_R^{\mu\nu})_{\dot{b}\dot{a}}$. According to equations 2.40 and 2.42,

$$(S_L^{01})_a{}^b = \frac{1}{2}i\sigma_1 = \frac{i}{2}\epsilon^{231}\sigma_1 = i(S_L^{23})_a{}^b \Rightarrow (S_L^{\mu\nu})_a{}^b = +\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}(S_{L\rho\sigma})_a{}^b \tag{2.82}$$

where the factor of 1/2 is required since S and ϵ are antisymmetric with respect to exchange of indices ρ and σ. Taking the conjugate and using equation 2.49 we also have

$$(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}(S_{R\rho\sigma})_{\dot{a}}{}^{\dot{b}} \tag{2.83}$$

Now, consider $G_{ab}(x)$ to be a field in the $(1,0)$ representation which can be mapped into a *self-dual* antisymmetric tensor:

$$G^{\mu\nu}(x) \equiv (S_L^{\mu\nu})^{ab}G_{ab}(x) \tag{2.84}$$

A self-dual tensor is one which obeys

$$G^{\mu\nu}(x) = +\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}(x) \tag{2.85}$$

Taking the hermitian conjugate and applying equation 2.49 we get

$$G^{\dagger\mu\nu}(x) = -(S_R^{\mu\nu})^{\dot{a}\dot{b}}G^{\dagger}_{\dot{a}\dot{b}}(x) \tag{2.86}$$

which obeys the *anti-self dual* relation

$$G^{\dagger\mu\nu}(x) = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}G^{\dagger}_{\rho\sigma}(x) \tag{2.87}$$

From which one can extract the self-dual and anti-self-dual parts of the tensor field $A^{\mu\nu}$

$$G^{\mu\nu}(x) = \frac{1}{2}A^{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma} \tag{2.88}$$

$$G^{\dagger\mu\nu}(x) = \frac{1}{2}A^{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma} \tag{2.89}$$

The proof of equation 2.88 is as follows: Starting from the RHS

$$\begin{aligned}\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\left[\frac{1}{2}A_{\rho\sigma}+\frac{i}{4}\epsilon_{\rho\sigma\alpha\beta}A^{\alpha\beta}\right]&=\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}-\frac{1}{8}\left[-2\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}-\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}\right)\right]A^{\alpha\beta}\\=&\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}+\frac{1}{4}\left(A^{\mu\nu}-A^{\nu\mu}\right)=\frac{1}{2}A^{\mu\nu}+\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}\end{aligned} \tag{2.90}$$

which is self-dual, as required. The proof of equation 2.89 is similar. **Q.E.D**
Therefore

$$A^{\mu\nu}=G^{\mu\nu}+G^{\dagger\mu\nu} \tag{2.91}$$

The fields $G^{\mu\nu}$ and $G^{\dagger\mu\nu}$ are in $(1,0)$ and $(0,1)$ representations respectively.

2.3 Manipulating spinor indices

In the last section we said that the combination $\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}}=2\eta^{\mu\nu}$ is invariant (equation 2.75). We will now provide the proof of this statement:
One can easily verify the following

1. $\mu=\nu \implies$
   - $\mu=\nu=0$: On the LHS we get $\mathrm{Tr}(\mathbb{1})=2$ which is equal to the RHS *i.e.* $2\eta^{00}=2$ in our metric.
   - $\mu=\nu=i=1,2,3$: it can be checked that the LHS is -2 for all i. The RHS is $2g^{ii}=-2$, as required.
2. $\mu\neq\nu \implies$ The result on both side of the equation is zero since $\eta^{\mu\nu}$ is diagonal.

Q.E.D

Let us now find more informations about matrices $(S_L^{\mu\nu})_a{}^a$ and $(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}}$. First, we use the the invariance of the symbols ϵ_{ab}, $\epsilon_{\dot{a}\dot{b}}$ and $\sigma^{\mu}_{a\dot{a}}$.

$$\epsilon_{ab}=L(\Lambda)_a{}^cL(\Lambda)_b{}^d\epsilon_{cd} \tag{2.92}$$

On the other hand for an infinitesimal Lorentz transformation we have, as before $\Lambda^{\mu}{}_{\nu}=\delta^{\mu}{}_{\nu}+\omega^{\mu}{}_{\nu}$ and so

$$L_a{}^b(1+\omega)=\delta_a{}^b-\frac{i}{2}\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b \tag{2.93}$$

substituting this into equation 2.92 we get to first order in ω

$$\begin{aligned}\epsilon_{ab}=&\epsilon_{ab}-\frac{i}{2}\omega_{\mu\nu}\left((S_L^{\mu\nu})_a{}^c\epsilon_{cb}+(S_L^{\mu\nu})_b{}^d\epsilon_{ad}\right)\\=&\epsilon_{ab}-\frac{i}{2}\omega_{\mu\nu}\left(-(S_L^{\mu\nu})_{ab}+(S_L^{\mu\nu})_{ba}\right)\end{aligned} \tag{2.94}$$

and so the object in the bracket must be equal to zero. Hence $(S_L^{\mu\nu})_{ab} = (S_L^{\mu\nu})_{ba}$ *i.e.* it is symmetric with respect to the spinor indices. Note that this result was derived in the previous section using a different method. Similarly $(S_R^{\mu\nu})_{\dot{a}\dot{b}} = (S_R^{\mu\nu})_{\dot{b}\dot{a}}$. Now, consider $\sigma^\mu_{a\dot{a}}$ for which we have

$$\sigma^\rho_{a\dot{a}} = \Lambda^\rho{}_\tau L(\Lambda)_a{}^b R(\Lambda)_{\dot{a}}{}^{\dot{b}} \sigma^\tau_{b\dot{b}} \tag{2.95}$$

So for infinitesimal transformations one has

$$\Lambda^\rho{}_\tau = \delta^\rho{}_\tau - \frac{i}{2}\omega_{\mu\nu}(S_{\mathrm{v}}^{\mu\nu})^\rho{}_\tau \tag{2.96}$$

$$L_a{}^b(1+\omega) = \delta_a{}^b - \frac{i}{2}\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b \tag{2.97}$$

$$R_{\dot{a}}{}^{\dot{b}}(1+\delta\omega) = \delta_{\dot{a}}{}^{\dot{b}} - \frac{i}{2}\delta\omega_{\mu\nu}(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} \tag{2.98}$$

with

$$(S_{\mathrm{v}}^{\mu\nu})^\rho{}_\tau \equiv i(\eta^{\mu\rho}\delta^\nu{}_\tau - \eta^{\nu\rho}\delta^\mu{}_\tau) \tag{2.99}$$

The proof of equation 2.96 is presented below:

$$\begin{aligned}\Lambda^\rho{}_\tau =& \delta^\rho{}_\tau + \frac{1}{2}\omega_{\mu\nu}(\eta^{\mu\rho}\delta^\nu{}_\tau - \eta^{\nu\rho}\delta^\mu{}_\tau) = \delta^\rho{}_\tau + \frac{1}{2}(\omega_{\mu\tau}\eta^{\mu\rho} - \omega_{\tau\nu}\eta^{\nu\rho}) \\ =& \delta^\rho{}_\tau + \frac{1}{2}(\omega_{\mu\tau}\eta^{\mu\rho} - \omega_{\tau\mu}\eta^{\mu\rho}) = \delta^\rho{}_\tau + \omega^\rho{}_\tau\end{aligned} \tag{2.100}$$

as expected. We can now substitute equations 2.96 to 2.99 into equation 2.95

$$\begin{aligned}\sigma^\rho_{a\dot{a}} =& \left(\delta^\rho{}_\tau + \frac{1}{2}\omega_{\mu\nu}(\eta^{\mu\rho}\delta^\nu{}_\tau - \eta^{\nu\rho}\delta^\mu{}_\tau)\right)\left(\delta_a{}^b - \frac{i}{2}\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b\right)\left(\delta_{\dot{a}}{}^{\dot{b}} - \frac{i}{2}\omega_{\mu\nu}(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}}\right)\sigma^\tau_{b\dot{b}} \\ =& \left(\delta^\rho{}_\tau\delta_a{}^b\delta_{\dot{a}}{}^{\dot{b}} - \frac{i}{2}\omega_{\mu\nu}\delta^\rho{}_\tau\delta_a{}^b(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} - \frac{i}{2}\omega_{\mu\nu}\delta^\rho{}_\tau\delta_{\dot{a}}{}^{\dot{b}}(S_L^{\mu\nu})_a{}^b + \frac{1}{2}\omega_{\mu\nu}(\eta^{\mu\rho}\delta^\nu{}_\tau - \eta^{\nu\rho}\delta^\mu{}_\tau)\delta_a{}^b\delta_{\dot{a}}{}^{\dot{b}}\right)\sigma^\tau_{b\dot{b}}\end{aligned} \tag{2.101}$$

Extracting the coefficients of $\omega_{\mu\nu}$ implies

$$(\eta^{\mu\rho}\delta^\nu{}_\tau - \eta^{\nu\rho}\delta^\mu{}_\tau)\sigma^\tau_{a\dot{a}} - i(S_L^{\mu\nu})_a{}^b\sigma^\rho_{b\dot{a}} - i(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}}\sigma^\rho_{a\dot{b}} = 0 \tag{2.102}$$

Multiplying by $\sigma_{\rho c\dot{c}}$ one gets

$$\sigma^\mu_{c\dot{c}}\sigma^\nu_{a\dot{a}} - \sigma^\nu_{c\dot{c}}\sigma^\mu_{a\dot{a}} - i(S_L^{\mu\nu})_a{}^b\sigma^\rho_{b\dot{a}}\sigma_{\rho c\dot{c}} - i(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}}\sigma^\rho_{a\dot{b}}\sigma_{\rho c\dot{c}} = 0 \tag{2.103}$$

Then using 2.74 we have

$$\begin{aligned}&\sigma^\mu_{c\dot{c}}\sigma^\nu_{a\dot{a}} - \sigma^\nu_{c\dot{c}}\sigma^\mu_{a\dot{a}} - 2i(S_L^{\mu\nu})_a{}^b\epsilon_{bc}\epsilon_{\dot{a}\dot{c}} - 2i(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}}\epsilon_{ac}\epsilon_{\dot{b}\dot{c}} \\ =&\sigma^\mu_{c\dot{c}}\sigma^\nu_{a\dot{a}} - \sigma^\nu_{c\dot{c}}\sigma^\mu_{a\dot{a}} + 2i(S_L^{\mu\nu})_{ac}\epsilon_{\dot{a}\dot{c}} + 2i(S_R^{\mu\nu})_{\dot{a}\dot{c}}\epsilon_{ac} = 0\end{aligned} \tag{2.104}$$

Now multiply the equation by $\epsilon^{\dot{a}\dot{c}}$ and use the fact that $\epsilon^{\dot{a}\dot{c}}(S_R^{\mu\nu})_{\dot{a}\dot{c}} = 0$ and $\epsilon^{\dot{a}\dot{c}}\epsilon_{\dot{a}\dot{c}} = -2$ to get

$$(S_L^{\mu\nu})_{ac} = \frac{i}{4}\epsilon^{\dot{a}\dot{c}}\left(\sigma^{\mu}_{c\dot{c}}\sigma^{\nu}_{a\dot{a}} - \sigma^{\nu}_{c\dot{c}}\sigma^{\mu}_{a\dot{a}}\right) \tag{2.105}$$

Similarly, multiplying equation 2.104 by ϵ^{ac} results in

$$(S_R^{\mu\nu})_{\dot{a}\dot{c}} = \frac{i}{4}\epsilon^{ac}(\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{c\dot{c}} - \sigma^{\nu}_{a\dot{a}}\sigma^{\mu}_{c\dot{c}}) \tag{2.106}$$

Define

$$\bar{\sigma}^{\mu\dot{a}a} \equiv \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{b\dot{b}} \tag{2.107}$$

Numerically,

$$\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma}) \tag{2.108}$$

Proof:
First note that

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_2 \tag{2.109}$$

Using the above result we have

$$\bar{\sigma}^{\mu\dot{a}a} = -\epsilon^{ab}\sigma^{\mu}_{b\dot{b}}\epsilon^{\dot{b}\dot{a}} = -\left[(i\sigma_2)(\sigma^{\mu})(i\sigma_2)\right]^{a\dot{a}} = \left[(\sigma_2\sigma^{\mu}\sigma_2)^T\right]^{\dot{a}a} \tag{2.110}$$

Try $\mu = 1$:

$$\bar{\sigma}^1 = \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]^T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1 \tag{2.111}$$

The method is similar for the other components. Hence, $\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma})$. **Q.E.D**

Multiplying equations 2.105 and 2.106 by ϵ^{bc} and $\epsilon^{\dot{b}\dot{c}}$ respectively and using $\bar{\sigma}^{\mu\dot{a}a}$ we get

$$(S_L^{\mu\nu})_a{}^b = \frac{i}{4}\left(\sigma^{\mu}_{a\dot{a}}\bar{\sigma}^{\nu\dot{a}b} - \sigma^{\nu}_{a\dot{a}}\bar{\sigma}^{\mu\dot{a}b}\right) = \frac{i}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})_a{}^b \tag{2.112}$$

and

$$\begin{aligned} (S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} &= \frac{i}{4}\left(\sigma^{\mu}_{a\dot{a}}\bar{\sigma}^{\nu\dot{b}a} - \sigma^{\nu}_{a\dot{a}}\bar{\sigma}^{\mu\dot{b}a}\right) = \frac{i}{4}\left(\bar{\sigma}^{\nu\dot{b}a}\sigma^{\mu}_{a\dot{a}} - \bar{\sigma}^{\mu\dot{b}a}\sigma^{\nu}_{a\dot{a}}\right) \\ \Rightarrow (S_R^{\mu\nu})^{\dot{b}}{}_{\dot{a}} &= -\frac{i}{4}\left(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu}\right)^{\dot{b}}{}_{\dot{a}} \end{aligned} \tag{2.113}$$

We will now adopt the following convention, a missing pair of contracted undotted indices, and a missing pair of contracted dotted indices will be written as

$$\chi\psi = \chi^a\psi_a \text{ and } \chi^{\dagger}\psi^{\dagger} = \chi^{\dagger}_{\dot{a}}\psi^{\dagger\dot{a}} \tag{2.114}$$

respectively.
We also know that the Weyl fields describe spin-$\frac{1}{2}$ particles *i.e.* fermions. Hence the corresponding fields *anticommute* rather than commute:

$$\chi_a(x)\psi_b(y) = -\psi_b(y)\chi_a(x) \tag{2.115}$$

Note that switching ${}_a{}^a$ to ${}^a{}_a$ will pick up a minus sign.

Proof: We know that $\psi^a = \epsilon^{ab}\psi_b$ and $\chi_a = \epsilon_{ac}\chi^c$. So

$$\psi^a\chi_a = \epsilon^{ab}\epsilon_{ac}\psi_b\chi^c = -\epsilon^{ba}\epsilon_{ac}\psi_b\chi^c = -\delta^b{}_c\psi_b\chi^c = -\psi_b\chi^b = -\psi_a\chi^a \tag{2.116}$$

Q.E.D

Using this equation 2.114 can be written as

$$\chi\psi = \chi^a\psi_a = -\psi_a\chi^a = \psi^a\chi_a = \psi\chi \tag{2.117}$$

and the hermitian conjugate

$$(\chi\psi)^\dagger = (\chi^a\psi_a)^\dagger = (\psi_a)^\dagger(\chi^a)^\dagger = \psi^\dagger_{\dot{a}}\chi^{\dagger\dot{a}} = \psi^\dagger\chi^\dagger \tag{2.118}$$

also $\psi^\dagger\chi^\dagger = \chi^\dagger\psi^\dagger$ by 2.117. Note that when the indices are suppressed, a left-handed field is always written without a dagger and a right handed one is written with a dagger.

