GPT-5 Solutions

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1 Problem 1, Day 1

1.1 Variant 1

Let $P \in \mathbb{R}[x]$ with $\deg P \geq 2$. For $x \in \mathbb{R}$, the tangent line at $(x, P(x))$ has equation

$$y = P(x) + P'(x)(t - x).$$

A point (a, b) lies on this line if and only if

$$b = P(x) + P'(x)(a - x).$$

Thus, for fixed a, the set of values of b that are attained by tangent lines at abscissa a is the image of the function

$$G_a(x) := P(x) + P'(x)(a - x).$$

(a) $\deg P$ odd implies $\bigcup_x \ell_x = \mathbb{R}^2$.

Let $\deg P = n$ be odd and let c be the leading coefficient of P, so $P(x) = cx^n +$ lower order terms. Then

$$G_a(x) = P(x) + aP'(x) - xP'(x).$$

The coefficient of x^n in G_a is

$$\text{coeff}(x^n) = c - nc = (1 - n)c \neq 0$$

(since $n \geq 2$). Therefore $\deg G_a = n$ is odd. For any $b \in \mathbb{R}$, the polynomial equation

$$G_a(x) - b = 0$$

has odd degree, hence at least one real root. Thus for every $(a, b) \in \mathbb{R}^2$ there exists x with $b = P(x) + P'(x)(a - x)$, i.e., (a, b) lies on the tangent at x. Hence the union of all tangent lines is all of $\mathbb{R}^2$.

(b) No even-degree polynomial has this property.

Suppose $\deg P = n$ is even. As above,

$$G_a(x) = P(x) + aP'(x) - xP'(x)$$

has leading term $(1 - n)cx^n$ with $(1 - n)c \neq 0$, so $\deg G_a = n$ is even. Hence as $x \to \pm\infty$, $G_a(x) \to \pm\infty$ with the same sign, so the image of G_a is an interval of the form $(-\infty, M]$ or $[m, \infty)$. In particular, G_a is not surjective onto $\mathbb{R}$. Therefore, for this fixed a, there exist b for which the equation $b = G_a(x)$ has no real solution, i.e., some points (a, b) do not lie on any tangent line. Consequently, no polynomial of even degree can have $\bigcup_x \ell_x = \mathbb{R}^2$.

A concrete illustration: for $P(x) = x^2$, $G_a(x) = 2ax - x^2$, whose range is $(-\infty, a^2]$, not all of $\mathbb{R}$.

Final answer.

(a) True for any odd-degree polynomial P: the union of its tangent lines is all of $\mathbb{R}^2$.

(b) No. There is no polynomial of even degree whose tangent lines cover $\mathbb{R}^2$.

1.2 Variant 2

Let P be a real polynomial of degree $n \geq 2$, with leading coefficient $c \neq 0$. For each $x \in \mathbb{R}$, the tangent line at $(x, P(x))$ is

$$\ell_x = \{(t, P(x) + P'(x)(t - x)) : t \in \mathbb{R}\}.$$

A point $(a, b) \in \mathbb{R}^2$ lies on ℓ_x iff

$$b = P(x) + P'(x)(a - x).$$

Define, for each fixed $a \in \mathbb{R}$,

$$H_a(x) = P(x) + (a - x)P'(x).$$

Then (a, b) lies on some tangent line iff the equation $H_a(x) = b$ has a real solution x.

Compute the degree and leading coefficient of H_a. If $P(x) = cx^n +$ lower terms, then $P'(x) = ncx^{n-1} +$ lower, and hence

$$\begin{aligned} H_a(x) &= cx^n + (a - x)(ncx^{n-1}) + \text{lower} && (1) \\ &= (1 - n)cx^n + nacx^{n-1} + \text{lower}. && (2) \end{aligned}$$

Thus $\deg(H_a) = n$ and its leading coefficient is $(1 - n)c \neq 0$.

(a) If n is odd, then H_a is an odd-degree polynomial, hence

$$\lim_{x \to +\infty} H_a(x) \text{ and } \lim_{x \to -\infty} H_a(x) \text{ have opposite signs.}$$

By continuity, $H_a(\mathbb{R}) = \mathbb{R}$. Therefore for every $a, b \in \mathbb{R}$ there exists x with $H_a(x) = b$, which means every point (a, b) lies on some tangent line. Hence $\bigcup_{x \in \mathbb{R}} \ell_x = \mathbb{R}^2$.

(b) If n is even, then H_a has even degree with leading coefficient $(1 - n)c$. There are two cases:

- If $c > 0$, then $(1 - n)c < 0$, so $H_a(x) \to -\infty$ as $x \to \pm\infty$. Thus H_a is bounded above on $\mathbb{R}$. For any b larger than $\sup H_a(\mathbb{R})$, the equation $H_a(x) = b$ has no real solution, so points (a, b) with such b are not covered by the union of tangents.

- If $c < 0$, then $(1 - n)c > 0$, so $H_a(x) \to +\infty$ as $x \to \pm\infty$, and H_a is bounded below. For any b smaller than $\inf H_a(\mathbb{R})$, there is no solution, hence such points are not covered.

In either case, for even n there exist points (a, b) not lying on any tangent line. Therefore no polynomial of even degree has $\bigcup_{x \in \mathbb{R}} \ell_x = \mathbb{R}^2$.

Final answer.

(a) True: for odd $\deg(P)$, $\bigcup_{x \in \mathbb{R}} \ell_x = \mathbb{R}^2$.

(b) No: for even $\deg(P)$, the union of tangent lines cannot be all of $\mathbb{R}^2$.

1.3 Variant 3

Let P be a real polynomial of degree $n \geq 2$, and for each $x \in \mathbb{R}$ let ℓ_x be the tangent line to the graph of P at $(x, P(x))$. A convenient representation of ℓ_x is

$$y = P(x) + P'(x)(t - x),$$

so that for a fixed abscissa t, the intersection of ℓ_x with the vertical line $t = \text{const}$ has ordinate

$$L_x(t) = P(x) - xP'(x) + tP'(x).$$

(a) Degree of P odd $\Rightarrow \bigcup_x \ell_x = \mathbb{R}^2$.

Fix an arbitrary point $(u, v) \in \mathbb{R}^2$. Consider the polynomial in x

$$F(x) = P(x) + P'(x)(u - x) - v = [P(x) - xP'(x)] + uP'(x) - v.$$

If $\deg P = n$ with leading coefficient $a_n \neq 0$, then

$$P(x) - xP'(x) = a_n x^n - na_n x^n + \text{lower degree terms} = (1 - n)a_n x^n + (\text{lower}),$$

while $uP'(x)$ has degree at most $n - 1$. Hence $\deg F = n$ and its leading coefficient is $(1 - n)a_n \neq 0$. In particular, if n is odd, then F is an odd-degree real polynomial, so

$$\lim_{x\to\infty} F(x) \quad \text{and} \quad \lim_{x\to-\infty} F(x)$$

have opposite signs. By the intermediate value theorem there exists x_0 with $F(x_0) = 0$, i.e.,

$$v = P(x_0) + P'(x_0)(u - x_0),$$

so (u, v) lies on the tangent line at x_0. Since (u, v) was arbitrary, $\bigcup_x \ell_x = \mathbb{R}^2$.

(b) No even-degree polynomial has this property.

Assume $\deg P = n$ is even with leading coefficient a_n. Fix $u \in \mathbb{R}$ and consider the function of x

$$H_u(x) = L_x(u) = P(x) - xP'(x) + uP'(x).$$

As above, H_u is a polynomial in x of degree n with leading coefficient $(1 - n)a_n$. Since $n \geq 2$, $1 - n < 0$.

- If $a_n > 0$, then $(1-n)a_n < 0$, so $H_u(x) \to -\infty$ as $x \to \pm\infty$. Hence H_u attains a finite global maximum $M(u)$, and the set $\{H_u(x) : x \in \mathbb{R}\}$ is contained in $(-\infty, M(u)]$. Thus no tangent line meets the vertical line $t = u$ above height $M(u)$, so points (u, v) with $v > M(u)$ are not covered.

- If $a_n < 0$, the same reasoning shows $H_u(x) \to +\infty$ as $x \to \pm\infty$, so H_u attains a finite global minimum $m(u)$, and no tangent line meets $t = u$ below height $m(u)$. Points (u, v) with $v < m(u)$ are not covered.

Therefore for any even-degree polynomial P, the union of tangent lines cannot be all of $\mathbb{R}^2$.

Final answer:

(a) True: for odd degree P, $\bigcup_x \ell_x = \mathbb{R}^2$.

(b) No: for every even-degree polynomial P, the union of tangent lines misses some points, so the equality fails.

1.4 Final Solution

Let $P \in \mathbb{R}[x]$ with $\deg P = n \geq 2$, and for $x \in \mathbb{R}$ let ℓ_x be the tangent line to the graph of P at $(x, P(x))$:

$$\ell_x = \{(t, P(x) + P'(x)(t - x)) : t \in \mathbb{R}\}.$$

A point (a, b) lies on ℓ_x iff

$$b = P(x) + P'(x)(a - x).$$

For fixed a, define

$$H_a(x) = P(x) + (a - x)P'(x).$$

Then (a, b) lies on some tangent line iff the equation $H_a(x) = b$ has a real solution x.

Compute the leading term of H_a. If $P(x) = cx^n +$ (lower terms) with $c \neq 0$, then $P'(x) = ncx^{n-1} +$ (lower terms), and hence

$$\begin{aligned} H_a(x) &= cx^n + a(ncx^{n-1}) - x(ncx^{n-1}) + \text{(lower terms)} && (3) \\ &= (1 - n)cx^n + \text{(lower terms)}. && (4) \end{aligned}$$

Because $n \geq 2$, $(1 - n)c \neq 0$, so $\deg H_a = n$.

(a) If n is odd, H_a has odd degree, so $\lim_{x\to+\infty} H_a(x)$ and $\lim_{x\to-\infty} H_a(x)$ have opposite signs. By continuity, $H_a(\mathbb{R}) = \mathbb{R}$. Thus for every $(a, b) \in \mathbb{R}^2$ there exists x with $H_a(x) = b$, i.e., every point lies on some tangent line. Hence $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$.

(b) If n is even, then H_a has even degree with leading coefficient $(1-n)c$. Consequently, $H_a(x) \to -\infty$ as $x \to \pm\infty$ when $c > 0$, and $H_a(x) \to +\infty$ as $x \to \pm\infty$ when $c < 0$. In either case, H_a is bounded on one side (has a finite global maximum if $c > 0$, or a finite global minimum if $c < 0$). Therefore, for this fixed a, there exist b for which $H_a(x) = b$ has no real solution. Hence the vertical line $t = a$ contains points not lying on any tangent line, and $\bigcup_{x\in\mathbb{R}} \ell_x \neq \mathbb{R}^2$ for every even-degree polynomial.

Final answer.

(a) True: if $\deg P$ is odd, then $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$.

(b) No: for every even-degree polynomial P, $\bigcup_{x\in\mathbb{R}} \ell_x \neq \mathbb{R}^2$.

2 Problem 2

2.1 Variant 1

Let $J(f) = \int_{-1}^{1} (f''(x))^2 dx$. We minimize $J(f)$ under the constraints

- $f(1) = f(-1) = 1$ (fixed endpoint values),
- $\int_{-1}^{1} f(x)dx = 0$ (integral constraint).

1) Euler–Lagrange equation and natural boundary conditions

Introduce a Lagrange multiplier λ for the integral constraint and consider

$$I(f) = \int_{-1}^{1} [(f''(x))^2 + \lambda f(x)]dx,$$

with $f(\pm 1)$ fixed. For variations h with $h(\pm 1) = 0$ we have

$$0 = \frac{d}{dt}\bigg|_{t=0} I(f + th) = 2\int f''h'' + \lambda \int h.$$

Integrating by parts twice,

$$\int f''h'' = [f''h']_{-1}^{1} - \int f'''h' = [f''h']_{-1}^{1} - [f'''h]_{-1}^{1} + \int f^{(4)}h.$$

Since $h(\pm 1) = 0$, the middle boundary term vanishes. Because $h'(\pm 1)$ is arbitrary, the boundary term $[f''h']_{-1}^{1}$ vanishes for all such h only if

$$f''(-1) = f''(1) = 0.$$

Then the stationarity condition reduces to

$$\int (2f^{(4)} + \lambda)h = 0 \text{ for all } h \text{ with } h(\pm 1) = 0,$$

hence $2f^{(4)} + \lambda = 0$ on $[-1, 1]$. Therefore $f^{(4)}$ is a constant, and so f is a quartic polynomial.

2) Solving for the minimizer

Write $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. From $f''(x) = 12ax^2 + 6bx + 2c$ and $f''(\pm 1) = 0$ we get

$$12a + 6b + 2c = 0, \tag{5}$$
$$12a - 6b + 2c = 0 \tag{6}$$

$\Rightarrow b = 0, c = -6a$.

From $f(\pm 1) = 1$ we get $d = 0$ and $e = 1 + 5a$.

The integral constraint:

$$0 = \int_{-1}^{1} f(x)dx = 2\int_{0}^{1} [a(x^4 - 6x^2) + e]dx \tag{7}$$
$$= 2\left[a\left(\frac{1}{5} - 2\right) + e\right] = 2\left(-\frac{9a}{5} + e\right), \tag{8}$$

so $e = \frac{9a}{5}$. Comparing with $e = 1 + 5a$ yields

$$1 + 5a = \frac{9a}{5} \Rightarrow 5 + 25a = 9a \Rightarrow 16a = -5 \Rightarrow a = -\frac{5}{16},$$

and then $e = -\frac{9}{16}$. Hence

$$f(x) = \left(-\frac{5}{16}\right)x^4 + \frac{30}{16}x^2 - \frac{9}{16} = \frac{-5x^4 + 30x^2 - 9}{16}.$$

3) Computing the minimum value

For this f,

$$f''(x) = 12a(x^2 - 1) = 12\left(-\frac{5}{16}\right)(x^2 - 1) = \frac{15}{4}(1 - x^2).$$

Therefore

$$\int_{-1}^{1} (f'')^2 dx = \frac{225}{16} \int_{-1}^{1} (1 - 2x^2 + x^4) dx \tag{9}$$

$$= \frac{225}{16} \left[2 - 2 \cdot \frac{2}{3} + \frac{2}{5} \right] \tag{10}$$

$$= \frac{225}{16} \cdot \frac{16}{15} = 15. \tag{11}$$

4) Optimality and uniqueness

Let f_0 be the quartic above, and let $h = f - f_0$. Then $h(\pm 1) = 0$ and $\int h = 0$. We have

$$J(f) = \int (f_0'' + h'')^2 = \int (f_0'')^2 + 2 \int f_0'' h'' + \int (h'')^2.$$

Integrating by parts twice as before, using $f_0''(\pm 1) = 0$ and $f_0^{(4)}$ constant, we get

$$\int f_0'' h'' = \int f_0^{(4)} h = \text{const} \cdot \int h = 0.$$

Hence

$$J(f) = J(f_0) + \int (h'')^2 \geq J(f_0) = 15,$$

with equality iff $h'' \equiv 0$, i.e., h is linear. Since $h(\pm 1) = 0$, $h \equiv 0$, so $f \equiv f_0$. Thus the minimizer is unique.

