Derivation of Virasoro Algebra from Dilatation Invariance

Based on Martin Lüscher's talk (Munich, December 1988)

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1 Aim of the Talk

From statistical physics we know that conformal invariance at the critical point is a stronger condition (and not very plausible) compared to dilatation invariance which is a natural consequence of the renormalization group ideas. Here it is shown that in 2-dimensions, dilatation invariance plus the existence of an energy-momentum tensor $\theta_{\mu\nu}$ imply general conformal invariance on the level of the field algebra.

2 Assumptions

Before we begin, let us list all the assumptions involved:

- Wightman axioms, a set of axioms used to formulate quantum field theory with a precise mathematical framework.

  I. Hilbert space $\mathcal{H}$ of physical states
  II. Unitary representation $U(\Lambda, a)$ of the Poincaré group
  III. Forward light cone $P_0 \geq 0$, $P^2 \geq 0$ and a unique ground
  IV. Local fields $\phi(x)$, $\psi(y)$, ... which obey Bose or Fermi statistics depending on their spin.

- Dilatation symmetry: Unitary operators $V(\lambda)$ with $\lambda > 0$, such that

$$V(\lambda)|0\rangle = |0\rangle \tag{1}$$

$$V(\lambda)\phi(x)V(\lambda)^{-1} = \lambda^d \phi(\lambda x) \tag{2}$$

- Energy-momentum tensor $\theta_{\mu\nu}$, is a field of dimension 2 with the following properties

$$\theta_{\mu\nu} = \theta_{\nu\mu} \tag{3}$$

$$\theta^\dagger_{\mu\nu} = \theta_{\mu\nu} \tag{4}$$

$$\partial^\mu \theta_{\mu\nu} = 0 \tag{5}$$

The fact that the energy-momentum tensor is of dimension 2 when working in 2-D is expected since,

$$P_\mu = \int dx^1 \ \theta_{0\mu} \tag{6}$$

and we know $[P] = 1$ and $[x^1] = -1$.

Using the operator for translations

$$U(1, \epsilon)\phi(x)U^{-1}(1, \epsilon) = \phi(x + \epsilon) \tag{7}$$

and expanding on both sides, with $U(1, \epsilon) = 1 + i\epsilon_\mu P^\mu$ yields

$$[P^\mu, \phi] = -i\partial^\mu \phi(x) \tag{8}$$

which means that $\theta_{\mu\nu}$ generates the translations,

$$\int dx^1 \ [\theta_{0,\mu}(x), \phi(y)] = -i\partial_\mu \phi(y) \tag{9}$$

The results that will follow from now on are deduces from the above assumptions.

Finally, note that the metric is of the form

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

3 $\theta_{\mu\nu}$ is Traceless

In order to prove that $\theta_{\mu\nu}$ is traceless, it is more convenient to introduce light-cone coordinates.

$$x^+ = x^0 + x^1 \tag{11}$$

$$x^- = x^0 - x^1 \tag{12}$$

In other words, $x^0 = \frac{1}{2}(x^+ + x^-)$ and $x^1 = \frac{1}{2}(x^+ - x^-)$. The light-cone coordinates are at 45 degrees to the space-time axis. In fact, they are the worldline of light in $\pm x^1$ direction emitted at $x^0 = 0$.

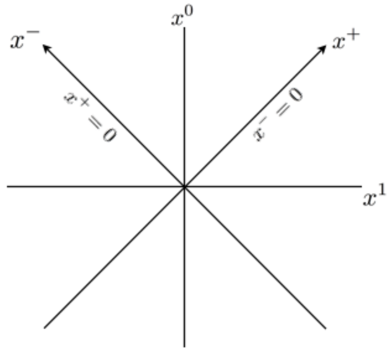

Figure 1: Light-cone Coordinate System

Note that

$$\partial_+ = \frac{\partial}{\partial x^+} = \frac{\partial x^0}{\partial x^+}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x^+}\frac{\partial}{\partial x^1} = \frac{1}{2}(\partial_0 + \partial_1) \tag{13}$$

Similarly,

$$\partial_- = \frac{1}{2}(\partial_0 - \partial_1) \tag{14}$$

then the metric in the new coordinates transform as

$$\eta'_{+-} = \frac{\partial x^\rho}{\partial x^+}\frac{\partial x^\sigma}{\partial x^-}\eta_{\rho\sigma} = \frac{1}{2} \tag{15}$$

In the same way, one can easily show that $\eta'_{++} = \eta'_{--} = 0$ and $\eta'_{-+} = \frac{1}{2}$ *i.e.*

$$\eta' = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{16}$$

Then

$$x_- = \eta_{-+}x^+ = \frac{1}{2}(x^0 + x^1) = \frac{1}{2}(x_0 - x_1) \tag{17}$$

$$x_+ = \eta_{+-}x^- = \frac{1}{2}(x^0 - x^1) = \frac{1}{2}(x_0 + x_1) \tag{18}$$

As a result the components energy-momentum tensor transform in the following way

$$\theta_{++} = \frac{\partial x_+}{\partial x_\mu}\frac{\partial x_+}{\partial x_\nu}\theta_{\mu\nu} = \frac{1}{4}(\theta_{00} + 2\theta_{01} + \theta_{11}) \tag{19}$$

$$\theta_{--} = \frac{\partial x_-}{\partial x_\mu}\frac{\partial x_-}{\partial x_\nu}\theta_{\mu\nu} = \frac{1}{4}(\theta_{00} - 2\theta_{01} + \theta_{11}) \tag{20}$$

$$\theta_{+-} = \theta_{-+} = \frac{1}{4}(\theta_{00} - \theta_{11}) \tag{21}$$

Using these we have

$$\partial_-\theta_{++} + \partial_+\theta_{-+} = 0 \tag{22}$$

$$\partial_-\theta_{+-} + \partial_+\theta_{--} = 0 \tag{23}$$

The Lorentz boost can be written in the following form

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \tag{24}$$

Then we would have

$$U(\Lambda)\theta_{++}U(\Lambda^{-1}) = e^{2\chi}\theta_{++}(e^\chi x^+, e^{-\chi}x^-) \tag{25}$$

Proof: We know that under Lorentz, tensors transform as

$$U(\Lambda)T^{\mu\nu}(x)U^{-1}(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma T^{\rho\sigma}(\Lambda^{-1}x) \tag{26}$$

This can be contracted with η several times to give

$$U(\Lambda)T_{\mu\nu}U(\Lambda)^{-1} = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma T_{\rho\sigma}(\Lambda^{-1}x) \tag{27}$$

For the argument $\Lambda^{-1}x$ we have

$$\begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} \cosh\chi\ x^0 + \sinh\chi\ x^1 \\ \sinh\chi\ x^0 + \cosh\chi\ x^1 \end{pmatrix} \tag{28}$$

but in terms of light-cone coordinates it has components $\cosh\chi\ x^0 + \sinh\chi\ x^1 + \sinh\chi\ x^0 + \cosh\chi\ x^1 = e^\chi(x^0 + x^1) = e^\chi x^+$ and $e^{-\chi}x^-$.