To understand these conventions more, let us take the hermitian conjugate of the following

$$\psi^\dagger\bar{\sigma}^\mu\chi = \psi^\dagger_{\dot{a}}\bar{\sigma}^{\mu\dot{a}c}\chi_c \tag{2.119}$$

Its transformations under Lorentz is

$$U(\Lambda)\left[\psi^\dagger\bar{\sigma}^\mu\chi\right]U^{-1}(\Lambda) = \Lambda^\mu{}_\nu\left[\psi^\dagger\bar{\sigma}^\nu\chi\right] \tag{2.120}$$

i.e. like a vector field. The hermitian conjugate is

$$\left[\psi^\dagger\bar{\sigma}^\mu\chi\right]^\dagger = \left[\psi^\dagger_{\dot{a}}\bar{\sigma}^{\mu\dot{a}c}\chi_c\right]^\dagger = \chi^\dagger_{\dot{c}}\bar{\sigma}^{\mu\dot{c}a}\psi_a = \chi^\dagger\bar{\sigma}^\mu\psi \tag{2.121}$$

where we have used the fact that $\bar{\sigma}^\mu = (I, \vec{\sigma})$ is hermitian. Note the way induces are summer on each side of the equation. Also remember that since we are dealing with it fields here, ψ and χ are the operators.

2.4 The Dirac equation

The aim of this section is to derive the Dirac equation. This equation describes spin-1/2 particles in quantum field theory and is therefore consistent with both special relativity and quantum mechanics. In order that see the big picture more easily, we have summaried the steps as follows:

- Build the algebra starting from
$$\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu] \tag{2.122}$$
and prove that $[\sigma^{\mu\nu}, \sigma^{\rho\sigma}]$ satisfies the Lorentz algebra for the generator of boosts and rotations seen before. Conclude the full representation, $S(\Lambda) = \exp[-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}]$, using the relation $S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu$.

- Show explicitly that the components of $\sigma^{\mu\nu}$ are indeed infinitesimal generators of boosts and rotations and so will yield the full space-time Lorentz transformations when exponentiated.

- Use the two dimensional representation of Lorentz for spin-1/2 particles derived previously including left handed and right handed spinors - the key point here is that at this stage, we are sure that we are describing spin-1/2 particles in a framework that is consistent with special relativity.

- Finally, consider different combination of spinors that remain invariant under such a transformation leading to the Dirac bilinears, the invariant action and Lagrangian of the theory, from which we derive the well-known Dirac equation.

Consider a set of matrices γ^μ, γ^ν which satisfy the following algebraic property:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{2.123}$$

There are different representation of the Dirac matrices that satisfy this relation, the choice here is the Weyl (Chiral) basis

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \text{and} \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.124}$$

Now, let us define
$$\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{2.125}$$

It is possible to prove that it satisfies the Lorentz algebra of the generator of boosts and rotations:
$$\boxed{[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = -i(\eta^{\mu\rho}\sigma^{\nu\sigma} - \eta^{\mu\sigma}\sigma^{\nu\rho} - \eta^{\nu\rho}\sigma^{\mu\sigma} + \eta^{\nu\sigma}\sigma^{\mu\rho})} \tag{2.126}$$

Proof:

$$\begin{aligned}
&[\sigma^{\mu\nu},\sigma^{\rho\sigma}] = -\frac{1}{16}\left([\gamma^\mu,\gamma^\nu],[\gamma^\rho,\gamma^\sigma]\right) = \\
&-\frac{1}{16}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho \\
&-\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu) \\
=&-\frac{1}{16}(2\eta^{\nu\rho}[\gamma^\mu,\gamma^\sigma] - \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma + \gamma^\sigma\gamma^\nu\gamma^\rho\gamma^\mu - 2\eta^{\nu\sigma}[\gamma^\mu,\gamma^\rho] + \gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu \\
&-2\eta^{\mu\rho}[\gamma^\nu,\gamma^\sigma] + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma - \gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu + 2\eta^{\mu\sigma}[\gamma^\nu,\gamma^\rho] - \gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu) \\
=&-\frac{1}{16}(2\eta^{\nu\rho}[\gamma^\mu,\gamma^\sigma] - 2\eta^{\nu\sigma}[\gamma^\mu,\gamma^\rho] - 2\eta^{\mu\rho}[\gamma^\nu,\gamma^\sigma] + 2\eta^{\mu\sigma}[\gamma^\nu,\gamma^\rho] \\
&+\ 2\eta^{\nu\rho}[\gamma^\mu,\gamma^\sigma] - 2\eta^{\nu\sigma}[\gamma^\mu,\gamma^\rho] - 2\eta^{\mu\rho}[\gamma^\nu,\gamma^\sigma] + 2\eta^{\mu\sigma}[\gamma^\nu,\gamma^\rho] \\
&+\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\mu\gamma^\sigma\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu \\
&-\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\mu\gamma^\sigma\gamma^\rho\gamma^\nu + \gamma^\sigma\gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma) \\
=&\frac{-1}{16}(4\eta^{\nu\rho}[\gamma^\mu,\gamma^\sigma] - 4\eta^{\nu\sigma}[\gamma^\mu,\gamma^\rho] - 4\eta^{\mu\rho}[\gamma^\nu,\gamma^\sigma] + 4\eta^{\mu\sigma}[\gamma^\nu,\gamma^\rho]) \\
=&-i(\eta^{\mu\rho}\sigma^{\nu\sigma} - \eta^{\mu\sigma}\sigma^{\nu\rho} - \eta^{\nu\rho}\sigma^{\mu\sigma} + \eta^{\nu\sigma}\sigma^{\mu\rho})
\end{aligned} \tag{2.127}$$

where, going from the second equality to the third one, we have used the anticommutation relation to swap the two middle gammas in each term and manipulated the result. **Q.E.D**

Therefore, for boosts we have

$$\sigma^{0i} = \frac{i}{4}[\gamma^0,\gamma^i] = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \frac{i}{2}\gamma^0\gamma^i \tag{2.128}$$

Note that it is non-unitary, *i.e.* $(S^{0i})^\dagger \neq (S^{0i})^{-1}$. For rotations,

$$\sigma^{ij} = \frac{i}{4}[\gamma^i,\gamma^j] = -\frac{i}{4}\begin{pmatrix} 2i\epsilon_{ijk}\sigma_k & 0 \\ 0 & 2i\epsilon_{ijk}\sigma_k \end{pmatrix} = \frac{1}{2}\epsilon_{ijk}\Sigma_k \tag{2.129}$$

where $\underline{\Sigma} = \begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix}$ which is hermitian and unitary.

Before we go any further, first consider the Lorentz transformation

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = -i(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha) \tag{2.130}$$

where μ and ν determine which of the six matrices for boost and rotation we are dealing with and α and β run through the elements of that matrix. These matrices act on ordinary

Lorentz for vectors. One can see this by parameterising the infinitesimal transformation for a 4-vector V,

$$V'^{\alpha} = \left(\delta^{\alpha}{}_{\beta} - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}\right) V^{\beta} \tag{2.131}$$

Let the angle of rotation be $\omega_{12} = -\omega_{21} = \theta$ with all the other components of ω being zero. Using equation 2.130 we get

$$V' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V \tag{2.132}$$

Which is a rotation about the z-axis (in xy-plane) by a small angle θ so that $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$. Similarly for a boost, let $\omega_{01} = -\omega_{10} = \phi$ to get an infinitesimal boost in the x direction

$$V' = \begin{pmatrix} 1 & -\phi & 0 & 0 \\ -\phi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V \tag{2.133}$$

Now that we know $\mathcal{J}^{\mu\nu}$ describes a Lorentz transformation let us prove the following relation which will soon become useful in deriving the full spinor representation of the Lorentz group:

$$\boxed{[\sigma^{\rho\sigma}, \gamma^{\mu}] = (\mathcal{J}^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu}} \tag{2.134}$$

Proof: On the LHS we have,

$$\begin{aligned}
[\gamma^{\mu}, \sigma^{\rho\sigma}] =& \frac{i}{4}[\gamma^{\mu}, [\gamma^{\rho}, \gamma^{\sigma}]] = \frac{i}{4}[\gamma^{\mu}, \gamma^{\rho}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\rho}] = \frac{i}{4}(\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma} - \gamma^{\mu}\gamma^{\sigma}\gamma^{\rho} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu} + \gamma^{\sigma}\gamma^{\rho}\gamma^{\mu}) \\
=& \frac{i}{4}(2\eta^{\mu\rho}\gamma^{\sigma} - \gamma^{\rho}\gamma^{\mu}\gamma^{\sigma} - 2\eta^{\sigma\mu}\gamma^{\rho} + \gamma^{\sigma}\gamma^{\mu}\gamma^{\rho} - 2\eta^{\sigma\mu}\gamma^{\rho} + \gamma^{\rho}\gamma^{\mu}\gamma^{\sigma} + 2\eta^{\mu\rho}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}) \\
=& i(\eta^{\mu\rho}\gamma^{\sigma} - \eta^{\sigma\mu}\gamma^{\rho})
\end{aligned} \tag{2.135}$$

while on the RHS

$$\begin{aligned}
(\mathcal{J}^{\rho\sigma})^{\mu}{}_{\beta}\gamma^{\beta} =& \eta^{\mu\alpha}(\mathcal{J}^{\rho\sigma})_{\alpha\beta}\gamma^{\beta} = -i\eta^{\mu\alpha}\left(\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta} - \delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}\right)\gamma^{\beta} \\
=& -i\eta^{\mu\alpha}\left(\delta^{\rho}{}_{\alpha}\gamma^{\sigma} - \delta^{\sigma}{}_{\alpha}\gamma^{\rho}\right) = -i\eta^{\mu\beta}\gamma^{\sigma} + i\eta^{\sigma\mu}\gamma^{\rho} = -\text{LHS}
\end{aligned} \tag{2.136}$$

Hence the result. **Q.E.D**

To first order in ω this relation can be written as

$$\left(1 - \frac{i}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right)\gamma^{\mu}\left(1 + \frac{i}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right) = \left(1 - \frac{i}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right)^{\mu}{}_{\nu}\gamma^{\nu} \tag{2.137}$$

which is the infinitesimal version of

$$\underbrace{\left(e^{-\frac{i}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}}\right)}_{S(\Lambda)}\gamma^{\mu}\underbrace{\left(e^{\frac{i}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}}\right)}_{S(\Lambda)^{-1}} = \underbrace{\left(e^{-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}}\right)^{\mu}{}_{\nu}}_{\Lambda^{\mu}{}_{\nu}}\gamma^{\nu} \tag{2.138}$$

i.e. the gamma matrices satisfy

$$\boxed{S(\Lambda)\gamma^{\mu}S(\Lambda)^{-1} = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}} \tag{2.139}$$

where $S(\Lambda) = \left(e^{-\frac{i}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}}\right)^{\mu}{}_{\nu}$ is the full spinor representation of the Lorentz group.

It is possible to write down $S(\Lambda)$ explicitly for the case of rotations and boost. Recall that for rotations we had $\sigma^{ij} = \frac{1}{2}\epsilon_{ijk}\Sigma_k$ which we can insert in $S_{\mathrm{Rot}}(\Lambda) = e^{-\frac{i}{2}\omega_{ij}\sigma^{ij}}$. Now, let $\omega_{ij} = -\epsilon_{iim}\theta^m$ and use the identity $\epsilon_{ijm}\epsilon_{ijk} = 2\delta_{jm}$ to get

$$\boxed{S_{\mathrm{Rot}}(\Lambda) = \exp\left(i\frac{\Sigma.\theta}{2}\right)} \tag{2.140}$$

It would be interesting to actually see the $S(\Lambda)$ *is* different from the Lorentz transformation Λ. Take for example rotation about the z-axis by an angle $\omega_{12} = -\omega_{21} = -\theta_3$, then $S_{\mathrm{Rot}}(\Lambda) = \exp i\Sigma_3\frac{\theta_3}{2}$. Now, let the angle $\theta = 2\pi$. Then, for a general Lorentz transformation acting on a 4-vector one would have

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} \tag{2.141}$$

while

$$S(\Lambda) = \begin{pmatrix} e^{i\pi\sigma_3} & 0 \\ 0 & e^{i\pi\sigma_3} \end{pmatrix} = \begin{pmatrix} \cos\pi + i\Sigma_3\sin\pi & 0 \\ 0 & \cos\pi + i\Sigma_3\sin\pi \end{pmatrix} = -\mathbb{1} \tag{2.142}$$

which means that rotating a vector by 2π will result in the original one while in the case of a spinor it picks up a minus sign. In other words, for spinors, one needs to rotate by 720 degrees to obtain the original result.

Here is a summery of rotations about the x, y and z axes through an angle $-\theta$ for a two-component spinor:

- $$\boxed{\exp\left(-i\frac{\sigma_1\theta}{2}\right) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}} \tag{2.143}$$

  Proof:

  $$\begin{aligned} &\exp\left(-i\frac{\theta}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \\ =&\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(-i\frac{\theta}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2!}\left(-i\frac{\theta}{2}\right)^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3!}\left(-i\frac{\theta}{2}\right)^3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4!}\left(-i\frac{\theta}{2}\right)^4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + ... \\ =&\begin{pmatrix} 1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 - ... & -i\theta/2 + i\frac{1}{3!}\left(\frac{\theta}{2}\right)^3 - ... \\ = -i\theta/2 + i\frac{1}{3!}\left(\frac{\theta}{2}\right)^3 - ... & 1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 - ... \end{pmatrix} \\ =&\begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \end{aligned}$$

- $$\boxed{\exp\left(-i\frac{\sigma_2\theta}{2}\right) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}} \tag{2.144}$$

  Proof: Similar to above. Note this has exactly the same form as the usual rotation matrix in 2 dimensions except it is by half the original angle.

- $$\boxed{\exp\left(-i\frac{\sigma_3\theta}{2}\right) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}} \tag{2.145}$$

  Proof: Follows straight form the fact that σ_3 is diagonal, so and one can simply exponential each element on the diagonal.

We can derive a similar relation to 2.140 for boosts by recalling that $\sigma^{0i} = \frac{i}{2}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} = \frac{i}{2}\gamma^0\gamma^i = \frac{i}{2}\alpha^i$. Now, let $\omega_{0i} = -\omega_{i0} = \phi_i$ and substitute into $S(\Lambda) = e^{-\frac{i}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}}$ to get

$$\boxed{S_{\text{Boost}}(\Lambda) = \exp\left(\frac{\underline{\alpha}.\underline{\phi}}{2}\right)} \tag{2.146}$$

where $\alpha = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix}$ and $\underline{\phi} = \phi\hat{\underline{\phi}}$.

