Second Olympiad for NUP team selection

May 2025

Problem 1. (10 points) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any real numbers $a < b$, the image $f([a, b])$ is a closed interval of length $b - a$.

Solution. The functions $f(x) = x + c$ and $f(x) = -x + c$ with some constant c obviously satisfy the condition of the problem. We will prove now that these are the only functions with the desired property.

Let f be such a function. Then f clearly satisfies $|f(x) - f(y)| \le |x - y|$ for all x, y; therefore, f is continuous. Given x, y with $x < y$, let $a, b \in [x, y]$ be such that $f(a)$ is the maximum and $f(b)$ is the minimum of f on $[x, y]$. Then $f([x, y]) = [f(b), f(a)]$; hence

$$y - x = f(a) - f(b) \le |a - b| \le y - x$$

This implies $\{a, b\} = \{x, y\}$, and therefore f is a monotone function. Suppose f is increasing. Then $f(x)-f(y) = x-y$ implies $f(x) - x = f(y) - y$, which says that $f(x) = x + c$ for some constant c. Similarly, the case of a decreasing function f leads to $f(x) = -x + c$ for some constant c.

Problem 2. (10 points) Let A be an $n \times n$ real matrix such that $3A^3 = A^2 + A + I$ (I is the identity matrix). Show that the sequence A^k converges to an idempotent matrix. (A matrix B is called idempotent if $B^2 = B$.)

Solution. The minimal polynomial of A is a divisor of $3x^3 - x^2 - x - 1$. This polynomial has three different roots. This implies that A is diagonalizable: $A = C^{-1}DC$ where D is a diagonal matrix. The eigenvalues of the matrices A and D are all roots of polynomial $3x^3 - x^2 - x - 1$. One of the three roots is 1, the remaining two roots have smaller absolute value than 1. Hence, the diagonal elements of D^k, which are the k-th powers of the eigenvalues, tend to either 0 or 1 and the limit $M = \lim D^k$ is idempotent. Then $\lim A^k = C^{-1}MC$ is idempotent as well.

Problem 3. (10 points) Let $a_1, a_2, \dots, a_{51}$ be non-zero elements of a field. We simultaneously replace each element with the sum of the 50 remaining ones. In this way we get a sequence $b_1, \dots, b_{51}$. If this new sequence is a permutation of the original one, what can be the characteristic of the field? (The characteristic of a field is p, if p is the smallest positive integer such that

$$\underbrace{x + x + \cdots + x}_{p} = 0$$

for any element x of the field. If there exists no such p, the characteristic is 0.)

Solution. Let $S = a_1 + a_2 + \cdots + a_{51}$. Then $b_1 + b_2 + \cdots + b_{51} = 50S$. Since $b_1, b_2, \dots, b_{51}$ is a permutation of $a_1, a_2, \cdots, a_{51}$, we get $50S = S$, so $49S = 0$. Assume that the characteristic of the field is not equal to 7. Then $49S = 0$ implies that $S = 0$. Therefore $b_i = -a_i$ for $i = 1, 2, \dots, 51$. On the other hand, $b_i = a_{\varphi(i)}$ where $\varphi \in S_{51}$. Therefore, if the characteristic is not 2, the sequence $a_1, a_2, \cdots, a_{51}$ can be partitioned into pairs $\{a_i, a_{\varphi(i)}\}$ of additive inverses. But this is impossible, since 51 is an odd number. It follows that the characteristic of the field is 7 or 2.

The characteristic can be either 2 or 7. For the case of 7, $x_1 = \cdots = x_{51} = 1$ is a possible choice. For the case of 2, any elements can be chosen such that $S = 0$, since then $b_i = -a_i = a_i$.

Problem 4. (10 points) Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f(b) - f(a) = (b - a)f'\left(\sqrt{ab}\right) \quad \text{for all} \quad a, b > 0. \tag{1}$$

Solution. First, we show that it f is infinitely many times differentiable. By substituting $a = \frac{1}{2}t$ and $b = 2t$ in (1),

$$f'(t) = \frac{f(2t) - f\left(\frac{1}{2}t\right)}{\frac{3}{2}t}. \tag{2}$$

Inductively, if f is k times differentiable then the right-hand side of (2) is k times differentiable, so the $f'(t)$ on the left-hand side is k times differentiable as well; hence f is $k + 1$ times differentiable.

Now substitute $b = e^h t$ and $a = e^{-h} t$ in (1), differentiate three times with respect to h then take limits with $h \to 0$:

$$f(e^h t) - f(e^{-h} t) - (e^h t - e^{-h} t)f'(t) = 0$$

$$\left(\frac{\partial}{\partial h}\right)^3 \big(f(e^h t) - f(e^{-h}t) - (e^h t - e^{-h}t)f'(t)\big) = 0$$
$$e^{3ht}t^3 f'''(e^{ht}) + 3e^{2ht}t^2 f''(e^{ht}) + e^{ht}tf'(e^{ht}) + e^{-3ht}t^3 f'''(e^{-ht}) + 3e^{-2ht}t^2 f''(e^{-ht}) + e^{-ht}tf'(e^{-ht})$$
$$-(e^{ht} + e^{-ht})f'(t) = 0$$
$$2t^3 f'''(t) + 6t^2 f''(t) = 0$$
$$tf'''(t) + 3f''(t) = 0$$
$$(tf(t))''' = 0.$$

Consequently, $tf(t)$ is at most a quadratic polynomial of t, and therefore

$$f(t) = C_1 t + \frac{C_2}{t} + C_3 \tag{3}$$

with some constants C_1, C_2, and C_3.

