8th IMC 2001, July 19 – July 25, Prague, Czech Republic, Second day

Problem 1

Let $r, s \geq 1$ be integers and $a_0, a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{s-1}$ be real non-negative numbers such that

$$(a_0 + a_1 x + \cdots + a_{r-1} x^{r-1} + x^r)(b_0 + b_1 x + \cdots + b_{s-1} x^{s-1} + x^s) = 1 + x + x^2 + \cdots + x^{r+s}.$$

Prove that each a_i and each b_j equals either 0 or 1.

Problem 2

Let $a_0 = \sqrt{2}$, $b_0 = 2$, and define

$$a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}, \quad b_{n+1} = \frac{2b_n}{2 + \sqrt{4 + b_n^2}}.$$

a) Prove that the sequences (a_n) and (b_n) are decreasing and converge to 0.

b) Prove that $(2^n a_n)$ is increasing, $(2^n b_n)$ is decreasing, and that these two sequences converge to the same limit.

c) Prove that there exists a constant $C > 0$ such that for all n:

$$0 < b_n - a_n < \frac{C}{8^n}.$$

Problem 3

Find the maximum number of points on the unit sphere in $\mathbb{R}^n$ such that the distance between any two points is strictly greater than $\sqrt{2}$.

Problem 4

Let $A = (a_{k,\ell})_{k,\ell=1}^n$ be an $n \times n$ complex matrix such that for each $1 \leq m \leq n$ and each $1 \leq j_1 < \cdots < j_m \leq n$, the determinant

$$\det(a_{j_k, j_\ell})_{k,\ell=1}^m = 0.$$

Prove that $A^n = 0$ and that there exists a permutation $\sigma \in S_n$ such that the permuted matrix $(a_{\sigma(k),\sigma(\ell)})$ is strictly upper-triangular.

Problem 5

Prove that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$ satisfying

$$f(x+y) \geq f(x) + y f(f(x)) \quad \forall x, y \in \mathbb{R}.$$

Problem 6

For each positive integer n, let

$$f_n(\theta) = \sin(\theta)\sin(2\theta)\cdots\sin(2^n\theta).$$

Prove that for all real θ and all n:

$$|f_n(\theta)| \leq \frac{2}{\sqrt{3}}\, |f_n(\pi/3)|.$$