Consider now the commutator $[\sigma^{\rho\sigma}, \gamma^\mu] = -i(\eta^{\mu\rho}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho)$. For rotations,

$$[\sigma^{ij}, \gamma^0] = 0 \tag{2.147}$$

i.e. γ^0 commutes with the generator of rotations. Also, since rotations are unitary, one can conclude that the full representation for rotations trivially satisfies

$$S_{\text{Rot}}(\Lambda)^{-1} = \gamma^0 S_{\text{Rot}}(\Lambda)^\dagger \gamma^0 \tag{2.148}$$

This is not the case for boosts because

$$[\sigma^{0i}, \gamma^0] = -i\gamma^i \neq 0 \tag{2.149}$$

However, if we take the anticommutator and set $\rho = 0$, $\sigma = i$, $\mu = 0$

$$\{\sigma^{\rho\sigma}, \gamma^\mu\} = \frac{i}{4}\{[\gamma^\rho, \gamma^\sigma], \gamma^\mu\} = \frac{i}{4}(\gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\sigma\gamma^\rho\gamma^\mu + \gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\rho) \Rightarrow = 0 \tag{2.150}$$

i.e. γ^0 anticommutes with the generator of boosts. As well as that, $S_{\text{Boost}}(\Lambda)^{-1} = \exp\left(-\frac{\alpha.\phi}{2}\right)$ while, $S_{\text{Boost}}(\Lambda)^\dagger = S_{\text{Boost}}(\Lambda)$. By simply expanding the exponential and using the above relation one can easily show that

$$S_{\text{Boost}}(\Lambda)^{-1} = \gamma^0 S_{\text{Boost}}(\Lambda)^\dagger \gamma^0 \tag{2.151}$$

So in general,

$$\boxed{S(\Lambda)^{-1} = \gamma^0 S(\Lambda)^\dagger \gamma^0} \tag{2.152}$$

So far, using special relativity, we have built the spinor representation of the Lorentz group and examined its properties. The next goal is to show that this representation is indeed related to physical spin-1/2 particles and consistent with quantum mechanics. However, this has already been discussed in great detail in the previous section when we built the $SU(2) \times SU(2)$ representation. Recall the following:

$$[N_i, N_j] = i\epsilon_{ijk}N_k \tag{2.153a}$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger \tag{2.153b}$$

$$[N_i, N_j^\dagger] = 0 \tag{2.153c}$$

where $K_i = i(N_i - N_i^\dagger)$, $J_i = N_i + N_i^\dagger$ and we label the two angular momenta by (j, j'). Here though, we are concerned with the left and right handed spinor representations *i.e.* $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Consider a general Lorentz transformation consisting of both a boost and a rotation,

$$\Lambda = \exp(i\underline{J}.\underline{\theta} + i\underline{K}.\underline{\phi}) \tag{2.154}$$

which can be written as

$$\Lambda = \exp\Big(i\underline{N}.(\underline{\theta} + i\underline{\phi})) + i\underline{N}^\dagger.(\underline{\theta} - i\underline{\phi}))\Big) \tag{2.155}$$

For the left handed spinor $(\frac{1}{2}, 0)$, the field corresponding to $\underline{N}$, say ψ_L transforms as a spinor while the other one, ψ_R, transforms as a scalar. This implies that we have to take $\underline{N}^\dagger = 0$ which means $J = -iK$. But also, $\underline{J} = \frac{1}{2}\underline{\sigma}$ (see 2.40) which implies that $K = \frac{i}{2}\underline{\sigma}$. We therefore have,

$$\psi'_L = \exp\left(\frac{i}{2}(\underline{J} - i\underline{K}).(\underline{\theta} + i\underline{\phi})\right)\psi_L = \exp\left(\frac{i}{2}\underline{\sigma}.(\underline{\theta} + i\underline{\phi})\right)\psi_L \tag{2.156}$$

Similarly for $(0, \frac{1}{2})$, we have $\underline{N} = 0$ so that $J = iK$. In this case

$$\psi'_R = \exp\left(\frac{i}{2}(\underline{J} + i\underline{K}).(\underline{\theta} - i\underline{\phi})\right)\psi_R = \exp\left(\frac{i}{2}\underline{\sigma}.(\underline{\theta} - i\underline{\phi})\right)\psi_R \tag{2.157}$$

Thus, putting them together as $\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$ the full spinor transforms as

$$\begin{pmatrix}\psi'_L \\ \psi'_R\end{pmatrix} = \begin{pmatrix}\exp\left(\frac{i}{2}\underline{\sigma}.(\underline{\theta} + i\underline{\phi})\right) & 0 \\ 0 & \exp\left(\frac{i}{2}\underline{\sigma}.(\underline{\theta} - i\underline{\phi})\right)\end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \tag{2.158}$$

Finally, we are ready to build the invariant Lagrangian. There are several combinations of the spinor field ψ and γ matrices that have certain invariance properties under Lorentz transformations. This will result in the notion of Dirac bilinears which in turn can be used to construct a suitable Lagrangian for the spinor field. Here we will only limit ourselves to the simplest case. It is easy to see that $\bar{\psi}\psi$ transforms as a scalar using equation 2.152:

$$\begin{aligned}\bar{\psi}'(x')\psi'(x') =& \psi'^\dagger(x')\gamma^0\psi'(x') = \psi^\dagger(x)S^\dagger(\Lambda)\gamma^0 S(\Lambda)\psi(x) \\ =& \psi^\dagger(x)\gamma^0 S(\Lambda)^{-1}S(\Lambda)\psi(x) = \bar{\psi}(x)\psi(x)\end{aligned} \tag{2.159}$$

Another possibility is $\bar{\psi}\gamma_\mu\partial^\mu\psi$. First note that equation 2.139 implies $\gamma_\mu = \Lambda_\mu{}^\nu S(\Lambda)^{-1}\gamma_\nu S(\Lambda)$. Then

$$\begin{aligned}\bar{\psi}'(x')\gamma_\nu\partial'^\nu\psi'(x') =& \psi'^\dagger(x')\gamma^0\gamma_\nu\partial'^\nu\psi'(x') = \psi^\dagger(x)S^\dagger(\Lambda)\gamma^0\gamma_\nu\Lambda_\mu{}^\nu\partial^\mu S(\Lambda)\psi(x) \\ \overset{2.152}{=}& \bar{\psi}(x)\underbrace{S(\Lambda)^{-1}\gamma_\nu\Lambda_\mu{}^\nu S(\Lambda)}\partial^\mu\psi(x) = \bar{\psi}(x)\gamma_\mu\partial^\mu\psi(x)\end{aligned} \tag{2.160}$$

The invariant Lagrangian can therefore be written as:

$$\boxed{\mathcal{L} = \bar{\psi}(x)\left(i\gamma_\mu\partial^\mu - m\right)\psi(x)} \tag{2.161}$$

which, by making use of Euler-Lagrange equations, will imply the Dirac equation:

$$\boxed{(i\gamma_\mu\partial^\mu - m)\psi(x) = 0} \tag{2.162}$$

Chapter 3

Supersymmetry

In this chapter we will first discuss the Coleman-Mandula theorem and the reasons why the extension to graded algebras are required. Then we will present the supersymmetric algebras and finally their representations for different types of particles.

3.1 Extension to graded algebras

There are two different types of symmetries for elementary particles:

1. *Space-time symmetries:* These symmetries correspond to the transformation of space-time coordinates. For example rotations and translations *i.e.* Poincaré transformations.

2. *Internal symmetries:* In a field theory, these symmetries are related to the transformations of the fields *e.g.* $\psi_a(x) \rightarrow \psi'_a(x) = L_a{}^b \psi_b(x)$.

In 1971, a *no-go theorem* by Coleman and Mandula stated that irreducible multiplets of the Poincaré symmetry group cannot contain particles of different mass or spin. They showed that the general symmetry of the S-matrix is the direct product of the Poincaré symmetry with an internal symmetry group. In other words, the general Lie algebra of the symmetries of the S-matrix contains

- The energy-momentum operator P^μ
- The Lorentz rotation generator $M^{\mu\nu}$
- A finite number of Lorentz scalar operators B_ℓ which must belong to the Lie algebra of a compact Lie group.

In 1975, however, Haag, Sohnius and Lopuszanski extended the results of Coleman and Mandula to include symmetry operations obeying fermi statistics and anticommutator relations defined by $\{A, B\} = AB + BA$. They found that space-time symmetries and internal symmetries can only be related to each other by a fermonic symmetry operator Q of spin-$\frac{1}{2}$ (not $\frac{3}{2}$ or higher). They concluded that the multiplets can contain particles of different spins only in the presence of supersymmetry. Supersymmetry (SUSY), by definition, is a symmetry between fermions and bosons. Having said that, since spin is related to behaviour under spacial rotations, it is really a space-time symmetry. SUSY has therefore the property of combining internal and space-time symmetries in order to attempt to unify all fundamental interactions.

Let Q, Q' and Q'' represent the fermionic (anticommuting) part of the algebra while X, X' and X'' represent the bosonic (commuting) part. Then, schematically we have

$$\{\text{fermionic}, \text{fermionic}\} = \text{bosonic} \implies \{Q, Q'\} = X \tag{3.1}$$

$$[\text{bosonic}, \text{bosonic}] = \text{bosonic} \implies [X, X'] = X'' \tag{3.2}$$

$$[\text{fermionic}, \text{bosonic}] = \text{fermionic} \implies [Q, X] = Q'' \tag{3.3}$$

The bosonic operators X are determined by the Coleman-Mandula theorem *i.e.* they are either elements of the Poincaré algebra $\mathscr{P} = \{P^\mu, M^{\mu\nu}\}$ or the elements of the compact Lie algebra $\mathscr{A}$ which are Lorentz invariant. The algebra $\mathscr{A}$ is a direct sum of a semi-simple algebra $\mathscr{A}_1$ and an Abelian algebra $\mathscr{A}_2$ *i.e.* $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$.

Let us now explore the compact Lie group and the relation between the internal symmetries in more detail. Given a compact Lie group one can define a *connected Lie gorup* which has special importance as we shall see. These are groups of transformations $T(\theta)$, described by a finite set of real continuous parameters, θ^a, such that every element of the group is connected to the identity by a path within the group. The elements of the group must then satisfy the composition rule

$$T(\bar{\theta})T(\theta) = T\left(f(\bar{\theta}, \theta)\right) \tag{3.4}$$

where $f^a(\bar{\theta}, \theta)$ is a function of $\bar{\theta}$s and θs. By taking $\theta^a = 0$ to be the coordinates of the identity, we get

$$f^a(\theta, 0) = f^a(0, \theta) = \theta^a \tag{3.5}$$

One can then taylor expand the unitary operator corresponding to this transformation in the neighbourhood of the identity as follows:

$$U(T(\theta)) = 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + \ldots \tag{3.6}$$

where t_a, $t_{bc} = t_{cb}$ are operators which are independent of θs and the factor of i exist in order to make sure that t_a is a hermitian. Explicitly,

$$t_a = -i\frac{\partial}{\partial\theta^a}U(T(\theta))\Big|_{\theta=0} \tag{3.7}$$

$$t_{bc} = (-i)(-i)\frac{\partial^2}{\partial\theta^b\partial\theta^c}U(T(\theta))\Big|_{\theta=0} \tag{3.8}$$

which is why t_{bc} is symmetric with respect of its two indices. Suppose $U(T(\theta))$ form an ordinary representation of this group so that

$$U\left(T(\bar\theta)\right)U\left(T(\theta)\right) = U\left(T\left(f(\bar\theta,\theta)\right)\right) \tag{3.9}$$

Now expanding $f^a(\bar\theta,\theta)$ to second order and using equation 3.5 we get

$$\begin{aligned} f^a(\bar\theta,\theta) =& \underbrace{f(0,0)}_{0} + \underbrace{\frac{\partial f}{\partial\theta^a}}_{1}\theta^a + \underbrace{\frac{\partial f}{\partial\bar\theta^a}}_{1}\bar\theta^a + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial\theta^2}}_{0}\theta^2 + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial\bar\theta^2}}_{0}\bar\theta^2 + \frac{1}{2}\frac{\partial^2 f}{\partial\bar\theta^b\theta^c} + \dots \\ =& \theta^a + \bar\theta^a + f^a{}_{bc}\bar\theta^b\theta^c + \dots \end{aligned} \tag{3.10}$$

since the presence of θ^2 and $\bar\theta^2$ would violate equation 3.5. Note that the coefficients $f^a{}_{bc}\bar\theta^b\theta^c$ are real and there is a summation between indices. The left hand side of equation 3.9 can now be written as

$$\begin{aligned} &\left[1 + i\bar\theta^a t_a + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc} + \dots\right]\left[1 + i\theta^{a'}t_{a'} + \frac{1}{2}\theta^{b'}\theta^{c'}t_{b'c'} + \dots\right] \\ =& 1 + i\theta^{a'}t_{a'} + \frac{1}{2}\theta^{b'}\theta^{c'}t_{b'c'} + i\bar\theta^a t_a - \bar\theta^a\theta^{a'}t_a t_{a'} + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc} + \dots \end{aligned} \tag{3.11}$$

and the RHS

$$\begin{aligned} &1 + i\left(\theta^a + \bar\theta^a + f^a{}_{bc}\bar\theta^b\theta^c + \dots\right)t_a + \frac{1}{2}\left(\theta^b + \bar\theta^b + \dots\right)\left(\theta^c + \bar\theta^c + \dots\right)t_{bc} \\ =& 1 + i\theta^a t_a + i\bar\theta^a t_a + if^a{}_{bc}\bar\theta^b\theta^c t_a + \frac{1}{2}\left(\theta^b\theta^c + \theta^b\bar\theta^c + \bar\theta^b\theta^c + \bar\theta^b\bar\theta^c\right)t_{bc} + \dots \end{aligned} \tag{3.12}$$

Equating, relabelling and using the fact that $t_{bc} = t_{cb}$, we get

$$-\bar\theta^b\theta^c t_b t_c = if^a{}_{bc}\bar\theta^b\theta^c t_a + \frac{1}{2}\bar\theta^b\theta^c t_{bc} + \frac{1}{2}\bar\theta^b\theta^c t_{bc} \tag{3.13}$$

Comparing coefficients results in

$$t_{bc} = -t_b t_c - if^a{}_{bc}t_a \tag{3.14}$$

Thus, given the function $f(\bar\theta,\theta)$, which determines the structure of the group, and the quadratic coefficients $f^a{}_{bc}$ one can calculate the second order terms in $U(T(\theta))$ from the generators t_a which appear in the first order terms. We also must have $t_{bc} = t_{cb}$. Then

$$\begin{aligned} [t_b, t_c] = t_b t_c - t_c t_b =& -t_{bc} - if^a{}_{bc}t_a + t_{cb} + if^a{}_{cb}t_a = -if^a{}_{bc}t_a + if^a{}_{cb}t_a \\ \equiv& iC^a{}_{bc}t_a \end{aligned} \tag{3.15}$$

where ${C^a}_{bc}$ are a set of constants known as the *structure constatns*.
Therefore, provided that we know the first order term generators t_a, we can derived the complete power series $U(T(\theta))$ using an infinite sequence of 3.14.

Now, consider the case where the function $f(\theta, \bar{\theta})$ obeys

$$f^a(\theta, \bar{\theta}) = \theta^a + \bar{\theta}^a \tag{3.16}$$

i.e. is additive. In this case the structure constants vanish and the generators commute:

$$[t_b, t_c] = 0 \tag{3.17}$$

we then say that the group is commutative or *Abelian*.

