8th IMC 2001, July 19 – July 25, Prague, Czech Republic, First day

Problem 1

Let n be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, \ldots, n^2$ written in order starting top left and moving along each row in turn left–to–right. We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

Problem 2

Let r, s, t be positive integers which are pairwise relatively prime. If a and b are elements of a commutative multiplicative group with unity element e, and $a^r = b^s = (ab)^t = e$, prove that $a = b = e$.

Does the same conclusion hold if a and b are elements of an arbitrary non-commutative group?

Problem 3

Find

$$\lim_{t \to 1^-} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n}$$

Problem 4

Let $k \in \mathbb{N}$. Let $p(x)$ be a polynomial of degree n with coefficients in $\{-1, 0, 1\}$, and divisible by $(x-1)^k$. Let q be prime such that

$$\frac{q}{\ln q} < \frac{k}{\ln(n+1)}$$

Prove that all complex qth roots of unity are roots of $p(x)$.

Problem 5

Let A be an $n \times n$ complex matrix such that $A \neq \lambda I$ for any $\lambda \in \mathbb{C}$. Prove that A is similar to a matrix with at most one non-zero entry on the main diagonal.

Problem 6

Suppose differentiable functions $a(x), b(x), f(x), g(x) : \mathbb{R} \to \mathbb{R}$ satisfy:

- $f(x) \geq 0,\ f'(x) \geq 0,\ g(x) > 0,\ g'(x) > 0$
- $\lim_{x\to\infty} a(x) = A > 0,\ \lim b(x) = B > 0$
- $\lim f(x) = \lim g(x) = \infty$
- $\frac{f'}{g'} + a(x) \cdot \frac{f}{g} = b(x)$

Prove:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{B}{A+1}$$