For the tensor itself,

$$U(\Lambda)\theta_{++}(x)U^{-1}(\Lambda) = \frac{1}{4}U(\Lambda)\left(\theta_{00}(\Lambda^{-1}x) + 2\theta_{01}(\Lambda^{-1}x) + \theta_{11}(\Lambda^{-1}x)\right)U^{-1}(\Lambda) \tag{29}$$

So, for example for the θ_{00} component

$$U(\Lambda)\theta_{00}(x)U^{-1}(\Lambda) = \Lambda_0{}^\rho\Lambda_0{}^\sigma\theta_{\rho\sigma}(\Lambda^{-1}x) = \cosh^2\chi\ \theta_{00} + 2\sinh\chi\ \cosh\chi\ \theta_{01} + \sinh^2\chi\ \theta_{11} \tag{30}$$

Adding together with the results of θ_{11} and $\theta_{01} = \theta_{10}$ and writing the hyperbolic functions in terms of exponentials gives the above result.

Q.E.D

For dilatations

$$V(\lambda)\theta_{++}(x)V(\lambda)^{-1} = \lambda^2\theta_{++}(\lambda x) \tag{31}$$

since the field has dimension 2. Choosing $\lambda = e^{-\chi}$ and combining equations 25 and 31 gives

$$U(\Lambda)V(\lambda)\theta_{++}(x)V(\lambda)^{-1}U(\Lambda)^{-1} = U(\Lambda)e^{-2\chi}\theta_{++}(e^{-\chi}x)U(\Lambda)^{-1} = \theta_{++}(x^+, \lambda^2 x^-) \tag{32}$$

Hence

$$\boxed{\langle 0|\theta_{++}(x)\theta_{++}(y)|0\rangle = A(x^+ - y^+ - i\epsilon)^{-4}} \tag{33}$$

Proof:

To convince oneself that this is the correct form for the two-point function note the following:

- Indeed, the dimensions on both sides of the equation match since $[\theta] = 2$ and $[x^+] = -1$ therefore the dimension is 4 in total.
- The choice of the relative minus sign between x^+ and y^+ is in order to preserve translational invariance.
- Recalling the action of dilatation and Lorentz operators on the vacuum

  $$V(\lambda)|0\rangle = |0\rangle \quad , \quad U(\Lambda)|0\rangle = |0\rangle \tag{34}$$

  equation 32 gives

  $$\langle 0|UV\theta_{++}(x^+)\underbrace{V^{-1}U^{-1}UV}_{\mathbb{1}}\theta_{++}(y^+)V^{-1}U^{-1}|0\rangle = \langle 0|\theta_{++}(x^+, \lambda^2 x^-)\theta_{++}(y^+, \lambda^2 y^-)|0\rangle \tag{35}$$

  since Lorentz and dilatations are symmetries of the system, the above equation must remain invariant under such transformations. It is clear that the x^+ component remains invariant while the x^- changes. Therefore, the result can only depend on x^+.

Q.E.D

Now, one can apply $\partial_{-_x}\partial_{-_y}$ on both sides of the equation. The RHS immediately gives 0. Then using positivity condition,

$$\partial_-\theta_{++}(x)|0\rangle = 0 \tag{36}$$

Thus by the Reeh-Schlieder theorem (Appendix A)

$$\partial_-\theta_{++} = 0 \tag{37}$$

In the same way, one can show

$$\partial_+\theta_{--} = 0 \tag{38}$$

Finally, using 22 and 23 we get

$$\partial_+\theta_{+-} = \partial_-\theta_{+-} = 0 \quad \Rightarrow \quad \theta_{+-} = \text{constant} \tag{39}$$

because θ_{+-} is a field of dimension 2, it follows that $\theta_{+-} = 0$ *i.e.* $\theta_{00} = \theta_{11}$ then

$$\theta^\mu{}_\mu = \eta^{\nu\mu}\theta_{\nu\mu} = \theta_{00} - \theta_{11} = 0 \tag{40}$$

Proving the claim that the energy-momentum tensor θ is traceless.

4 Commutators

Equation 37 implies that θ_{++} only depends on x^+ and similarly θ_{--} depends on x^- only. From locality we conclude

$$[\theta_{++}(x), \theta_{--}(y)] = 0 \qquad \text{for all } x,\ y \tag{41}$$

Now, we will discuss the commutator of θ_{++} with itself using a simplified notation:

$$\theta_{++} \to \theta \qquad , \qquad x^+ \to x \in \mathbb{R} \tag{42}$$

<u>Theorem:</u> There exists a number $c \geq 0$ such that

$$\boxed{[\theta(x), \theta(y)] = \frac{c}{6\pi} i^3 \delta'''(x-y) + 4i\delta'(x-y)\theta(y) - 2i\delta(x-y)\partial_y\theta(y)} \tag{43}$$

c is called the "central charge".

<u>Proof:</u>

By locality, $[\theta(x), \theta(y)] = 0$ for $x \neq y$. Define O_k to be a $d = 3-k$ dimensional, local hermitian field of the form

$$O_k(x) = \frac{i}{k!} \int dz\ z^k [\theta(x+z), \theta(x)] \qquad k = 0, 1, 2, ... \tag{44}$$

Then,

$$\begin{aligned} \langle 0|O_k(x)O_k(y)|0\rangle =& A_k(x-y-i\epsilon)^{2k-6} \\ \underset{k\geq 3}{=}& (-1)^{k-3} A_k \int \frac{dp}{2\pi} e^{-ip(x-y)} \delta^{(2k-6)}(p) \end{aligned} \tag{45}$$

where in the last equality for $k \geq 3$ we have used used the fact that for a δ-distribution

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x) = (-1)^n\ f^{(n)}(x)\Big|_{x=0} \tag{46}$$

Before we continue, let us define a state in the following form

$$|\psi\rangle = \int dx\ O_k(x) f(x)|0\rangle \tag{47}$$

Then by positive-definiteness, since $\langle\psi|\psi\rangle \geq 0$, we must have

$$\langle 0| \int dx\ dy\ O_k(x)O_k(y) f(x) f(y)^* |0\rangle \geq 0 \tag{48}$$

for all test functions $f(x)$ in the Schwartz space. However for $k \geq 3$ we have

$$\begin{aligned} &\int dx\ dy\ \frac{dp}{2\pi}(-1)^{k-3} A_k\ e^{-ip(x-y)} \delta^{(2k-6)}(p) f(x) f^*(y) \\ =& (-1)^{k-3} A_k \int \frac{dp}{2\pi}\ \delta^{(2k-6)}(p) \int dx\ f(x) e^{-ipx} \int dy\ f(y)^* e^{ipy} \\ =& (-1)^{k-3} A_k\ \frac{d^{2k-6}}{dp^{2k-6}} \parallel \tilde{f}(p) \parallel^2 \Big|_{p=0} \end{aligned} \tag{49}$$

which is not necessarily greater or equal to zero for all test functions since it involves its derivatives. Thus

$$O_k = 0 \qquad k > 3 \tag{50}$$

For $k = 3$, it is clear that $O_3(x)$ is independent of x. By locality, this would imply that O_3 commutes with all the fields and therefore it is proportional to the unit operator:

$$O_3 = -\frac{c}{6\pi} \tag{51}$$

On the other hand, since $\theta(x)$ generates translations

$$\int dx\ [\theta(x), \theta(y)] = -2i\partial_y \theta(y) \tag{52}$$

Proof:

From equations 19 and 20, $\theta_{++} = \theta_{--} + \theta_{01}$. As well as that,

$$\partial_- \theta_{++}(x) = 0 \quad \Rightarrow \quad \frac{1}{2}(\partial_0 - \partial_1)\theta_{++} = 0 \quad \Rightarrow \quad \partial_0 \theta_{++} = \partial_1 \theta_{++} \tag{53}$$