Going back to our discussion of the internal symmetry group, recall that we said B_ℓ are a finite number of Lorentz scalar operators that belong to the Lie algebra of a compact Lie group. Hence the following relations can be concluded:

$$\boxed{[B_\ell, B_m] = i{C_{\ell m}}^k B_k} \tag{3.18}$$

$$[B_\ell, P_\mu] = [B_\ell, M_{\mu\nu}] = 0 \tag{3.19}$$

since we are taking a direct product.

3.2 Introducing supersymmetric algebras

Supersymmetric transformations are generated by quantum operators Q. When acting on a fermionic state by Q, it will be change to a bosonic state and vice versa, *i.e.*

$$Q|\text{fermionic}\rangle = |\text{bosonic}\rangle \quad , \quad Q|\text{bosonic}\rangle = |\text{fermionic}\rangle \tag{3.20}$$

i.e. Qs change the statistic and thus the spin of the particles by $\frac{1}{2}$.
We expect Q not to be invariant under rotations since the effect of rotations on fermions is different with that on the bosons. For example, let U be a unitary operator in Hilbert space representing some rotation by 360° about an arbitrary axis. We know that

$$U|\text{fermion}\rangle = -|\text{fermion}\rangle \quad , \quad U|\text{boson}\rangle = |\text{boson}\rangle \tag{3.21}$$

Then

$$\begin{aligned} UQ|\text{boson}\rangle =& UQU^{-1}U|\text{boson}\rangle = UQU^{-1}|\text{boson}\rangle \\ \equiv& U|\text{fermion}\rangle = -|\text{fermion}\rangle = -Q|\text{boson}\rangle \end{aligned} \tag{3.22}$$

and so

$$UQU^{-1} = -Q \tag{3.23}$$

Hence, the rotated SUSY generator picks ups minus sign just like a fermionic state under rotation. Our aim is now to generalise this notion and show that Q transforms precisely like a *spinor operator* under Lorentz *i.e.* as a spin-$\frac{1}{2}$ object. We will then prove that Q's do not commute with Lorentz transformations. In other words, applying a Lorentz transformation and then a supersymmetry transformations is different with that in the reverse order.

First, we will decompose the generators Q ito a sum of irreducible representations of the homogeneous Lorentz group:

$$Q = \sum Q_{\underline{\alpha_1 \ldots \alpha_a}, \underline{\dot{\alpha}_1 \ldots \dot{\alpha}_b}} \tag{3.24}$$

where the $Q_{\underline{\alpha_1 \ldots \alpha_a}, \underline{\dot{\alpha}_1 \ldots \dot{\alpha}_b}}$ are antisymmetric with respect to the underlined indices belonging to irreducible spin-$\frac{1}{2}(a+b)$ representations of the Lorentz group. Also, note that the greek indices run from one to two. Since Q's anticommute, one can conclude that $a+b$ must be odd because of the relations between spin and statistics. The goal is to show that $a+b=1$. For this purpose, we will make the following assumptions:

- The Hilbert space in which the operators Q act, has a positive definite metric:

$$\langle ..|\{Q, Q^\dagger\}|..\rangle = \langle ..|QQ^\dagger|..\rangle + \langle ..|Q^\dagger Q|..\rangle = \Big|Q^\dagger|..\rangle\Big|^2 + \Big|Q|..\rangle\Big|^2 \geq 0 \tag{3.25}$$

  with equality only when $Q = 0$.

- Both Q and its hermitian conjugate $\bar{Q}$ belong to the algebra.

Now, consider the anticommutator

$$\{Q_{\underline{\alpha_1 \ldots \alpha_a}, \underline{\dot{\alpha}_1 \ldots \dot{\alpha}_b}}, \bar{Q}_{\underline{\dot{\beta}_1 \ldots \dot{\beta}_a}, \underline{\beta_1 \ldots \beta_b}}\} \tag{3.26}$$

when all indices are equal to one, we get the product

$$Q_{\underbrace{\underline{1 \ldots 1}}_{a}, \underbrace{\underline{\dot{1} \ldots \dot{1}}}_{b}} \bar{Q}_{\underbrace{\underline{\dot{1} \ldots \dot{1}}}_{a}, \underbrace{\underline{1 \ldots 1}}_{b}} \tag{3.27}$$

to be a member of spin-(a+b)representation of the Lorentz group. Therefore,

$$\{Q_{\underbrace{\underline{1 \ldots 1}}_{a}, \underbrace{\underline{\dot{1} \ldots \dot{1}}}_{b}}, \bar{Q}_{\underbrace{\underline{\dot{1} \ldots \dot{1}}}_{a}, \underbrace{\underline{1 \ldots 1}}_{b}}\} \tag{3.28}$$

will close into a bosonic element of the algebra with spin $(a+b)$. From Coleman-Mandula theorem we know that this element is either zero or a component of P_μ. But for Poincaré $a+b \leq 1$, so for $a+b > 1$ the result must be zero. On the other hand, equation 3.25 implies that $Q_{\underbrace{1\ldots1}_{a},\underbrace{\dot{1}\ldots\dot{1}}_{b}} = 0$. We also said $Q_{\underline{\alpha_1\ldots\alpha_a},\underline{\dot{\alpha}_1\ldots\dot{\alpha}_b}}$ are irreducible under Lorentz transformations so they must all vanish for $a+b>1$. Therefore, $a+b=1$ which in turn implies that the odd part of the SUSY algebra is completely made of spin-$\frac{1}{2}$ fermionic operators ${Q_\alpha}^L$ and its conjugate ${\bar{Q}_{\dot{\alpha}}}^M$. More precisely, ${Q_\alpha}^L$ belongs to the $(\frac{1}{2},0)$ representation and ${\bar{Q}_{\dot{\alpha}}}^M$ belongs to the $(0,\frac{1}{2})$ representation of the algebra.

The anticommutator relation

$$\{{Q_\alpha}^L, \bar{Q}_{\dot{\alpha}M}\} \quad\in\quad (\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2}) \tag{3.29}$$

where $(\frac{1}{2},\frac{1}{2})$ is the vector representation which means that the anticommutator must correspond to the 4-vector operator P_μ. So we need to use a mapping to write this in the spinor representation corresponding to $(\frac{1}{2},\frac{1}{2})$ on the right hand side of the equation. Recall that this type of mapping has already been discussed in the previous chapter, explicitly we can refer to equation 2.68. *i.e.* the anticommmutator closes into $P_{\alpha\dot{\alpha}} = {\sigma_{\alpha\dot{\alpha}}}^\mu P_\mu$ as a linear combination. Thus

$$\{{Q_\alpha}^L, \bar{Q}_{\dot{\alpha}M}\} = P_{\alpha\dot{\alpha}} {C^L}_M \tag{3.30}$$

Where ${C^L}_M$ is a finite dimensional hermitian matrix.

Proof: We write $(Q_{\dot{\alpha}M})^\dagger = {Q_\alpha}^M$ and $\left({Q_\alpha}^L\right)^\dagger = Q_{\dot{\alpha}L}$. Then if we take the hermitian conjugate of equation 3.30 we get

$$\begin{aligned}\{{Q_\alpha}^L, \bar{Q}_{\dot{\alpha}M}\}^\dagger =& \left({Q_\alpha}^L\bar{Q}_{\dot{\alpha}M}\right)^\dagger + \left(\bar{Q}_{\dot{\alpha}M}{Q_\alpha}^L\right)^\dagger = \left(\bar{Q}_{\dot{\alpha}M}\right)^\dagger\left({Q_\alpha}^L\right)^\dagger + \left({Q_\alpha}^L\right)^\dagger\left(\bar{Q}_{\dot{\alpha}M}\right)^\dagger \\ =&{Q_\alpha}^M\bar{Q}_{\dot{\alpha}L} + \bar{Q}_{\dot{\alpha}L}{Q_\alpha}^M = \{{Q_\alpha}^M, \bar{Q}_{\dot{\alpha}L}\} = P_{\alpha\dot{\alpha}}{C^M}_L\end{aligned} \tag{3.31}$$

but the right hand side is

$$P_{\alpha\dot{\alpha}}\left({C^L}_M\right)^\dagger = P_{\alpha\dot{\alpha}}\left({C_L}^M\right)^* \tag{3.32}$$

Comparing, we get

$$\left({C_L}^M\right)^* = {C^M}_L \quad\Longrightarrow\quad \left(({C^M}_L)^t\right)^* = {C^M}_L \quad\Longrightarrow\quad C^\dagger = C \tag{3.33}$$

is hermitian. **Q.E.D**

Therefore, ${C^L}_M$ can be diagonalised by a unitary transformation (Spectral theorem). As well as that, because $\{{Q_1}^L, \bar{Q}_{\dot{1}L}\}$ is positive definite, ${C^L}_M$ has positive definite eigenvalues. Hence we can choose a basis such that

$$\boxed{\{{Q_\alpha}^L, \bar{Q}_{\dot{\alpha}M}\} = 2P_{\alpha\dot{\alpha}}{\delta^L}_M} \tag{3.34}$$

One can check that this makes sense by considering

$$0 < \sum_{\alpha=1}^{2}\{{Q_\alpha}^L, \left({Q_\alpha}^L\right)^\dagger\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu{\delta^L}_L = 2\mathrm{Tr}\left(\sigma^\mu P_\mu\right)_{\alpha\dot{\alpha}} = 4P^0. \tag{3.35}$$

which confirms that $E = P^0$ is positive definite, as required.

Before we proceed to build the rest of the algebra, let us generalise the ordinary *Jacobi identity* introduced in quantum mechanics, to include anticommutators as well as commutators. This relation will be very useful, as we shall see.

$$(-1)^{\eta_c\eta_a}[[A,B\},C\} + (-1)^{\eta_a\eta_b}[[B,C\},A\} + (-1)^{\eta_b\eta_c}[[C,A\},B\} = 0 \tag{3.36}$$

where $\eta = 1$ for fermionic (odd) operators and $\eta = 0$ for bosonic (even) ones. Then, for example, for $[A,B\} = C$ we would have $\eta^c = \eta^a + \eta^b$ (mod 2). To see this, consider the following:

$$\begin{array}{lcl} \{\text{fermionic}, \text{fermionic}\} = \text{bosonic} & \longleftrightarrow & 1+1 \equiv 0 \\ [\text{bosonic}, \text{bosonic}] = \text{bosonic} & \longleftrightarrow & 0+0 \equiv 0 \\ [\text{fermionic}, \text{bosonic}] = \text{fermionic} & \longleftrightarrow & 1+0 \equiv 1 \end{array}$$

as expected. So, for a more general operator $O = ABC...$ we have $\eta(0) = \eta_a + \eta_b + \eta_c + ...$ mod(2). Note that

$$[O,O'\} = OO' - (-1)^{\eta(O)\eta(O')}O'O = -(-1)^{\eta(O)\eta(O')}[O',O\} \tag{3.37}$$

We are now ready to prove the generalised Jacobi identity:
The combination ABC appears twice in this order, *i.e.* from the first and second terms. ACB also appears twice but from the second and third terms in equation 3.36. It is sufficient to prove that the coefficients of these vanish, since the rest of the terms are simply the cyclic permutations of these two which implies that the result will be zero.
For coefficient of ABC we have $(-1)^{\eta_c\eta_a}$ from the first term. As for the second term,

$$(-1)^{\eta_a\eta_b}[[B,C\},A\} = (-1)^{\eta_a\eta_b}(-)(-1)^{\eta_a(\eta_b+\eta_c)}[A,[B,C\}\} \tag{3.38}$$

so for the coefficient of ABC we have

$$(-1)^{\eta_c\eta_a} - (-1)^{\eta_a\eta_b}(-1)^{\eta_a(\eta_b+\eta_c)} = 0 \tag{3.39}$$

one can easily check that this is indeed 0 for all possibilities $\eta = 0, 1$. As for the coefficients of ACB, from the second and third terms in 3.36 we have

$$\begin{aligned} &(-1)^{\eta_a\eta_b}(-)(-1)^{\eta_a(\eta_b+\eta_c)}[A,[B,C\}\} + (-1)(-1)^{\eta_b\eta_c}(-1)^{\eta_c\eta_a}[[A,C\},B\} \\ =& -(-1)^{\eta_a\eta_b}(-1)^{\eta_a(\eta_b+\eta_c)}(-)(-1)^{\eta_b\eta_c}[A,[C,B\}\} - (-1)^{\eta_b\eta_c}(-1)^{\eta_c\eta_a}[[A,C\},B\} \Longrightarrow \\ &(-1)^{\eta_a\eta_b}(-1)^{\eta_a(\eta_b+\eta_c)}(-1)^{\eta_b\eta_c} - (-1)^{\eta_b\eta_c}(-1)^{\eta_c\eta_a} = (-1)^{\eta_c\eta_a}(-1)^{\eta_b\eta_c}\left((-1)^{2\eta_a\eta_b} - 1\right) \\ =&0 \quad \forall \quad \text{possibilities.} \end{aligned} \tag{3.40}$$

Q.E.D

Our task is now to prove the SUSY generators commute with momenta. We know that the commutator could contain $(1, \frac{1}{2})$ and $(0, \frac{1}{2})$ representations since

$$[Q_\alpha{}^L, P_\mu] \in \underbrace{(\frac{1}{2}, 0)}_{\text{dim=2}} \otimes \underbrace{(\frac{1}{2}, \frac{1}{2})}_{\text{dim=4}} = \underbrace{(1, \frac{1}{2})}_{\text{dim=6}} \oplus \underbrace{(0, \frac{1}{2})}_{\text{dim=2}\,\checkmark} \tag{3.41}$$

However, there are no $(1, \frac{1}{2})$ generators present therefore we can only get a combination of $\bar{Q}$'s on belonging to the $(0, \frac{1}{2})$ representation on the right hand side. So we can deduce that

$$[P_{\alpha\dot{\alpha}}, Q_\gamma{}^L] = Z^L{}_M \epsilon_{\alpha\gamma} \bar{Q}_{\dot{\alpha}}{}^M \tag{3.42}$$

where $Z^L{}_M$ are some set of numbers and $P_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}{}^\mu P_\mu$. We make use of the Jacobi identity:

$$[P_{\beta\dot{\beta}}, [P_{\alpha\dot{\alpha}}, Q_\gamma{}^L]] + [P_{\alpha\dot{\alpha}}, [Q_\gamma{}^L, P_{\beta\dot{\beta}}] + [Q_\gamma{}^L, [P_{\beta\dot{\beta}}, P_{\alpha\dot{\alpha}}]] = 0 \tag{3.43}$$

which implies, using the fact that momenta commute,

$$Z^L{}_M \epsilon_{\alpha\gamma} [P_{\beta\dot{\beta}}, \bar{Q}_{\dot{\alpha}}{}^M] - Z^L{}_M \epsilon_{\beta\gamma} [P_{\alpha\dot{\alpha}}, \bar{Q}_{\dot{\beta}}{}^M] = 0 \tag{3.44}$$