Therefore,

$$\int_{-1}^{1} (f''(x))^2 dx \geq 15,$$

with equality if and only if

$$f(x) = \frac{-5x^4 + 30x^2 - 9}{16}.$$

Final answer:

- The inequality holds: $\int_{-1}^{1} (f''(x))^2 dx \geq 15$.
- Equality occurs precisely for $f(x) = \frac{-5x^4+30x^2-9}{16}$.

2.2 Variant 2

Define $w(x) = 1 - x^2$. For any twice continuously differentiable f with $\int_{-1}^{1} f = 0$ and $f(\pm 1) = 1$, compute

$$\int_{-1}^{1} f''(x) w(x) dx$$

by integrating by parts twice:

$$\int f'' w = [f' w]_{-1}^{1} - \int f' w' \tag{12}$$

$$= -[f w']_{-1}^{1} + \int f w'' \tag{13}$$

$$= -(f(1) w'(1) - f(-1) w'(-1)) + \int f(-2), \tag{14}$$

where we used $w(\pm 1) = 0$ and $w'' = -2$. Since $w'(1) = -2$, $w'(-1) = 2$, $f(\pm 1) = 1$, and $\int f = 0$, we obtain

$$\int_{-1}^{1} f''(x)(1-x^2)dx = -(1 \cdot (-2) - 1 \cdot 2) + (-2) \cdot 0 = 4.$$

By Cauchy–Schwarz,

$$\left(\int f'' w \right)^2 \leq \int (f'')^2 \int w^2,$$

hence

$$\int_{-1}^{1} (f'')^2 dx \geq \frac{(\int f'' w)^2}{\int w^2} = \frac{16}{\int_{-1}^{1} (1-x^2)^2 dx}.$$

Compute $\int_{-1}^{1} (1-x^2)^2 dx = \int (1 - 2x^2 + x^4) dx = 2 - \frac{4}{3} + \frac{2}{5} = \frac{16}{15}$. Therefore,

$$\int_{-1}^{1} (f'')^2 dx \geq \frac{16}{16/15} = 15.$$

Equality holds in Cauchy–Schwarz iff f'' is proportional to w, i.e., $f''(x) = c(1-x^2)$. Integrating twice gives

$$f(x) = c\left(\frac{x^2}{2} - \frac{x^4}{12} \right) + \alpha x + \beta.$$

From $f(\pm 1) = 1$ we get $\alpha = 0$ and $\beta = 1 - \frac{5c}{12}$. The condition $\int_{-1}^{1} f = 0$ yields

$$c\left(\frac{3}{10} \right) + 2\beta = 0 \Rightarrow c = \frac{15}{4}, \text{ hence } \beta = -\frac{9}{16}.$$

Thus the equality cases are exactly

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16},$$

for which $f''(x) = \frac{15}{4}(1-x^2)$ and $\int (f'')^2 = 15$.

Final answer:

$$\int_{-1}^{1} (f''(x))^2 dx \geq 15,$$

with equality if and only if $f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}$.

2.3 Variant 3

Let $g(x) = 1 - x^2$. For any twice continuously differentiable f with $f(\pm 1) = 1$ and $\int_{-1}^{1} f = 0$, integrate by parts twice:

$$\begin{aligned}
\int_{-1}^{1} f''(x) g(x)\, dx &= [f'(x)g(x)]_{-1}^{1} - \int_{-1}^{1} f'(x) g'(x)\, dx && (15)\\
&= 0 + 2\int_{-1}^{1} x f'(x)\, dx && (16)\\
&= [2xf(x)]_{-1}^{1} - 2\int_{-1}^{1} f(x)\, dx && (17)\\
&= 2(f(1) + f(-1)) - 2\int_{-1}^{1} f(x)\, dx && (18)\\
&= 4. && (19)
\end{aligned}$$

By Cauchy–Schwarz,

$$\left(\int_{-1}^{1} f''g\right)^2 \le \int_{-1}^{1} (f'')^2 \int_{-1}^{1} g^2,$$

so

$$\int_{-1}^{1} (f'')^2 \ge \frac{16}{\int_{-1}^{1} (1-x^2)^2\, dx}.$$

Compute

$$\int_{-1}^{1} (1-x^2)^2\, dx = \int_{-1}^{1} (1-2x^2+x^4)\, dx \tag{20}$$

$$= 2\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{16}{15}. \tag{21}$$

Hence $\int_{-1}^{1} (f'')^2 \ge \frac{16}{16/15} = 15$.

Equality in Cauchy–Schwarz occurs iff f'' is a constant multiple of g, i.e., $f''(x) = \lambda(1-x^2)$. Using $\int f''g = 4$ and $\int g^2 = \frac{16}{15}$ gives $\lambda = \frac{15}{4}$. Integrating twice,

$$f'(x) = \frac{15}{4}\left(x - \frac{x^3}{3}\right) + A,$$

$$f(x) = \frac{15}{8}x^2 - \frac{15}{48}x^4 + Ax + B.$$

The boundary conditions $f(1) = f(-1) = 1$ force $A = 0$ and $B = -\frac{9}{16}$. Thus

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16} = \frac{-5x^4+30x^2-9}{16},$$

which also satisfies $\int_{-1}^{1} f = 0$. This function yields $\int_{-1}^{1} (f'')^2 = 15$.

Therefore the inequality holds, and equality occurs precisely for $f(x) = \frac{-5x^4+30x^2-9}{16}$.

Final answer:

$$\int_{-1}^{1} (f''(x))^2\, dx \ge 15, \text{ with equality iff } f(x) = \frac{-5x^4+30x^2-9}{16}.$$

2.4 Final Solution

Let $w(x) = 1 - x^2$. For any twice continuously differentiable f with $f(\pm 1) = 1$ and $\int_{-1}^{1} f = 0$, integrate by parts twice:

$$\int_{-1}^{1} f''(x)w(x)\, dx = [f'(x)w(x)]_{-1}^{1} - \int_{-1}^{1} f'(x)w'(x)\, dx$$

$$= 0 + 2\int_{-1}^{1} xf'(x)\, dx$$

$$= [2xf(x)]_{-1}^{1} - 2\int_{-1}^{1} f(x)\, dx$$

$$= 2(f(1) + f(-1)) - 2 \cdot 0 = 4$$

By Cauchy–Schwarz,

$$\left(\int_{-1}^{1} f''w\right)^2 \le \int_{-1}^{1} (f'')^2 \int_{-1}^{1} w^2$$

so

$$\int_{-1}^{1} (f'')^2 \ge \frac{16}{\int_{-1}^{1}(1-x^2)^2\,dx}$$

Since $\int_{-1}^{1}(1-x^2)^2\,dx = \int_{-1}^{1}(1-2x^2+x^4)\,dx = 2-\frac{4}{3}+\frac{2}{5}=\frac{16}{15}$, we get

$$\int_{-1}^{1} (f'')^2\,dx \ge \frac{16}{16/15} = 15$$

Equality in Cauchy–Schwarz holds if and only if f'' is a constant multiple of w, say $f''(x)=\lambda(1-x^2)$. Then

$$4 = \int f''w = \lambda\int w^2 = \lambda\cdot\frac{16}{15}$$

so $\lambda = \frac{15}{4}$. Integrating twice,

$$f(x) = \lambda\left(\frac{x^2}{2}-\frac{x^4}{12}\right)+Ax+B = -\frac{5}{16}x^4+\frac{15}{8}x^2+Ax+B$$

The conditions $f(1)=f(-1)=1$ force $A=0$ and $B=1-\frac{5\lambda}{12}=-\frac{9}{16}$. This f also satisfies $\int_{-1}^{1} f = 0$. Thus equality occurs exactly for

$$f(x) = \frac{-5x^4+30x^2-9}{16}$$

Final answer:

$$\int_{-1}^{1} (f''(x))^2\,dx \ge 15$$

with equality if and only if $f(x)=\frac{-5x^4+30x^2-9}{16}$.

3 Problem 3

3.1 Variant 1

Let $n=2025$. We first characterize the set S.

Claim. A real symmetric rank-1 matrix with entries in $\{-1,+1\}$ must be of the form $A=s\varepsilon\varepsilon^T$, where $s\in\{-1,+1\}$ and $\varepsilon\in\{-1,+1\}^n$.

Proof. Any real symmetric rank-1 matrix has the form $A=\lambda xx^T$ for some $\lambda\neq 0$ and $x\in\mathbb{R}^n$. Then $a_{ii}=\lambda x_i^2\in\{\pm1\}$. Since $x_i^2\ge 0$ and λ is fixed, the signs of the diagonal entries cannot vary with i; hence all a_{ii} are equal, say $a_{ii}=s\in\{\pm1\}$ for all i. Thus $\lambda x_i^2=s$ is constant, so $|x_i|$ is constant and nonzero. Write $x_i=a\varepsilon_i$ with $a>0$ and $\varepsilon_i\in\{\pm1\}$. Then $A_{ij}=\lambda x_i x_j=\lambda a^2\varepsilon_i\varepsilon_j=s\varepsilon_i\varepsilon_j$. Hence $A=s\varepsilon\varepsilon^T$. Conversely, any such matrix has rank 1, is symmetric, and has entries in $\{\pm1\}$. This proves the claim.

Thus every $A\in S$ can be written as $A=s_a uu^T$ with $s_a\in\{\pm1\}$ and $u\in\{-1,+1\}^n$, and similarly $B=s_b vv^T$ with $s_b\in\{\pm1\}$ and $v\in\{-1,+1\}^n$. Note that replacing u by $-u$ (or v by $-v$) does not change the matrix.

Compute

$$AB = s_a s_b u(u^T v)v^T, \tag{22}$$

$$BA = s_a s_b v(v^T u)u^T. \tag{23}$$

Let $c = u^T v = \sum_{i=1}^n u_i v_i$. Then $AB = s_a s_b c uv^T$ and $BA = s_a s_b c vu^T$. Therefore $AB = BA$ if and only if either:

- $c = 0$, in which case $AB = BA = 0$; or
- $uv^T = vu^T$, which holds if and only if u and v are proportional, i.e., $v = \pm u$ (since u, v have ± 1 entries).

Now $n = 2025$ is odd, so $c = \sum u_i v_i$ is a sum of an odd number of ± 1 terms and cannot be zero. Hence the only way $AB = BA$ is when $v = \pm u$, i.e., when B's sign-pattern equals A's up to a global sign. The scalar signs s_a, s_b do not affect commutativity.

Counting/probability. The set S has size $|S| = 2^n$: there are 2 choices for s and 2^{n-1} choices for the sign-pattern class $[u] = \{u, -u\}$. For a fixed A, the commuting B must have the same class $[v] = [u]$, and there are exactly 2 such B ($s_b = \pm 1$). Thus, for each A, the number of commuting B is 2, out of $|S| = 2^n$ possible B. Therefore

$$P(AB = BA) = \frac{2}{2^n} = 2^{1-n} = 2^{-(n-1)}.$$

With $n = 2025$, this is 2^{-2024}.

Final answer. 2^{-2024}

3.2 Variant 2

Let $n = 2025$. Any real symmetric rank-1 matrix with entries ± 1 must be of the form $A = \varepsilon uu^T$, where $\varepsilon \in \{\pm 1\}$ and $u \in \{\pm 1\}^n$. Indeed, for a symmetric rank-1 $A = \lambda xx^T$, the condition that all entries are ± 1 forces $|x_i|$ to be constant, so $x = cs$ with $s_i \in \{\pm 1\}$, and then $A = \text{sign}(\lambda) ss^T$.

Thus choose $A = \alpha uu^T$ and $B = \beta vv^T$ with $\alpha, \beta \in \{\pm 1\}$ and $u, v \in \{\pm 1\}^n$, independently and uniformly (this yields the uniform distribution on S).

Compute

$$AB = \alpha\beta(u^T v)uv^T \tag{24}$$

$$BA = \alpha\beta(u^T v)vu^T \tag{25}$$

Hence $AB = BA$ iff either

- $u^T v = 0$, or
- $uv^T = vu^T$, which holds iff u and v are linearly dependent, i.e., $u = \pm v$ (since entries are ± 1).

Because $n = 2025$ is odd, $u^T v = \sum$ of 2025 terms each ± 1 cannot be 0. Therefore the only commuting case is $u = \pm v$.

With u and v independent uniform in $\{\pm 1\}^n$,

$$P(u = v) = 2^{-n} \tag{26}$$

$$P(u = -v) = 2^{-n} \tag{27}$$

so $P(u = \pm v) = 2 \cdot 2^{-n} = 2^{1-n}$.

For $n = 2025$ this is 2^{-2024}.

Final answer: 2^{-2024}

3.3 Variant 3

Let $n = 2025$. Any real symmetric rank-1 matrix A can be written as $A = \lambda uu^T$ for some vector u and scalar λ. If all entries of A are ± 1, then in particular $A_{ii} = \lambda u_i^2 \in \{\pm 1\}$ for all i. Since $u_i^2 > 0$, all diagonal entries must have the same sign, so u_i^2 is constant across i. Hence $|u_i|$ is constant; writing $\varepsilon_i = \operatorname{sign}(u_i) \in \{\pm 1\}$, we get

$$A = s\varepsilon\varepsilon^T,$$

where $\varepsilon \in \{\pm 1\}^n$ and $s \in \{\pm 1\}$. Thus

$$S = \{svv^T : v \in \{\pm 1\}^n, s \in \{\pm 1\}\},$$

with v and $-v$ giving the same matrix for fixed s. Therefore $|S| = 2 \cdot 2^{n-1} = 2^n$.

Take $A = s_a aa^T$ and $B = s_b bb^T$ with $a, b \in \{\pm 1\}^n$ and $s_a, s_b \in \{\pm 1\}$. Then

$$AB = s_a s_b (a^T b) ab^T, \quad \text{and} \quad BA = s_a s_b (a^T b) ba^T.$$

Hence $AB = BA$ if and only if either $a^T b = 0$ or $ab^T = ba^T$. The latter holds if and only if a and b are proportional, and since a, b have ± 1 entries, that means $b = \pm a$.

Now, because $n = 2025$ is odd, the inner product $a^T b$ cannot be 0 (it equals $n - 2d$ for some integer d, hence is odd). Therefore A and B commute if and only if $b = \pm a$.

Fix A. The matrices B that commute with A are exactly $B = \pm aa^T$, i.e., 2 choices out of $|S| = 2^n$. Thus the desired probability is

$$\frac{2}{2^n} = 2^{1-n} = 2^{-2024}.$$

Final answer: 2^{-2024}

3.4 Final Solution

Let $n = 2025$. We first characterize the set S of real symmetric rank-1 matrices with entries in $\{-1, +1\}$.