Also, $x^1 = \frac{x^+ - x^-}{2}$ and so $\int dx^+ = 2\int dx^1$ since the integrand is independent of x_-. Then

$$\begin{aligned}&\int dx[\theta_{++}(x), \theta_{++}(y)] = 2\int dx^1\ [\theta_{--}(x) - \theta_{01}, \theta_{++}(y)] \overset{41}{=} 2\int dx^1[\theta_{01}, \theta_{++}(y)]\\ &= 2(-i)\partial_1\theta_{++}(y) \overset{53}{=} -i(\partial_1 + \partial_0)\theta_{++}(y) = -2i\partial_y\theta_{++}(y)\end{aligned} \tag{54}$$

Q.E.D

Therefore,

$$O_0(x) = i\int dz'[\theta(z'), \theta(x)] = 2\partial_x \theta(x) \tag{55}$$

Now, take an arbitrary state $|\psi\rangle \in \mathcal{H}$. Then,

$$\langle\psi|[\theta(x+z), \theta(x)]|0\rangle = \sum_{k=0}^{K} \delta^{(k)}(z)\psi_k(x) \tag{56}$$

where $K < \infty$ and $\psi_k(x)$ are some distributions. Now, if we multiply both sides of the above equation by z^m and integrate with respect to z from $-\infty$ to ∞, on the LHS we get

$$m!(-i)\langle 0|O_m(x)|0\rangle \tag{57}$$

and on the RHS

$$\sum_{k=0}^{K} \int dz\ z^m \delta^{(k)}(z)\psi_k(x) \tag{58}$$

Note that for $m > k$, we after perfuming the integration by parts in delta, we will have extra factors of z and so it vanishes when $z = 0$. On the other hand, if $m < k$, as some stage we will get derivative of 1 which again vanishes. Hence the equation is only non-zero when $m = k$ implying

$$\psi_k(x) = -i(-1)^k \langle\psi|O_k(x)|0\rangle \tag{59}$$

In particular, $\psi_k = 0$ for $k > 3$. Therfore,

$$[\theta(x+z), \theta(x)] = -i \sum_{k=0}^{3} (-1)^k \delta^{(k)}(z) O_k(x) \tag{60}$$

holds for the vacuum and, as a consequence of the Reeh-Schlieder theorem, as an operator identity.

In summary we have shown that

$$[\theta(x), \theta(y)] = \frac{c}{6\pi} i^3 \delta'''(x-y) + 4i\delta'(x-y)O_1(y) - 2i\delta(x-y)\partial_y\theta(y) \tag{61}$$

It remains to find O_1. For this, we use the fact that $[\theta(x), \theta(y)] + [\theta(y), \theta(x)] = 0$ in the above equations and integrate with respect to x. The terms involving δ''' immediately vanish and the ones with $\delta(x-y)$ add up to give $-4i\partial_y\theta(y)$. Also,

$$\int dx\ \delta'(x-y)O_1(y) = 0 \tag{62}$$

while

$$i \int dx\ \delta'(y-x)O_1(x) = +i\partial_y O_1(y) \tag{63}$$

using the fact that $\delta'(x) = -\delta'(-x)$. Hence

$$\partial_x O_1(x) = 4\partial_x\theta(x) \qquad \Rightarrow \qquad O_1 = 4\theta \tag{64}$$

by locality and dilatation invariance.

The only part left to complete the proof of the theorem is to show that $c \geq 0$. The commutation rule 43, implies that

$$\langle 0|[\theta(x), \theta(y)]|0\rangle = \frac{c}{2\pi} i^3 \delta'''(x-y) \tag{65}$$

since $\langle 0|\theta(x)|0\rangle = 0$ by translation and dilatation invariance. On the other hand, recall that

$$\langle 0|\theta(x)\theta(y)|0\rangle = A(x-y-i\epsilon)^{-4} \tag{66}$$

which gives

$$\begin{aligned} \langle 0|[\theta(x), \theta(y)]|0\rangle =& A\left\{(x-y-i\epsilon)^{-4} - (y-x-i\epsilon)^{-4}\right\} \\ =& A\left\{(x-y-i\epsilon)^{-4} - (x-y+i\epsilon)^{-4}\right\} \\ =& A\left(\frac{-2\pi i}{6}\right)\delta'''(x-y) \end{aligned} \tag{67}$$

where we have used (Appendix B)

$$\delta'''(x) = \frac{-6}{2\pi i}\left\{(x-i\epsilon)^{-4} - (x+i\epsilon)^{-4}\right\} \tag{68}$$

Equating 65 and 67 gives

$$A = \frac{c}{2\pi^2} \tag{69}$$

The Fourier representation of 65 is of the form

$$\langle 0|\theta(x)\theta(y)|0\rangle = \frac{c}{12\pi^2}\int_0^\infty dp\ p^3 e^{-ip(x-y)} \tag{70}$$

Therefore, $c \geq 0$ is necessary to insure positivity of the 2-point functions.

Q.E.D

5 Interpretation

For any smooth rapidly decaying test-function $f(x)$ define

$$\theta(f) = \int dx\ f(x)\theta(x) \tag{71}$$

Then,

$$\boxed{[\theta(f), \theta(g)] = i^3 \frac{c}{12\pi} \int dx\ (fg''' - gf''') + 2i\theta([f,g])} \tag{72}$$

where

$$[f,g](x) = f(x)g'(x) - g(x)f'(x) \tag{73}$$

Proof:

$$\begin{aligned} [\theta(f), \theta(g)] &= \int dx\ dy\ f(x)g(y)\ [\theta(x), \theta(y)] \\ &= \int dx\ dy\ f(x)g(y) \left\{ \frac{c}{6\pi} i^3 \delta'''(x-y) + 4i\delta'(x-y)\theta(y) - 2i\delta(x-y)\partial_y\theta(y) \right\} \end{aligned} \tag{74}$$

Let us compute the above expression term by term by performing the integral over y. The first term can be written as

$$\frac{i^3 c}{12\pi} \int dx\ dy\ f(x)g(y)\delta'''(x-y) + f(y)g(x)\delta'''(y-x) \tag{75}$$

Integrating over y and noting $\delta'''(x) = -\delta'''(-x)$, gives

$$\frac{i^3 c}{12\pi} \int dx\ f(x)g'''(x) - g(x)f'''(x) \tag{76}$$

For the second term, again,

$$\begin{aligned} 4i \int dx\ dy\ f(x)g(y)\delta'(x-y)\theta(y) &= 2i \int dx\ dy\ f(x)g(y)\delta'(x-y)\theta(y) + f(y)g(x)\delta'(y-x)\theta(x) \\ =& 2i \int dx\ f(x) \int dy\ \frac{d}{dy}(g(y)\theta(y))\delta(x-y) - 2i \int dx\ g(x)f'(x)\theta(x) \\ =& 2i \int dx\ f(x)\frac{d}{dx}(g(x)\theta(x)) - 2i \int dx\ g(x)f'(x)\theta(x) \\ =& 2i \left\{ \int dx\ f(x)g'(x)\theta(x) + f(x)g(x)\theta'(x) - g(x)f'(x)\theta(x) \right\} \end{aligned} \tag{77}$$

and finally, the third term,

$$-2i \int dy\ dx\ f(x)g(y)\delta(x-y)\partial_y\theta(y) = -2i \int dx\ \theta'(x)f(x)g(x) \tag{78}$$

which putting it all together give the required result.