We need to find $[P_{\alpha\dot{\alpha}}, \bar{Q}_{\dot{\beta}}{}^M]$. This can be done by taking the conjugate of equation 3.42 *i.e.* $[\bar{Q}_{\dot{\gamma}}{}^L, P_{\alpha\dot{\alpha}}] = \bar{Z}^L{}_M \epsilon_{\dot{\alpha}\dot{\gamma}} Q_\alpha{}^M$ so that the above equation will now read

$$\begin{aligned} & Z^L{}_M \epsilon_{\alpha\gamma} \left(-\bar{Z}^M{}_K \epsilon_{\dot{\beta}\dot{\alpha}} Q_\beta{}^K\right) - Z^L{}_M \epsilon_{\beta\gamma} \left(-\bar{Z}^M{}_K \epsilon_{\dot{\alpha}\dot{\beta}} Q_\alpha{}^K\right) \\ \Rightarrow & Z^L{}_M \bar{Z}^M{}_K \epsilon_{\dot{\alpha}\dot{\beta}} \left(\epsilon_{\alpha\gamma} Q_\beta{}^K + \epsilon_{\beta\gamma} Q_\alpha{}^K\right) = 0 \\ \Rightarrow & Z\bar{Z} = 0 \Rightarrow |Z| = 0 \end{aligned} \tag{3.45}$$

and so $Z^L{}_M$ must vanish resulting in $[P_{\alpha\dot{\alpha}}, Q_\gamma{}^L] = 0$. Hence

$$\boxed{[P_\mu, Q_\alpha{}^L] = 0} \tag{3.46}$$

and

$$\boxed{[P_\mu, \bar{Q}_{\dot{\alpha}L}] = 0} \tag{3.47}$$

We will now consider the anticommutator of two SUSY generators with undotted indices. Since

$$\{Q_\alpha{}^L, Q_\beta{}^M\} \in (\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (0, 0)_A \oplus (1, 0)_S \tag{3.48}$$

By Coleman-Mandula theorem, the antisymmetric part $(0,0)$ has spin zero and so it correspond to a scalar while the symmetric part $(1,0)$ corresponds to the self-dual part of the Lorentz generator $M_{\mu\nu}$ denoted by $M_{\underline{\alpha\beta}}$. Let us explore this in more detail. Because the left hand side of the equation involves spinors, we need to find a mapping for the right hand side that takes us from $M_{\mu\nu}$ to $M_{\underline{\alpha\beta}}$ which is symmetric with respect to exchange of its two spinor indices. As presented in the previous chapter, this mapping is provided by equations 2.84 and 2.88. So we expect the anticommutator to close into

$$\{Q_\alpha{}^L, Q_\beta{}^M\} = \epsilon_{\alpha\beta} X^{\underset{\vee}{LM}} + M_{\underline{\alpha\beta}} Y^{\underline{LM}} \tag{3.49}$$

then using the Jacobi identity we get

$$[P_{\gamma\dot{\gamma}}, \{Q_\alpha{}^L, Q_\beta{}^M\}] + \{Q_\alpha{}^L, [Q_\beta{}^M, P_{\gamma\dot{\gamma}}]\} + \{Q_\beta{}^M, [P_{\gamma\dot{\gamma}}, Q_\alpha{}^L]\} = 0 \tag{3.50}$$

but the last two terms vanish since Q and P commute so that

$$[P_{\gamma\dot{\gamma}}, \{Q_\alpha{}^L, Q_\beta{}^m\}] = 0 \tag{3.51}$$

implying

$$[P_{\gamma\dot{\gamma}}, \epsilon_{\gamma\beta} X^{\underset{\vee}{LM}} + M_{\underline{\alpha\beta}} Y^{\underline{LM}}] = 0 \tag{3.52}$$

but $P_{\gamma\dot{\gamma}}$ commutates with the scalar $X^{\underset{\vee}{LM}}$ meaning

$$[P_{\gamma\dot{\gamma}}, M_{\underline{\alpha\beta}} Y^{\underline{LM}}] = 0 \Rightarrow Y^{\underline{LM}} \underbrace{[P_{\gamma\dot{\gamma}}, M_{\underline{\alpha\beta}}]}_{\text{non-zero}} = 0 \Rightarrow Y^{\underline{LM}} = 0 \tag{3.53}$$

Also, one can prove explicitly why $X^{\underset{\vee}{LM}}$ must be antisymmetric with respect to L and M:

$$Q_\alpha{}^L Q_\beta{}^M + Q_\beta{}^M Q_\alpha{}^L = \epsilon_{\alpha\beta} X^{LM} \tag{3.54}$$

Relabelling $\alpha \leftrightarrow \beta$ and $L \leftrightarrow M$ implies

$$\begin{aligned} &Q_\beta{}^M Q_\alpha{}^L + Q_\alpha{}^L Q_\beta{}^M = \epsilon_{\beta\alpha} X^{ML} \\ \Rightarrow &Q_\alpha{}^L Q_\beta{}^M + Q_\beta{}^M Q_\alpha{}^L = -\epsilon_{\alpha\beta} X^{ML} \end{aligned} \tag{3.55}$$

So $X^{\underset{\vee}{LM}} = -X^{\underset{\vee}{ML}}$. **Q.E.D**
Therefore,

$$\boxed{\{Q_\alpha{}^L, Q_\beta{}^M\} = \epsilon_{\alpha\beta} X^{\underset{\vee}{LM}} = \epsilon_{\alpha\beta} a^{\ell,\underset{\vee}{LM}} B_\ell} \tag{3.56}$$

where B_ℓ is a hermitian element a member of $\mathscr{A}_1 \oplus \mathscr{A}_2$ and and $a^{\ell,\underset{\vee}{LM}}$ is antisymmetric in L and M.

Taking the hermitian conjugate of 3.56, the LHS is

$$\begin{aligned}\{Q_\alpha{}^L Q_\beta{}^M\}^\dagger =&\left(Q_\alpha{}^L Q_\beta{}^M\right)^\dagger + \left(Q_\beta{}^M Q_\alpha{}^L\right)^\dagger = \left(Q_\beta{}^M\right)^\dagger + \left(Q_\alpha{}^L\right)^\dagger + \left(Q_\alpha{}^L\right)^\dagger\left(Q_\beta{}^M\right)^\dagger \\ =&Q_{\dot\beta M} Q_{\dot\alpha L} + Q_{\dot\alpha L} Q_{\dot\beta M} = \{Q_{\dot\alpha L}, Q_{\dot\beta M}\}\end{aligned} \tag{3.57}$$

and the RHS

$$\left(\epsilon_{\alpha\beta} a^{\ell,\underset{\vee}{LM}} B_\ell\right)^\dagger = \epsilon_{\dot\alpha\dot\beta} a^*_{\ell,\underset{\vee}{LM}} B^\ell = \epsilon_{\dot\alpha\dot\beta} X^\dagger{}_{\underset{\vee}{LM}} \tag{3.58}$$

The next relation we're going to consider, knowing that B_ℓ is a Lorentz scalar, is

$$[Q_\alpha{}^L, B_\ell] \quad \in \quad (\frac{1}{2},0)\otimes(0,0) = (\frac{1}{2},0) \tag{3.59}$$

so that one expects the right hand side to be a sum of Q's:

$$\boxed{[Q_\alpha{}^L, B_\ell] = S_\ell{}^L{}_M Q_\alpha{}^M} \tag{3.60}$$

with $S_\ell{}^L{}_M$, some complex constants. Again, taking the hermitian conjugate the LHS yields

$$\begin{aligned}[Q_\alpha{}^L, B_\ell]^\dagger =&\left(Q_\alpha{}^L B_\ell\right)^\dagger - \left(B_\ell Q_\alpha{}^L\right)^\dagger \\ =&\left(B_\ell\right)^\dagger\left(Q_\alpha{}^L\right)^\dagger - \left(Q_\alpha{}^L\right)^\dagger\left(B_\ell\right)^\dagger \\ =&B^\ell \bar Q_{\dot\alpha L} - \bar Q_{\dot\alpha L} B^\ell \\ =&[B^\ell, \bar Q_{\dot\alpha L}]\end{aligned} \tag{3.61}$$

Hence

$$\boxed{[B^\ell, \bar Q_{\dot\alpha L}] = S^{*\ell}{}_L{}^M \bar Q_{\dot\alpha M}} \tag{3.62}$$

Let us write the Jacobi identity for B_ℓ, $Q_\alpha{}^L$ and $\bar Q_{\dot\beta M}$:

$$\begin{aligned}&[B_\ell, \{Q_\alpha{}^L, \bar Q_{\dot\beta M}\} + \{Q_\alpha{}^L, [\bar Q_{\dot\beta M}, B_\ell]\} - \{\bar Q_{\dot\beta M}, [B_\ell, Q_\alpha{}^L]\} = 0 \\ \Rightarrow&[B_\ell, 2\sigma_{\alpha\dot\beta}{}^\mu P_\mu \delta^L{}_M] + \{Q_\alpha{}^L, -S^{*\ell}{}_M{}^K \bar Q_{\dot\beta K}\} - \{\bar Q_{\dot\beta M}, -S_\ell{}^L{}_K Q_\alpha{}^K\}\end{aligned} \tag{3.63}$$

The first terms is equal to zero since B_ℓ and P commute. Then

$$\begin{aligned}&-2P_{\alpha\dot\beta}\delta^L{}_K S^{*\ell}{}_M{}^K + 2P_{\alpha\dot\beta}\delta^K{}_M S_\ell{}^L{}_K = 0 \\ \Rightarrow\;\; &2P_{\alpha\dot\beta}\left[S^{*\ell}{}_M{}^L - S_\ell{}^L{}_M\right] = 0\end{aligned} \tag{3.64}$$

So that the term in the brackets must vanish

$$S^{*\ell}{}_M{}^L = S_\ell{}^L{}_M \tag{3.65}$$

meaning $S_\ell{}^L{}_M$ is hermitian.
Note that generators $X^{\overset{LM}{\vee}} = a^{\overset{\ell,LM}{\vee}} B_\ell$ form an invariant subalgebra of $\mathscr{A}_1 \oplus \mathscr{A}_2$.
Proof:

$$\begin{aligned}
&[B_\ell, \{Q_\alpha{}^L, Q_\beta{}^M\}] + \{Q_\alpha{}^L, [Q_\beta{}^M, B_\ell]\} - \{Q_\beta{}^M [B_\ell, Q_\alpha{}^L]\} = 0 \\
\Rightarrow &[B_\ell, \epsilon_{\alpha\beta} X^{\overset{LM}{\vee}}] + \{Q_\alpha{}^L, S_\ell{}^M{}_K Q_\alpha{}^K\} + \{Q_\beta{}^M, S_\ell{}^L{}_K Q_\alpha{}^K\} = 0 \\
\Rightarrow &[B_\ell, \epsilon_{\alpha\beta} X^{\overset{LM}{\vee}}] + \{Q_\alpha{}^L, Q_\alpha{}^K\} S_\ell{}^M{}_K + \{Q_\beta{}^M, Q_\alpha{}^K\} S_\ell{}^L{}_K = 0 \\
\Rightarrow &\epsilon_{\alpha\beta}[B_\ell, X^{\overset{LM}{\vee}}] + \epsilon_{\alpha\beta} X^{\overset{LK}{\vee}} S_\ell{}^M{}_K + \epsilon_{\beta\alpha} X^{\overset{MK}{\vee}} S_\ell{}^L{}_K = 0 \\
\Rightarrow &\epsilon_{\alpha\beta}\{[B_\ell, X^{\overset{LM}{\vee}}] + S_\ell{}^M{}_K X^{\overset{LK}{\vee}} - S_\ell{}^L{}_K X^{\overset{MK}{\vee}}\} = 0 \\
\Rightarrow &[B_\ell, X^{\overset{LM}{\vee}}] = S_\ell{}^L{}_K X^{\overset{MK}{\vee}} - S_\ell{}^M{}_K X^{\overset{LK}{\vee}}
\end{aligned} \tag{3.66}$$

i.e. $[B_\ell, X^{\overset{LM}{\vee}}]$ closes into a set of generators $X^{\overset{LM}{\vee}}$. As well as that, the$X^{\overset{LM}{\vee}}$ are linear combinations of the B_ℓ. Hence $X^{\overset{LM}{\vee}}$ form an invariant subalgebra of $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$.

It is possible to show that generators $X^{\overset{LM}{\vee}}$ commute with all the generators of $\mathscr{A}$ using the identity

$$\begin{aligned}
&[Q_\alpha{}^L, \{Q_\beta{}^M, \bar{Q}_{\dot\gamma K}\}] + [Q_\beta{}^M, \{\bar{Q}_{\dot\gamma K}, Q_\alpha{}^L\}] + [\bar{Q}_{\dot\gamma K}, Q_\alpha{}^L, Q_\beta{}^M] = 0 \\
\Rightarrow \quad &[Q_\alpha{}^L, 2\sigma_{\beta\dot\gamma}{}^\mu P_\mu \delta^M{}_K] + [Q_\beta{}^M, 2\sigma_{\alpha\dot\gamma}{}^\mu P_\mu \delta^L{}_K] + [\bar{Q}_{\dot\gamma K}, \epsilon_{\alpha\beta} X^{\overset{LM}{\vee}}] = 0 \\
\Rightarrow \quad &\epsilon_{\alpha\beta}[\bar{Q}_{\dot\gamma k}, X^{\overset{LM}{\vee}}] = 0
\end{aligned} \tag{3.67}$$

Now we need to show that $X^{\overset{LM}{\vee}}$ commute with P_μ using equation 3.56 and the fact that P and Q commute with each other. We start with

$$\begin{aligned}
[P_\mu, \{Q_\alpha{}^L, Q_\beta{}^M\}] &= [P_\mu, Q_\alpha{}^L Q_\beta{}^M + Q_\beta{}^M Q_\alpha{}^L] \\
&= P_\mu Q_\alpha{}^L Q_\beta{}^M + P_\mu Q_\beta{}^M Q_\alpha{}^L - Q_\alpha{}^L Q_\beta{}^M P_\mu - Q_\beta{}^M Q_\alpha{}^L P_\mu \\
&= [P_\mu, Q_\alpha{}^L Q_\beta{}^M] + [P_\mu, Q_\beta{}^M Q_\alpha{}^L] = 0
\end{aligned} \tag{3.68}$$

So the commutator of the right hand side of equation 3.56 yields

$$\epsilon_{\alpha\beta}[P_\mu, X^{\overset{LM}{\vee}}] = 0 \quad \Rightarrow \quad \boxed{[P_\mu, X^{\overset{LM}{\vee}}] = 0} \tag{3.69}$$

i.e. P_μ commutes with $X^{\overset{LM}{\vee}}$. Again, we use the identity

$$[X^{\overset{AB}{\vee}},\{Q_\alpha{}^L,\bar{Q}_{\dot{\beta}M}\}]+\{Q_\alpha{}^L,[\bar{Q}_{\dot{\beta}M},X^{\overset{AB}{\vee}}]\}-\{\bar{Q}_{\dot{\beta}M},[X^{\overset{AB}{\vee}},Q_\alpha{}^L]\}=0 \tag{3.70}$$

where the first two terms vanish since $X^{\overset{AB}{\vee}}$ commutes with P. So we must have

$$\{\bar{Q}_{\dot{\beta}M},[X^{\overset{AB}{\vee}},Q_\alpha{}^L]\}=0 \tag{3.71}$$

