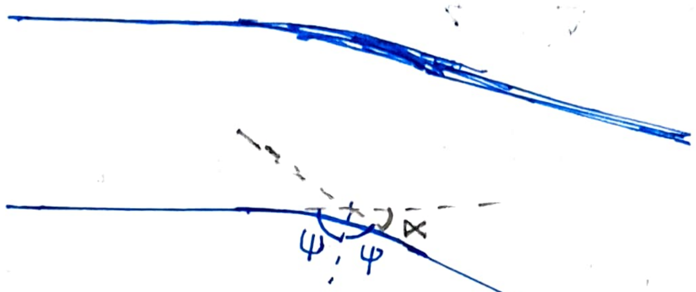
Date: ☐☐/☐☐/☐☐☐☐ ASB -6

Q No. Exercise No. ☐☐ Solved Problems: Sub ☐ Obj ☐

Derivation of Deflection Angle



Gravitational Potential $= \phi(r) \leftarrow$ O

$\alpha = \pi - 2\psi$ where α is the deflection angle and ψ is the angle between asymptotic direction and direction of closest approa

$-\nabla\phi = -\frac{GM}{|R|^2}\hat{R}$

Let $f = -\frac{\partial V}{\partial r}$

Kinetic energy of ~~particle~~ photon $= E = \sqrt{m^2c^4 + p^2c^2} = pc$

$$\alpha = \int_{r_0}^{r_m} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2m\phi}{l^2} - \frac{1}{r^2}}} \qquad r_m \to \infty \quad r_0 = 0 \qquad \text{target equation}$$

Assume a particle of mass m $\quad KE = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow m\ddot{r} - mr\dot{\theta}^2 = f$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow mr^2\dot{\theta} = l \qquad \Rightarrow \dot{\theta}^2 = \frac{l^2}{m^2r^4}$$

as $\frac{d}{dt}(\ ) = 0$

$$m\ddot{r} = -\frac{d\phi}{dr} - \frac{d}{dr}\left(\frac{l^2}{2mr^2}\right) = -\frac{d}{dr}\left(\phi + \frac{l^2}{2mr^2}\right)$$

$$\Rightarrow m\ddot{r}\frac{dr}{dt} = -\frac{d}{dt}\left(\phi + \frac{l^2}{2mr^2}\right) = \frac{d}{dt}\left(\frac{m\dot{r}^2}{2}\right) \Rightarrow \frac{m\dot{r}^2}{2} + \phi + \frac{l^2}{2mr^2} = E$$

$$\Rightarrow \frac{m\dot{r}^2}{2} = -\left(\phi + \frac{l^2}{2mr^2}\right) \Rightarrow \dot{r} = \sqrt{-\frac{2}{m}\left(\phi + \frac{l^2}{2mr^2} - E\right)}$$

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Point Mass Lens

$$\alpha = \frac{2}{c^2}\int_{-\infty}^{\infty} \nabla_\perp \phi \, dz$$

In the weak field limit $\phi = -\frac{GM}{r}$

As photon is deflected from the z axis, we can ~~consider~~ take the central gravitational entity in the x-y plane at z=0

$r = \sqrt{x^2+y^2+z^2}$

$$\nabla\phi = \partial_x \phi \hat{i} + \partial_y \phi \hat{j} = \frac{GM}{r^3}(x\hat{i} + y\hat{j})$$

$$\frac{d}{dr}\frac{GM}{r} = \frac{-GM}{r^2} \qquad \frac{d}{dx}\frac{GM}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}GM\frac{2x}{(x^2+y^2+z^2)^{3/2}} = \frac{-GMx}{r^3}$$

$$\Rightarrow \alpha = \frac{2}{c^2}\int_{-\infty}^{\infty}\frac{GM}{r^3}(x\hat{i}+y\hat{j})\,dz$$

$$= R_s(x\hat{i}+y\hat{j}) \underbrace{\int_{-\infty}^{\infty}\frac{dz}{(x^2+y^2+z^2)^{3/2}}}_{\text{from calculator}}$$

$$= \frac{z}{(x^2+y^2)\sqrt{x^2+y^2+z^2}}$$

$$= \frac{2}{(x^2+y^2)}$$

$$= 2R_s\frac{(x\hat{i}+y\hat{j})}{(x^2+y^2)}$$

$$|\alpha| = \frac{2R_s}{(x^2+y^2)}$$

Shapiro Delay

$$\Delta t = \int \Delta\left(\frac{dl}{v}\right) = \int \frac{dl}{c'} - \frac{dl}{c} = \frac{1}{c}\int dl \left(\frac{c}{c'} - 1\right) = \int \frac{(n-1)}{c} dl$$

$$n = 1 - \frac{2\phi}{c^2} \quad \Rightarrow \quad \Delta t = -\frac{2}{c^3}\int \phi \, dl$$

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{-2}{m}\left(\left\{\phi + \frac{l^2}{2mr^2}\right\} - E\right)} \qquad \dot{\theta} = \frac{d\theta}{dt} = \frac{l}{mr^2}$$

$$\frac{dr}{d\theta} = \frac{\sqrt{-\frac{2}{m}\left(\left\{\phi + \frac{l^2}{2mr^2}\right\} - E\right)}}{l/mr^2}$$

$$\Rightarrow d\theta = \frac{l\, dr}{r^2\sqrt{-2m\left(\left\{\phi + \frac{l^2}{2mr^2}\right\} - E\right)}}$$

$$\int_{\theta_0}^{\theta} d\theta = \int_{r_0}^{r_\infty} \frac{dr}{r^2\sqrt{2m\left(\frac{E}{l^2} - \frac{\phi}{l^2} - \frac{1}{2mr^2}\right)}}$$

r_0 = periapsis

$$\Rightarrow \psi = \int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{\frac{2m(E-\phi)}{l^2} - \frac{1}{r^2}}}$$

E is the total energy $T + \phi$

$\alpha = \pi - 2\psi$

$$\alpha = \pi - 2\int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{\quad}}$$

$$\ddot{x} = \frac{1}{n}\bar{\nabla} n - \bar{\nabla} \ln n$$

$$= \frac{1}{n}\bar{\nabla} n + \bar{\nabla}\frac{2\phi}{c^2} = \frac{1}{n}\bar{\nabla} n + \frac{2}{c^2}\nabla\phi \longrightarrow -\nabla\phi = a$$

$$= \frac{1}{n}\bar{\nabla} n - \frac{2a}{c^2} \approx -\frac{2a}{c^2}$$

Total deflection angle $\bar{\alpha} = -\int_{\lambda_A}^{\lambda_B} \ddot{x}\, d\lambda = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \nabla\phi \, d\lambda$

Similar to scattering theory since $\frac{\phi}{c^2} \ll 1$

$$\boxed{\theta = \pi - 2b\int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{1-\left(\frac{b}{r}\right)^2 - \frac{2\phi}{c^2}}}}$$

$\Rightarrow \alpha \ll \frac{\pi}{2}$