Math Analysis Olympiad Problems

February 2025

Problem 1. Continuity and Differentiability at a Point. Consider the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

1. Prove that f is continuous at $x = 0$.
2. Decide whether f is differentiable at $x = 0$ and, if so, compute $f'(0)$.

Problem 2. Lipschitz Continuity via Bounded Derivative. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function with a bounded derivative; that is, there exists $M > 0$ such that

$$|f'(x)| \leq M \quad \text{for all } x \in (a, b).$$

Prove that f is Lipschitz continuous on $[a, b]$; i.e., show that for all $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq M|x - y|.$$

Problem 3. A Telescoping Series with Alternating Terms. Study the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{\sqrt{n+1}} - \frac{(-1)^n}{\sqrt{n}} \right)$$

Determine whether the series converges absolutely, conditionally or diverges. Provide a complete justification.

Problem 4. Let $f \in C^1(a, b)$. Suppose

$$\lim_{x \to a^+} f(x) = +\infty, \quad \lim_{x \to b^-} f(x) = -\infty,$$

and

$$f'(x) + f^2(x) \geq -1 \quad \text{for all } x \in (a, b).$$

Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

Problem 5. Let $f \in C^1[a, b]$, $f(a) = 0$ and suppose that $\lambda \in \mathbb{R}$, $\lambda > 0$, is such that

$$|f'(x)| \leq \lambda |f(x)|$$

for all $x \in [a, b]$. Is it true that $f(x) = 0$ for all $x \in [a, b]$?

Problem 6. Find

$$\lim_{N \to \infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N - k)}.$$

Note that ln denotes the natural logarithm.

Problem 7. Let $F : (1, \infty) \to \mathbb{R}$ be the function defined by

$$F(x) := \int_x^{x^2} \frac{dt}{\ln t}$$

Show that F is one-to-one (i.e., injective) and find the range (i.e., set of values) of F.

Problem 8. Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers such that $b_0 = 1$,

$$b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1 + \sqrt{b_{n-1}}}.$$

Calculate

$$\sum_{n=1}^{\infty} b_n 2^n.$$

Problem 9. Evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}\, dx,$$

where n is a natural number.

Problem 10.

(i) Prove that

$$\lim_{x\to+\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = \frac{1}{2}.$$

(ii) Prove that there is a positive constant c such that for every $x \in [1, \infty)$ we have

$$\left| \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2} \right| \le \frac{c}{x}.$$