On the other hand, $[X^{\overset{AB}{\vee}},Q_\alpha{}^L]$ is in the $(\frac{1}{2},0)$ representation. In other words the commutator will result to a linear combination of the Q's *i.e.*

$$[X^{\overset{AB}{\vee}},Q_\alpha{}^L]=T_{ABLR}Q_\alpha{}^R \tag{3.72}$$

and so equation 3.71 will now read

$$T_{ABLR}\{\bar{Q}_{\dot{\beta}M},Q_\alpha{}^R\}=T_{ABLR}\underbrace{\left(2\sigma_{\alpha\dot{\beta}}{}^\mu P_\mu\delta^R{}_M\right)}_{\neq 0}=0 \quad\Rightarrow\quad T_{ABLR}=0 \tag{3.73}$$

Hence $X^{\overset{AB}{\vee}}$ and Q commute:

$$\boxed{[X^{\overset{AB}{\vee}},Q_\alpha{}^L]=0} \tag{3.74}$$

As well as that,

$$\boxed{[X^{\overset{KN}{\vee}},X^{\overset{LM}{\vee}}]=\frac{1}{2}\epsilon^{\beta\alpha}[\{Q_\alpha{}^K,Q_\beta{}^N\},X^{\overset{LM}{\vee}}]=0} \tag{3.75}$$

Since $X^{\overset{LM}{\vee}}$ and Q commute. The above equation can easily be verified:

$$\begin{aligned}\frac{1}{2}\epsilon^{\beta\alpha}[\epsilon_{\alpha\beta}X^{\overset{KN}{\vee}},X^{\overset{LM}{\vee}}]&=\frac{1}{2}\epsilon^{\beta\alpha}\epsilon_{\alpha\beta}[X^{\overset{KN}{\vee}},X^{\overset{LM}{\vee}}]\\ =&\frac{1}{2}(\epsilon^{12}\epsilon^{21}+\epsilon^{21}\epsilon^{12})[X^{\overset{KN}{\vee}},X^{\overset{LM}{\vee}}]=[X^{\overset{KN}{\vee}},X^{\overset{LM}{\vee}}] \quad ✔\end{aligned} \tag{3.76}$$

What $[X^{\overset{KN}{\vee}},X^{\overset{LM}{\vee}}]=0$ really means is that $X^{\overset{LM}{\vee}}$ is Abelian and so forms an Abelian (invariant) subalgebra of $\mathscr{A}$. On the other hand, we know that $\mathscr{A}_1$ is semi-simple, hence non-abelian. Therefore, $X^{\overset{LM}{\vee}}$ is a member of $\mathscr{A}_2$ and commute with all the generator of $\mathscr{A}$

$$\boxed{[X^{\overset{LM}{\vee}},B_\ell]=0} \tag{3.77}$$

This is why they are called *central charges i.e.* they are in the *centre* of the supersymmetric algebra.
Using this relation, equation 3.66 yields:

$$S_\ell{}^M{}_K X^{\overset{LK}{\vee}}-S_\ell{}^L{}_K X^{\overset{MK}{\vee}}=0 \tag{3.78}$$

substituting $X^{\overset{LM}{\vee}} = a^{\ell,\overset{LM}{\vee}} B_\ell$ we get

$$S_\ell{}^M{}_K a^{k,\overset{LK}{\vee}} - S_\ell{}^L{}_K a^{k,\overset{MK}{\vee}} = 0 \tag{3.79}$$

then, from the fact that $S_\ell{}^M{}_K$ is hermitian and $a_k^{\overset{MK}{\vee}}$ is antisymmetric, we get

$$S_\ell{}^M{}_K a^{k,\overset{KL}{\vee}} = -a^{k,\overset{MK}{\vee}} S^{*\ell}{}_K{}^L \tag{3.80}$$

On the other hand, $S_\ell{}^M{}_K$ form a representation of compact Lie algebra $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$.
Proof:

$$\begin{aligned} &[B_\ell, [B_m, Q_\alpha{}^\ell]] + [B_m, [Q_\alpha{}^L, B_\ell]] + [Q_\alpha{}^L, [B_\ell, B_m]] = 0 \\ \Rightarrow &- [B_\ell, S_m{}^L{}_M] + [B_m, S_\ell{}^L{}_M Q_\alpha{}^M] + [Q_\alpha{}^L, iC_{\ell m}{}^K B_k] = 0 \\ \Rightarrow &+ S_\ell{}^L{}_M S_\ell{}^M{}_K Q_\alpha{}^K - S_\ell{}^L{}_M S_m{}^M{}_K Q_\alpha{}^K + iC_{\ell m}{}^k S_K{}^L{}_M Q_\alpha{}^M = 0 \end{aligned} \tag{3.81}$$

Comparing the coefficients of $Q_\alpha{}^K$ implies

$$(S_m S_\ell)^L{}_K - (S_\ell S_m)^L{}_K = -iC_{\ell m}{}^K S^L{}_K \tag{3.82}$$

Hence

$$\boxed{[S_\ell, S_m] = iC_{m\ell}{}^k S_k} \tag{3.83}$$

and the conjugate

$$\boxed{[S^*{}_m, S^*{}_\ell] = -iC_{m\ell}{}^K S^*{}_k} \tag{3.84}$$

as required by the Lie algebra of the compact Lie group. **Q.E.D**
Equation 3.80 states that matrices a_k intertwine the representations S_ℓ with its complex conjugate $-S_\ell^*$. So if central charges exist, they must be of the form $X^{\overset{LM}{\vee}} = a^{\ell,\overset{LM}{\vee}} B_\ell$.

Finally, we need to understand how the Supersymmetry generators transform under Lorentz transformations. We will need to use the fact the Q_α can be transformed as a spinor and as well as an operator.

1. Spinor: The Lorentz transformation of a spinor in the $(\frac{1}{2}, 0)$ representation is $Q'_\alpha = e^{\left(-\frac{i}{2}\omega_{\mu\nu}S_L^{\mu\nu}\right)_\alpha{}^\beta} Q_\beta$ or for infinitesimal trasformations,

$$Q'_\alpha = \left(1 - \frac{i}{2}\omega_{\mu\nu}S_L^{\mu\nu}\right)_\alpha{}^\beta Q_\beta \tag{3.85}$$

   where $\left(S_L^{\mu\nu}\right)_\alpha{}^\beta = -\left(S_L^{\nu\mu}\right)_\alpha{}^\beta$ are, as before, a set of 2×2 antisymmetric matrices obeying the same commutation relations as the Lorentz generators $M^{\mu\nu}$.

2. Operator: We have, under Lorentz transformations $Q'_\alpha = U(\Lambda)Q_\alpha U^{-1}(\Lambda)$, so that for infinitesimal transformations we get

$$Q'_\alpha = \left(1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) Q_\alpha \left(1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \tag{3.86}$$

Now we can set equal the results of (1) and (2):

$$\begin{aligned} \delta_\alpha{}^\beta Q_\beta - \left(\frac{i}{2}\omega_{\mu\nu}S_L^{\mu\nu}\right)_\alpha{}^\beta Q_\beta &= Q_\alpha - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}Q_\alpha + \frac{i}{2}\delta\omega_{\mu\nu}Q_\alpha M^{\mu\nu} + O(\omega^2) \\ \Rightarrow \quad -\frac{i}{2}\omega_{\mu\nu}\left(S_L^{\mu\nu}\right)_\alpha{}^\beta Q_\beta &= [Q_\alpha, M^{\mu\nu}]\left(\frac{i}{2}\omega_{\mu\nu}\right) \end{aligned} \tag{3.87}$$

Therefore, comparing the coefficient of $\omega_{\mu\nu}$ on both sides of the equation,

$$\boxed{[M^{\mu\nu}, Q_\alpha] = \left(S_L^{\mu\nu}\right)_\alpha{}^\beta Q_\beta} \tag{3.88}$$

Similarly for a spinor in $\bar{Q}_{\dot\alpha}$ in $(0, \frac{1}{2})$ we have

$$\boxed{[M^{\mu\nu}, \bar{Q}_{\dot\alpha}] = \left(S_R^{\mu\nu}\right)_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}} \tag{3.89}$$

where, as we have already proved in the previous chapter, $(S_R^{\mu\nu})_{\dot a}{}^{\dot b} = -[(S_L^{\mu\nu})_a{}^b]^*$. Note that $(S_L^{12})_a{}^b = \frac{1}{2}\sigma_3$ and since σ_3 is real, we also have $(S_R^{\mu\nu})_{\dot a}{}^{\dot b} = -\frac{1}{2}\sigma_3$.

3.3 Irreducible representations of Supersymmetry algebra

Our aim of this section is to derive the Supersymmetric representation of the graded algebra for both massless and massive cases. Before we proceed to discuss this in detail, let us build an overall picture of such representations.

Informally, a representation consists of linear operators on some vector space, with the operators having the same 'properties' as the objects they are going to represent. We know that SUSY transformations related bosons and fermions so one can divide the representations space into two parts: a bosonic and a fermionic one *i.e.*

$$\text{Representation space} \quad = \quad \text{Bosonic subspace} \ + \ \text{Fermionic subspace} \tag{3.90}$$

The bosonic generators $P^\mu, M^{\mu\nu}$ and B_ℓ , which form the even part of the algebra, map the subspaces into themselves, or rather, onto a proper subspace of the original space.

$$\text{bosonic(fermionic)} \xrightarrow{P^\mu, M^{\mu\nu}, B_\ell} \text{bosonic(fermionic)} \tag{3.91}$$

However, the Supersymmetry generators Q, which form the odd part of the algebra, map the bosonic subspace into the fermionic one and vice-versa.

$$\text{bosonic(fermionic)} \xrightarrow{Q} \text{fermionic(bosonic)} \tag{3.92}$$

For two subsequent odd mappings, which is the case with the anticommutator relations, we have

$$\text{bosonic(fermionic)} \xrightarrow{\text{First } Q} \text{fermionic(bosonic)} \xrightarrow{\text{Second } Q} \text{bosonic(fermionic)} \tag{3.93}$$

Therefore, it can be said that what $\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu$ really means is that there is a mapping from one subspace to the other and back to the original one such that the total result is as if we we had acted with P^μ on the original subspace.
There is a theorem that states that the fermionic and bosonic subspaces of the representations space of SUSY algebra have the same dimensions. This is because the result of no mapping has dimensions higher than the original and also in the mappings like $Q\bar{Q} + \bar{Q}Q$ no dimensions get lost. We will later on prove rigorously that SUSY representations contain equal numbers of bosonic and fermionic states.

We know that the SUSY generators Q commute with energy-momentum 4-vector P^μ. Thus P^2, the mass operator, is a Casimir operator of the algebra. Since $[Q, P^2] = 0$ we conclude that the irreducible representations of the supersymmetry algebra are of equal mass. However, SUSY multiplets contain different spins and so W^2 is no longer a Casimir operator *i.e.* $[Q, W^2] \neq 0$.

Let us now introduce a fermion number operator N_F such that $(-)^{N_F}$ has eigenvalue +1 and -1 on bosonic and fermionic states respectively.

$$(-)^{N_F}\ |\text{bosonic}\rangle = +1\ |\text{bosonic}\rangle \quad , \quad (-)^{N_F}\ |\text{fermionic}\rangle = -1\ |\text{fermionic}\rangle \tag{3.94}$$

Therefore,

$$(-)^{N_F} Q_\alpha = -Q_\alpha (-)^{N_F} \tag{3.95}$$

which can easily be checked:

$$\begin{aligned} \left[(-)^{N_F} Q + Q(-)^{N_F}\right] |\text{bosonic}\rangle =& (-)^{N_F} |\text{fermionic}\rangle + Q|\text{bosonic}\rangle \\ =& -|\text{fermionic}\rangle + |\text{fermionic}\rangle = 0 \end{aligned} \tag{3.96}$$

Similarly,

$$\begin{aligned}\left[(-)^{N_F}Q + Q(-)^{N_F}\right]|\text{fermionic}\rangle =&(-)^{N_F}|\text{bosonic}\rangle - Q|\text{fermionic}\rangle \\ =&+|\text{bosonic}\rangle - |\text{bosonic}\rangle = 0\end{aligned} \tag{3.97}$$

as required.

For any finite dimensional representation of the algebra, *i.e.* the trace is well-defined, we have

$$\text{Tr}\left[(-)^{N_F}\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\}\right] = \text{Tr}\left[(-)^{N_F}\left(Q_\alpha{}^A\bar{Q}_{\dot{\beta}B} + \bar{Q}_{\dot{\beta}B}Q_\alpha{}^A\right)\right] \tag{3.98}$$

Note that trace is cyclic *e.g.* $\text{Tr}(ABCD) = \text{Tr}(BCDA) = \text{Tr}(CDAB = \text{Tr}(DABC)$ and so $\text{Tr}(AB) = \text{Tr}(BA)$. We get

$$= \text{Tr}\left(-Q_\alpha{}^A(-)^{N_F}\bar{Q}_{\dot{\beta}B} + Q_\alpha{}^A(-)^{N_F}\bar{Q}_{\dot{\beta}B}\right) = \text{Tr}(0) = 0 \tag{3.99}$$

On the other hand $\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\} = 2\sigma_{\alpha\dot{\beta}}{}^\mu P_\mu \delta^A{}_B$ which implies

$$\text{Tr}\left[(-)^{N_F}\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\}\right] = 2\sigma_{\alpha\dot{\beta}}{}^\mu \delta^A{}_B \text{Tr}\left[(-)^{N_F}P_\mu\right] = 0 \tag{3.100}$$

For a non-zero momentum P_μ we must then have

$$\text{Tr}(-)^{N_F} = 0 \tag{3.101}$$

implying that there are equal number of eigenvalues +1 and -1. In other words, there are equal number of bosonic and fermionic states. **Q.E.D**

3.3.1 Massless case

Now we are ready to build the representation of the supersymmetric algebra for massless particles. We start with $P^2 = 0$. One can boost to a like-like reference frame $P^\mu = (E, 0, 0, E)$. Then, the space-time properties of the state are determined by its energy E and helicity λ which is the projection of spin into the direction of motion, here being the 3-direction. Here, we have $W^0|E,\lambda\rangle = \lambda P^0|E,\lambda\rangle$ (see equation 1.143), or because of Lorentz covariance

$$W^\mu|E,\lambda\rangle = \lambda P^\mu|E,\lambda\rangle \tag{3.102}$$

Now, since $[Q, P_\mu]$=0, when Q acts on a state, the energy and momentum remain unchanged. As for helicity, we need to consider the following:

$$W^0 Q_\alpha|E,\lambda\rangle = Q_\alpha \underbrace{W^0|E,\lambda\rangle}_{E\lambda|E,\lambda\rangle} + [W^0, Q_\alpha]|E,\lambda\rangle \tag{3.103}$$

but we need to determine $[W^0, Q_\alpha]$. Consider

$$\begin{aligned}
[W^\mu, Q_\alpha] =& \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[M_{\nu\rho}P_\sigma, Q_\alpha] \\
=& \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left([M_{\nu\rho}, Q_\alpha]P_\sigma + M_{\nu\rho}\underbrace{[P_\sigma, Q_\alpha]}_{0}\right) \\
=& \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(S_{L\nu\rho})_\alpha{}^\beta Q_\beta P_\sigma
\end{aligned} \tag{3.104}$$