Q.E.D

Equation 73 is the Lie bracket associated with the group of diffeomorphisms[1] of $\mathbb{R}$. To see this, first note that

$$x \to x + \epsilon f(x) \qquad \text{with } \epsilon \text{ infinitesimal} \tag{79}$$

defines an infinitesimal diffeomorphism. Then for the function $F(x)$ we have

$$F(x) \to F(x + \epsilon f(x)) = F(x) + \epsilon f(x) F'(x) \tag{80}$$

so that the associated variation δ_f is defined through

$$\delta_f F(x) = f(x) F'(x) \tag{81}$$

Then,

$$[\delta_f, \delta_g] = \delta_{[f,g]} \tag{82}$$

Proof: On the LHS

$$\begin{aligned}
[\delta_f, \delta_g]F(x) =& \delta_f\left(\delta_g F(x)\right) - \delta_g\left(\delta_f F(x)\right) = \delta_f\left(g(x)F'(x)\right) - \delta_g\left(f(x)F'(x)\right) \\
=& f(x)\left(g(x)F'(x)\right)' - g(x)\left(f(x)F'(x)\right)' \\
=& f(x)g(x)F''(x) + f(x)g'(x)F'(x) - g(x)f(x)F''(x) - g(x)f'(x)F'(x) \\
=& f(x)g'(x)F'(x) - g(x)f'(x)F'(x)
\end{aligned} \tag{83}$$

while the RHS gives

$$\delta_{[f,g]}F(x) = [f(x), g(x)]F'(x) \tag{84}$$

which by definition of the Lie bracket are equal.

Q.E.D

Now, by observing equation 72, we can see that $\theta(f)$ take the form of a representation. Namely, a representation of a central extension[2] of the Lie algebra of the group of diffeomorphisms of $\mathbb{R}$. Recall that $\theta(x)$ is a local hermitian operator. Therefore, this algebra is represented by local hermitian automorphism[3] of the algebra of fields and in this sense can be thought of a symmetry of the system.

6 n-Point Functions of $\theta(x)$

The algebra 43 completely determines the vacuum expectation values

$$W_n(x_1, ..., x_n) = \langle 0|\theta(x_1)...\theta(x_n)|0\rangle \tag{85}$$

In order to see this, let us define

$$\theta^{(-)}(x) = \int_0^\infty \frac{dp}{2\pi} e^{-ipx}\, \tilde{\theta}(p) \tag{86}$$

$$\theta^{(+)}(x) = \theta(x) - \theta^{(-)}(x) \tag{87}$$

[1] "A diffeomorphism is a map between manifolds which is differentiable and has a differentiable inverse."

[2] According to Weinberg, the term on the RHS of the Lie commutation relation proportional to the unit element is called the "central charge".

[3] An automorphism is an isomorphism of an object to itself.

where

$$\tilde{\theta}(p) = \int dx\; e^{ipx}\; \theta(x) \tag{88}$$

Due to the spectrum condition[4]

$$\theta^{(-)}(x)|0\rangle = 0 \qquad \text{and since } \; \theta^{(+)}(x) = \theta^{(-)}(x)^{\dagger} \qquad \langle 0|\theta^{(+)}(x) = 0 \tag{89}$$

Then 43 gives

$$[\theta^{(-)}(x), \theta(y)] = \frac{c}{2\pi^2}(x-y-i\epsilon)^{-4} - \frac{2}{\pi}(x-y-i\epsilon)^{-2}\theta(y) - \frac{1}{\pi}(x-y-i\epsilon)^{-1}\partial_y\theta(y) \tag{90}$$

Therefore, the n-point function W_n take the form

$$W_n = \langle 0|[\theta^{(-)}(x_1), \theta(x_2)...\theta(x_n)]|0\rangle \tag{91}$$

Using equation 90, W_n can be written in terms of W_{n-1} and W_{n-2} (Appendix C). Continuing in the same way, we obtain W_n as a sum of products of

$$(x_i - x_j - i\epsilon)^{-k} \quad , \quad i<j \quad , \quad k = 0, 1, ..., 5 \tag{92}$$

Note that W_n can be thought of as a boundary value of a meromorphic[5] function $\hat{W}_n(z_1, ..., z_n)$, with $z_k \in \mathbb{C}$, which is totally symmetric (Appendix D) and regular for $z_i \neq z_k$.

7 Conformal Invariance

Take $\mathcal{O}$ to be any polynomial of local fields smeared with test functions of compact support.[6] Then,

$$[\theta(x), \mathcal{O}] \qquad \text{for large } x \tag{93}$$

Now, define the conformal variations

$$\delta_k\mathcal{O} = \frac{1}{2}\int dx\; x^k\; [\theta(x), \mathcal{O}] \qquad k = 0, 1, 2 \tag{94}$$

to see this, recall our discussion in section 5. Let infinitesimal diffeomorphisms be of the form

$$x \to x + \epsilon x^k \tag{95}$$

In other words we have $f(x) = x^k$. Therefore, one can associate $k = 0$ to translations, $k = 1$ to dilatations and $k = 2$ to the a complicated transformation, *i.e.* the special conformal. Then we simply use

$$\theta(x^k) = \int dx\; x^k\theta(x) \tag{96}$$

and the fact that the infinitesimal variation in an operator is represented by a commutator of the operator with object that generates the variation, implying equation 94.

[4] The condition uses the fact that the momenta are in the forward light cone V^+, and shows that for any $p \notin V^+$ the Fourier transform of the n-point function vanishes.

[5] A meromorphic function is a function that is holomorphic on the domain except for a set of poles.

[6] The support of a function is the set of points for which the function is not zero. A function has compact support if it is zero outside a comport set *i.e.* its support is compact.

These transformations generate the group $SL(2,\mathbb{R})$, the group of 2×2 real matrices with determinant 1. It act on x via

$$x \to \frac{ax+b}{cx+d} \qquad , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \tag{97}$$

It is possible to convince one-self that this is the case by expanding the above transformation near the identity (as it will be done in the next section). Note that by x we really mean x^+, meaning that there is another $SL(2,\mathbb{R})$ acting on x^-.

This group is referred to as the (restricted) conformal group.

The claim now is that the correlation function of $\theta(x)$ are always conformally invariant. In other words,

$$\boxed{\sum_{j=1}^{n} \langle 0|\theta(x_1)...\delta_k\theta(x_j)...\theta(x_n)|0\rangle = 0} \tag{98}$$

In order to prove we first need to show

$$\delta_k\theta(x) = -iD_k\theta(x) \qquad , \qquad D_k = 2kx^{k-1} + x^k\partial_x \tag{99}$$

Proof:

- $k=0$

$$\delta_0\theta(x) = \frac{1}{2}\int dy\ [\theta(y),\theta(x)] = -i\partial_x\theta(x) = -iD_0\theta(x) \tag{100}$$

- $k=1$

$$\begin{aligned}\delta_1\theta(x) =& \frac{1}{2}\int dy\ y\ [\theta(y),\theta(x)] = \frac{1}{2}\int dy\ y\ \left(\frac{c}{6\pi}i^3\delta'''(y-x) + 4i\delta'(y-x)\theta(x) - 2i\delta(y-x)\partial_x\theta(x)\right)\\ =& \frac{1}{2}\int (dy\ (-4i)\ \delta(y-x)\theta(x)) - ix\partial_x\theta(x) = (-2i - ix\partial_x)\,\theta(x) = -iD_1\theta(x)\end{aligned} \tag{101}$$

- $k=2$

$$\begin{aligned}\delta_2\theta(x) =& \frac{1}{2}\int dy\ y^2\ [\theta(y),\theta(x)] = \frac{1}{2}\int dy\ y^2\ \left(\frac{c}{6\pi}i^3\delta'''(y-x) + 4i\delta'(y-x)\theta(x) - 2i\delta(y-x)\partial_x\theta(x)\right)\\ =& \left(-4ix - ix^2\partial_x\right)\theta(x) = -iD_2\theta(x)\end{aligned} \tag{102}$$

Q.E.D

Then we have, knowing that W_n only depends on $(x_i - x_j)^{-k}$ due to 92,

$$\boxed{\sum_{j=1}^{n} D_k^{(j)}\ W_n(x_1,...,x_n) = 0} \tag{103}$$

where $D_k^{(j)}$ denotes the operator D_k with rep respect to x_j. Once can easily check the above expression is true for the 2-point functions for $k=0,1,2$. Since we have already shown that W_n can be written in terms of W_{n-1} and W_{n-2} (Appendix C), one can continue in the same way, and prove the above claim by induction. Equation 103 is the conformal Ward identity ("infinitesimal or weak conformal invariance").

