Taller 2

1,3 Encuentre la funcion de dencidad espectral (transformada de Furier) para las siguientes señales

a) $e^{-a|t|}$ $a>0$

$$\int_{-\infty}^{\infty} e^{-a|t|} e^{-jwt} \, dt$$

$$X(w) = \int_{-\infty}^{0} e^{at} e^{-jwt} \, dt + \int_{0}^{\infty} e^{-at} e^{-jwt} \, dt$$

$$X(w) = \int_{-\infty}^{0} e^{(a-jw)t} \, dt + \int_{0}^{\infty} e^{-(a+jw)t} \, dt$$

$$X(w) = \left. \frac{e^{(a-jw)t}}{a-jw} \right|_{-\infty}^{0} + \left. \frac{e^{-(a+jw)t}}{a+jw} \right|_{0}^{\infty}$$

$$X(w) = \frac{1}{a-jw} - \underbrace{\cancel{\frac{1}{(a-jw)e^{\infty}}}}_{0} + \underbrace{\cancel{\frac{1}{(a+jw)e^{\infty}}}}_{0} - \frac{1}{-(a+jw)}$$

$$X(w) = \frac{1}{a-jw} + \frac{1}{a+jw}$$

$$X(w) = \frac{a \cancel{-jw} + a \cancel{+jw}}{a^2 + \cancel{ajw} - \cancel{ajw} - (jw)^2}$$

$$X(w) = \frac{2a}{a^2 - (-1)w^2} = \boxed{\frac{2a}{a^2 + w^2}}$$

b) $\cos(w_c t)$

$$\int_{-\infty}^{\infty} \cos(w_c t)\, e^{-jwt}\, dt$$

$$\int_{-\infty}^{\infty} \left[\frac{e^{jw_c t} + e^{-jw_c t}}{2}\right] e^{-jw\,t}\, dt \qquad \cos(w_c t) = \frac{e^{jw_c t} + e^{-jw_c t}}{2}$$

$$\frac{1}{2}\int_{-\infty}^{\infty} \left(e^{jw_c t} + e^{-jw_c t}\right) e^{-jwt}\, dt$$

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{jw_c t} e^{-jwt} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-jw_c t} e^{-jwt}\, dt$$

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{(w_c - w)jt}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(w_c + w)t}\, dt$$

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-j(w - w_c)t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(w_c + w)t}\, dt$$

Propiedad

$$\int_{-\infty}^{\infty} e^{j(w_0 - w)t}\, dt = 2\pi\delta(w_0 - w)$$

$$= \frac{1}{\cancel{2}}\cancel{2}\pi\,\delta(w - w_c) + \frac{1}{\cancel{2}}\cancel{2}\pi\,\delta(w_c + w)$$

$$= \pi\delta(w - w_c) + \pi\delta(w_c + w)$$

$$\boxed{= \pi\left(\delta(w - w_c) + \delta(w_c + w)\right)}$$

c) $Sen(W_s t)$

$$X(w) = \int_{-\infty}^{\infty} Sen(w_s t)\, e^{-jwt}\, dt \; ; \; Sen(w_s t) = \frac{e^{jw_s t} - e^{-jw_s t}}{2j}$$

$$X(w) = \int_{-\infty}^{\infty} \left[\frac{e^{jw_s t} - e^{-jw_s t}}{2j} \right] e^{-jwt}\, dt$$

$$X(w) = \frac{1}{2j} \int_{-\infty}^{\infty} e^{jw_s t} e^{-jwt} - \frac{1}{2j} \int_{-\infty} e^{-jw_s t} e^{-jwt}\, dt$$

$$X(w) = \frac{1}{2j} \int_{-\infty}^{\infty} e^{-j(w-w_c)t} - \frac{1}{2j} \int_{-\infty}^{\infty} e^{-j(w+w_c)t}\, dt$$

Por propiedad

$$\int_{-\infty}^{\infty} e^{-j(w-w_0)t}\, dt = 2\pi\, \delta(w-w_0)$$

$$X(w) \quad \frac{1}{\cancel{2}j} \cancel{2}\pi\, \delta(w-w_c) - \frac{1}{\cancel{2}j} \cancel{2}\pi\, \delta(w+w_c)$$

$$\boxed{W(t) = \frac{\pi}{j} \left(\delta(w-w_c) - \delta(w+w_c) \right)}$$

d) $f(t)\, Cos(W_c t)$

$$X(w) = \int_{-\infty}^{\infty} f(t)\, cos(W_c t)\, e^{-jwt}\, dt$$

$$X(w) = \frac{1}{2} \int_{-\infty}^{\infty} f(t)\, e^{jw_c t} e^{-jwt}\, dt + \frac{1}{2} \int_{-\infty} f(t) e^{-jw_c t} e^{-jwt}$$

$$X(w) = \frac{1}{2}\int_{-\infty}^{\infty} f(t)\, e^{-j(w-w_c)t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} f(t)\, e^{-j(w+w_c)t}\, dt$$