For $\mu = 0$:

$$\frac{E}{2}\left(\epsilon^{0123}(S_{12})_\alpha{}^\beta Q_\beta + \epsilon^{0213}(S_{21})_\alpha{}^\beta Q_\beta\right) = E(S_{12})_\alpha{}^\beta Q_\beta \tag{3.105}$$

on the other hand

$$(S_{ij})_\alpha{}^\beta = \frac{1}{2}\epsilon_{ijk}\sigma^k \Rightarrow (S_{12})_\alpha{}^\beta = \frac{1}{2}\epsilon_{123}\sigma^3 = \frac{1}{2}\sigma^3 \tag{3.106}$$

Therefore, we get from 3.103

$$W^0 Q_\alpha|\lambda, E\rangle = E\left(\lambda\mathbb{1} + \frac{1}{2}\sigma^3\right)_\alpha^{\ \beta} Q_\beta|E, \lambda\rangle \tag{3.107}$$

Noting that when $\alpha \neq \beta$ the result is zero since $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, there can only be two cases: Either $\alpha = \beta = 1$, then

$$\boxed{Q_1 \text{ raises the helicity by } \tfrac{1}{2}}$$

or $\alpha = \beta = 2$ which corresponds to -1 so that

$$\boxed{Q_2 \text{ lowers the helicity by } \tfrac{1}{2}}$$

As already mentioned, $[M^{\mu\nu}, \bar{Q}_{\dot{\alpha}}] = \left(S_R^{\mu\nu}\right)_{\dot{\alpha}}^{\ \dot{\beta}}\bar{Q}_{\dot{\beta}}$ with $(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} = -\frac{1}{2}\sigma_3$ which only differs to the previous case by a minus sign. Hence

$$W^0 Q_{\dot{\alpha}}|\lambda, E\rangle = E\left(\lambda\mathbb{1} - \frac{1}{2}\sigma^3\right)_{\dot{\alpha}}^{\ \dot{\beta}} Q_{\dot{\beta}}|E, \lambda\rangle \tag{3.108}$$

$$\boxed{Q_{\dot{1}} \text{ lowers the helicity by } \tfrac{1}{2}}$$

and

$$\boxed{Q_{\dot{2}} \text{ raises the helicity by } \tfrac{1}{2}}$$

Recall that we boosted to a frame where $P^\mu = (E,0,0,E)$ *i.e.* $P_\mu = (E,0,0,-E)$, then

$$\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\} = 2\sigma_{\alpha\dot{\beta}}{}^\mu P_\mu \delta^A{}_B \tag{3.109}$$

where

$$\sigma^\mu P_\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix} \tag{3.110}$$

Therefore,

$$\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\} = 2\begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}\delta^A{}_B \tag{3.111}$$

$$\{Q_\alpha{}^A, Q_\beta{}^B\} = \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = 0 \tag{3.112}$$

in the absence of central charges.

Note that because of the positivity condition 3.25, and the fact that the anticommutator of Q and $\bar{Q}$ is only non zero when $\alpha = \beta = 2$ (due to 3.111), we must have

$$Q_1{}^A = \bar{Q}_{\dot{1}A} = 0 \tag{3.113}$$

Rescaling the Q's

$$\begin{aligned} a^A &= \frac{1}{2\sqrt{E}} Q_2{}^A \\ a^\dagger_A &= \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{2}}{}^A = \left(a^A\right)^\dagger \end{aligned} \tag{3.114}$$

where A runs from 1 to N. So we have N creation and annihilation operators satisfying

$$\begin{aligned} \{a^A, a^\dagger_B\} &= \delta^A{}_B \\ \{a^A, a^B\} &= \{a^\dagger_A, a^\dagger_B\} = 0 \end{aligned} \tag{3.115}$$

There are N fermionic degrees of freedom. Any irreducible representation of this algebra is characterised by a Clifford ground state (vacuum) denoted by $|E, \lambda_0\rangle \equiv \Omega_{\underline{\lambda}}$ where $\underline{\lambda}$ is the state of lowest helicity. Thus, applying the annihilation operator on this state

$$a^A \Omega_{\underline{\lambda}} = 0 \tag{3.116}$$

Other states are generated by successive application of N operators $\left(a^A\right)^\dagger$.

$$a^\dagger_i |E, \lambda_0\rangle = |E, \lambda_0 + \frac{1}{2}, i\rangle \tag{3.117}$$

and

$$a_i^\dagger a_j^\dagger |E, \lambda_0\rangle = |E, \lambda_0 + 1, ij\rangle \tag{3.118}$$

Note that this is antisymmetric since they are fermionic operators that anticommute *i.e.* $a_i^\dagger a_j^\dagger = a_j^\dagger a_i^\dagger$ implying $|E, \lambda_0 + 1, ij\rangle = -|E, \lambda_0 + 1, ji\rangle$.

Therefore in general,

$$\Omega^{(n)}_{\underline{\lambda}+\frac{1}{2}n,\ A_1 \ldots A_n} = \frac{1}{\sqrt{n!}} a^\dagger_{A_n} \ldots a^\dagger_{A_1} \Omega_{\underline{\lambda}} \tag{3.119}$$

The states $\Omega^{(n)}_{\underline{\lambda}+\frac{1}{2}n,\ A_1 \ldots A_n}$ have helicity $\underline{\lambda} + \frac{1}{2}n$ and are antisymmetric in $A_1 \ldots A_n$. We now have states with the same energy but different helicity which are $\binom{N}{n}$-times degenerate. The state of highest helicity is

$$a_1^\dagger a_2^\dagger \ldots a_N^\dagger |E, \lambda_0\rangle = |E, \lambda_0 + \frac{N}{2}, 12 \ldots N\rangle \tag{3.120}$$

implying that the highest helicity is $\overline{\lambda} = \underline{\lambda} + \frac{N}{2}$ which occurs exactly once. The dimension of representations is

$$\sum_{n=0}^{N} \binom{N}{n} = (1+1)^N = 2^N \tag{3.121}$$

using binomial expansion. So we have

<u>helicity</u>:	λ_0	$\lambda_0 + \frac{1}{2}$	$\lambda_0 + 1$	...	$\lambda_0 + \frac{N}{2}$
<u>No. of possible states</u>:	$\binom{N}{0} = 1$	$\binom{N}{1} = N$	$\binom{N}{2}$	...	$\binom{N}{N} = 1$

One can check that the number of fermions and boson are indeed the same:

$$\sum_{n=0}^{N/2} \binom{N}{2n} - \sum_{n=0}^{N/2} \binom{N}{2n+1} = 2^{N-1} - 2^{N-1} = 0 \ \checkmark \tag{3.122}$$

The results are summarised in the tables below.

hel. vs λ	-2	-3/2	-1	-1/2	0	1/2	1	3/2
2								1
3/2							1	1
1						1	1	
1/2					1	1		
0				1	1			
-1/2			1	1				
-1		1	1					
-3/2	1	1						
-2	1							

Table 3.1: N=1 case

hel vs. λ	-2	-3/2	-1	-1/2	0	1/2	1
2							1
3/2						1	2
1					1	2	1
1/2				1	2	1	
0			1	2	1		
-1/2		1	2	1			
-1	1	2	1				
-3/2	2	1					
-2	1						

Table 3.2: N=2 case

hel vs. λ	-2	-3/2	-1	-1/2	0	1/2
2						1
3/2					1	3
1				1	3	3
1/2			1	3	3	1
0		1	3	3	1	
-1/2	1	3	3	1		
-1	3	3	1			
-3/2	3	1				
-2	1					

Table 3.3: N=3 case

hel vs. λ	-2	-3/2	-1	-1/2	0
2					1
3/2				1	4
1			1	4	6
1/2		1	4	6	4
0	1	4	6	4	1
-1/2	4	6	4	1	
-1	6	4	1		
-3/2	4	1			
-2	1				

Table 3.4: N=4 case

3.3.2 Massive case

In order to determine the representation of the supersymmetric algebra for a massive particle, we first boost to the rest frame of the particle: $P^\mu = (M, 0, 0, 0) = P_\mu$. Now we have,

$$\sigma^\mu P_\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \tag{3.123}$$

and so

$$\begin{aligned} \{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}\beta}\} &= 2M\delta_{\alpha\dot{\beta}}\delta^A{}_B \\ \{Q_\alpha{}^A, Q_\beta{}^B\} &= \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}\beta}\} = 0 \end{aligned} \tag{3.124}$$

where A and B run to 1 to N. We can rescale the Q's such that

$$a_\alpha{}^A = \frac{1}{\sqrt{2M}} Q_\alpha{}^A \tag{3.125}$$

$$\left(a_\alpha{}^A\right)^\dagger = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot{\alpha}A} \tag{3.126}$$

Then,

$$\frac{1}{\sqrt{2M}} \times \frac{1}{\sqrt{2M}} \{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\} = \delta_{\alpha\dot{\beta}} \delta^A{}_B \tag{3.127}$$

which implies

$$\begin{aligned} &\{a_\alpha{}^A, \left(a_\beta{}^B\right)^\dagger\} = \delta_\alpha{}^\beta \delta^A{}_B \\ &\{a_\alpha{}^A, \left(a_\beta{}^B\right) = \{\left(a_\alpha{}^A\right)^\dagger, \left(a_\beta{}^B\right)^\dagger\} = 0 \end{aligned} \tag{3.128}$$

Our aim from now on is to prove that $a_\alpha{}^A$ and $\left(a_\alpha{}^A\right)^\dagger$ are in fact the 2N fermionic creation and annihilation operators.

We will first consider the case where $N = 1$. Using relations 3.88 and 3.89 we get

$$\boxed{[J_i, Q_\alpha] = \frac{1}{2} (\sigma_i)_\alpha{}^\beta Q_\beta} \tag{3.129}$$

implying

$$J_i Q_\alpha = Q_\alpha J_i + \frac{1}{2} \left(\sigma^i\right)_\alpha{}^\beta Q_\beta \tag{3.130}$$

taking the hermitian conjugate and using $(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} = -[(S_L^{\mu\nu})_a{}^b]^*$,

$$\boxed{[J_i, \bar{Q}_{\dot{\alpha}}] = -\frac{1}{2} \left[\left(\sigma^i\right)_\alpha{}^\beta\right]^* \bar{Q}_{\dot{\beta}}} \tag{3.131}$$

When $i = 3$ *i.e.* J_3, we have from 3.130, and taking the normalisation into accout,

$$a_1 |j_3\rangle = |j_3 + \frac{1}{2}\rangle \tag{3.132}$$

$$a_2 |j_3\rangle = |j_3 - \frac{1}{2}\rangle \tag{3.133}$$

Similarly for $\bar{Q}_{\dot{\beta}}$

$$a_1^\dagger |j_3\rangle = |j_3 - \frac{1}{2}\rangle \tag{3.134}$$

$$a_2^\dagger|j_3\rangle = |j_3 + \frac{1}{2}\rangle \tag{3.135}$$

Consider now **j=0:**

$$[\bar{Q}_{\dot\alpha}, J^2] = [\bar{Q}_{\dot\alpha}, Ji]J_i + Ji[\bar{Q}_{\dot\alpha}, J_i] = +\frac{1}{2}\left(\sigma_i\right)_{\dot\alpha}{}^{\beta} Q_{\dot\beta} J_i + J_i \left(\frac{1}{2}\sigma_i\right)_{\dot\alpha}^{\ \dot\beta} \bar{Q}_{\dot\beta} \tag{3.136}$$

replacing

$$J_i \bar{Q}_{\dot\beta} = \bar{Q}_{\dot\beta} J_i - \frac{1}{2}\left(\sigma_i\right)_{\dot\beta}{}^{\dot\gamma} \bar{Q}_{\dot\gamma} \tag{3.137}$$

we get

$$\begin{aligned} &\frac{1}{2}\left(\sigma_i\right)_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta} J_i + \frac{1}{2}\left(\sigma_i\right)_{\dot\alpha}{}^{\dot\beta}\left(\bar{Q}_{\dot\beta} J_i - \frac{1}{2}\left(\sigma_i\right)_{\dot\beta}{}^{\dot\gamma} \bar{Q}_{\dot\gamma}\right) \\ &= \left(\sigma_i\right)_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta} J_i - \frac{1}{4}\left(\sigma_i\right)_{\dot\alpha}{}^{\dot\beta}\left(\sigma_i\right)_{\dot\beta}{}^{\dot\gamma} \bar{Q}_{\dot\gamma} \\ &= \left(\sigma_i\right)_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta} J_i - \frac{3}{4}\bar{Q}_{\dot\alpha} \end{aligned} \tag{3.138}$$

Therefore,

$$[\bar{Q}_{\dot\alpha}, J^2] = -\frac{3}{4}\bar{Q}_{\dot\alpha} + \left(\sigma_i\right)_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta} J_i \tag{3.139}$$

Let $\dot\alpha = 1$ and $\dot\beta = 2$ then

$$\begin{aligned} [\bar{Q}_{\dot\alpha}\bar{Q}_{\dot\beta}, J_3] =& [\bar{Q}_{\dot 1}\bar{Q}_{\dot 2}, J_3] = [\bar{Q}_{\dot 1}, J_3]\bar{Q}_{\dot 2} + \bar{Q}_{\dot 1}[\bar{Q}_{\dot 2}, J_3] \\ =& \frac{1}{2}\left(\sigma_3\right)_{\dot 1}{}^{\dot\gamma} \bar{Q}_{\dot\gamma}\bar{Q}_{\dot 2} + \frac{1}{2}\bar{Q}_{\dot 1}\left(\sigma_3\right)_{\dot 2}{}^{\dot\lambda} \bar{Q}_{\dot\lambda} \end{aligned} \tag{3.140}$$

However, the first term only contributes to the sum if $\dot\gamma = 1$ and the second term only does so when $\dot\lambda = 2$

$$\frac{1}{2}\bar{Q}_{\dot 1}\bar{Q}_{\dot 2} + \bar{Q}_{\dot 1}\left(\frac{1}{2}\right) \times (-1)\, \bar{Q}_{\dot 2} = \frac{1}{2}\bar{Q}_{\dot 1}\bar{Q}_{\dot 2} - \frac{1}{2}\bar{Q}_{\dot 1}\bar{Q}_{\dot 2} = 0 \tag{3.141}$$

$$\boxed{[a_1^\dagger a_2^\dagger, J_3] = 0} \tag{3.142}$$

$$\begin{aligned} [\bar{Q}_{\dot 1}\bar{Q}_{\dot 2}, J^2] =& [\bar{Q}_{\dot 1}, J^2]\bar{Q}_{\dot 2} + \bar{Q}_{\dot 1}\left[\bar{Q}_{\dot 2}, J^2\right] \\ =& -\frac{3}{4}\bar{Q}_{\dot 1}\bar{Q}_{\dot 2} + \left(\sigma_i\right)_{\dot 1}{}^{\dot\alpha} \bar{Q}_{\dot\alpha} J_i \bar{Q}_{\dot 2} - \frac{3}{4}\bar{Q}_{\dot 1}\bar{Q}_{\dot 2} + \left(\sigma_i\right)_{\dot 2}{}^{\dot\beta} \bar{Q}_{\dot 1}\bar{Q}_{\dot\beta} J_i \\ =& -\frac{3}{2}\bar{Q}_{\dot 1}\bar{Q}_{\dot 2} + \left(\sigma_i\right)_{\dot 1}{}^{\dot\alpha} \bar{Q}_{\dot\alpha} J_i \bar{Q}_{\dot 2} + \left(\sigma_i\right)_{\dot 2}{}^{\dot\beta} \bar{Q}_{\dot 1}\bar{Q}_{\dot\beta} J_i \end{aligned} \tag{3.143}$$

