# Linear Algebra Olympiad Problems

February 2025

**Problem 1.** Given the matrix

$$A_n = \begin{pmatrix} 1 & \frac{2}{n} & \frac{6}{n} \\ \frac{3}{n} & 1 & 0 \\ -\frac{1}{n} & 0 & 1 \end{pmatrix}$$

Calculate $\lim_{n\to\infty} A_n^n$.

**Problem 2.** Let $A$ and $B$ be nonzero matrices of size $n \times n$, where $n \geq 2$. It is known that $ABA = A$. Does it follow that $BAB = B$?

**Problem 3.** Suppose that for matrices $A$ and $B$, the equality

$$AB = \alpha A + \beta B$$

holds, where $\alpha$ and $\beta$ are some nonzero numbers. Prove that $BA = AB$.

**Problem 4.** Let $A, B, C$ be square matrices of the same size, where $A$ is non-singular. Prove that if

$$(A - B)CA = B,$$

then

$$AC(A - B) = B.$$

**Problem 5.** Let $A$ and $B$ be distinct real square matrices. If

$$A^3 = B^3 \quad \text{and} \quad A^2B = B^2A,$$

can the matrix $A^2 + B^2$ be invertible?

**Problem 6.** All elements of a determinant of order three are squares of odd numbers. Prove that the determinant is divisible by 64.

**Problem 7.** Let $A$ and $B$ be square matrices such that

$$A^2 = A, \quad B^2 = B, \quad AB = BA.$$

What values can the determinant of the matrix $A - B$ take?

**Problem 8.** The matrices $A$ and $B$ do not commute, i.e., $AB \neq BA$. Can it happen that the matrices $A^2$ and $B^2$ commute?

**Problem 9.** Compute the determinant of the matrix $A = (a_{i,j})$ of size $n \times n$ with the general term $a_{i,j} = \max(i, j)$.

**Problem 10.** Compute the determinant:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & \frac{1}{2} & 0 & \dots & 0 \\ 1 & 0 & 0 & \frac{1}{3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \dots & \frac{1}{n} \end{vmatrix}$$