Por propiedad

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$$

$$\boxed{X(w) = \frac{1}{2} F(w-w_c) + \frac{1}{2} F(w+w_c)}$$

e) $e^{-a|t|^2}$

$$t^2 = |t|^2$$

$$X(w) = \int_{-\infty}^{\infty} e^{-at^2}\, e^{-jwt}\, dt$$

$$-at^2 - jwt = -a\left[t^2 - \frac{jwt}{a}\right]$$

$$X(w) = e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(t+\frac{jw}{2a}\right)^2}\, dt$$

$$t^2 - \frac{jwt}{a} = \left(t + \frac{jw}{2a}\right)^2 - \left(\frac{jw}{2a}\right)^2$$

$$\left(\frac{jw}{2a}\right)^2 = -\frac{w^2}{4a^2}$$

$$u = t + \frac{jw}{2a}$$

entonces

$$t^2 - \frac{jwt}{a} = \left(t + \frac{jw}{2a}\right)^2 - \frac{w^2}{4a^2}$$

$$du = dt$$

$$X(w) = e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-au^2}\, du$$

$$-at^2 - jwt = -a\left[\left(t + \frac{jw}{2a}\right)^2 + \frac{w^2}{4a^2}\right]$$

$$= a\left(t + \frac{jw}{2a}\right)^2 + \frac{w^2}{4a}$$

$$\boxed{X(w) = e^{-\frac{w^2}{4a}} \sqrt{\frac{\pi}{a}}}$$

f) A rect$_d$ (t) $A, d \in \mathbb{R}$

rect d (t) $\begin{cases} 1, & \text{si } |t| \leq \frac{d}{2} \\ 0, & \text{en otro caso} \end{cases}$ $x(t) = A$ si $|t| = \frac{d}{2}$

$$X(w) = \int_{-\infty}^{\infty} A\, rect_d(t) \cdot e^{-jwt}\, dt$$

Acotando los limites de integración

$$X(w) = A \int_{-d/2}^{d/2} e^{-jwt}\, dt = A \left[\frac{e^{-jwt}}{-jw} \Big|_{-d/2}^{d/2} \right]$$

$$= A \cdot \frac{1}{jw} \left(e^{-jw(d/2)} - e^{-jw(-d/2)} \right)$$

Aplicando la identidad trigonometrica $e^{-j\theta} - e^{j\theta} = -2j\,\text{sen}\,\theta$

$$X(w) = \frac{A}{-jw} \left(-2j\,\text{Sen}\,(wd/2)\right) = \frac{A}{w} \cdot 2\,\text{Sen}\,(wd/2)$$

$$\text{Sin}(x) = \frac{\text{Sen}(x)}{x} \cdot X(w) \quad A \cdot d\, \frac{\text{Sen}\,(wd/2)}{(wd/2)}$$

$$\boxed{X(w) = A \cdot d \cdot \text{Sen}\left(\frac{wd}{2}\right)}$$

1,4 Aplicar las Propiedades de la transformada de Fourier para resolver.

a) $F\{e^{-jw_1 t}\cos(w_c t)\}$

Aplicando la identidad trigonometrica $\cos(w_c t) = \frac{1}{2}(e^{jw_c t} - e^{-jw_c t})$

$$e^{-jwt}\cos(w_c t) = e^{-jw_1 t} \cdot \frac{1}{2}(e^{jw_c t} - e^{-jw_c t})$$

$$= \frac{1}{2}(e^{-j(w_1 - w_c)t} + e^{-j(w_1 + w_c)t})$$

Aplicando la transformada de Fourier

$$F\{e^{-jw_0 t}\} = 2\pi\delta(w - w_0)$$

$$F\{e^{-jw_1 t}\cos(w_c t)\} = \frac{1}{2}\left[F\{e^{-j(w_1 - w_c)t}\} + F\{e^{-j(w_1 + w_c)t}\}\right]$$

$$= \frac{1}{2}\left[2\pi\delta(w - (w_1 - w_c)) + 2\pi\delta(w - (w_1 + w_c))\right.$$

$$= \pi\left[\delta(w - (w_1 - w_c)) + 2\pi\delta(w - (w_1 + w_c))\right]$$

b) $F\{u(t)\cos^2(w_c t)\}$ $\quad w_c \in \mathbb{R}$ $\quad u(t) =$ Funcion Escalon

