PH 303 - Report 1

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July 2021

1 Introduction

Packages Used

1. `numpy`
2. `matplotlib: pyplot`
3. `scipy: optimize.curve_fit, optimize.brentq, integrate.quad`
4. `PIL`

Constants Used

- Speed of light: $c = 3 \cdot 10^8 \frac{m}{s}$
- Astronomical unit: $AU = 1.5 \cdot 10^{11} m$
- Gravitational constant: $G = 6.67 \cdot 10^{-11} \frac{m^3}{kg\ s}$
- Parsec: $pc = 3 \cdot 10^{16} m$
- Giga parsec: $Gpc = 3 \cdot 10^{25} m$
- Hubble's constant: $H_0 = 70 \cdot 10^3 \frac{m}{s\ Mpc}$
- ¡Line break¿
- Mass of sun: $M_\odot = M\_sun = 2 \cdot 10^{30} kg$
- Radius of sun: $r_\odot = r\_sun = 7 \cdot 10^8 m$

2 Redshift Distribution

Scanned the plot given in the paper using an online plot digitizer

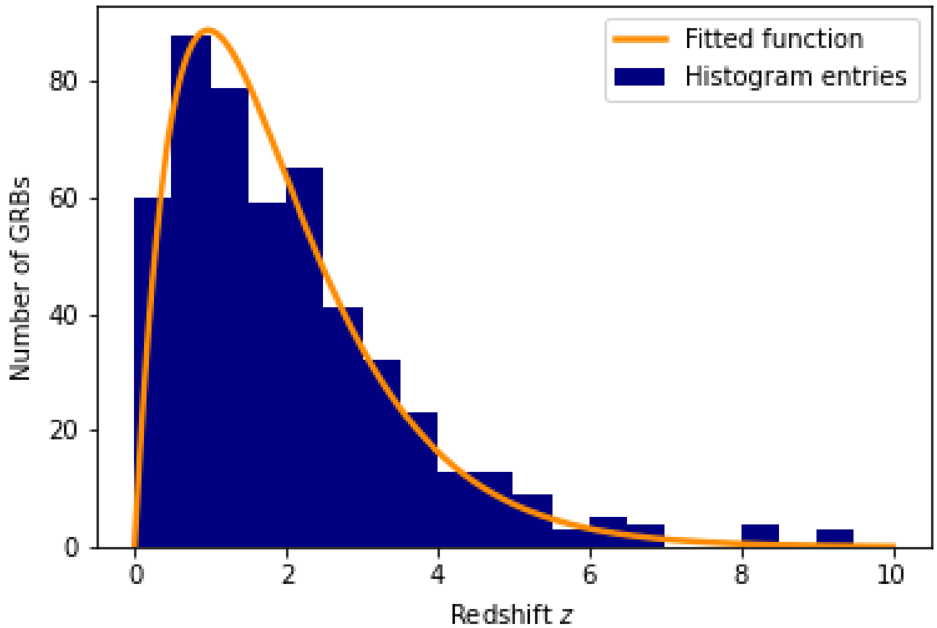

Figure 1: Fitted Distribution

Points were then sampled by normalising the curve to a probabilistic distribution function and randomly sampling points on the curve. Covered 99.88% of the sample space by discretization

For $DS \sim \frac{c}{H_0}$,

$$D_S = \frac{z_S c}{H_0} \tag{1}$$

After obtaining a random distribution of GRB progenitors and their redshifts, distance from the observer to the source were calculated using the above expression. For the sake of simplicity, z_S was taken to be 2 throughout the reminder of this section although the basis for generalization has been set. Typical distance D was taken to be $1Gpc$ as given in the paper. Once again, for the sake of simplicity, preliminary analysis was done using mass of lens, $M = 10^{-13} M_\odot$

The Einstein angle can now be calculated as

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{1}{D}} \tag{2}$$

This has a dedicated function `Einstein_angle(M,D)`

3 Benchmark for GRB Lensing Parallax Detection

The *conservative* values were taken for present analysis. Nevertheless, this can be adjusted by changing the variable values later

- Radiometric resolution: $\epsilon = 0.1$
- Detected frequency: $\nu = 1.2 \cdot 10^{20} Hz$ $(0.5 MeV)$
- Number of GRBs: $NGRB = 1000$
- Number of clearly detectable GRBs: $NGRB_{tiny} = 0.03 * 1000 = 30$ ($i.e.$, radius of source $r_S < 0.1 r_\odot$

4 Magnification

Function `image_magnification(u)` calculates the magnification and brightness flux of each image. The input consists of $u = \frac{\beta}{\theta_E}$ where β is the angular position of the source. The function returns two values, the magnifications of each image produced

$$\mu_{\pm} = \left|0.5 \pm \frac{u^2+2}{2u\sqrt{u^2+4}}\right|$$

Total magnification is then calculated by `total_magnification(u)` as

$$A = \mu_+ + \mu_-$$

Finally, a function `delta_A` is defined to calculate the resolution of measured magnification

$$\delta_A = \frac{|A_1 - A_2|}{\frac{A_1+A_2}{2}} \tag{3}$$

4.1 Finite Source Size Effect

This paper takes into account the second-order terms for magnification calculation. These result in slightly different expressions to calculate magnification and is solved under `finite_source_size_effect(r_S, D, theta_E, u)`