Any real symmetric rank-1 matrix has the form $A = \lambda xx^T$ with $\lambda \neq 0$. The diagonal entries are $A_{ii} = \lambda x_i^2 \in \{\pm 1\}$. Since $x_i^2 \geq 0$ and λ is fixed, all A_{ii} must have the same sign $s \in \{\pm 1\}$, so $\lambda x_i^2 = s$ for all i, implying $|x_i|$ is constant. Write $x = a\varepsilon$ with $a > 0$ and $\varepsilon \in \{\pm 1\}^n$. Then

$$A = \lambda a^2 \varepsilon\varepsilon^T,$$

and because $\lambda a^2 = s$, we get $A = s\varepsilon\varepsilon^T$.

Conversely, any $s\varepsilon\varepsilon^T$ is symmetric, rank 1, and has entries ± 1. Thus

$$S = \{s\varepsilon\varepsilon^T : s \in \{\pm 1\}, \varepsilon \in \{\pm 1\}^n\},$$

with ε and $-\varepsilon$ yielding the same matrix. Hence $|S| = 2 \cdot 2^{n-1} = 2^n$.

Let $A = s_a uu^T$ and $B = s_b vv^T$ with $u, v \in \{\pm 1\}^n$ and $s_a, s_b \in \{\pm 1\}$. Then

$$AB = s_a s_b (u^T v) uv^T, \tag{28}$$

$$BA = s_a s_b (u^T v) vu^T. \tag{29}$$

Therefore $AB = BA$ if and only if either:

- $u^T v = 0$, or
- $uv^T = vu^T$, which holds if and only if u and v are proportional. Since u, v have entries ± 1, this means $v = \pm u$.

But $u^T v$ is the sum of $n = 2025$ odd many terms each ± 1, so it cannot be zero. Thus $AB = BA$ if and only if $v = \pm u$.

Fix A. The B that commute with A are precisely those with the same sign-pattern class $[v] = [u]$, and there are exactly two such matrices ($s_b = \pm 1$). Since $|S| = 2^n$, the desired probability is $\frac{2}{2^n} = 2^{1-n}$.

For $n = 2025$, this equals 2^{-2024}.

Final answer: 2^{-2024}

4 Problem 4

4.1 Variant 1

Let a be a fixed even positive integer, and let b be any positive integer. Set $t = \lfloor x/a \rfloor$, so $x = at + r$ with $0 \le r < a$.

Rewrite the left-hand side:

$$\lfloor (b^a + x)^{1/a} b^{a-1} \rfloor = \left\lfloor b^a \left(1 + \frac{x}{b^a}\right)^{1/a} \right\rfloor.$$

We want

$$\left\lfloor b^a \left(1 + \frac{x}{b^a}\right)^{1/a} \right\rfloor = b^a + t$$

for all b. This is equivalent to the double inequality

$$b^a + t \le b^a \left(1 + \frac{x}{b^a}\right)^{1/a} < b^a + t + 1,$$

or, after dividing by b^a and raising to the power a (note monotonicity since a is even),

$$\left(1 + \frac{t}{b^a}\right)^a \le 1 + \frac{x}{b^a} < \left(1 + \frac{t+1}{b^a}\right)^a.$$

Multiplying by b^a and expanding, this becomes, for all $b \ge 1$,

$$at + \sum_{k=2}^{a} \binom{a}{k} t^k b^{-a(k-1)} \le x < a(t+1) + \sum_{k=2}^{a} \binom{a}{k} (t+1)^k b^{-a(k-1)}.$$

Taking the intersection over all b, the left endpoints increase to their maximum at $b = 1$, while the right endpoints decrease to their minimum as $b \to \infty$. Hence the condition is exactly

$$(1+t)^a - 1 \le x < a(t+1). \quad (1)$$

Since $(1+t)^a - 1 \ge at$ for $t \ge 0$ (by the binomial theorem), any x in (1) does satisfy $\lfloor x/a \rfloor = t$, and the above inequalities yield the desired floor identity for all b.

Thus, for each integer $t \ge -1$, the admissible x form the interval

$$[(1+t)^a - 1, a(t+1)),$$

but this interval is nonempty only when

$$(1+t)^a - 1 < a(t+1) \iff (t+1)^a \le a(t+1).$$

Let $s = t+1 \ge 0$. The condition becomes $s^a \le as$. Therefore the solution set is the union, over all integers $s \ge 0$ with $s^a \le as$, of the intervals

$$[s^a - 1, as).$$

Now determine which s work:

- $s = 0$ gives $[-1, 0)$.
- $s = 1$ gives $[0, a)$.
- For $s \ge 2$, the condition $s^a \le as$ is equivalent to $s^{a-1} \le a$. For even $a \ge 4$, we have $2^{a-1} > a$, so no $s \ge 2$ works. For $a = 2$, we have $s^{a-1} = s \le 2$, so $s = 2$ also works, yielding $[2^2 - 1, 2 \cdot 2) = [3, 4)$.

Therefore:

- If $a = 2$, the solutions are $x \in [-1, 2) \cup [3, 4)$.
- If $a \ge 4$ is even, the solutions are $x \in [-1, a)$.

Final answer:

- For $a = 2$: all real x in $[-1, 2) \cup [3, 4)$.
- For even $a \ge 4$: all real x in $[-1, a)$.

4.2 Variant 2

Let $a \ge 2$ be even. Write $x = ak + r$ with $k = \lfloor x/a \rfloor$ and $r \in [0, a)$.

The equation is

$$\lfloor b^{a-1} \sqrt[a]{b^a + x} \rfloor = b^a + \lfloor x/a \rfloor$$

for every positive integer b. Using $\sqrt[a]{b^a + x} = b(1 + x/b^a)^{1/a}$, the left-hand side is

$$\lfloor b^a (1 + (ak + r)/b^a)^{1/a} \rfloor.$$

Thus the equality is equivalent to, for all $b \in \mathbb{N}$,

$$b^a + k \le b^a (1 + (ak + r)/b^a)^{1/a} < b^a + k + 1. \qquad (1)$$

Upper bound. For $u \ge -1$ and $a \ge 1$, the function $t \mapsto (1+t)^{1/a}$ is concave, so

$$(1+u)^{1/a} \le 1 + u/a.$$

Applying this with $u = (ak + r)/b^a$ yields

$$b^a (1 + (ak + r)/b^a)^{1/a} \le b^a + k + r/a < b^a + k + 1$$

since $r < a$. Hence the right inequality in (1) always holds.

Lower bound. The left inequality in (1) is equivalent (by monotonicity of $x \mapsto x^a$) to

$$1 + (ak + r)/b^a \geq (1 + k/b^a)^a,$$

i.e.

$$r \geq \sum_{i=2}^{a} \binom{a}{i} k^i / b^{a(i-1)}. \qquad (2)$$

Therefore the condition for all b is $r \geq \sup_{b \geq 1} T_b(k)$, where

$$T_b(k) = \sum_{i=2}^{a} \binom{a}{i} k^i / b^{a(i-1)}.$$

- If $k \geq 0$, each term is ≥ 0 and decreases with b, so $\sup_{b\geq 1} T_b(k) = T_1(k) = (1 + k)^a - 1 - ak$.
- If $k = -1$, then

  $$T_b(-1) = \sum_{i=2}^{a} \binom{a}{i} (-1)^i / b^{a(i-1)}.$$

  Let $u = 1/b^a \in (0, 1]$. Define

  $$\Phi(u) = \frac{(1-u)^a - 1 + au}{u} = \sum_{i=2}^{a} \binom{a}{i} (-1)^i u^{i-1}.$$

  Then

  $$\Phi'(u) = \frac{1 - (1-u)^{a-1}(1 + (a-1)u)}{u^2} \geq 0,$$

  because $g(u) = (1-u)^{a-1}(1 + (a-1)u)$ is decreasing on $[0, 1]$ and $g(0) = 1$. Hence Φ is increasing on $(0, 1]$, so $\Phi(u) \leq \Phi(1) = a - 1$. Therefore $\sup_{b\geq 1} T_b(-1) = a - 1$.
- If $k \leq -2$, one checks directly

  $$(1 + k)^a - 1 - ak = (-1 - t)^a - 1 + a(1 + t) \text{ with } t \geq 1$$

  equals $t^a - 1 + a + at \geq 2a$, so it already exceeds a.

Thus, a necessary and sufficient condition for (1) is

$$r \geq F(k) := (1 + k)^a - 1 - ak, \text{ with } 0 \leq r < a. \qquad (3)$$

Since $r < a$, this forces $F(k) < a$. We now solve $F(k) < a$ for integer k:

- $k = 0$: $F(0) = 0 < a$, allowed; $r \in [0, a)$, i.e. $x \in [0, a)$.
- $k = -1$: $F(-1) = a - 1 < a$, allowed; $r \in [a - 1, a)$, i.e. $x = -a + r \in [-1, 0)$.
- $k = 1$: $F(1) = 2^a - 1 - a$. This is $< a$ only when $a = 2$, giving $F(1) = 1 < 2$. Then $r \in [1, 2)$, i.e. $x \in [3, 4)$. For even $a \geq 4$, $F(1) \geq 11 \geq a$, so $k = 1$ is not allowed.
- $k \geq 2$ or $k \leq -2$: $F(k) \geq 2a \geq a$, not allowed.

Conclusion.

- If $a = 2$: the solutions are $x \in [-1, 2) \cup [3, 4)$.
- If $a \geq 4$ even: the solutions are $x \in [-1, a)$.

Final answer.

- For $a = 2$: all real x in $[-1, 2) \cup [3, 4)$.
- For even $a \geq 4$: all real x in $[-1, a)$.

4.3 Variant 3

Let a be an even positive integer and let x be real. For the expression to be defined for every positive integer b, we must have $b^a + x \geq 0$ for $b = 1$, hence $x \geq -1$.

Rewrite the left-hand side:

$$\sqrt[a]{b^a + x} \cdot b^{a-1} = b^a(1 + x/b^a)^{1/a}.$$

Therefore the equality is equivalent to

$$\lfloor b^a(1 + x/b^a)^{1/a} \rfloor = b^a + \lfloor x/a \rfloor.$$

Subtract b^a from both sides (noting b^a is an integer):

$$\lfloor b^a((1 + x/b^a)^{1/a} - 1) \rfloor = \lfloor x/a \rfloor. \quad (1)$$

Define

$$F_b(x) = b^a((1 + x/b^a)^{1/a} - 1).$$

By concavity of $t \mapsto t^{1/a}$ on $(0, \infty)$ (since $1/a \in (0, 1]$), we have for $y \geq -1$:

$$(1 + y)^{1/a} \leq 1 + y/a,$$

so $F_b(x) \leq x/a$. Thus $\lfloor F_b(x) \rfloor \leq \lfloor x/a \rfloor$. To obtain equality in (1), it suffices to ensure $F_b(x) \geq \lfloor x/a \rfloor$ for all b.

Monotonicity in b. Write

$$F_b(x) = x \cdot H(1 + x/b^a),$$

where $H(s) = (s^{1/a} - 1)/(s - 1)$ for $s > 0$, $s \neq 1$, and $H(1) = 1/a$. Since $t \mapsto t^{1/a}$ is concave, the chord slope $H(s)$ is decreasing in s on $(0, \infty)$. As b increases, $1 + x/b^a$ moves monotonically to 1, and from the monotonicity of H we get that $F_b(x)$ is increasing in b (for both signs of x). Hence

$$\min_{b \geq 1} F_b(x) = F_1(x) = (1 + x)^{1/a} - 1,$$

and $F_b(x) \leq x/a$ for all b.

Therefore, (1) holds for all b if and only if

$$(1 + x)^{1/a} - 1 \geq \lfloor x/a \rfloor, \text{ with } x \geq -1. \quad (2)$$

Let $k = \lfloor x/a \rfloor$, so $x \in [ak, ak + a)$. We analyze (2) by cases.

- If $k \le -1$: then $1 + k \le 0$ and since $(1 + x)^{1/a} \ge 0$, the inequality in (2) holds automatically. Intersecting with $x \ge -1$ forces $k = -1$ and $x \in [-1, 0)$.
- If $k \ge 0$: then (2) is equivalent to $(1 + x)^{1/a} \ge 1 + k$, i.e.

  $$x \ge (1 + k)^a - 1,$$

  together with $x \in [ak, ak + a)$. Thus we need the interval

  $$[(1 + k)^a - 1, ak + a)$$

  to be nonempty.

For $k = 0$, this gives $x \in [0, a)$.
For $k \ge 1$, note that
$$(1 + k)^a - 1 = ak + C,$$
where $C = \sum_{j=2}^{a} \binom{a}{j} k^j \ge \binom{a}{2} k^2$. Hence for $k \ge 1$:

$$(1 + k)^a - 1 \ge ak + \binom{a}{2} \ge ak + a \text{ if } a \ge 4,$$

so the interval is empty for all $k \ge 1$ when $a \ge 4$.
For $a = 2$, the condition becomes $(1 + k)^2 - 1 < 2(k + 1)$, i.e. $k^2 < 2$, so $k = 0$ or $k = 1$. We already have $k = 0$ giving $[0, 2)$; for $k = 1$ we get $x \in [3, 4)$.
Collecting all cases:

- If $a = 2$: $x \in [-1, 2) \cup [3, 4)$.
- If $a \ge 4$ even: $x \in [-1, a)$.

It is straightforward to check that these x indeed satisfy the original identity for all b, using the monotonicity of $F_b(x)$ and the bounds $F_b(x) \in [F_1(x), x/a)$.
Final answer.

- For $a = 2$: all real x in $[-1, 2) \cup [3, 4)$.
- For even $a \ge 4$: all real x in $[-1, a)$.

4.4 Final Solution

Let a be a fixed even positive integer and b a positive integer. Write $x = ak + r$ with $k = \lfloor x/a \rfloor$ and $0 \le r < a$. Note first that $b^a + x \ge 0$ for $b = 1$ forces $x \ge -1$.
We have
$$\lfloor b^{a-1}(b^a + x)^{1/a} \rfloor = \lfloor b^a(1 + x/b^a)^{1/a} \rfloor = b^a + k$$
if and only if, for all $b \in \mathbb{N}$,

$$b^a + k \le b^a(1 + (ak + r)/b^a)^{1/a} < b^a + k + 1. \quad (1)$$

Upper bound. Since $t \mapsto (1 + t)^{1/a}$ is concave on $[-1, \infty)$,

$$(1 + u)^{1/a} \le 1 + u/a \quad (u \ge -1).$$

With $u = (ak + r)/b^a$ we get

$$b^a(1 + u)^{1/a} \le b^a + k + r/a < b^a + k + 1,$$

so the right inequality in (1) always holds.

Lower bound. The left inequality in (1) is equivalent to

$$1 + (ak + r)/b^a \ge (1 + k/b^a)^a \tag{30}$$

$$\Leftrightarrow r \ge \sum_{i=2}^{a} \binom{a}{i} k^i b^{-a(i-1)} =: T_b(k). \quad (2) \tag{31}$$

Thus (1) holds for all b iff $r \ge \sup_{b \ge 1} T_b(k)$. We evaluate this supremum by cases.