8 The Compact Picture

As a consequence of the weak conformal invariance 103, we aim to show that correlation functions of θ can be analytically extended to $\bar{\mathbb{R}} \cong S^1$, the compactification of $\mathbb{R}$.

Equation 103 is the infinitesimal expression of the following symmetry of $W_n(x_1, ...x_n)$. Let,

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \in \quad SL(2, \mathbb{R}) \tag{104}$$

be close to the identity and define

$$g.x = \frac{\alpha x + \beta}{\gamma x + \delta} \tag{105}$$

Then, provided $\gamma x_j + \delta > 0$ for all j, one has

$$W_n(g.x_1, ..., g.x_n) = \prod_{j=1}^{n} (\gamma x_j + \delta)^4 W_n(x_1, ..., x_n) \tag{106}$$

Proof: We would like

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{107}$$

to be near the identity. To this end, we take α, β to be of order 1 *i.e.* $\alpha = 1 + \epsilon_1$ and $\delta = 1 + \epsilon_2$ while β and γ are taken to be close to zero.

Before we continue note that one can use the Ward identity and $D_k = 2kx^{k-1} + x^k \partial_x$ to get, for each x_i, the following:

$$\partial_x W = 0 \tag{108}$$

for $k = 0$. For $k = 1$,

$$x \partial_x W = -2W \tag{109}$$

and finally for $k = 2$,

$$x^2 \partial_x W = -4xW \tag{110}$$

Also, since the determinant is equal to 1,

$$(1 + \epsilon_1)(1 + \epsilon_2) - 2\beta\gamma = 1 \quad \Rightarrow \quad \epsilon_1 + \epsilon_2 = 0 \tag{111}$$

where we have ignored terms of the form $\beta\gamma$ and $\epsilon_1\epsilon_2$ to this order. Then, the coordinate transformation 105 can be written as

$$\begin{aligned} \frac{(1 + \epsilon_1)x + \beta}{\gamma x + (1 + \epsilon_2)} =& [(1 + \epsilon_1)x + \beta][1 - \epsilon_2 - \gamma x] \\ =& x + (\epsilon_1 - \epsilon_2)x - \gamma x^2 + \beta \end{aligned} \tag{112}$$

Taylor expanding W gives

$$\begin{aligned} W\left(x + \beta + (\epsilon_1 - \epsilon_2)x - \gamma x^2\right) =& W(x) + \left(\beta + (\epsilon_1 - \epsilon_2)x - \gamma x^2\right) \partial_x W \\ =& W(x) + \left(-2(\epsilon_1 - \epsilon_2) + 4\gamma x\right) W(x) \\ =& \left(1 + 4(\epsilon_2 + \gamma x)\right) W(x) \end{aligned} \tag{113}$$

where in the second equality we have used the 3 equations from the Ward identity derived above to remove the derivate acting on W and in the last equality 111 has been used. This is just the infinitesimal version of

$$(1 + \epsilon_2 + \gamma x)^4 = (\delta + \gamma x)^4 \tag{114}$$

completing the proof.

Q.E.D

There is an obvious problem with this symmetry. $SL(2, \mathbb{R})$ does not act in a regular way on $\mathbb{R}$ because when $\gamma x + \delta = 0$ is not well-defined. However, in this context, one can see that $SL(2, \mathbb{R})$ acts in a regular way on

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \tag{115}$$

which can be regarded as the one-point compactification of $\mathbb{R}$ through the stereographic projection (see figure 2)

$$z \to x = \frac{\mathrm{Re}(z)}{1 - \mathrm{Im}(z)} \tag{116}$$

Parameterising z by

$$z = -ie^{i\tau} \quad , \quad -\pi < \tau < \pi \tag{117}$$

then

$$x = \frac{\sin(\tau)}{1 + \cos \tau} = \frac{2 \sin(\tau/2) \cos(\tau/2)}{2 \cos^2(\tau/2)} = \tan(\tau/2) \tag{118}$$

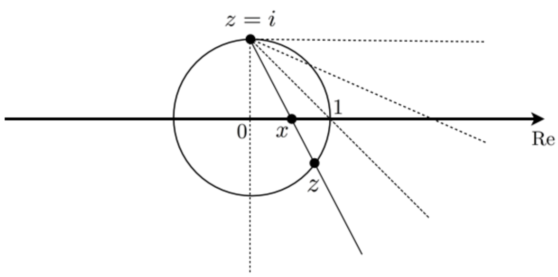

Figure 2: Stereographic Projection of $z \in S^1 \ \to \ x \in \mathbb{R}$

As it is clear from the figure, for every point on the real line, there is a unique corresponding point on the circle except of the $\pm\infty$ which correspond to the so-called pole located at $z = i$ *i.e.* $\tau = \pi$. The associate point can be found by drawing a straight line from the pole. The line will intersect each the x-axis and the circle at a point. We then say that these points are equivalent. By trying a few points (such as $0, 1, -1$), it is easy convince oneself that the mapping from the real axis to the circle is of the form

$$x \longmapsto \frac{z + i}{iz + 1} \tag{119}$$

The correponding matrix of the form $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Using

$$\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} \alpha + i\beta - i\gamma + \delta & i\alpha + \beta + \gamma - i\delta \\ -i\alpha + \beta + \gamma + i\delta & \alpha - i\beta + i\gamma + \delta \end{pmatrix} \tag{120}$$

the action of $g \in SL(2, \mathbb{R})$ on z is written as

$$g.z = \frac{az + b}{b^* z + a*} \tag{121}$$

where $a = \frac{1}{2}(\alpha + i\beta - i\gamma + \delta)$ and $b = \frac{1}{2}(i\alpha + \beta + \gamma - i\delta)$. Note that the mapping matrix $\begin{pmatrix} a & b \\ b^* & a* \end{pmatrix}$ has determinant 1 also.

Equation 121 is non-singular.

Proof: For $b \neq 0$ the denominator is zero when $z = \frac{-a^*}{b^*}$. If $z = \frac{-a^*}{b^*}$ is possible, then we must have $|z| = 1$ but,

$$\left|\frac{a^*}{b^*}\right|^2 = \left|\frac{\alpha - i\beta + i\gamma + \delta}{-i\alpha + \beta + \gamma + i\delta}\right|^2 \tag{122}$$

i.e. we would need, $\left|\frac{a^*}{b^*}\right|^2 = 1$:

$$\frac{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2\alpha\delta - 2\beta\gamma}{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - 2\alpha\delta + 2\beta\gamma} = \frac{\chi^2 + 2}{\chi^2 - 2} \tag{123}$$

taking $\chi^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \in \mathbb{R}$ and using $\alpha\delta - \beta\gamma = 1$. Of course, the above equation can never be equal to 1 and we reach contradiction. Hence, $z \neq \frac{a^*}{b^*}$ making equation 121 non-singular. Note that for the case $b = 0$, $g.z = \frac{a}{a^*} z = \frac{re^{i\phi}}{re^{-i\phi}} z = e^{2i\phi} z$ which is just a rotation.