Using

$$[\bar{Q}_{\dot{2}}, J_i] = \bar{Q}_{\dot{2}} J_i - J_i \bar{Q}_{\dot{2}} = \frac{1}{2} (\sigma_i)_{\dot{2}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} \tag{3.144}$$

to substitute for $\left(J_i \bar{Q}_{\dot{2}}\right)$, we get

$$= -\frac{3}{2} \bar{Q}_{\dot{1}} \bar{Q}_{\dot{2}} + (\sigma_i)_{\dot{1}}{}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \bar{Q}_{\dot{2}} J_i - \frac{1}{2} (\sigma_i)_{\dot{1}}{}^{\dot{\alpha}} (\sigma_i)_{\dot{2}}{}^{\dot{\beta}} \bar{Q}_{\dot{\alpha}} \bar{Q}_{\dot{\beta}} + (\sigma_i)_{\dot{2}}{}^{\dot{\beta}} \bar{Q}_{\dot{1}} \bar{Q}_{\dot{\beta}} J_i \tag{3.145}$$

Summing over $i = 1, 2, 3$ results in

$$\begin{aligned} -\frac{3}{2} \bar{Q}_{\dot{1}} \bar{Q}_{\dot{2}} + & \bar{Q}_{\dot{2}} \bar{Q}_{\dot{2}} J_1 - \frac{1}{2} \times 1 \times 1 \times \bar{Q}_{\dot{2}} \bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}} \bar{Q}_{\dot{1}} J_1 \\ & - i \bar{Q}_{\dot{2}} \bar{Q}_{\dot{2}} J_2 - \frac{1}{2} \times (-i) \times (+i) \bar{Q}_{\dot{2}} \bar{Q}_{\dot{1}} + i \bar{Q}_{\dot{1}} \bar{Q}_{\dot{1}} J_2 \\ & + \bar{Q}_{\dot{1}} \bar{Q}_{\dot{2}} J_3 - \frac{1}{2} \times 1 \times (-1) \bar{Q}_{\dot{1}} \bar{Q}_{\dot{2}} + (-1) \bar{Q}_{\dot{1}} \bar{Q}_{\dot{2}} J_3 \\ & = \bar{Q}_{\dot{2}} \bar{Q}_{\dot{2}} J_- + \bar{Q}_{\dot{1}} \bar{Q}_{\dot{1}} J_+ \end{aligned} \tag{3.146}$$

where J_- decreases J_3 by 1 and J_+ increases it by 1. On the other hand, $\bar{Q}_{\dot{2}} \bar{Q}_{\dot{2}}$ acting on any state gives zero. This is because $\{\bar{Q}_{\dot{2}}, \bar{Q}_{\dot{2}}\} = 0 \Rightarrow \bar{Q}_{\dot{2}} \bar{Q}_{\dot{2}} = -\bar{Q}_{\dot{2}} \bar{Q}_{\dot{2}}$, can only be if $\bar{Q}_{\dot{2}} \bar{Q}_{\dot{2}} = 0$. Similarly for $\bar{Q}_{\dot{1}} \bar{Q}_{\dot{1}}$. Hence,

$$\boxed{[\bar{Q}_{\dot{1}} \bar{Q}_{\dot{2}}, J^2] = 0} \tag{3.147}$$

Recall

$$\begin{aligned} [\bar{Q}_{\dot{\alpha}}, J^2] &= -\frac{3}{4} \bar{Q}_{\dot{\alpha}} + (\sigma_i)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} J_i \\ \Rightarrow J^2 \bar{Q}_{\dot{\alpha}} &= \bar{Q}_{\dot{\alpha}} J^2 + \frac{3}{4} \bar{Q}_{\dot{\alpha}} - (\sigma_i)_{\dot{1}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} J_i \\ \Rightarrow J^2 \bar{Q}_{\dot{1}} |\Omega\rangle &= \bar{Q}_{\dot{1}} J^2 + \frac{3}{4} \bar{Q}_{\dot{1}} - (\sigma_i)_{\dot{1}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} J_i \end{aligned} \tag{3.148}$$

Also

$$\begin{aligned} (\sigma_i)_{\dot{1}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} J_i &= (\sigma_1)_{\dot{1}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} J_1 + (\sigma_2)_{\dot{1}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} J_2 + (\sigma_3)_{\dot{1}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} J_3 \\ &= \bar{Q}_{\dot{2}} J_1 - i \bar{Q}_{\dot{2}} J_2 + \bar{Q}_{\dot{1}} J_3 \\ &= \bar{Q}_{\dot{2}} \underbrace{(J_1 - i J_2)}_{J_-} + \bar{Q}_{\dot{1}} J_3 \end{aligned} \tag{3.149}$$

Let the ground state be $|\Omega\rangle = |m, 0, 0\rangle$, then

$$J^2 \bar{Q}_{\dot{1}} |\Omega\rangle = \left(\bar{Q}_{\dot{1}} J^2 + \frac{3}{4} \bar{Q}_{\dot{1}} \bar{Q}_{\dot{2}} - \bar{Q}_{\dot{2}} J_- - \bar{Q}_{\dot{1}} J_3 \right) |\Omega\rangle \tag{3.150}$$

The first term and the last two terms vanish since both j and j_3 are zero for this state. But the left hand side of the above equation is

$$J^2 \underbrace{\bar{Q}_{\dot{1}}|\Omega\rangle}_{\text{spin j state}} := j(j+1)\underbrace{\bar{Q}_{\dot{1}}|\Omega\rangle} \tag{3.151}$$

So here we must have

$$j(j+1)\bar{Q}_{\dot{1}}|\Omega\rangle = \frac{3}{4}\bar{Q}_{\dot{1}}|\Omega\rangle \tag{3.152}$$

which immediately implies that $j = \frac{1}{2}$. On the other hand, we had we also had, for general j, $\bar{Q}_{\dot{1}}|j_3\rangle = |j_3 - \frac{1}{2}\rangle$ (ignoring the normalisation). Here we have a state with $j = 0 \Rightarrow j_3 = 0$ therefore, $\left(a_1^\dagger|\Omega\rangle\right)$ has $j = \frac{1}{2}$ and $j_3 = -\frac{1}{2}$ *i.e.*

$$\boxed{a_1^\dagger|\Omega\rangle = |m, \frac{1}{2}, -\frac{1}{2}\rangle} \tag{3.153}$$

Similarly for $\bar{Q}_{\dot{2}}$

$$J^2\bar{Q}_{\dot{2}}|\Omega\rangle = \bar{Q}_{\dot{2}}J^2 + \frac{3}{4}\bar{Q}_{\dot{2}} - (\sigma_i)_{\dot{2}}{}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}J_i \tag{3.154}$$

and

$$\begin{aligned}(\sigma_i)_{\dot{2}}{}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}J_i =&(\sigma_1)_{\dot{2}}{}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}J_1 + (\sigma_2)_{\dot{2}}{}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}J_2 + (\sigma_3)_{\dot{2}}{}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}J_3\\ =&\bar{Q}_{\dot{2}}J_1 + i\bar{Q}_{\dot{2}}J_2 - \bar{Q}_{\dot{2}}J_3\\ =&\bar{Q}_{\dot{2}}\left(J_1 + iJ_2\right) - \bar{Q}_{\dot{2}}J_3\end{aligned} \tag{3.155}$$

resulting in 3.154 to be equal to

$$\left(\bar{Q}_{\dot{2}}J^2 + \frac{3}{4}\bar{Q}_{\dot{2}} - \bar{Q}_{\dot{2}}J_+ + \bar{Q}_{\dot{2}}J_3\right)|\Omega\rangle \tag{3.156}$$

and because the total j of the state is zero while $a_2^\dagger|j_3\rangle = |j_3 + \frac{1}{2}\rangle$ we conclude

$$\boxed{a_2^\dagger|\Omega\rangle = |m, \frac{1}{2}, +\frac{1}{2}\rangle} \tag{3.157}$$

It is important to mention that $a_1^\dagger$ and $a_2^\dagger$ are *grassmann variables*, *i.e.* they satisfy the following:

$$a_\alpha^\dagger a_\beta^\dagger = -a_\beta^\dagger a_\alpha^\dagger \tag{3.158}$$

and

$$\{a_2^\dagger, a_2^\dagger\} = \{a_1^\dagger, a_1^\dagger\} = 0 \tag{3.159}$$

so

$$a_2^\dagger a_2^\dagger = -a_2^\dagger a_2^\dagger \Rightarrow a_2^\dagger a_2^\dagger |\Omega\rangle = 0 \tag{3.160}$$

Similarly,

$$a_1^\dagger a_1^\dagger |\Omega\rangle = 0 \tag{3.161}$$

Therefore, it only remains to consider $a_2^\dagger a_1^\dagger |\Omega\rangle$ which is an object with the same spin value as Ω *i.e.* if Ω has spin j then this state is also a spin j object.

For the case of $j = 0$ we then have:

$$|\Omega\rangle = |m, j = 0, P^\mu, j_3 = 0\rangle \tag{3.162}$$

$$a_{1,2}^\dagger |\Omega\rangle = |m, j = \frac{1}{2}, P^\mu, j_3 = \pm\frac{1}{2}\rangle \tag{3.163}$$

$$a_1^\dagger a_2^\dagger |\Omega\rangle = |m, j = 0, P^\mu, j_3 = 0\rangle \tag{3.164}$$

So we have two states of spin 0 and one of spin $\frac{1}{2}$.
However, 3.162 and 3.164 look the same. The questions now is that how do we know these states are actually different different. They are different since if we apply parity the former transforms as a scalar while the latter transforms as a pseudovector.

Also note that $a_1^\dagger a_2^\dagger |\Omega\rangle$ belongs to the $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ representation, but since $a_1^\dagger a_2^\dagger = -a_2^\dagger a_1^\dagger$ is antisymmetric, we only take the antisymmetric part of the representation *i.e.* $(0, 0)$ meaning that acting on $|\Omega\rangle$ by $a_1^\dagger a_2^\dagger$ does not change the spin of the state so it remains zero.

Let us now consider the case where $\underline{\mathbf{j \neq 0}}$:

- We know that $a_2^\dagger$ $(a_1^\dagger)$ acting on the state will increase (decrease) the third component of spin, J_3 by $\frac{1}{2}$. (Operator)
- Also by the addition of angular momenta $j_1 \otimes j_2 = |j_1 - j_2|, ..., |j_1 - j_2|$. (Spinor)

Here we have the product of a spinor $(a^\dagger)$ and a state which can be a spinor or otherwise

$$\frac{1}{2} \otimes j = |j - \frac{1}{2}|, j + \frac{1}{2} \tag{3.165}$$

$$\boxed{a_1^\dagger |\Omega\rangle = k_1 |m, j + \frac{1}{2}, P^\mu, j_3 - \frac{1}{2}\rangle + k_2 |m, j - \frac{1}{2}, P^\mu, j_3 - \frac{1}{2}\rangle} \tag{3.166}$$

$$\boxed{a_2^\dagger|\Omega\rangle = k_3|m, j+\frac{1}{2}, P^\mu, j_3+\frac{1}{2}\rangle + k_4|m, j-\frac{1}{2}, P^\mu, j_3+\frac{1}{2}\rangle} \tag{3.167}$$

Where the k's are the Clebsch-Gordon coefficients.

Recall that $a_1^\dagger$ and $a_2^\dagger$ are Grassmann variables so $a_2^\dagger a_1^\dagger = -a_2^\dagger a_1^\dagger$ *i.e.* antisymmetric which implies that we only take the antisymmetric part of the representation $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. Hence

$$a_2^\dagger a_1^\dagger|\Omega\rangle = -a_2^\dagger a_1^\dagger|\Omega\rangle \quad \text{which is a spin } j \text{ object like } \Omega \tag{3.168}$$

The multiplet has spins $(j, j+\frac{1}{2}, j-\frac{1}{2}, j)$ or explicitly

$$\begin{aligned} 2\times\ |m, j, P^\mu, j_3\rangle &\xrightarrow{\text{Total number of states}} 2\times(2j+1) = 4j+2 \\ |m, j-\frac{1}{2}, P^\mu, j_3\rangle &\xrightarrow{\hspace{3cm}} 2\ (j+\frac{1}{2}) = 2j+2 \\ |m, j+\frac{1}{2}, P^\mu, j_3\rangle &\xrightarrow{\hspace{3cm}} 2\ (j-\frac{1}{2}) = 2j \end{aligned} \tag{3.169}$$

Note that if the two spin j state are fermions, the other two states are boson (since they differ by $\frac{1}{2}$) and vice-versa. This is in agreement with what we proved before that the number of bosons and fermions in a multiplet is equal.
For the case of $N = 1$ we have

Spin	Ω_0	$\Omega_{\frac{1}{2}}$	Ω_1	$\Omega_{\frac{3}{2}}$
0	2	1		
1/2	1	2	1	
1		1	2	1
3/2			1	2
2				1

Table 3.5: Massive particle N=1 case

In general, it can be said that, all the states are generated by successive application of $a^\dagger$'s on the ground state characterised by $|m, s_0, s_3\rangle$. Thus a typical state may be written as

$$|\ \ \rangle = \left(a_{\alpha_1}{}^{A_1}\right)^\dagger ... \left(a_{\alpha_n}{}^{A_n}\right)^\dagger |m, s_0, s_3\rangle \tag{3.170}$$

which is totally antisymmetric.

Maximal spin: Knowing that $A = 1, ..., N$ and $a_2^\dagger$ increases the spin by $\frac{1}{2}$ we get $a_1^{\dagger\ 1} ... a_1^{\dagger\ N}|m, s_0\rangle$

which mean that $s_{\max} = s_0 + \frac{N}{2}$. This is because there are N of $a_2^\dagger$'s.

Minimal spin: We must have $s_{\min} \geq 0$ therefore

$$\begin{cases} 0 & \text{for } \frac{N}{2} \geq s_0 \\ s_0 - \frac{N}{2} & \text{Otherwise} \end{cases} \tag{3.171}$$

Top state: One can reach this state y applying all $2N$ fermionic operators once. Its spin is the same as that of the original state *i.e.* s_0. This is because the result of applying all N of $a_1^\dagger$'s, each of them decreasing the spin by $\frac{1}{2}$ is then cancelled by applying all N of $a_2^\dagger$'s, each increasing it by $\frac{1}{2}$. Hence the net result will be the original spin.

Bibliography

[1] Steven Weinberg, *The quantum theory of fields: Foundations*. volume 1 Cambridge University Press, Cambridge, 2005.

[2] Mark Allen Srednicki, *Quantum field theory*, Cambridge University Press, Cambridge, 2007

[3] Julius Wess and Jonathan Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton, N.J., 2nd ed., rev. and expanded edition, 1992.

[4] Martin F. Sohnius, *Introducing supersymmetry*, Cambridge, 1985.

[5] Wu-Ki Tung, *Group theory in physics*, World Scientific, Philadelphia, 1985.

[6] Steven Weinberg, *The quantum theory of fields: Supersymmetry*. volume 3 Cambridge University Press, Cambridge, 2005.