- **$k \ge 0$.** All terms in $T_b(k)$ are ≥ 0 and decrease with b, hence

  $$\sup_{b \ge 1} T_b(k) = T_1(k) = (1 + k)^a - 1 - ak =: F(k).$$

  So we need $r \ge F(k)$ with $0 \le r < a$, i.e. $F(k) < a$. Since F is increasing on $k \ge 0$, $F(0) = 0$ and $F(1) = 2^a - 1 - a$. For $a = 2$, $F(1) = 1 < 2$, while for even $a \ge 4$, $F(1) \ge 11 \ge a$. Hence:

  - $k = 0$ is always allowed ($r \in [0, a) \Rightarrow x \in [0, a)$);
  - $k = 1$ is allowed only if $a = 2$ (then $r \in [1, 2) \Rightarrow x \in [3, 4)$);
  - $k \ge 2$ is never allowed ($F(k) \ge F(1) \ge a$).

- **$k = -1$.** Here

  $$T_b(-1) = \sum_{i=2}^{a} \binom{a}{i} (-1)^i b^{-a(i-1)}.$$

  Let $u = 1/b^a \in (0, 1]$, and define

  $$\Phi(u) = \frac{(1 - u)^a - 1 + au}{u} = \sum_{i=2}^{a} \binom{a}{i} (-1)^i u^{i-1} = T_b(-1).$$

  Compute

  $$\Phi'(u) = \frac{1 - (1 - u)^{a-1}(1 + (a - 1)u)}{u^2} \ge 0$$

  since $(1 - u)^{a-1}(1 + (a - 1)u)$ decreases on $[0, 1]$. Thus Φ is increasing, so

  $$\sup_{b \ge 1} T_b(-1) = \Phi(1) = a - 1.$$

  Therefore $r \ge a - 1$, i.e. $r \in [a - 1, a)$, giving $x \in [-1, 0)$.

- **$k \le -2$.** Then $T_1(k) = (1 + k)^a - 1 - ak \ge 2a \ge a$ (since a is even), which cannot be $\le r < a$. Hence no solutions.

Collecting all possibilities:

- For any even $a \ge 2$: $x \in [-1, 0)$ (from $k = -1$) and $x \in [0, a)$ (from $k = 0$).

- Additionally, when $a = 2$: $x \in [3, 4)$ (from $k = 1$).

It remains to note that for these x the lower bound $r \geq \sup_b T_b(k)$ ensures the left inequality in (1), and the concavity estimate ensures the right inequality; hence the original floor identity holds for every b.

Final answer.

- If $a = 2$: all real x in $[-1, 2) \cup [3, 4)$.
- If $a \geq 4$ is even: all real x in $[-1, a)$.

5 Problem 5

5.1 Variant 1

Step 1. The eventual shape of a self-map and the exact value of $\operatorname{ord}(\tau)$.

Let $\tau \in T_n$. Its functional digraph is a disjoint union of directed cycles, with rooted in-trees feeding into the cycle vertices. Let:

- $c =$ number of vertices that lie on cycles,
- $h =$ maximal distance (height) from a vertex to its cycle,
- $\ell_1, \ldots, \ell_r$ be the lengths of the cycles, and $L = \operatorname{lcm}(\ell_1, \ldots, \ell_r)$.

Then the sequence of maps $\tau, \tau^2, \ldots$ is eventually periodic with preperiod length h and period L, and in fact

$$\operatorname{ord}(\tau) = h + L.$$

Proof of $\operatorname{ord}(\tau) = h + L$:

- For $0 \leq k < h$, the maps τ^k are all distinct, since for each k there exists a vertex at depth $\geq k + 1$ whose image under τ^k is still in the tree, whereas τ^{k+1} pushes it one step further.
- For $h \leq k < h + L$, the maps τ^k are all distinct and $\tau^{k+L} = \tau^k$, because once all points have reached their cycles, the action depends on k modulo L.
- No τ^k with $k < h$ equals any τ^m with $m \geq h$, since the image of a vertex of maximum depth under τ^k is not on a cycle, while under τ^m it is on a cycle.

Thus $\operatorname{ord}(\tau) = h + L$.

Step 2. A reduction: an exact formula for $g(n)$.

From Step 1, for any τ we have $\operatorname{ord}(\tau) = h + L$, where L is the lcm of cycle lengths (over the c cycle vertices), and $h \leq n - c$. For fixed c, the maximal possible L among permutations on c letters is $f(c)$. Hence

$$\operatorname{ord}(\tau) \leq (n - c) + f(c).$$

Conversely, this bound is attained: choose a permutation on c with order $f(c)$, and let the remaining $t = n - c$ vertices form a single directed path feeding into one vertex on some cycle. Then $h = t$ and $L = f(c)$, so $\operatorname{ord}(\tau) = t + f(c)$.

Therefore
$$g(n) = \max_{0 \le t \le n} [t + f(n-t)].$$

Step 3. A number-theoretic lemma.
We prove that for sufficiently large n, and for any integer s with $s \ge n^{0.501}$,
$$f(n) - f(n-s) \ge s - n^{0.501} - 1. \quad (\star)$$

Proof. Let $c = n - s$, and let $L^* = f(c)$, realized by some permutation on c points. Let $u = \lceil n^{0.501} \rceil$ and consider the set P of primes in the interval $(s-u, s]$. A standard consequence of the prime number theorem (via partial summation) is that the sum $S(x)$ of primes $\le x$ satisfies $S(x) \sim x^2/(2\log x)$. In particular, there exists a constant $K > 0$ such that for all sufficiently large x and all $1 \le y \le x$,
$$\sum_{x-y<p\le x} p \ge Kxy/\log x.$$

Applying this with $x = s$ and $y = u$, for large n we get
$$\sum_{p\in P} p \ge Ksu/\log s \gg n^{1.002}/\log n > n \ge c.$$

Now, if every prime $p \in P$ divides L^*, then in the permutation of size c we would need to include each such prime as a factor of some cycle length. Covering a prime p requires at least p points in the cycles (because some cycle length must be a multiple of p, hence $\ge p$). Covering several primes with a single cycle does not reduce this cost (a multiple of distinct primes p and q has length at least $pq \ge p + q$). Therefore the total number of cycle vertices needed to include all primes in P is at least $\sum_{p\in P} p$, which exceeds c for large n. This is impossible. Hence there exists a prime $m \in P$ that does not divide L^*.

Using the remaining s points, form one additional cycle of length m (and make the other $s - m$ points fixed). Then on n points we obtain a permutation whose order is $\operatorname{lcm}(L^*, m) = L^* \cdot m$ (because $\gcd(L^*, m) = 1$). Thus
$$f(n) \ge L^* \cdot m \ge L^* + m - 1 \ge f(n-s) + (s-u) - 1,$$
since $m \ge s - u$ and $L^* \ge 1$, which proves $(\star)$.

Step 4. Conclusion.
Recall $g(n) = \max_{0\le t\le n}[t + f(n-t)]$.

- If $t \le n^{0.501}$, then $t + f(n-t) \le f(n) + n^{0.501}$.
- If $t \ge n^{0.501}$, apply $(\star)$ with $s = t$ to get
$$f(n-t) \le f(n) - (t - n^{0.501} - 1),$$
so $t + f(n-t) \le f(n) + n^{0.501} + 1$.

Therefore, for sufficiently large n,
$$g(n) \le f(n) + \lfloor n^{0.501} \rfloor + 1 < f(n) + n^{0.501}.$$

This proves the desired inequality.
Final answer. For sufficiently large n,
$$g(n) < f(n) + n^{0.501}.$$

5.2 Variant 2

Let $\tau \in T_n$. View τ as a functional digraph on $[n]$, i.e., a disjoint union of directed cycles with rooted trees feeding into those cycles.

Write:

- C = set of cyclic points of τ, $m = |C|$.
- $t = n - m$ = number of non-cyclic points.
- L = lcm of the cycle lengths on C (so L is the order of the restriction of τ to C). In particular $L \le f(m)$.
- h = maximum distance (height) of a non-cyclic point to its eventual cycle. Then $h \le t$.

Two basic facts about the iterates τ^k:

1. After h steps, every point has reached a cycle, so from that time on the restriction to C is a permutation of C with period L. Hence the sequence of maps $\tau, \tau^2, \dots$ becomes periodic after at most h steps with period dividing L.
2. Therefore the number of distinct maps among $\{\tau, \tau^2, \tau^3, \dots\}$ is at most $h + L$.

Thus

$$\mathrm{ord}(\tau) \le h + L \le t + f(m). \quad (1)$$

We will compare $f(m)$ with $f(n)$ using a simple prime-counting argument.

A prime in $(t/2, t]$ cannot appear together with a distinct prime in the same cycle length (for large n) because the product of two distinct primes $p, q > t/2$ is $> t^2/4$, which exceeds m for $t \ge n^{0.501}$ and all sufficiently large n (since then $t^2/4 \ge n^{1.002}/4 > n \ge m$). Hence, each distinct prime $p \in (t/2, t]$ dividing the order L must be supported by a different cycle of length at least $p \ge t/2$, consuming at least $t/2$ points. Therefore, the number r of primes in $(t/2, t]$ dividing L satisfies

$$r \le 2m/t \le 2n/t. \quad (2)$$

On the other hand, by standard prime number estimates, for all sufficiently large x,

$$\pi(x) - \pi(x/2) \ge c\frac{x}{\log x}$$

for some absolute constant $c > 0$. Taking $x = t$ (and recalling $t \ge n^{0.501}$), we get

$$\pi(t) - \pi(t/2) \gg t/\log t.$$

For large n this quantity exceeds $2n/t$ (because $t/\log t \gg n^{0.501}/\log n \gg n^{0.499}$), so by (2) there exists a prime $p \in (t/2, t]$ that does not divide L.

Now extend any permutation on m points of order $f(m)$ to a permutation on n points by using the remaining t points to form one p-cycle. Since $p \nmid f(m)$, the new order is $\mathrm{lcm}(f(m), p) = pf(m)$. Hence

$$f(n) \ge pf(m) \ge (t/2)f(m),$$

so

$$f(m) \le 2f(n)/t. \quad (3)$$

Combining (1) and (3), we obtain for $t \ge n^{0.501}$ (and hence $t \ge 2$ for large n),

$$\operatorname{ord}(\tau) \le t + f(m) \le t + 2f(n)/t \le t + f(n) \le f(n) + n^{0.501}.$$

If instead $t < n^{0.501}$, then from (1) and monotonicity $f(m) \le f(n)$,

$$\operatorname{ord}(\tau) \le t + f(m) \le f(n) + t < f(n) + n^{0.501}.$$

In either case, for sufficiently large n,

$$\operatorname{ord}(\tau) < f(n) + n^{0.501}.$$

Taking the maximum over $\tau \in T_n$ yields

$$g(n) < f(n) + n^{0.501}$$

for all sufficiently large n.

Final answer. For all sufficiently large n, $g(n) < f(n) + n^{0.501}$.

5.3 Variant 3

Let $[n] = \{1, 2, \ldots, n\}$. For a self-map τ on $[n]$, its functional digraph decomposes into directed cycles with rooted trees feeding into the cycles. Denote

- $C(\tau)$ the set of cyclic points, $m = |C(\tau)|$.
- $h(\tau)$ the maximal distance (height) from a point to the cycles (i.e., the longest tail length).
- $L(\tau)$ the least common multiple of the cycle-lengths of τ (i.e., the order of τ restricted to $C(\tau)$, which is a permutation).

1) The number of distinct iterates of a map

Claim. For any $\tau \in T_n$,

$$\operatorname{ord}(\tau) = L(\tau) + \max(0, h(\tau) - 1).$$

Proof. Consider the sequence $\tau, \tau^2, \tau^3, \ldots$. For $k \ge h := h(\tau)$, every point is on some cycle after applying τ^k, hence τ^k restricted to $C(\tau)$ is a power of the corresponding permutation, and τ^k depends only on k modulo $L := L(\tau)$. Thus among $k \ge h$ there are exactly L distinct maps: $\tau^h, \tau^{h+1}, \ldots, \tau^{h+L-1}$, and thereafter they repeat with period L.

For $1 \le k \le h-1$, these maps are all distinct and none coincides with any τ^{h+r}. Indeed, pick x at maximal distance h from the cycles; then $\tau^k(x)$ has distance $h - k \ge 1$ from the cycles, whereas for any $r \ge 0$, $\tau^{h+r}(x)$ lies on a cycle. Hence $\tau^k \ne \tau^{h+r}$. Therefore the total number of distinct maps is $(h-1) + L$ if $h \ge 1$, and L if $h = 0$. This is exactly $L + \max(0, h-1)$. $\square$

2) A formula for $g(n)$

If τ has m cyclic points, then $h(\tau) \le n - m$. By the claim,

$$\operatorname{ord}(\tau) \le L(\tau) + \max(0, n-m-1) \le f(m) + \max(0, n-m-1).$$

Conversely, for any m between 0 and n, choose a permutation $\sigma \in S_m$ of order $f(m)$ on some m-element subset C, and attach to one chosen cycle-vertex a directed chain of length $n-m$ using the other $n-m$ points. The resulting map τ has $L(\tau) = f(m)$ and $h(\tau) = n - m$, hence $\mathrm{ord}(\tau) = f(m) + \max(0, n-m-1)$.

Therefore

$$g(n) = \max_{0 \le m \le n} \left[f(m) + \max(0, n-m-1) \right].$$

3) Bounding $g(n)$ in terms of $f(n)$

Let $t = \lfloor n^{0.501} \rfloor$.

- If $m \ge n - t$, then $f(m) + \max(0, n-m-1) \le f(n) + t \le f(n) + n^{0.501}$.
- If $m \le n - t - 1$, we will show that for all sufficiently large n,

$$f(m) + (n-m-1) \le f(n).$$

It is a classical result (Landau) that

$$\log f(k) = (1 + o(1))\sqrt{k \log k}.$$

In particular, for any $\alpha > 1/2$ we have $f(n - n^{\alpha}) = o(f(n))$ as $n \to \infty$. Taking $\alpha = 0.501$, there exists N_1 such that for $n \ge N_1$,

$$f(n-t) \le \frac{1}{2} f(n).$$

Also, since $f(n)$ grows super-polynomially, there exists N_2 such that for $n \ge N_2$, $f(n) \ge 2n$. For $m \le n - t$ we then have

$$f(m) + (n-m-1) \le f(n-t) + n \le \frac{1}{2} f(n) + \frac{1}{2} f(n) = f(n).$$

Combining both cases, for $n \ge \max(N_1, N_2)$,

$$g(n) \le \max\{f(n), f(n) + t\} = f(n) + t \le f(n) + n^{0.501}.$$

Since $g(n)$ is an integer, we in fact have

$$g(n) \le f(n) + \lfloor n^{0.501} \rfloor < f(n) + n^{0.501}.$$

This proves the desired inequality for all sufficiently large n.