Q.E.D

Recalling how tensors transform:

$$T'_{\mu\nu} = \frac{\partial x'_\mu}{\partial x'^\rho} \frac{\partial x'_\nu}{\partial x'^\sigma} T_{\rho\sigma} \tag{124}$$

we can now lift the energy-momentum tensor $\theta(x)$ to a field on S^1. Therefore

$$T(z) = (\cos\frac{\tau}{2})^{-4}\ \theta(\tan\frac{\tau}{2}) \tag{125}$$

As a non-trivial consequence of conformal invariance, this field extend to an operator valued distribution on S^1 such that

$$V_n(z_1, ..., z_n) = \langle 0|T(z_1)...T(z_n)|0\rangle \tag{126}$$

is a boundary value of a function which is analytic for

$$0 < |z_1| < |z_2| < ... < |z_n| \tag{127}$$

Also, v_n is globally invariant under $SL(2,\mathbb{R})$ *i.e.*

$$\boxed{V_n(g.z_1,...,g.z_n) = \prod_{j=1}^{n}(b^* z_j + a^*)^4 V_n(z_1,...,z_n)} \tag{128}$$

This transformation is completely regular. Note that V_n is invariant under a translation of the variable τ_j.

9 The Virasoro Algebra

Equation 43 now implies (Appendix E)

$$[T(z),T(w)] = \frac{8c}{3\pi} i^3 \left(\delta'''(\tau-\omega) + \delta'(\tau-\omega)\right) + 16\ i\ \delta'(\tau-\omega)T(w) - 8\ i\ \delta(\tau-\omega)\ \partial_\omega T(w) \tag{129}$$

If we define the Fourier components L_n through

$$L_n = \frac{1}{8i}\oint_{S^1} dz\ z^{-n-1}\ T(z) \tag{130}$$

we find that (Appendix)

$$\boxed{[L_n, L_m] = \frac{c}{12}(n-n^3)\delta_{n+m} - (n-m)L_{n+m}} \tag{131}$$

This is the **Virasoro** algebra. Recall that the operators L_n are obtained by smearing a Wightman field with an admissible test function and so the operators L_n can be arbitrarily multiplied.

Before we conclude, it is worth mentioning that at this stage one can go further and show using $L_n^\dagger = L_{-n}$ and

$$\langle 0|T(z)T(w)|0\rangle = \frac{8c}{\pi^2}\frac{z^2w^2}{(z-w+\epsilon z)^4} \tag{132}$$

that

$$\langle 0|L_n^\dagger L_n|0\rangle = 0 \qquad \text{if}\ \ n \leq 1 \tag{133}$$

and hence

$$L_n|0\rangle = 0 \qquad \text{for}\ \ n \leq 1 \tag{134}$$

For $n \leq -1$ this result can be regarded as a consequence of the spectrum condition such that $L_n|0\rangle$ would be a state with negative energy. For $n = 0, \pm 1$, the L_n's are the generator of $SL(2,\mathbb{R})$.

Also, operators L_n act as raising and lowering operators with respect to the "conformal Hamiltonian" L_0 *i.e.*

$$[L_0, L_n] = nL_n \tag{135}$$

10 Conclusion

Starting from Wightman axioms, assuming Poincare and dilatation invariance and the existence of an energy momentum tensor we have shown general conformal invariance on the level of the field algebra in 2-dimensions. The invariance of the ground state $|0\rangle$ however, is a separate issue and was not addressed here. We have also shown, without any further assumptions, that the correlation functions of the energy-momentum tensor $\theta_{\mu\nu}(x)$ are exactly calculable.

Appendix A Reeh-Schlieder Theorem

This section is based on Streater and Wightman's book "PCT, Spin and Statistics, and All That". In what follows I have tried to write the relevant definitions and theorems that have been used in the talk. The reader is encouraged to refer to the book for a more rigorous presentation.

Definition: For a relativistic quantum field theory, satisfying the usual axioms, the vacuum state is *cyclic* for the smeared fields, *i.e.* "if the polynomials in the smeared field components $P(\phi_1(f), \phi_2(g), ...)$, when applied to the vacuum state ψ_0, yield a set D_0 of vectors which is dense in the Hilbert space of states."

Let $\mathscr{P}(\mathcal{O})$ be the set of all polynomials of the form

$$c + \sum_{j=1}^{N} \phi\left(f_1^{(j)}\right) ... \phi\left(f_j^{(j)}\right) \tag{136}$$

where $f_k^{(j)}$ are test functions with support in the open set $\mathcal{O}$ of space-time and $c \in \mathbb{C}$ is arbitrary. If p and q are such polynomials, it implies that $p+q, \alpha p, p^*, pq$ are also polynomials of this type. In other words, $\mathscr{P}(\mathcal{O})$ is the polynomial algebra of $\mathcal{O}$.

Then, the following theorem states that "the algebra associated with *any* open set $\mathcal{O}$ has ψ_0 as a cyclic vector. It is due to Reeh and Schlieder."

Theorem: Let $\mathcal{O}$ be an open set of space-time. Then the vacuum ψ_0 is a cyclic vector for $\mathscr{P}(\mathcal{O})$, if it is a cyclic vector for $\mathscr{P}$ over all space-time. In other words, vectors of the form

$$\sum_{j=1}^{N} \phi\left(f_1^{(j)}\right) ... \phi\left(f_j^{(j)}\right) \psi_0 \tag{137}$$

with support$(f_k^{(j)}) \subset \mathcal{O}$ are dense in the Hilbert space of states $\mathscr{H}$.

The proof of this theorem will not be presented here.

The next theorem is exactly what we have used in section 3.

Theorem: Let $\mathcal{O}$ be an open set, with a certain boundedness condition, and $T \in \mathscr{P}(\mathcal{O})$. Then,

$$T\psi_0 = 0 \qquad \Longrightarrow \qquad T = 0 \tag{138}$$

The proof is rathe short and makes use of the previous theorem (see Wightman's book, page 139).

Appendix B A Useful Definition of the Delta Function

It is useful to see an alternative way to represent the delta function:

$$\begin{aligned} \delta(x) &= \lim_{\epsilon\to 0} \left(\int_{-\infty}^{0} \frac{dk}{2\pi} e^{ikx} e^{\epsilon k} + \int_{0}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{-\epsilon k} \right) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \left(\frac{1}{ix+\epsilon} - \frac{1}{ix-\epsilon} \right) \\ &= \lim_{\epsilon\to 0} \frac{1}{2\pi i} \left((x-i\epsilon)^{-1} - (x+i\epsilon)^{-1} \right) \end{aligned} \tag{139}$$

It then follows that

$$\delta'(x) = \frac{-1}{2\pi i}\left((x-i\epsilon)^{-2} - (x+i\epsilon)^{-2}\right) \tag{140}$$

$$\delta''(x) = \frac{-1}{\pi i}\left(-(x-i\epsilon)^{-3} + (x+i\epsilon)^{-3}\right) \tag{141}$$

$$\delta'''(x) = \frac{-6}{2\pi i}\left((x-i\epsilon)^{-4} + (x+i\epsilon)^{-4}\right) \tag{142}$$

Clearly, $\delta'''(x) = -\delta'''(-x)$. Same is true with $\delta'(x)$.