Aplicando la identidad trigonometrica

$$\cos^2(w_c t) = \frac{1}{2} + \frac{1}{2}\cos(2w_c t)$$

$$u(t)\cos^2(w_c t) = u(t) \cdot \left[\frac{1}{2} + \frac{1}{2}\cos(2w_c t)\right]$$

$$= \frac{1}{2}u(t) + \frac{1}{2}u(t)\cos(2w_c t)$$

Aplicando Propiedades lineales de la transformada de Fourier

$$F\{u(t)\cos^2(w_c t)\} = \frac{1}{2}F\{u(t)\} + \frac{1}{2}\{u(t)\cos(2w_c t)\}$$

La transformada de Fourier de u(t) es

$$F\{u(t)\} = \pi\delta(w) + \frac{1}{jw}$$

Propiedad de Modulacion

$$F\{u(t)\cos(2w_c t) = \frac{1}{2}\left[F\{u(t)e^{jw_0 t}\} + F\{u(t)e^{-jw_0 t}\}\right.$$

Ademas

$$F\{u(t)e^{\pm jw_0 t}\} = \pi\delta(w \mp w_0) + \frac{1}{j(w \mp w_0)}$$

Entonces

$$F\{u(t)\cos^2(w_c t)\} = \frac{1}{2}\left[\pi\delta(w-2w_c) + \frac{1}{j(w-2w_c)} + \right.$$

$$\pi\delta(w+2w_c) + \frac{1}{j(w+2w_c)}$$

$$F\{u(t)\cos^2(w_c t)\} = \frac{1}{2}\left(\pi\delta(w) + \frac{1}{jw}\right) + \frac{1}{4}\left[\pi\delta(w-2w_c) + \right.$$

$$\left.\frac{1}{j(w-2w_c)} + \pi\delta(w+2w_c) + \frac{1}{j(w+2w_c)}\right]$$

$$= \frac{\pi}{2}\delta(w)\frac{1}{2jw} + \frac{\pi}{4}\left[\delta(w-2w_c) + \delta(w+2w_c)\right] +$$

$$\frac{1}{4j}\left[\frac{1}{w-2w_c} + \frac{1}{w+2w_c}\right]$$

c) $F^{-1}\left\{ \dfrac{7}{w^2+6w+45} * \dfrac{10}{(8+jw/3)^2} \right\}$

Aplicando Teorema de Convolución Para la transformada de Fourier.

$$F^{-1}\{F(w) * G(w)\} = 2\pi\ F(t)\cdot g(t)$$

Donde

$$f(t) = F^{-1}\{F(w)\}\ ,\quad g(t) = F^{-1}\{\varphi(w)\}$$

Calculamos la transformada inversa de la Primera Función

$$F(w) = \frac{7}{w^2+6w+45} \qquad w^2+6w+45 = (w^2+6w+9)+36$$

$$= (w+3)^2+6^2$$

$F(w) = \dfrac{7}{(w+3)^2+6^2}$, Aplicando la propiedad Par de transformada

$$F\{e^{-a|t|}\} = \frac{2a}{a^2+w^2} \rightarrow H(w) = \frac{7}{w^2+6^2} \rightarrow a = 6$$

$F^{-1}\left\{ \dfrac{2(6)}{6^2+w^2} \right\} = e^{-6|t|}$, ajustando las Constantes

$$H(w) = \frac{7}{12}\cdot\frac{12}{w^2+6^2} \rightarrow h(t) = F^{-1}\{H(w)\}$$

$$f(t) = \frac{7}{12}e^{-6|t|}$$

Calcular la transformada inversa de la Segunda funcion $g(t)$

$$G(\omega) = \frac{10}{(8 + j\omega)^2}$$

$$F\{t\, e^{-at} u(t)\} = \frac{1}{(a + j\omega)^2}$$

rescribiendo

$$G(\omega) = \frac{10}{\left(\frac{1}{3}\left(24 + j\omega\right)\right)^2} = \frac{10}{\frac{1}{9}(24 + j\omega)^2}$$

la expresión $\frac{90}{(24 + j\omega)^2}$ Coincide Con la forma $\frac{1}{(a + j\omega)^2}$

$$g(t) = F^{-1}\left\{\frac{90}{(24 + j\omega)^2}\right\} = 90 \cdot F^{-1}\left\{\frac{1}{(24 + j\omega)^2}\right\}$$