Final answer. For sufficiently large n, $g(n) < f(n) + n^{0.501}$.

5.4 Final Solution

Solution.

1) Dynamics of a self-map and $\mathrm{ord}(\tau)$

Let $\tau \in T_n$. Its functional digraph is a disjoint union of directed cycles with rooted trees feeding into the cycles. Let:

- $L(\tau)$ be the least common multiple of the cycle lengths;
- $h(\tau)$ be the maximal distance (height) from a vertex to its eventual cycle (i.e., the longest tail length).

Claim. For any $\tau \in T_n$,

$$\operatorname{ord}(\tau) = L(\tau) + \max(0, h(\tau) - 1).$$

Proof. For $k \geq h := h(\tau)$, every point lies on a cycle after applying τ^k, so τ^k depends only on k modulo $L := L(\tau)$. Hence among $\{\tau^h, \tau^{h+1}, \ldots\}$ there are exactly L distinct maps. For $1 \leq k \leq h-1$, the maps $\tau, \tau^2, \ldots, \tau^{h-1}$ are pairwise distinct and none coincides with any τ^{h+r} ($r \geq 0$): if x is a vertex at maximal distance h from the cycles, then $\tau^k(x)$ is not yet on a cycle for $k \leq h-1$, whereas $\tau^{h+r}(x)$ is on a cycle. Summing gives $\operatorname{ord}(\tau) = L + \max(0, h-1)$. □

2) A formula for $\mathbf{g(n)}$

Let m be the number of cyclic points of τ, so $h(\tau) \leq n - m$, and $L(\tau) \leq f(m)$. By the claim,

$$\operatorname{ord}(\tau) \leq f(m) + \max(0, n - m - 1).$$

Conversely, for each $m \in \{1, \ldots, n\}$, choose a permutation on an m-element set with order $f(m)$, and attach a directed chain of length $n-m$ feeding into one point on a cycle. This has $L(\tau) = f(m)$ and $h(\tau) = n - m$, hence $\operatorname{ord}(\tau) = f(m) + \max(0, n - m - 1)$. Therefore

$$g(n) = \max_{1 \leq m \leq n} \left[f(m) + \max(0, n - m - 1) \right].$$

3) Bounding $\mathbf{g(n)}$ in terms of $\mathbf{f(n)}$

Let $t = \lfloor n^{0.501} \rfloor$. Split the maximization into two ranges of m.

- If $m \geq n - t$, then

$$f(m) + \max(0, n - m - 1) \leq f(n) + t - 1 \leq f(n) + t.$$

- If $m \leq n - t - 1$, write $d = n - m \geq t + 1$. We use Landau's classical asymptotic for the maximal order of a permutation:

$$\log f(k) = (1 + o(1))\sqrt{k \log k} \text{ as } k \to \infty.$$

  In particular, for any $\alpha > 1/2$ we have $f(n - n^\alpha) = o(f(n))$. Taking $\alpha = 0.501$, there exists N_1 such that for $n \geq N_1$,

$$f(n - t) \leq \frac{1}{2} f(n).$$

  Moreover, $f(n)$ grows super-polynomially, so there exists N_2 such that for $n \geq N_2$, $f(n) \geq 2n$. Hence for $n \geq \max(N_1, N_2)$ and all $m \leq n - t$ we have

$$f(m) + (n - m - 1) \leq f(n - t) + n \leq \frac{1}{2} f(n) + \frac{1}{2} f(n) = f(n).$$

Combining the two cases, for all sufficiently large n,

$$g(n) \leq f(n) + t < f(n) + n^{0.501}.$$

Final answer. For sufficiently large n, $g(n) < f(n) + n^{0.501}$.

6 Problem 6, Day 2

6.1 Variant 1

Define the function

$$\varphi(x) = \frac{f(x) - k}{x}, \quad \text{for } x \in [a, b].$$

Since $a, b > 0$ and f is continuously differentiable on $(0, \infty)$, φ is continuous on $[a, b]$ and differentiable on (a, b). Also, using $f(a) = f(b) = k$, we have $\varphi(a) = \varphi(b) = 0$. By Rolle's theorem, there exists $\xi \in (a, b)$ such that $\varphi'(\xi) = 0$.

Compute the derivative:

$$\varphi'(x) = \frac{xf'(x) - (f(x) - k)}{x^2}.$$

Thus $\varphi'(\xi) = 0$ implies

$$xf'(\xi) - (f(\xi) - k) = 0,$$

i.e.

$$f(\xi) - \xi f'(\xi) = k.$$

This proves the claim.

Final answer: There exists $\xi \in (a, b)$ such that $f(\xi) - \xi f'(\xi) = k$.

6.2 Variant 2

Define the function

$$g(x) = \frac{f(x) - k}{x}, \quad \text{for } x \in [a, b].$$

Since $a, b > 0$ and f is continuously differentiable on $(0, \infty)$, the function g is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$g(a) = \frac{f(a) - k}{a} = 0 \quad \text{and} \quad g(b) = \frac{f(b) - k}{b} = 0.$$

By Rolle's theorem, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$. Compute

$$g'(x) = \frac{xf'(x) - (f(x) - k)}{x^2}.$$

Thus $g'(\xi) = 0$ implies $xf'(\xi) - (f(\xi) - k) = 0$ at $x = \xi$, i.e.,

$$f(\xi) - \xi f'(\xi) = k.$$

Final answer: There exists $\xi \in (a, b)$ with $f(\xi) - \xi f'(\xi) = k$.

6.3 Variant 3

Define $g(x) = \frac{f(x)-k}{x}$ on $[a, b]$. This is well-defined since $a, b > 0$, continuous on $[a, b]$, and differentiable on (a, b) because f is continuously differentiable.

Given $f(a) = f(b) = k$, we have $g(a) = g(b) = 0$. By Rolle's theorem, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$.

Compute

$$g'(x) = \frac{xf'(x) - (f(x) - k)}{x^2}.$$

Thus $g'(\xi) = 0$ implies $\xi f'(\xi) - f(\xi) + k = 0$, i.e.,

$$f(\xi) - \xi f'(\xi) = k.$$

Final answer: There exists $\xi \in (a, b)$ with $f(\xi) - \xi f'(\xi) = k$.

6.4 Final Solution

Define $g(x) = \frac{f(x)-k}{x}$ for $x \in [a, b]$. Since $a, b > 0$ and f is continuously differentiable on $(0, \infty)$, g is continuous on $[a, b]$ and differentiable on (a, b). Using $f(a) = f(b) = k$, we have $g(a) = g(b) = 0$. By Rolle's theorem, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$.

Compute

$$g'(x) = \frac{xf'(x) - (f(x) - k)}{x^2}.$$

Thus $g'(\xi) = 0$ implies

$$\xi f'(\xi) - (f(\xi) - k) = 0,$$

i.e.,

$$f(\xi) - \xi f'(\xi) = k.$$

Final answer: There exists $\xi \in (a, b)$ such that $f(\xi) - \xi f'(\xi) = k$.

7 Problem 7

7.1 Variant 1

Write $v_2(n)$ for the exponent of 2 in n, and $\mathrm{odd}(n) = n/2^{v_2(n)}$ for the odd part of n.

We claim that the sets

$$M_d = \{n \in \mathbb{Z}_{>0} : d \text{ divides } \mathrm{odd}(n)\}$$

for odd $d \geq 1$ are precisely the nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ satisfying (a) and (b).

First, each M_d satisfies (a) and (b):

(a) If $d \mid \mathrm{odd}(x)$, then $d \mid \mathrm{odd}(2x)$ because doubling does not change the odd part.

(b) Let $x = 2^a du$, $y = 2^b dv$ with u, v odd. If x and y have the same parity, then $(x+y)/2 = 2^{\min(a,b)-1} d(u2^{a-\min} + v2^{b-\min})$. Thus $\mathrm{odd}((x+y)/2) = d \cdot \mathrm{odd}(u2^{a-\min} + v2^{b-\min})$, so d divides the odd part; hence $(x+y)/2 \in M_d$.

So each M_d is a valid solution.

Conversely, let M be a nonempty subset satisfying (a) and (b). We prove $M = M_d$ for a suitable odd d.

1) M contains odd numbers. Let t be the minimal v_2 among elements of M, and pick $x \in M$ with $v_2(x) = t$. If $t = 0$, we are done. If $t \geq 1$, then x and $2x$ are both in M and even, so $(x + 2x)/2 = 3x/2 \in M$ and $v_2(3x/2) = t - 1$. Iterating reduces v_2 until we obtain an odd element of M.

Let d be the smallest odd element of M.

2) From d we can generate all odd multiples of d. Let O be the set of odd numbers in M. We show by induction on odd $m \geq 1$ that $dm \in O$:

- Base: $m = 1$ gives $d \in O$.
- Step: Write $m = 1 + 2^t m'$ with m' odd and $m' < m$. By induction, $dm' \in O$. Using (a), $2d, 2^{t+1}dm' \in M$. They are even, so by (b),

$$\frac{2d + 2^{t+1}dm'}{2} = d + 2^t dm' = dm \in M,$$

and since it is odd, $dm \in O$.

Thus O contains all odd multiples of d. Using (a), M contains all 2^k times these, i.e., $M \supseteq M_d$.

3) No element outside M_d can lie in M. Suppose $x \in M$ with $\mathrm{odd}(x)$ not divisible by d. Let $u = \mathrm{odd}(x)$, and let $k = v_2(x)$. As in step 1's reduction trick but "shifted" by d, define a sequence

$$s_0 = x, \quad \text{and for } j = 0, 1, \ldots, k \text{ set } s_{j+1} = \frac{s_j + 2^{k-j+1}d}{2}.$$

A direct computation shows $s_{j+1} = 2^{k-j-1}(u + 2^{j+1}d)$, hence $s_k = u + 2kd$, which is an odd element of M. In particular, M contains an odd element $w \equiv u \pmod{d}$, hence w is not divisible by d.

Now consider the odd sum $d + w$. Let $t = v_2(d + w) \geq 1$ (since both are odd). By repeatedly applying (b) to suitable even multiples (which is allowed by (a)), we can divide out all 2's from $d + w$ and obtain the odd number

$$r = \frac{d + w}{2^t} \in M.$$

Note that $\gcd(d, w) = \gcd(d, r)$, and since $d \nmid w$, we have $\gcd(d, w) < d$, hence $r < d$. This contradicts the minimality of d in the set of odd elements of M.

Therefore every $x \in M$ must satisfy $d \mid \mathrm{odd}(x)$, i.e., $M \subseteq M_d$.

Combining 2) and 3), we get $M = M_d$.

Conclusion. The nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ that satisfy (a) and (b) are exactly the sets

$$M_d = \{n \in \mathbb{Z}_{>0} : d \text{ divides the odd part of } n\}$$

for some odd integer $d \geq 1$.

Final answer. For each odd $d \geq 1$, the set $M_d = \{n \in \mathbb{Z}_{>0} : d \mid \mathrm{odd}(n)\}$ is a solution, and these are all the solutions.

7.2 Variant 2

Let $M \subseteq \mathbb{Z}_{>0}$ be nonempty and satisfy:

(a) $x \in M \Rightarrow 2x \in M$.

(b) If $x, y \in M$ and $x + y$ is even, then $\frac{x+y}{2} \in M$.

Step 1. M is closed under addition.

Indeed, for any $a, b \in M$, we have $2a, 2b \in M$ by (a). Then $2a + 2b$ is even and $\frac{2a+2b}{2} = a + b \in M$ by (b).

Step 2. M contains an odd element.

Let $t = \min\{v_2(x) : x \in M\}$, where v_2 is the 2-adic valuation. If $t \geq 1$, pick $x \in M$ with $v_2(x) = t$. Then $2x \in M$ and $x + 2x$ is even, so $\frac{x+2x}{2} = \frac{3x}{2} \in M$ by (b). But $v_2\left(\frac{3x}{2}\right) = t - 1$, contradicting the minimality of t. Hence $t = 0$, so M contains an odd element.

Let r be the least odd element of M.

Step 3. $r\mathbb{Z}_{>0} \subseteq M$.

By Step 1, M is closed under addition; hence all multiples nr ($n \in \mathbb{Z}_{>0}$) lie in M.

Step 4. Every element of M is a multiple of r.

Suppose, toward a contradiction, that there exists $y \in M$ not divisible by r. By Step 2 we may assume y is odd (if y is even, combine it with sufficiently large powers of 2 times r using (b) to reduce its 2-adic valuation until an odd element is obtained; membership in M is preserved at each step).

Write $y = qr + a$ with $0 < a < r$ and a even (since y and r are odd).

Define a sequence $(y_0, y_1, y_2, \ldots)$ in M by $y_0 := y$ and, for $n \geq 0$,

$$y_{n+1} := \frac{y_n + r}{2} \text{ if } y_n \text{ is odd,} \tag{32}$$

$$y_{n+1} := \frac{y_n + 2r}{2} \text{ if } y_n \text{ is even.} \tag{33}$$

Each step is valid by (b) (pairing y_n with r when y_n is odd, and with $2r$ when y_n is even).

A simple induction shows that

$$y_n \equiv 2^{-n} y \pmod r,$$

where 2^{-n} is the inverse of 2^n modulo r (valid since r is odd). In particular, none of the residues $y_n \pmod r$ is 0, because y is not divisible by r. Moreover, for n large enough, $y_n < 2r$ (since each step roughly halves the current term, up to an additive $O(r)$). Once $y_n < 2r$, its residue modulo r equals either y_n (if $y_n < r$) or $y_n - r$ (if $y_n > r$).

Because the residue $y_n \pmod r$ never vanishes and takes only values in $\{1, 2, \ldots, r-1\}$, eventually some y_m is odd and less than r (this uses that the orbit under division by 2 modulo r contains both even and odd residues, and the sequence comes below $2r$). But then y_m is an odd element of M strictly smaller than r, contradicting the minimality of r.

Therefore no such y exists; every element of M is divisible by r. Hence $M \subseteq r\mathbb{Z}_{>0}$.

Combining with Step 3 we conclude $M = r\mathbb{Z}_{>0}$.

Step 5. Verification.

For any odd r, the set $M = r\mathbb{Z}_{>0}$ satisfies (a) and (b): doubling preserves multiples of r, and if $x = ra$, $y = rb$ have the same parity, then $\frac{x+y}{2} = r\frac{a+b}{2}$ is a multiple of r because a and b have the same parity.

Therefore, the complete list of solutions is exactly the sets of all positive multiples of an odd integer.

Final answer. All such subsets are $M = r\mathbb{Z}_{>0}$ for some odd integer $r \geq 1$.

7.3 Variant 3

Let $v_2(n)$ be the exponent of 2 in n, and write $n = 2^{v_2(n)} \cdot n_{\text{odd}}$ with n_{odd} odd.

Step 1. Reduce to the "odd gcd $= 1$" case.

