9th IMC 2002, July 19 – 25, Warsaw, Poland, First day

First Day

Problem 1

A standard parabola is the graph of a quadratic polynomial $y = x^2 + ax + b$ with leading coefficient 1. Three standard parabolas with vertices V_1, V_2, V_3 intersect pairwise at points A_1, A_2, A_3. Let $A \mapsto s(A)$ be the reflection of the plane with respect to the x-axis.

Prove that standard parabolas with vertices $s(A_1), s(A_2), s(A_3)$ intersect pairwise at the points $s(V_1), s(V_2), s(V_3)$.

Problem 2

Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

Problem 3

Let n be a positive integer and let

$$a_k = \frac{1}{\binom{n}{k}}, \quad b_k = 2^{k-n}, \quad k = 1, 2, \ldots, n.$$

Show that

$$\sum_{k=1}^{n} \frac{a_k - b_k}{k} = 0.$$

Problem 4

Let $f : [a, b] \to [a, b]$ be a continuous function and let $p \in [a, b]$. Define $p_0 = p$ and $p_{n+1} = f(p_n)$ for $n \geq 0$. Suppose that the set

$$T_p = \{p_n : n = 0, 1, 2, \ldots\}$$

is closed, i.e. if $x \notin T_p$ then there is a $\delta > 0$ such that for all $x' \in T_p$ we have $|x' - x| \geq \delta$. Show that T_p has finitely many elements.

Problem 5

Prove or disprove the following statements:

(a) There exists a monotone function $f : [0, 1] \to [0, 1]$ such that for each $y \in [0, 1]$ the equation $f(x) = y$ has uncountably many solutions x.

(b) There exists a continuously differentiable function $f : [0, 1] \to [0, 1]$ such that for each $y \in [0, 1]$ the equation $f(x) = y$ has uncountably many solutions x?

Problem 6

For an $n \times n$ matrix M with real entries, let

$$\|M\| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Mx\|_2}{\|x\|_2},$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^n$.

Let A be an $n \times n$ real matrix satisfying

$$\|A^k - A^{k-1}\| \leq \frac{1}{2002k}$$

for all $k \geq 1$. Prove that $\|A^k\| \leq 2002$ for all positive integers k.