Appendix C Form of the n-Point Function W_n

$$\begin{aligned}
W_n =&\langle 0|[\theta^{(-)}(x_1), \theta(x_2)...\theta(x_n)]|0\rangle = \langle 0|\theta^{(-)}(x_1)\theta(x_2)...\theta(x_n)|0\rangle \\
=&\langle 0|[\theta_1^{(-)}, \theta_2]\theta_3...\theta_n|0\rangle + \langle 0|\theta_2\left(\theta^{(-)}\theta_3\right)...\theta_n|0\rangle \\
=&\frac{c}{2\pi^2}(x_1-x_2-i\epsilon)^{-4}\langle 0|\theta_3...\theta_n|0\rangle - \frac{2}{\pi}(x_1-x_2-i\epsilon)^{-2}\langle 0|\theta_2...\theta_n|0\rangle \\
-&\frac{1}{\pi}(x_1-x_2-i\epsilon)^{-1}\partial_{x_2}\langle 0|\theta_2...\theta_n|0\rangle + \langle 0|\theta_2\left(\theta^{(-)}\theta_3\right)...\theta_n|0\rangle \\
=&\frac{c}{2\pi^2}(x_1-x_2-i\epsilon)^{-4}W_{n-2} + \left(\frac{2}{\pi}(x_1-x_2-i\epsilon)^{-2} - \frac{1}{\pi}(x_1-x_2-i\epsilon)^{-1}\partial_2\right)W_{n-1} \\
+&\langle 0|\theta_2\left(\theta^{(-)}\theta_3\right)...\theta_n|0\rangle
\end{aligned} \tag{143}$$

Carrying on in exactly the same way for the last term, for the $n-1$ such iterations, one would need to compute

$$\begin{aligned}
&\langle 0|\theta_2\theta_3...\left(\theta_1^{(-)}\theta_n\right)|0\rangle = \langle 0|\theta_2\theta_3...[\theta_1^{(-)}\theta_n]|0\rangle + 0 \\
=&\frac{c}{2\pi}(x_1-x_n-i\epsilon)^{-4}W_{n-2} + \left(\frac{-2}{\pi}(x_1-x_n)^{-2} - \frac{1}{\pi}(x_1-x_n-i\epsilon)^{-1}\partial_n\right)W_{n-1}
\end{aligned} \tag{144}$$

Hence, overall we have,

$$\begin{aligned}
W_n =&\frac{c}{2\pi^2}\left\{(x_1-x_2-i\epsilon)^{-4} + (x_1-x_3-i\epsilon)^{-4} + .. + (x_1-x_n-i\epsilon)^{-4}\right\}W_{n-2} \\
=&\frac{1}{\pi}\left\{2(x_1-x_2-i\epsilon)^{-2} + ... + 2(x_1-x_n-i\epsilon)^{-2} + (x_1-x_2-i\epsilon)^{-1}\partial_2 + ... + (x_1-x_n-i\epsilon)^{-1}\partial_n\right\}W_{n-1}
\end{aligned} \tag{145}$$

Hence W_n can be written in terms of W_{n-1} and W_{n-2}.

Appendix D Example: The 3-Point Function W_3

$$
\begin{aligned}
W_3 =& \langle 0|\theta_1\theta_2\theta_3|0\rangle = \langle 0|\theta_1^{(-)}\theta_2\theta_3|0\rangle = \langle 0|[\theta_1^{(-)},\theta_2]\theta_3|0\rangle + \langle 0|\theta_2\theta_1^{(-)}\theta_3|0\rangle \\
=& \frac{c}{2\pi^2}(x_1-x_2-i\epsilon)^{-4}\langle 0|\theta_3|0\rangle - \frac{2}{\pi}(x_1-x_2-i\epsilon)^{-2}\langle 0|\theta_2\theta_3|0\rangle \\
-& \frac{1}{\pi}(x_1-x_2-i\epsilon)^{-1}\partial_2\langle 0|\theta_2\theta_3|0\rangle + \langle 0|\theta_2[\theta_1^{(-)},\theta_3]|0\rangle + 0 \\
=& \frac{-2}{\pi}(x_1-x_2-i\epsilon)^{-2}\langle\frac{c}{2\pi^2}(x_2-x_3-i\epsilon)^{-4}|0\rangle - \frac{1}{\pi}(x_1-x_2-i\epsilon)^{-1}\langle 0|\frac{c}{2\pi^2}\partial_2(x_2-x_3-i\epsilon)^{-4}|0\rangle \\
+& \langle 0|\theta_2\left(\frac{-2}{\pi}(x_1-x_3-i\epsilon)^{-2}\theta_3 - \frac{1}{\pi}(x_1-x_3-i\epsilon)^{-1}\partial_3\theta_3\right)|0\rangle
\end{aligned} \tag{146}
$$

where we have used $\langle 0|\theta|0\rangle = 0$. Applying the same for $\theta_2\theta_3$, we finally get,

$$
\begin{aligned}
W_3 =& \frac{-c}{\pi^3}(x_1-x_2-i\epsilon)^{-2}(x_2-x_3-i\epsilon)^{-4} + \frac{2c}{\pi^3}(x_1-x_2-i\epsilon)^{-1}(x_2-x_3-i\epsilon)^{-5} \\
+& \frac{-c}{\pi^3}(x_1-x_3-i\epsilon)^{-2}(x_2-x_3-i\epsilon)^{-4} - \frac{2c}{\pi^3}(x_1-x_3-i\epsilon)^{-1}(x_2-x_3-i\epsilon)^{-5}
\end{aligned} \tag{147}
$$

We expect this expression to be symmetric in its arguments due

$$
\langle 0|[\theta(x),\theta(y)]|0\rangle = \frac{c}{2\pi}i^3\delta'''(x-y) \tag{148}
$$

so that when $x \neq y$ it gives $\langle 0|[\theta(x),\theta(y)]|0\rangle = \langle 0|[\theta(y),\theta(x)]|0\rangle$. However, at first glance, equation 147 might no look symmetric therefore we need to do some manipulation to prove that it is. Note that since θ has dimension 2, the 3-point function is of dimension 6 and so it is best to factor out a product symmetric in $x_1,\ x_2,\ x_3$ that carries the correct dimensions and prove that its coefficient must be proportional to the identity. Before we continue, note that

$$
(x_1-x_2)+(x_2-x_3)+(x_3-x_1) = 0 \tag{149}
$$

squaring the above equation gives

$$
(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2+2(x_1x_2+x_1x_3+x_2x_3-x_1^2-x_2^2-x_3^2) = 0 \tag{150}
$$

Then for terms with powers of $-2,-4$ in 147 we get

$$
\begin{aligned}
&(x_1-x_2)^{-2}(x_2-x_3)^{-4}+(x_1-x_3)^{-2}(x_2-x_3)^{-4} \\
=&(x_1-x_2)^{-2}(x_2-x_3)^{-2}(x_1-x_3)^{-2}\left[\frac{(x_1-x_3)^2}{(x_2-x_3)^2}+\frac{(x_1-x_2)^2}{(x_2-x_3)^2}\right] \\
=&(x_1-x_2)^{-2}(x_2-x_3)^{-2}(x_1-x_3)^{-2}\left[\frac{-2(x_1x_2+x_1x_3+x_2x_3-x_1^2-x_2^2-x_3^2)}{(x_2-x_3)^2}-1\right]
\end{aligned} \tag{151}
$$

and for the terms involving $-1,-5$ powers,

$$
\begin{aligned}
&(x_1-x_2)^{-2}(x_2-x_3)^{-2}(x_1-x_3)^{-2}\left[\frac{(x_1-x_2)(x_1-x_3)^2}{(x_2-x_3)^3}-\frac{(x_1-x_3)(x_1-x_2)^2}{(x_2-x_3)^3}\right] \\
=&(x_1-x_2)^{-2}(x_2-x_3)^{-2}(x_1-x_3)^{-2}\frac{(x_1-x_2)(x_1-x_3)}{(x_2-x_3)^2}
\end{aligned} \tag{152}
$$

Putting them together with the correct factors,

$$
W_3 = \frac{-2c}{\pi^3}(x_1-x_2)^{-2}(x_2-x_3)^{-2}(x_1-x_3)^{-2} \tag{153}
$$

which is fully symmetric in x_1,x_2,x_3.