δ is defined as

$$\delta = \frac{\theta_S}{\theta_E} \tag{4}$$

where $\theta_S = r_S/D$

$$A(u,\delta) \approx \begin{cases} A_{in}(u,\delta) & \text{for } y < \delta \\ A_{out}(u,\delta) & \text{for } y > \delta \end{cases} \tag{5}$$

where

$$A_{in}(u,\delta) = \sqrt{1 + \frac{4}{delta^2}} - \frac{8}{\delta^3(\delta^2+4)^{1.5}}\frac{u^2}{2} - \frac{144(\delta^4+2\delta^2+2)}{\delta^5(\delta^2+4)^{3.5}}\frac{u^4}{24} \tag{6}$$

$$A_{out}(u,\delta) = \frac{2+u^2}{u\sqrt{u^2+4}} + \frac{8(u^2+1)}{u^3(u^2+4)^{2.5}}\frac{\delta^2}{2} + \frac{48(3u^6+6u^4+14u^2+12)}{u^5(u^2+4)^{4.5}}\frac{\delta^4}{24} \tag{7}$$

For $0.9\delta < u < 1.1\delta$, the two endpoints are linearly interpolated such that

$$A = A_{in}(0.9\delta,\delta)\frac{(u-0.9\delta)}{0.2} + A_{out}(1.1\delta,\delta)\frac{(1.1\delta-u)}{0.2} \tag{8}$$

5 Optical Depth and Constraints

This is the computationally intensive section of the code due to its high number of calculations and a rather poor use of numpy arrays. As D, r_S and θ_E are assumed to be constants for the time-being, a simplified function `calc_finite(u)` has been defined to invoke `finite_source_size_effect(r_S, D, theta_E, u)`

Tilts in the plane of the planetary motion *wrt* the GRB source position have been ignored due to no realistic values available for the tilt angle θ. Nevertheless, provisions have been set in place to use this value later.

A necessary condition for detectable magnification, that follows from eqn(3), is that either A_1 or A_2 should be greater than $1+\epsilon$ where $A_{1,2}$ are the magnifications at position of detectors $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. To calculate the upper threshold value of u, *i.e.* u_T, which can be detected for a particular ϵ, the function `brentq` was used but this gave unsatisfactory results. Hence, this was manually calculated for $M = 10^{-13}M_\odot$ and $D_S = 8.5Gpc$. Methods to incorporate Newton-Raphson method will be looked into later. u_T has been set at 1.719 (dimensionless quantity) by manual calculation.

Subsequently, the β_T value can be calculated as

$$u = \frac{\beta}{\theta_E} \Rightarrow \beta_T = u_T \cdot \theta_E \tag{9}$$

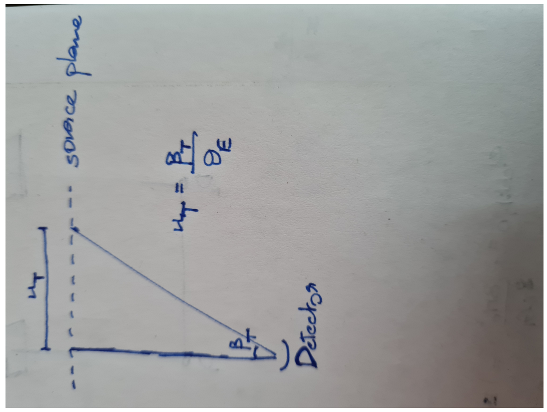

Figure 2: Caption

From this, the physical displacement on the lens plane y can be calculated in metres as

$$y = D_S \cdot \tan(\beta) \tag{10}$$

For the moment, the spatial separation of the detectors Δ_r were taken to be $2AU$. For the above selected set of parameter values, this means that the bounding box around the region $A_{1,2} > 1 + \epsilon$ would be much large than the small patches that can be magnified by a lens. An image is shown below for representational purposes

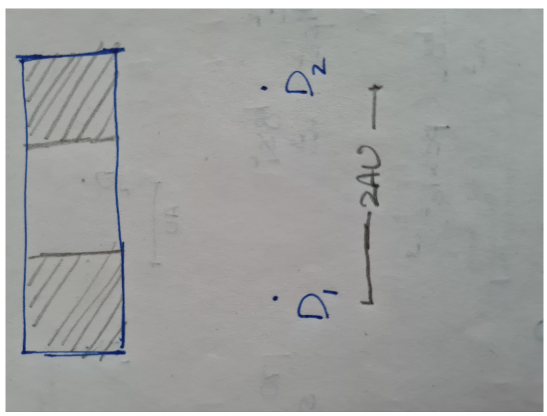

Figure 3: Caption

Now this rectangle is divided into an 80×80 grid (although the paper has suggested this number, I have used 200×200 due to its lack of precision at smaller mass ranges) and the $delta_A$ condition is applied at each cell. The number of cells that satisfies this condition are counted and the area of each cell is taken into account to calculate the comoving cross-section area σ

$$\sigma = count * area_o f_p ixel \tag{11}$$

The volume occupied by the possible lenses is then calculated as

$$V_L = D_S \int_0^1 \sigma dx \tag{12}$$

The paper has assumed the comoving number density to be uniformly distributed (with further notes under the appendix section) and constant

$$n = \rho_{crit,0} \Omega_{DM} \frac{f}{M} \tag{13}$$

And finally, the optical depth τ is calculated as the product of the comoving number density and comoving cross-section

$$\tau = n(f; M) \times V_L(D_S, \Delta r_\perp, r_S, \epsilon) \tag{14}$$

A Poissonian distribution of lenses are used with a single-lensing probability of $P_1 = \tau e^{-\tau}$. I believe I have found a discrepancy with the remaining code. I will rectify and accordingly update this report