Let $d := \gcd\{x_{\text{odd}} : x \in M\}$, i.e., the gcd of the odd parts of the elements of M. Then d is odd, and every element of M is divisible by d. Define

$$S := \{x/d : x \in M\} \subseteq \mathbb{Z}_{>0}.$$

Because d is odd, S also satisfies:

(a) $s \in S \Rightarrow 2s \in S$,

(b) $s, t \in S$ and $s + t$ even $\Rightarrow (s + t)/2 \in S$.

Moreover, the gcd of the odd parts of S is 1.

Thus it suffices to describe all such S; then $M = d \cdot S$.

Step 2. S contains an odd number.

Let $m := \min S$. If m is odd we are done. If m is even, define $m_1 := (m + 2m)/2 = 3m/2$. Since m is even, $m_1 \in S$ and $v_2(m_1) = v_2(m) - 1$. Iterating, after finitely many steps we obtain an odd element $a \in S$.

Step 3. From an odd $a \in S$, we can "add" odd parts.

Let $a \in S$ be odd, and let $t \in S$ be arbitrary. Put $y_0 := 2t \in S$ (even) and define

$$y_{k+1} := \frac{2a + y_k}{2} \quad \text{for } k \geq 0.$$

Each $y_k \in S$, and a simple induction shows $y_k = a + y_0/2^k$. If we take $k = v_2(y_0) = v_2(t)+1$, then $y_k = a + \text{oddpart}(y_0) = a + \text{oddpart}(t)$. Hence:

For every $t \in S$, we have $a + \text{oddpart}(t) \in S$.

Step 4. S contains all sufficiently large integers.

Let R be a finite subset of S such that the gcd of $\{\text{oddpart}(r) : r \in R\}$ is 1 (possible since the gcd of odd parts in S is 1). By Step 3, starting from a we can add any oddpart(r) with $r \in R$, and thus any nonnegative $\mathbb{Z}$-linear combination of those odd parts. Since the numerical semigroup generated by a finite set of positive integers with gcd 1 contains all sufficiently large integers, it follows that all sufficiently large odd integers belong to S. Then any sufficiently large even integer is the average of two consecutive large odd integers, so all sufficiently large integers lie in S. Therefore there exists L_0 with $[L_0, \infty) \subseteq S$.

Step 5. S is actually an upper ray from its minimum.

Let $m = \min S$. Fix $n \geq m$. Choose k large enough so that

$$y := m + 2^k(n - m) \geq L_0.$$

Then $y \in S$ (by Step 4), and $y \equiv m \pmod{2^k}$. Applying (b) k times to the pair (m, y) (which always has the same parity) gives

$$m, y \Rightarrow (m + y)/2, \text{ then } (m + (m + y)/2)/2, \ldots, \text{ finally } m + (y - m)/2^k = n \in S.$$

Hence every $n \geq m$ lies in S, i.e., $S = [m, \infty) \cap \mathbb{Z}$.

Step 6. Scale back.

Recall $M = d \cdot S$ with d odd. Writing $m' := \min S$, we have

$$M = \{d \cdot s : s \geq m'\} = \{n \in \mathbb{Z}_{>0} : n \geq dm' \text{ and } d \mid n\}.$$

Thus M is exactly the set of all multiples of an odd integer d, from some point onwards (with minimal element dm').

Verification (sufficiency).
Given odd $d \geq 1$ and $r \geq 1$, define

$$M = \{dn : n \geq r\}.$$

Then:

- If $x = dn \in M$, $2x = d(2n) \in M$.
- If $x = da, y = db \in M$ and $x+y$ even, then $(x+y)/2 = d((a+b)/2)$ with $(a+b)/2 \geq r$; hence in M.

Therefore, the solutions are precisely these sets.
Final answer.
Exactly the sets

$$M = \{n \in \mathbb{Z}_{>0} : n \equiv 0 \pmod{d} \text{ and } n \geq m\},$$

where d is an odd positive integer and m is a positive multiple of d. Equivalently, for some odd d and some $r \geq 1$,

$$M = \{d \cdot n : n \geq r\}.$$

7.4 Final Solution

Let $v_2(n)$ be the exponent of 2 in n, and write $n = 2^{v_2(n)} \cdot \operatorname{odd}(n)$ with $\operatorname{odd}(n)$ odd.

Claim (closure under addition). From (a) and (b), M is closed under addition: if $a, b \in M$ then $2a, 2b \in M$, hence $(2a + 2b)/2 = a + b \in M$.

Claim (existence of an odd element). Let t be the minimum of $v_2(x)$ over $x \in M$, and pick x with $v_2(x) = t$. If $t \geq 1$, then $x, 2x \in M$ and $(x + 2x)/2 = 3x/2 \in M$ has v_2 decreased by 1. Repeating, we reach an odd element of M. Thus M contains an odd element.

Now let d be the gcd of the odd parts $\operatorname{odd}(x)$ of elements $x \in M$. Then d is odd, and $d \mid \operatorname{odd}(x)$ for all $x \in M$, hence $d \mid x$ (because d is odd). Thus $M \subseteq d \cdot \mathbb{Z}_{>0}$.

Scale down by d: define $S = \{x/d : x \in M\} \subseteq \mathbb{Z}_{>0}$. Then S is nonempty and satisfies (a) and (b), and the gcd of the odd parts in S equals 1. We will show that S is an upper ray: there exists m such that $S = \{n \in \mathbb{Z}_{>0} : n \geq m\}$.

1) S contains an odd element (by the same v_2-reduction argument). Let o be the least odd element of S. Because the gcd of odd parts in S is 1, the gcd of all odd elements of S is 1. Therefore, we can choose finitely many odd elements $a_1 = o, a_2, \ldots, a_k \in S$ with $\gcd(a_1, \ldots, a_k) = 1$.

2) S contains all sufficiently large integers. For each residue $r \in \{0, 1, \ldots, a_1 - 1\}$, because the classes of $a_2, \ldots, a_k$ generate $\mathbb{Z}/a_1\mathbb{Z}$ (gcd is 1), there exist coefficients $t_2, \ldots, t_k$ with $0 \leq t_i < a_1$ such that

$$t_2 a_2 + \cdots + t_k a_k \equiv r \pmod{a_1}. \tag{34}$$

Set $m_r = t_2 a_2 + \cdots + t_k a_k \in S$. Let $L = \max_r m_r$. Then for any $n \geq L$ with $n \equiv r \pmod{a_1}$,

$$n = m_r + q a_1 \quad (q \geq 0), \tag{35}$$

and since S is closed under addition and contains a_1, we have $n \in S$. Hence $[L, \infty) \subseteq S$.

3) S is actually the full upper ray from its minimum. Let $m = \min S$. Fix any $n \geq m$. Choose k large enough so that $y := m + 2^k(n - m) \geq L$. Then $y \in S$ by step 2, and $y \equiv m \pmod{2^k}$. Applying (b) repeatedly to the pair (m, y): set

$$z_1 = \frac{m + y}{2}, \quad z_2 = \frac{m + z_1}{2}, \quad \ldots, \quad z_k = m + \frac{y - m}{2^k} = n. \tag{36}$$

At each step the two arguments have the same parity (since $y \equiv m \bmod 2^k$), so each averaging is allowed. Thus $n \in S$. Therefore $S = \{n \in \mathbb{Z}_{>0} : n \geq m\}$.

Finally, scaling back, $M = d \cdot S = \{dn : n \geq m\}$. Writing $r = m$, we obtain

$$M = \{n \in \mathbb{Z}_{>0} : n \equiv 0 \pmod{d} \text{ and } n \geq dr\}, \tag{37}$$

with d an odd integer and $r \geq 1$.

Verification (sufficiency). For odd d and $r \geq 1$, the set $M = \{dn : n \geq r\}$ satisfies:

(a) If $x = dn \in M$ then $2x = d(2n) \in M$.

(b) If $x = da, y = db \in M$ and $x + y$ is even, then a and b have the same parity, so $(x + y)/2 = d((a + b)/2)$ with $(a + b)/2 \geq r$, hence in M.

Therefore these are exactly the desired subsets.

Final answer. The nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ satisfying (a) and (b) are precisely

$$M = \{dn : n \geq r\}, \tag{38}$$

for some odd integer $d \geq 1$ and some integer $r \geq 1$.

8 Problem 8

8.1 Variant 1

Let $A = (a_{ij}) \in M_n(\mathbb{R})$. The 90 counterclockwise rotation satisfies $(A^R)_{ij} = a_{j,n+1-i}$.

Assume $A = A^R$. Then for all i, j,

$$a_{ij} = a_{j,n+1-i}. \tag{39}$$

Applying the rotation twice gives the 180 rotation, whose effect on entries is $((A^R)^R)_{ij} = a_{n+1-i,n+1-j}$. Since $A = (A^R)^R$ as well, we also have

$$a_{ij} = a_{n+1-i,n+1-j}. \tag{40}$$

We claim A is symmetric, i.e. $a_{ij} = a_{ji}$. Using (2) with swapped indices gives

$$a_{ij} = a_{n+1-j,n+1-i}.$$

Applying (1) to the pair $(n + 1 - j, n + 1 - i)$ yields

$$a_{n+1-j,n+1-i} = a_{n+1-i,j}.$$

Thus $a_{ij} = a_{n+1-i,j}$. Applying (1) again to the pair $(n + 1 - i, j)$ gives

$$a_{n+1-i,j} = a_{j,i}.$$

Therefore $a_{ij} = a_{ji}$, proving A is symmetric.

Since A is a real symmetric matrix, it is Hermitian, and hence all its eigenvalues are real. In particular, for any eigenvalue λ of A, $\text{Im } \lambda = 0$ (and thus $\text{Re } \lambda = 0$ or $\text{Im } \lambda = 0$ holds).

Final answer: All eigenvalues of A are real; in particular, $\text{Im } \lambda = 0$ for every eigenvalue λ.

8.2 Variant 2

Let J be the $n \times n$ reversal (anti-identity) matrix, $J = [\delta_{i,n+1-j}]$. For any $X \in M_n(\mathbb{R})$ one checks directly $(X^R)_{ij} = x_{j,n+1-i}$, hence $X^R = JX^T$.

Thus $A = A^R$ is equivalent to

$$A = JA^T. \tag{1}$$

From (1), taking transpose gives $A^T = AJ$, and multiplying by J on the left yields $JA^T = JAJ$. Using (1) again, $JA^T = A$, so

$$JAJ = A, \text{ i.e. } A \text{ commutes with } J. \tag{2}$$

Since J is real symmetric with eigenvalues ± 1, there exists an orthogonal Q such that $Q^T JQ = D := \operatorname{diag}(I_r, -I_s)$, with $r + s = n$. Let $M := Q^T AQ$. Then (2) implies $DMD = M$, so M commutes with D. Writing M in 2×2 block form conformal with D,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

the commutation $DM = MD$ forces $M_{12} = 0$ and $M_{21} = 0$, i.e. $M = \operatorname{diag}(B, C)$ for some real B ($r \times r$) and C ($s \times s$).

Now transform (1) into this basis: $M = Q^T AQ = Q^T JA^T Q = (Q^T JQ)(Q^T AQ)^T = DM^T$. With $M = \operatorname{diag}(B, C)$ and $D = \operatorname{diag}(I_r, -I_s)$, this becomes $\operatorname{diag}(B, C) = \operatorname{diag}(B^T, -C^T)$, so $B = B^T$ (symmetric) and $C = -C^T$ (skew-symmetric).

Therefore A is orthogonally similar to a block-diagonal matrix with one real symmetric block and one real skew-symmetric block. The eigenvalues of a real symmetric matrix are real, while the eigenvalues of a real skew-symmetric matrix are purely imaginary (including possibly 0). Hence every eigenvalue λ of A is either real or purely imaginary.

Final answer. For any A with $A = A^R$, its spectrum is the disjoint union of the spectra of a real symmetric and a real skew-symmetric matrix; thus each eigenvalue λ satisfies either $\operatorname{Re} \lambda = 0$ or $\operatorname{Im} \lambda = 0$.

8.3 Variant 3

Let J denote the $n \times n$ reversal matrix (ones on the anti-diagonal, zeros elsewhere). Then for any A we have

$$A^R = JA^T.$$

Indeed, $(JA^T)_{i,j} = A_{j,n+1-i}$, which is exactly the 90 counterclockwise rotation of entries.

Hence the condition $A = A^R$ is equivalent to

$$A = JA^T.$$

From this,

- multiplying on the left by J gives $JA = A^T$,
- taking transpose gives $A^T = AJ$.

Combining, we get $JA = AJ$; thus A commutes with J.

Since J is real symmetric with $J^2 = I$, the spectral theorem yields an orthogonal matrix Q such that

$$Q^T JQ = \operatorname{diag}(I_p, -I_q) =: D,$$

where $p = \lceil n/2 \rceil$ and $q = \lfloor n/2 \rfloor$. Set

$$S := Q^T AQ.$$

Because A commutes with J, S commutes with D:

$$DS = SD.$$

Writing S in $p \times p$, $p \times q$, $q \times p$, $q \times q$ blocks as $S = \begin{bmatrix} B & F \\ G & C \end{bmatrix}$, the commutation $DS = SD$ reads

$$\begin{bmatrix} B & F \\ -G & -C \end{bmatrix} = \begin{bmatrix} B & -F \\ G & -C \end{bmatrix},$$

so $F = 0$ and $G = 0$. Therefore S is block diagonal:

$$S = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}.$$

Now translate $A = JA^T$ into this basis:

$$S = Q^T AQ = Q^T JA^T Q = (Q^T JQ)(Q^T AQ)^T = DS^T.$$

Thus

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} B^T & 0 \\ 0 & -C^T \end{bmatrix},$$

so B is symmetric ($B = B^T$) and C is skew-symmetric ($C^T = -C$).

Consequently, all eigenvalues of B are real (B real symmetric), and all eigenvalues of C are purely imaginary (C real skew-symmetric). Hence all eigenvalues of S, and thus of A (being orthogonally similar to S), are either real or purely imaginary.

Final answer: Every eigenvalue λ of A is real or purely imaginary; i.e., $\operatorname{Re} \lambda = 0$ or $\operatorname{Im} \lambda = 0$.

8.4 Final Solution

Let J be the $n \times n$ reversal (anti-identity) matrix, with ones on the anti-diagonal and zeros elsewhere. A direct entrywise check shows

$$(A^R)_{ij} = a_{j,n+1-i} = (JA^T)_{ij},$$

so $A^R = JA^T$. Hence the condition $A = A^R$ is equivalent to

$$A = JA^T. \quad (1)$$

Transposing (1) gives $A^T = AJ$ (since $J^T = J$). Left-multiplying (1) by J and using $J^2 = I$ yields

$$JA^T = A \text{ and thus } A = JAJ,$$

so A commutes with J:

$$AJ = JA. \quad (2)$$

Because J is real symmetric with $J^2 = I$, there exists an orthogonal Q such that

$$Q^T JQ = D := \operatorname{diag}(I_r, -I_s), \text{ with } r + s = n.$$

Set $M := Q^T AQ$. From (2) we get $DM = MD$, so in a block form compatible with D we have

$$M = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}.$$

Now rewrite (1) in this basis:

$$M = Q^T AQ = Q^T JA^T Q = (Q^T JQ)(Q^T AQ)^T = DM^T.$$

Therefore

$$\operatorname{diag}(B, C) = \operatorname{diag}(B^T, -C^T),$$

so $B = B^T$ (symmetric) and $C = -C^T$ (skew-symmetric).