It is easy to verify that all functions of the form (3) satisfy the equation (1).

Problem 5. (10 points) Let $f(x)$ be a polynomial with real coefficients of degree n. Suppose that $\frac{f(k)-f(m)}{k-m}$ is an integer for all integers $0 \le k < m \le n$. Prove that $a - b$ divides $f(a) - f(b)$ for all pairs of distinct integers a and b.

Solution 1. We need the following

Lemma. Denote the least common multiple of $1, 2, \ldots, k$ by $L(k)$, and define

$$h_k(x) = L(k) \cdot \binom{x}{k} \quad (k = 1, 2, \ldots).$$

Then the polynomial $h_k(x)$ satisfies the condition, i.e. $a - b$ divides $h_k(a) - h_k(b)$ for all pairs of distinct integers a, b.

Proof. It is known that

$$\binom{a}{k} = \sum_{j=0}^{k} \binom{a-b}{j}\binom{b}{k-j}.$$

(This formula can be proved by comparing the coefficient of x^k in $(1+x)^a$ and $(1+x)^{a-b}(1+x)^b$.) From here we get:

$$h_k(a) - h_k(b) = L(k)\left(\binom{a}{k} - \binom{b}{k}\right) = L(k)\sum_{j=1}^{k}\binom{a-b}{j}\binom{b}{k-j} = (a-b)\sum_{j=1}^{k}\frac{L(k)}{j}\binom{a-b-1}{j-1}\binom{b}{k-j}.$$

On the right-hand side all fractions $\frac{L(k)}{j}$ are integers, so the right-hand side is a multiple of $(a-b)$. The lemma is proved.

Expand the polynomial f in the basis 1, $\binom{x}{1}$, $\binom{x}{2}, \ldots$ as

$$f(x) = A_0 + A_1\binom{x}{1} + A_2\binom{x}{2} + \cdots + A_n\binom{x}{n}. \tag{1}$$

We prove by induction on j that A_j is a multiple of $L(j)$ for $1 \le j \le n$. (In particular, A_j is an integer for $j \ge 1$.) Assume that $L(j)$ divides A_j for $1 \le j \le m-1$. Substituting m and some $k \in \{0, 1, \ldots, m-1\}$ in (1),

$$\frac{f(m) - f(k)}{m-k} = \sum_{j=1}^{m-1}\frac{A_j}{L(j)} \cdot \frac{h_j(m) - h_j(k)}{m-k} + \frac{A_m}{m-k}.$$

Since all other terms are integers, the last term $\frac{A_m}{m-k}$ is also an integer. This holds for all $0 \le k < m$, so A_m is an integer that is divisible by $L(m)$.

Hence, A_j is a multiple of $L(j)$ for every $1 \le j \le n$. By the lemma this implies the original statement.

Solution 2. The statement of the problem follows immediately from the following claim, applied to the polynomial

$$g(x, y) = \frac{f(x) - f(y)}{x - y}.$$

Claim. Let $g(x,y)$ be a real polynomial of two variables with total degree less than n. Suppose that $g(k,m)$ is an integer whenever $0 \le k < m \le n$ are integers. Then $g(k,m)$ is an integer for every pair k,m of integers.

Proof. Apply induction on n. If $n=1$ then g is a constant. This constant can be read from $g(0,1)$ which is an integer, so the claim is true.

Now suppose that $n \ge 2$ and the claim holds for $n-1$. Consider the polynomials

$$g_1(x,y) = g(x+1,y+1) - g(x,y+1) \quad \text{and} \quad g_2(x,y) = g(x,y+1) - g(x,y). \tag{1}$$

For every pair $0 \le k < m \le n-1$ of integers, the numbers $g(k,m)$, $g(k,m+1)$ and $g(k+1,m+1)$ are all integers, so $g_1(k,m)$ and $g_2(k,m)$ are integers, too. Moreover, in (1) the maximal degree terms of g cancel out, so $\deg g_1, \deg g_2 < \deg g$. Hence, we can apply the induction hypothesis to the polynomials g_1 and g_2 and we thus have $g_1(k,m), g_2(k,m) \in \mathbb{Z}$ for all $k,m \in \mathbb{Z}$.

In view of (1), for all $k,m \in \mathbb{Z}$, we have that

(a) $g(0,1) \in \mathbb{Z}$;

(b) $g(k,m) \in \mathbb{Z}$ if and only if $g(k+1,m+1) \in \mathbb{Z}$;

(c) $g(k,m) \in \mathbb{Z}$ if and only if $g(k,m+1) \in \mathbb{Z}$.

For arbitrary integers k,m, apply (b) $|k|$ times then apply (c) $|m-k-1|$ times as

$$g(k,m) \in \mathbb{Z} \Leftrightarrow \cdots \Leftrightarrow g(0,m-k) \in \mathbb{Z} \Leftrightarrow \cdots \Leftrightarrow g(0,1) \in \mathbb{Z}.$$

Hence, $g(k,m) \in \mathbb{Z}$. The claim has been proved.