Appendix E OPE of the Energy-Momentum Tensor as a Field in S^1

We would like to prove the expression:

$$[T(z),T(w)] = \frac{8c}{3\pi} i^3 \left(\delta'''(\tau-\omega)+\delta'(\tau-\omega)\right) + 16\ i\ \delta'(\tau-\omega)T(w) - 8\ i\ \delta(\tau-\omega)\ \partial_\omega T(w) \quad (154)$$

First note the following:

Using

$$\delta(g(x)) = \frac{\delta(x-x_0)}{|g'(x_0)|} \quad (155)$$

we get

$$\delta(\tan\frac{\tau}{2} - \tan\frac{\omega}{2}) = 2\ \delta(\tau-\omega)\cos^2(\frac{\omega}{2}) \quad (156)$$

Furthermore,

$$\frac{d}{d\tan\frac{\tau}{2}} = \frac{d\tau}{d\tan\frac{\tau}{2}}\frac{d}{d\tau} = 2\cos^2(\frac{\tau}{2})\frac{d}{d\tau} \quad (157)$$

Now,

$$\begin{aligned}
&[T(z),T(w)] = \left(\cos\frac{\tau}{2}\right)^{-4}\left(\cos\frac{\omega}{2}\right)^{-4}\left[\theta(\tan\frac{\tau}{2})\ ,\ \theta(\tan\frac{\omega}{2})\right] \\
&= \left(\cos\frac{\tau}{2}\right)^{-4}\left(\cos\frac{\omega}{2}\right)^{-4}\times \\
&\left\{\frac{c}{6\pi}i^3\delta'''(x-\tan\frac{\omega}{2}) + 4i\delta'(\tan\frac{\tau}{2}-\tan\frac{\omega}{2})\theta(\tan\frac{\omega}{2}) - 2i\delta(\tan\frac{\tau}{2}-\tan\frac{\omega}{2})\partial_{\tan\frac{\omega}{2}}\theta(\tan\frac{\omega}{2})\right\}
\end{aligned} \quad (158)$$

For the last term we have

$$\begin{aligned}
&\delta(\tan\frac{\tau}{2}-\tan\frac{\omega}{2})\partial_{\tan\frac{\omega}{2}}\theta(\tan\frac{\omega}{2}) = 2\delta(\tau-\omega)\cos^2(\frac{\omega}{2})2\cos^2(\frac{\omega}{2})\frac{d}{d\omega}\left(T(w)\cos^4(\frac{\omega}{2})\right) \\
&=4\delta(\tau-\omega)\cos^8(\frac{\omega}{2})\frac{d}{d\omega}T(w) + 4\delta(\tau-\omega)\cos^4(\frac{\omega}{2})T(w)4\cos^3(\frac{\omega}{2})(-)\sin(\frac{\omega}{2})\frac{1}{2} \\
&=4\delta(\tau-\omega)\cos^8(\frac{\omega}{2})\frac{d}{d\omega}T(w) - 8\delta(\tau-\omega)T(w)\cos^7(\frac{\omega}{2})\sin(\frac{\omega}{2})
\end{aligned} \quad (159)$$

The first term, when multiplied by the factor of $\left(\cos\frac{\tau}{2}\right)^{-4}\left(\cos\frac{\omega}{2}\right)^{-4}$ and $-2i$ give the last term on the RHS of 154. The second will cancel with a term in the following, which is the second term in 158,

$$\begin{aligned}
&\delta'(\tan\frac{\tau}{2}-\tan\frac{\omega}{2})\theta(\tan\frac{\omega}{2}) = \frac{d\tau}{d\tan\frac{\tau}{2}}\frac{d}{d\tau}\left(2\delta(\tau-\omega)\cos^2(\frac{\omega}{2})\right)\theta(\tan\frac{\omega}{2}) \\
&=4\cos^2(\frac{\omega}{2})\frac{d}{d\tau}\left(\delta(\tau-\omega)\cos^2(\frac{\omega}{2})\right)\theta(\tan\frac{\omega}{2}) \\
&=4\cos^4(\frac{\tau}{2})\delta(\tau-\omega)\theta(\tan\frac{\omega}{2}) + 4\cos^2(\frac{\tau}{2})\delta(\tau-\omega)2\cos(\frac{\omega}{2})(-\frac{1}{2})\sin(\frac{\omega}{2})\theta(\tan\frac{\omega}{2}) \\
&=4\cos^4(\frac{\tau}{2})\delta(\tau-\omega)\theta(\tan\frac{\omega}{2}) - 4\delta(\tau-\omega)\sin(\frac{\omega}{2})T(w)\cos^7(\frac{\omega}{2})
\end{aligned} \quad (160)$$

which when multiplied by $+4i$, cancels the term we wanted to go away in the above equation while its first term gives what is needed for 154.

The term involving δ''' is computed in the same way, by taking the appropriate derivatives of the above expansion, where terms involving δ'' vanish hence completing the proof.

Appendix F Computation of the Virasoro Algebra

In order to compute $[L_n, L_m]$, let us first consider the first term in

$$[T(z), T(w)] = \frac{8c}{3\pi} i^3 \left(\delta'''(\tau-\omega) + \delta'(\tau-\omega)\right) + 16\ i\ \delta'(\tau-\omega)T(w) - 8\ i\ \delta(\tau-\omega)\ \partial_\omega T(w) \quad (161)$$

and substitute it in

$$L_n = \frac{1}{8i}\oint_{S^1} dz\ z^{-n-1}\ T(z) \quad (162)$$

so we get,

$$\begin{aligned}
&\frac{-8ci^3}{64(3\pi)}\oint_{S^1} dz\ dw\ z^{-n-1}\ w^{-m-1}\ \left(\delta'''(\tau-\omega) + \delta'(\tau-\omega)\right)\\
=&\frac{-ci^3}{24\pi}\int_{-\pi}^{\pi} e^{i\tau}\ d\tau\ \oint_{S^1} dw\ \left(-ie^{i\tau}\right)^{-n-1} w^{-m-1}\left(\delta'''(\tau-\omega) + \delta'(\tau-\omega)\right)\\
=&\frac{-c\ i^3}{24\pi}\oint_{S^1} dw\ i\left[-(-in)^3(-ie^{i\omega})^{-n} - (-in)(-ie^{i\omega})^{-n}\right]\\
=&\frac{c}{24\pi}\oint_{S^1} dw\ w^{-m-1}\ (in^3w^{-n} - inw^{-n})\\
=&\frac{c\ i}{24\pi}(n^3-n)\oint_{S^1} dw\ \frac{1}{w^{(n+m)+1}} = \frac{c\ i}{24\pi}(n^3-n)\ 2\pi i\ \delta_{n+m,0}\\
=&\frac{c}{12\pi}(n-n^3)\delta_{n+m,0}
\end{aligned} \quad (163)$$

As required. The other term can be computed similarly to give the result.

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