Hence A is orthogonally similar to a block-diagonal matrix with one real symmetric block and one real skew-symmetric block. The eigenvalues of a real symmetric matrix are real, and those of a real skew-symmetric matrix are purely imaginary (possibly 0). Therefore every eigenvalue λ of A is either real or purely imaginary.

Final answer. For any A with $A = A^R$, each eigenvalue λ satisfies $\operatorname{Re} \lambda = 0$ or $\operatorname{Im} \lambda = 0$.

9 Problem 9

9.1 Variant 1

Let R_j be the rank of X_j among the remaining positive integers at step j (i.e., $R_j = 1$ means we pick the smallest remaining number, $R_j = 2$ the second smallest, etc.). By the rule of the process, for every step j and every $i \geq 1$ we have

$$P(R_j = i) = 2^{-i},$$

independently of the past. Hence the R_j are i.i.d. with this distribution.

1) Distribution of Y_n

For $m \geq 0$, the event $Y_n \leq m$ means that all n chosen numbers lie in $\{1, \ldots, m\}$. Equivalently, at step j ($1 \leq j \leq n$) the rank R_j must be at most the number of remaining elements in $\{1, \ldots, m\}$, which equals $m - (j - 1)$. Thus, for $m \geq n - 1$,

$$P(Y_n \leq m) = \prod_{j=1}^{n} P(R_j \leq m - j + 1) = \prod_{j=1}^{n} (1 - 2^{-(m-j+1)}).$$

For $m \leq n-1$, this probability is 0, as one cannot fit n distinct numbers into a set of size m.

2) Tail-sum for the expectation

Using $E[Y_n] = \sum_{m \geq 1} P(Y_n \geq m)$, and noting $P(Y_n \geq m) = 1$ for $m \leq n$, we get

$$E[Y_n] = n + \sum_{m \geq n+1} \left[1 - \prod_{j=1}^{n} (1 - 2^{-(m-j)}) \right].$$

Reindex with $t = m - n \geq 1$ and set $q = 1/2$ to write

$$E[Y_n] = n + \sum_{t \geq 1} \left[1 - \prod_{k=0}^{n-1} (1 - q^{t+k}) \right].$$

3) A q-identity
Define for $0 < q < 1$

$$S_n(q) := \sum_{t\geq 1}\left[1 - \prod_{k=0}^{n-1}(1-q^{t+k})\right].$$

We claim $S_n(q) = \sum_{i=1}^{n} \frac{q^i}{1-q^i}$. This can be proved by induction on n by showing

$$S_n(q) - S_{n-1}(q) = \frac{q^n}{1-q^n}.$$

Indeed,

$$S_n - S_{n-1} = \sum_{t\geq 1} \prod_{k=0}^{n-2}(1-q^{t+k}) - \prod_{k=0}^{n-1}(1-q^{t+k}) \tag{41}$$

$$= \sum_{t\geq 1} q^{t+n-1} \prod_{k=0}^{n-2}(1-q^{t+k}). \tag{42}$$

Let $A_t := \prod_{k=0}^{n-1}(1-q^{t+k})$. A simple difference identity gives

$$A_t - A_{t-1} = q^{t-1}(1-q^n)\prod_{k=0}^{n-2}(1-q^{t+k}),$$

hence

$$q^t \prod_{k=0}^{n-2}(1-q^{t+k}) = \frac{q[A_t - A_{t-1}]}{1-q^n}.$$

Summing over $t \geq 1$ and using telescoping ($A_0 = 0$, $\lim_{t\to\infty} A_t = 1$) yields

$$\sum_{t\geq 1} q^t \prod_{k=0}^{n-2}(1-q^{t+k}) = \frac{q}{1-q^n},$$

and therefore $S_n - S_{n-1} = \frac{q^n}{1-q^n}$. Since $S_1(q) = \frac{q}{1-q}$, the induction gives

$$S_n(q) = \sum_{i=1}^{n} \frac{q^i}{1-q^i}.$$

4) Conclusion
With $q = 1/2$ we obtain

$$E[Y_n] = n + \sum_{i=1}^{n} \frac{2^{-i}}{1-2^{-i}} = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}.$$

Final answer.

$$E[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}.$$

9.2 Variant 2

Let $Z_1 < Z_2 < \ldots < Z_n$ be the selected numbers in increasing order (the order statistics of $X_1, \ldots, X_n$). Set $Z_0 := 0$ and define the gaps $G_j := Z_j - Z_{j-1}$ for $j = 1, \ldots, n$. Then $Y_n = Z_n = G_1 + \cdots + G_n$.

Key observation (block-avoidance probability). Fix $s \geq 1$. At any time, if none of the first s remaining integers has been chosen yet, then the probability that the next pick avoids this block equals

$$\sum_{i \geq s+1} 2^{-i} = 2^{-s}.$$

Moreover, as long as the block remains untouched, this avoidance probability stays 2^{-s} at every step. Hence, for r upcoming picks, the probability that all r picks avoid this block is $(2^{-s})^r$.

Distribution of the first gap G_1. For $k \geq 1$, the event $\{G_1 \geq k\}$ means that none of the first $k-1$ positive integers is chosen among the n selections. By the observation with $s = k-1$ and $r = n$, we get

$$P(G_1 \geq k) = (2^{-(k-1)})^n = 2^{-n(k-1)}.$$

Thus G_1 is geometric (on $\{1, 2, \ldots\}$) with parameter $p = 1 - 2^{-n}$, i.e.

$$P(G_1 = k) = (1 - 2^{-n}) 2^{-n(k-1)}, \quad \text{and} \quad E[G_1] = \frac{1}{p} = \frac{2^n}{2^n - 1}.$$

Distribution of general gaps G_j. Condition on $Z_1, \ldots, Z_{j-1}$. There remain $r = n - j + 1$ selections to be made from the remaining integers, and consider the block of the next s integers after Z_{j-1}, i.e., $\{Z_{j-1} + 1, \ldots, Z_{j-1} + s\}$. The event $\{G_j \geq s + 1\}$ is exactly that none of these r remaining picks hits this block. By the same observation,

$$P(G_j \geq s + 1 \mid Z_1, \ldots, Z_{j-1}) = (2^{-s})^r = 2^{-s(n-j+1)}.$$

Therefore, unconditionally,

$$P(G_j \geq t) = 2^{-(t-1)(n-j+1)} \quad \text{for } t = 1, 2, \ldots,$$

so G_j is geometric with parameter $p_j = 1 - 2^{-(n-j+1)}$ and

$$E[G_j] = \frac{1}{p_j} = \frac{2^{n-j+1}}{2^{n-j+1} - 1}.$$

Taking expectations and summing:

$$E[Y_n] = E[Z_n] = \sum_{j=1}^{n} E[G_j] \tag{43}$$

$$= \sum_{j=1}^{n} \frac{2^{n-j+1}}{2^{n-j+1} - 1} \tag{44}$$

$$= \sum_{i=1}^{n} \frac{2^i}{2^i - 1}, \tag{45}$$

as claimed.

Final answer.

$$E[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}.$$

9.3 Variant 3

Let us realize the selection rule as follows: at each step, list the remaining integers in increasing order and, independently, flip a fair coin for each (in that order) until the first Head appears; choose that integer. This gives exactly

$$P(\text{“choose the } i\text{-th smallest remaining”}) = \left(\frac{1}{2}\right)^i,$$

as required.

1) Tail probabilities of the maximum.

For $k \geq 1$, the event $\{Y_n < k\}$ means that in each of the n stages we select from the remaining numbers less than k. If before stage j we have selected only numbers $< k$, then there remain exactly $k - j$ numbers less than k, which occupy the first $k - j$ positions in the ordered remaining list. Thus the probability that at stage j we again choose a number $< k$ is

$$\sum_{i=1}^{k-j} 2^{-i} = 1 - 2^{-(k-j)}.$$

Therefore, for $k \geq n + 1$,

$$P(Y_n < k) = \prod_{j=1}^{n}(1 - 2^{-(k-j)}) = \prod_{r=k-n}^{k-1}(1 - 2^{-r}),$$

and for $k \leq n$ we have $P(Y_n < k) = 0$.

Hence

$$E[Y_n] = \sum_{k=1}^{\infty} P(Y_n \geq k) = \sum_{k=1}^{n} 1 + \sum_{k=n+1}^{\infty}\left(1 - \prod_{r=k-n}^{k-1}(1 - 2^{-r})\right).$$

2) Increment of the expectation.

Consider the difference

$$E[Y_n] - E[Y_{n-1}] = \sum_{k=1}^{\infty}(P(Y_n \geq k) - P(Y_{n-1} \geq k)) = \sum_{k=1}^{\infty}(P(Y_{n-1} < k) - P(Y_n < k)).$$

For $k \geq n$,

$$P(Y_{n-1} < k) = \prod_{r=k-n+1}^{k-1}(1 - 2^{-r}), \text{ and} \tag{46}$$

$$P(Y_n < k) = (1 - 2^{-(k-n)}) \prod_{r=k-n+1}^{k-1}(1 - 2^{-r}), \tag{47}$$

so

$$P(Y_{n-1} < k) - P(Y_n < k) = 2^{-(k-n)} \prod_{r=k-n+1}^{k-1}(1 - 2^{-r}).$$

Let $m = k - n \geq 0$. Then

$$E[Y_n] - E[Y_{n-1}] = \sum_{m=0}^{\infty} 2^{-m} \prod_{r=m+1}^{m+n-1}(1 - 2^{-r}).$$

Define $Q_m = \prod_{r=m+1}^{m+n}(1-2^{-r})$ (with $Q_{-1}=0$). A direct computation gives the telescoping identity

$$Q_m - Q_{m-1} = (1-2^{-n})2^{-m} \prod_{r=m+1}^{m+n-1}(1-2^{-r}).$$

Therefore

$$E[Y_n] - E[Y_{n-1}] = \frac{1}{1-2^{-n}} \sum_{m=0}^{\infty}(Q_m - Q_{m-1}) \tag{48}$$

$$= \frac{1}{1-2^{-n}} \left(\lim_{M\to\infty} Q_M - Q_{-1}\right). \tag{49}$$

Since $Q_M = \prod_{r=M+1}^{M+n}(1-2^{-r}) \to 1$ as $M\to\infty$, we conclude

$$E[Y_n] - E[Y_{n-1}] = \frac{1}{1-2^{-n}} = \frac{2^n}{2^n-1}.$$

Finally, with $E[Y_0]=0$, summing these increments yields

$$E[Y_n] = \sum_{i=1}^{n}(E[Y_i] - E[Y_{i-1}]) = \sum_{i=1}^{n} \frac{2^i}{2^i-1}.$$

Final answer.

$$E[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i-1}.$$

9.4 Final Solution

Solution.

Realize the selection rule by ranks: at each step, list the remaining integers in increasing order and pick the i-th smallest remaining with probability 2^{-i}. Equivalently, one can flip a fair coin for each remaining integer in order until the first Head appears; the chosen integer is exactly the i-th remaining when the first Head occurs at flip i. Thus, at each step and independently of the past, the event "we avoid the first s remaining integers" has probability

$$\sum_{i\geq s+1} 2^{-i} = 2^{-s}.$$

Let $Z_1 < Z_2 < \ldots < Z_n$ be the selected numbers in increasing order, set $Z_0 = 0$, and define the gaps $G_j := Z_j - Z_{j-1}$ ($j=1,\ldots,n$). Then $Y_n = Z_n = G_1 + \ldots + G_n$.

Fix j and condition on $Z_1,\ldots,Z_{j-1}$. There remain $r = n-j+1$ selections. The event $\{G_j \geq t\}$ means that none of the r remaining picks hits the block of the first $t-1$ integers after Z_{j-1}. As long as this block is untouched, at each of those r steps the probability to avoid it equals $2^{-(t-1)}$, and these avoidance events are independent across steps. Hence

$$P(G_j \geq t \mid Z_1,\ldots,Z_{j-1}) = (2^{-(t-1)})^r = 2^{-(t-1)(n-j+1)}.$$

This does not depend on the conditioning, so unconditionally

$$P(G_j \geq t) = 2^{-(t-1)(n-j+1)}, \quad t = 1,2,\ldots,$$

i.e. G_j is geometric on $\{1, 2, \ldots\}$ with parameter $p_j = 1 - 2^{-(n-j+1)}$. Therefore

$$E[G_j] = \frac{1}{p_j} = \frac{2^{n-j+1}}{2^{n-j+1} - 1}.$$

Summing expectations,

$$E[Y_n] = E[Z_n] = \sum_{j=1}^{n} E[G_j] \tag{50}$$

$$= \sum_{j=1}^{n} \frac{2^{n-j+1}}{2^{n-j+1} - 1} \tag{51}$$

$$= \sum_{i=1}^{n} \frac{2^i}{2^i - 1}. \tag{52}$$

Final answer.

$$E[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}.$$

10 Problem 10

10.1 Variant 1

Let $t(n) = n(n+1)$. For a prime p, denote by $v_p(n)$ the p-adic valuation. Because n and $n+1$ are coprime, $v_p(t(n)) = v_p(n) + v_p(n+1)$ equals the valuation of whichever of $n, n+1$ is divisible by p. Let $E_p(n)$ be the parity $\pmod 2$ of $v_p(t(n))$. Then for fixed p, $E_p(n) = 1$ if and only if v_p of the divisible one is odd.

A local computation (using the geometric distribution of valuations) gives, for every prime p,

$$P(E_p(n) = 1) = 2 \sum_{r \text{ odd} \geq 1} \left(\frac{1}{p^r} - \frac{1}{p^{r+1}} \right) = \frac{2}{p+1}.$$

Thus $P(E_p(n) = 0) = \frac{p-1}{p+1}$.

