IMC 2025

Day 1

Problem 1

Let $P \in \mathbb{R}[x]$ be a polynomial with real coefficients, and suppose $\deg(P) \geq 2$. For every $x \in \mathbb{R}$, let $\ell_x \subset \mathbb{R}^2$ denote the line tangent to the graph of P at the point $(x, P(x))$.

(a) Suppose that the degree of P is odd. Show that

$$\bigcup_{x\in\mathbb{R}} \ell_x \;=\; \mathbb{R}^2.$$

(b) Does there exist a polynomial of even degree for which the above equality still holds?

Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function, and suppose that

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 0 \quad \text{and} \quad f(1) = f(-1) = 1.$$

Prove that

$$\int_{-1}^{1} \left(f''(x)\right)^2 \mathrm{d}x \;\geq\; 15,$$

and find all such functions for which equality holds.

Problem 3

Denote by $\mathcal{S}$ the set of all real symmetric 2025×2025 matrices of rank 1 whose entries take values -1 or $+1$. Let $A, B \in \mathcal{S}$ be matrices chosen independently uniformly at random. Find the probability that A and B commute, i.e., $AB = BA$.

Problem 4

Let a be an even positive integer. Find all real numbers x such that

$$\left\lfloor \sqrt[a]{b^a + x} \cdot b^{a-1} \right\rfloor \;=\; b^a + \left\lfloor \frac{x}{a} \right\rfloor$$

holds for every positive integer b. (Here $\lfloor x \rfloor$ denotes the largest integer that is no greater than x.)

Problem 5

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. Denote by S_n the set of all bijections from $[n]$ to $[n]$, and let T_n be the set of all maps from $[n]$ to $[n]$. Define the order $\operatorname{ord}(\tau)$ of a map $\tau \in T_n$ as the number of distinct maps in the set

$$\{\tau,\ \tau \circ \tau,\ \tau \circ \tau \circ \tau,\ \ldots\},$$

where $\circ$ denotes composition. Finally, let

$$f(n) \;=\; \max_{\tau \in S_n} \operatorname{ord}(\tau) \quad \text{and} \quad g(n) \;=\; \max_{\tau \in T_n} \operatorname{ord}(\tau).$$

Prove that

$$g(n) \;<\; f(n) \;+\; n^{0.501}$$

for sufficiently large n.

Day 2

Problem 6

Let $f : (0, \infty) \to \mathbb{R}$ be a continuously differentiable function, and let $b > a > 0$ be real numbers such that

$$f(a) = f(b) = k.$$

Prove that there exists a point $\xi \in (a, b)$ such that

$$f(\xi) - \xi f'(\xi) = k.$$

Problem 7

Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ satisfying both of the following properties:

(a) If $x \in M$, then $2x \in M$.

(b) If $x, y \in M$ and $x + y$ is even, then $\frac{x+y}{2} \in M$.

Problem 8

For an $n \times n$ real matrix $A \in M_n(\mathbb{R})$, denote by A^R its counter-clockwise $90°$ rotation. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^R = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}.$$

Prove that if $A = A^R$ then for any eigenvalue λ of A, we have $\Re\lambda = 0$ or $\Im\lambda = 0$.

Problem 9

Let n be a positive integer. Consider the following random process which produces a sequence of n distinct positive integers $X_1, X_2, \ldots, X_n$. First, X_1 is chosen randomly with

$$\mathbb{P}(X_1 = i) = 2^{-i} \quad \text{for every positive integer } i.$$

For $1 \leq j \leq n - 1$, having chosen $X_1, \ldots, X_j$, arrange the remaining positive integers in increasing order as $n_1 < n_2 < \cdots$, and choose X_{j+1} randomly with

$$\mathbb{P}(X_{j+1} = n_i) = 2^{-i} \quad \text{for every positive integer } i.$$

Let $Y_n = \max\{X_1, \ldots, X_n\}$. Show that

$$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i - 1},$$

where $\mathbb{E}[Y_n]$ is the expected value of Y_n.

Problem 10

For any positive integer N, let S_N be the number of pairs of integers $1 \leq a, b \leq N$ such that the number

$$(a^2 + a)(b^2 + b)$$

is a perfect square. Prove that the limit

$$\lim_{N \to \infty} \frac{S_N}{N}$$

exists and find its value.