$$g(t) = 90\, t\, e^{-24t}\, u(t)$$

Aplicando Teorema de Convolución

$$y(t) = 2\pi \cdot f(t) \cdot g(t)$$

$$y(t) = 2\pi \left(\frac{7}{12} e^{-j3t}\, e^{-6|t|}\right) \cdot \left(90\, t\, e^{-24t}\, u(t)\right)$$

$$y(t) = 105\, \pi\, e^{-j3t}\, e^{-6|t|}\, t\, e^{-24t}$$

$u(t) \rightarrow u(t)$ 0 para $t < 0$

$|t|$ t para $t \geq 0$

$$e^{-6|t|}\, e^{-24t} = e^{-6t} \cdot e^{-24t} = e^{-30t} \text{ (para } t \geq 0)$$

$$y(t) = 105\, \pi\, t\, e^{-(30 + j3)t}\, u(t)$$

d. $F\{3t^3\}$ Aplicando la propiedad de linealidad

$F\{3t^3\} = 3 \cdot F\{t^3\}$ Aplicando la Propiedad de diferenciación en frecuencia

$F\{t^n x(t)\} = j^n \dfrac{d^n}{dw^n} X(w) \rightarrow$ $n = 3$

$x(t) = 1$

$X(w) = F\{x(t) = 1\}$

$F\{1\} = 2\pi\, \delta(w)$

$F\{t^3\} = j^3 \dfrac{d^3}{dw^3}[2\pi\, \delta(w)] \rightarrow j^3 = j^2 \cdot j = (-1)j = -j$

$F\{t^3\} = -j \quad 2\pi \dfrac{d^3}{dw^3} \delta(w) \rightarrow \dfrac{d^3}{dw^3} \delta(w) = \delta^3(w)$

$F\{3t^3\} = 3 \cdot (-j\, 2\pi\, \delta^{(3)}(w)$

$$\boxed{F\{3t^3\} = -j\, 6 \cdot 2\pi\, \delta^{(3)}(w)}$$

e.) $\dfrac{B}{T} \sum_{n=-\infty}^{+\infty} \left(\dfrac{1}{a^2 + (w - nw_0)^2} + \dfrac{1}{a + j(w - nw_c)} \right.$

donde $n \in \{0, \pm 1, \pm 2, \dots\}$ $w_0 = 2\pi / T$

$B, T \in \mathbb{R}^+$

$X(w) = c \sum_{n=-\infty}^{+\infty} F(w - nw_0)$ Si $x(t) = f(t) \sum_{K=-\infty}^{\infty} \delta(t - KT)$

$$X(\omega) = \frac{1}{T} \sum_{K=\infty}^{\infty} F(\omega - n\omega_c)$$

la Constante de escala B:

la función de forma base en frecuencia es:

$$F(\omega) = \frac{1}{a^2 + \omega^2} + \frac{1}{a + j(\omega)}$$

$$f(t) = F^{-1}\{F(\omega)\}$$

$$F^{-1}\left\{\frac{1}{a^2 + \omega^2}\right\} + F\left\{\frac{1}{a + j\omega}\right\}$$

Para el primer termino aplicando la propiedad de Par de transformada

$$F^{-1}\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2} \rightarrow F^{-1}\left\{\frac{1}{a^2+\omega^2}\right\} = \frac{1}{2a} e^{-a|t|}$$

Para el Segundo termino aplicando la misma propiedad anterior

$$F^{-1}\{e^{-a|t|} u(t)\} = \frac{1}{a+j\omega} \rightarrow F^{-1}\left\{\frac{1}{a+j\omega}\right\} = e^{-at} u(t)$$

Entonces la funcion base f(t) es:

$$f(t) = \frac{1}{2a} e^{-a|t|} + e^{-at} u(t)$$

Ahora se construye la Señal final en el dominio del tiempo

$$x(t) = B \cdot f(t) \left(\sum_{n=-\infty}^{\infty} \delta(t - kT) \right)$$

$$x(t) = B \cdot \left(\frac{1}{2a} e^{-a|t|} + e^{-at} u(t) \right) \sum_{n=-\infty}^{+\infty} \delta(t-KT)$$

$$x(t) = B \sum_{n=-\infty}^{\infty} f(KT) \cdot \delta(t-KT), \quad f(KT) = \frac{1}{2a} e^{-a|KT|} + e^{-aKT} u(KT)$$

$$\boxed{x(t) = B \sum_{n=-\infty}^{\infty} \left(\frac{1}{2a} e^{-a|KT|} + e^{-aKT} u(KT) \right) \delta(t-KT)}$$