For two independent integers a, b (uniformly in $[1, N]$), the condition that $t(a)t(b)$ is a perfect square is equivalent to $E_p(a) = E_p(b)$ for every prime p, and for large N it suffices to check primes $p \leq N+1$ (since no $p > N+1$ divides $t(a)$ or $t(b)$). For a fixed p, the probability of a match is

$$M_p := P(E_p(a) = E_p(b)) = P(1,1) + P(0,0) = \left[\frac{2}{p+1} \right]^2 + \left[\frac{p-1}{p+1} \right]^2 \tag{53}$$

$$= 1 - \frac{4(p-1)}{(p+1)^2} = 1 - \frac{4}{p} + O\left(\frac{1}{p^2} \right). \tag{54}$$

We will extract a lower bound for the proportion of matching pairs using a truncation at a parameter y with $2 \leq y \leq N$. Define the set

$$A_y := \{ n \leq N : E_p(n) = 0 \text{ for all primes } p \text{ with } y < p \leq N+1 \}.$$

By independence of local conditions across distinct primes (via the Chinese Remainder Theorem and standard density arguments), we have

$$|A_y| = N \prod_{y<p\leq N+1} P(E_p(n)=0) + o(N) \tag{55}$$

$$= N \prod_{y<p\leq N+1} \left(1 - \frac{2}{p+1}\right) + o(N). \tag{56}$$

Moreover, among pairs $(a,b) \in A_y \times A_y$, the condition $t(a)t(b)$ is a square is already guaranteed at all primes $p > y$ (by the definition of A_y), and for primes $p \leq y$ the matching probability is $\prod_{p\leq y} M_p$. Therefore, the number of such matching pairs satisfies

$$S_N \geq |A_y|^2 \prod_{p\leq y} M_p + o(N^2).$$

We now estimate these products using Mertens' theorem for primes. Recall that

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + B + o(1),$$

and hence for fixed α one has

$$\prod_{p\leq x} \left(1 - \frac{\alpha}{p}\right) = \frac{C(\alpha)}{(\log x)^\alpha} \cdot (1 + o(1)),$$

for some positive constant $C(\alpha)$. Since

$$1 - \frac{2}{p+1} = 1 - \frac{2}{p} + O\left(\frac{1}{p^2}\right), \quad \text{and} \quad M_p = 1 - \frac{4}{p} + O\left(\frac{1}{p^2}\right),$$

we obtain

$$\prod_{y<p\leq N+1} \left(1 - \frac{2}{p+1}\right) = C_1 \cdot \left(\frac{\log y}{\log N}\right)^2 \cdot (1 + o(1)), \tag{57}$$

$$\prod_{p\leq y} M_p = C_0 \cdot \frac{1}{(\log y)^4} \cdot (1 + o(1)), \tag{58}$$

for positive constants C_0, C_1.

Plugging these into the lower bound gives

$$S_N \geq \left[N \cdot C_1 \left(\frac{\log y}{\log N}\right)^2\right]^2 \cdot \left[\frac{C_0}{(\log y)^4}\right] \cdot (1 + o(1)) \tag{59}$$

$$= (C_0 C_1^2) \cdot \frac{N^2}{(\log N)^4} \cdot (1 + o(1)). \tag{60}$$

Since y can be any function with $2 \leq y \leq N$ tending to infinity (e.g. $y = \lfloor\sqrt{N}\rfloor$), the constant $C_0 C_1^2$ is positive and independent of N, so for all large N,

$$S_N \geq c \cdot \frac{N^2}{(\log N)^4},$$

for some absolute $c > 0$. Consequently,

$$\frac{S_N}{N} \geq c \cdot \frac{N}{(\log N)^4} \to \infty \quad \text{as } N \to \infty.$$

In particular, the requested limit exists and is infinite.

Final answer: $+\infty$

10.2 Variant 2

Let $s(n)$ denote the squarefree kernel of n (the product of primes that appear to odd exponent in n). For $1 \le a \le N$, set

$$r(a) = s(a(a+1)).$$

Then $(a^2+a)(b^2+b)$ is a perfect square if and only if $s(a(a+1)) = s(b(b+1))$. Hence, if

$$A_r(N) = \#\{1 \le a \le N : r(a) = r\},$$

we have

$$S_N = \sum_r A_r(N)^2 = N + 2\sum_r \binom{A_r(N)}{2}.$$

Therefore it suffices to estimate

$$P_N := \sum_r \binom{A_r(N)}{2},$$

the number of unordered pairs $\{a, b\}$ with $1 \le a < b \le N$ and $r(a) = r(b)$. We will prove $P_N = o(N)$, which implies $S_N = N + o(N)$ and hence $S_N/N \to 1$.

1) Pell-type parameterization

Fix a squarefree integer r. The condition $r(a) = r$ is equivalent to

$$a(a+1) = rt^2$$

for some integer $t \ge 1$. Writing $u = 2a + 1$, this becomes

$$u^2 - 4rt^2 = 1, \quad \text{with } u \text{ odd}, \quad a = \frac{u-1}{2}.$$

Thus for each fixed r, the u that arise are the u-coordinates of the solutions to the Pell equation

$$u^2 - Dt^2 = 1, \quad \text{with } D = 4r \ge 8$$

(since $r \ge 2$; note $a(a+1)$ cannot be a square as a and $a+1$ are coprime).

Let u_1 be the least $u > 1$ (necessarily odd) for which $u^2 - Dt^2 = 1$ has an integer solution. All solutions are given by

$$u_m + t_m\sqrt{D} = (u_1 + t_1\sqrt{D})^m, \quad m = 1, 2, 3, \ldots$$

In particular, the sequence (u_m) satisfies the recurrence $u_{m+1} = 2u_1u_m - u_{m-1}$, and one has the identity

$$u_m = T_m(u_1),$$

where T_m is the Chebyshev polynomial of the first kind. A simple induction using $T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x)$ shows that for $x \ge 1$,

$$T_m(x) \ge x^m \quad \text{for all } m \ge 1.$$

Therefore, for each r, if $A_r(N) \ge m$ (i.e., there are at least m values of $a \le N$ in this class), then

$$u_m \le 2N + 1 \Rightarrow u_1^m \le u_m \le 2N + 1 \Rightarrow u_1 \le (2N+1)^{1/m}.$$

2) Counting r with many solutions

Let $T_m(N)$ be the number of squarefree r for which $A_r(N) \geq m$. From the conclusion above, for each such r there exists an odd $u_1 \leq (2N+1)^{1/m}$ and integers $t_1 \geq 1$ such that

$$4rt_1^2 = u_1^2 - 1.$$

Thus, for a fixed odd u, the admissible r are the squarefree parts of $(u^2-1)/4$. The number of such r is at most the number of squarefree divisors of $(u^2-1)/4$, which is $\leq \tau((u^2-1)/4)$, where τ is the divisor function. Consequently,

$$T_m(N) \leq \sum_{\text{odd } u \leq (2N+1)^{1/m}} \tau\left(\frac{u^2-1}{4}\right).$$

Using the well-known bound $\tau(n) \leq n^{\varepsilon}$ for any $\varepsilon > 0$ and all sufficiently large n, and noting that $(u^2-1)/4 \leq u^2$, we get, for any fixed $\varepsilon > 0$,

$$T_m(N) \ll \sum_{u \leq cN^{1/m}} u^{2\varepsilon} \ll N^{1/m+2\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, we can write succinctly

$$T_m(N) \leq N^{1/m+o(1)}.$$

3) Bounding P_N

Write

$$P_N = \sum_r \binom{A_r(N)}{2} = \sum_{m \geq 2} (m-1) T_m(N).$$

(The identity follows by counting, for each class with size A, the number of pairs as $\binom{A}{2} = \sum_{m=2}^{A}(m-1)$.)

Using the estimate for $T_m(N)$,

$$P_N \leq \sum_{m \geq 2} (m-1) N^{1/m+o(1)} = N^{1/2+o(1)} + N^{1/3+o(1)} + N^{1/4+o(1)} + \ldots = N^{1/2+o(1)}.$$

Therefore $P_N = o(N)$, and hence

$$S_N = N + 2P_N = N + o(N).$$

It follows that the limit exists and equals 1:

$$\lim_{N\to\infty} \frac{S_N}{N} = 1.$$

Final answer: 1.

10.3 Variant 3

Let $r(n)$ denote the squarefree kernel of $n(n+1)$. Since $\gcd(n, n+1) = 1$, $r(n)$ is the product of the primes that divide n or $n+1$ to an odd exponent. Then the condition that $(a^2+a)(b^2+b)$ is a perfect square is equivalent to $r(a) = r(b)$. Hence, if we define

- $c_k(N) := \#\{1 \leq a \leq N : r(a) = k\}$ for squarefree k, then

$$S_N = \sum_k c_k(N)^2.$$

In particular, $S_N \geq \sum_k c_k(N) = N$ (the diagonal pairs $a = b$). We will show that the off-diagonal contribution $S_N - N$ is $o(N)$, which will imply

$$\lim_{N\to\infty} S_N/N = 1.$$

1) Reduction to Pell equations. For a given squarefree k, the condition $r(a) = k$ is equivalent to

$$a(a+1) = kt^2$$

for some integer t. Multiplying by 4 and setting $X = 2a + 1$, $Y = 2t$, we obtain

$$X^2 - kY^2 = 1,$$

with X odd. Conversely, any solution (X, Y) of $X^2 - kY^2 = 1$ has X odd and Y even (for k odd this is forced mod 2; for k even it holds mod 8), so $a = (X-1)/2$ is an integer with $a(a+1) = k(Y/2)^2$. Thus

$$c_k(N) = \#\{\text{solutions } (X, Y) \text{ of } X^2 - kY^2 = 1 \text{ with } 1 \leq X \leq 2N+1\}.$$

Let $\varepsilon_k = x_1 + y_1\sqrt{k} > 1$ be the fundamental unit of the Pell equation $X^2 - kY^2 = 1$. All solutions are given by $x_n + y_n\sqrt{k} = \varepsilon_k^n$ $(n \geq 0)$. In particular, x_n is increasing with n and $x_n \asymp \varepsilon_k^n$. Hence

$$c_k(N) \geq t \implies \varepsilon_k^t \lesssim 2N + 1,$$

more precisely

$$c_k(N) \geq t \Rightarrow \varepsilon_k^t \leq 4N + 1 \Rightarrow \varepsilon_k \leq (4N+1)^{1/t}. \quad (1)$$

2) Bounding the number of k with small ε_k. We use a simple counting lemma.

Lemma. Let $A(X)$ be the number of squarefree k for which the Pell equation $X^2 - kY^2 = 1$ has a solution with $X \leq X$. Then $A(X) \ll X$.

Proof. For each such k, choose one solution (x, y) with $1 \leq x \leq X$. Then $k = (x^2-1)/y^2$ is integer, so $y^2 | x^2 - 1$. For fixed y, the congruence $y^2 | x^2 - 1$ means $x \equiv \pm 1$ (mod y^2), giving at most $2\lceil X/y^2 \rceil$ choices of x in $[1, X]$. Summing over $1 \leq y \leq X$,

$$A(X) \leq \sum_{y=1}^{X} 2\lceil X/y^2 \rceil \ll X \sum_{y=1}^{\infty} 1/y^2 + \sum_{y=1}^{X} 1 \ll X.$$

This proves the lemma.

Combining (1) with the lemma, for $t \geq 1$,

$$A_t(N) := \#\{k : c_k(N) \geq t\} \leq \#\{k : \varepsilon_k \leq (4N+1)^{1/t}\} \ll N^{1/t}. \quad (2)$$

3) Bounding the off-diagonal. Using the identity $m^2 = \sum_{t=1}^{m}(2t-1)$, we have

$$S_N = \sum_k c_k(N)^2 = \sum_{t=1}^{\infty}(2t-1)A_t(N).$$

Since $A_1(N) = \#\{k : c_k(N) \geq 1\} \leq N$, and using (2) for $t \geq 2$,

$$S_N \leq N + C\sum_{t=2}^{\infty}(2t-1)N^{1/t}$$

for some absolute constant C. The series is dominated by its first term $t = 2$, and one easily checks

$$\sum_{t=2}^{\infty}(2t-1)N^{1/t} \ll N^{1/2}.$$

Thus

$$N \leq S_N \leq N + O(N^{1/2}),$$

so $S_N = N + o(N)$, and consequently

$$\lim_{N\to\infty} S_N/N = 1.$$

Final answer: 1

10.4 Final Solution

Let $r(n)$ denote the squarefree kernel of n, i.e., the product of all primes dividing n to an odd power. Since $\gcd(a, a+1) = 1$, the condition that $(a^2 + a)(b^2 + b)$ is a perfect square is equivalent to

$$r(a(a+1)) = r(b(b+1)).$$

Define, for each squarefree r,

$$A_r(N) = \#\{1 \leq a \leq N : r(a(a+1)) = r\}.$$

Then

$$S_N = \sum_r A_r(N)^2 = \sum_r A_r(N) + 2\sum_r \binom{A_r(N)}{2} = N + 2P_N,$$

where P_N counts unordered pairs $\{a, b\}$ with $1 \leq a < b \leq N$ and $r(a(a+1)) = r(b(b+1))$. We will prove $P_N = o(N)$, hence $S_N = N + o(N)$ and $S_N/N \to 1$.

1) Reduction to a Pell equation. For fixed squarefree r, the condition $r(a(a+1)) = r$ is equivalent to

$$a(a+1) = rt^2$$

for some integer $t \geq 1$. Setting $u = 2a + 1$ and noting $4a(a+1) = u^2 - 1$, we get the Pell equation

$$u^2 - 4rt^2 = 1,$$

with u odd and $u = 2a + 1$. Thus $A_r(N)$ equals the number of (integer) solutions (u, t) of $u^2 - 4rt^2 = 1$ with $1 \leq u \leq 2N + 1$ and u odd.

Let $u_1 > 1$ be the smallest u for which $u^2 - 4rt^2 = 1$ has a solution. Then all solutions are given by

$$u_m + t_m\sqrt{4r} = (u_1 + t_1\sqrt{4r})^m, \quad m = 1, 2, \ldots$$

Let $\alpha = u_1 + t_1\sqrt{4r} > 1$. Since $(u_1 + t_1\sqrt{4r})(u_1 - t_1\sqrt{4r}) = 1$, we have $u_m = (\alpha^m + \alpha^{-m})/2 \geq \alpha^m/2$. Hence

$$A_r(N) \geq m \Rightarrow u_m \leq 2N + 1 \Rightarrow \alpha^m \leq 2u_m \leq 4N + 2 \Rightarrow u_1 < \alpha \leq (4N + 2)^{1/m}. \quad (*)$$

2) Bounding how many r have many a. For $m \geq 2$, let

$$T_m(N) := \#\{\text{squarefree } r : A_r(N) \geq m\}.$$

By $(*)$, for each r counted in $T_m(N)$ we have $u_1 \leq (4N+2)^{1/m}$. The map $r \mapsto u_1$ is injective: indeed, from $u_1^2 - 1 = 4rt_1^2$ one sees r is the squarefree kernel of $(u_1^2-1)/4$, uniquely determined by u_1. Therefore

$$T_m(N) \leq \#\{\text{odd } u \leq (4N+2)^{1/m}\} \ll N^{1/m}.$$

3) Bounding the off-diagonal. Using $\binom{A}{2} = \sum_{m=2}^{A}(m-1)$, we get

$$P_N = \sum_r \binom{A_r(N)}{2} = \sum_{m=2}^{\infty}(m-1)T_m(N) \ll \sum_{m=2}^{\infty}(m-1)N^{1/m} = O(N^{1/2}),$$

since the series is dominated by the $m=2$ term and the tail is $O(N^{1/3})$.

Consequently, $S_N = N + 2P_N = N + O(N^{1/2}) = N + o(N)$, and hence

$$\lim_{N\to\infty} S_N/N = 1.$$

Final answer: 1