GPT o3 Solutions

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Contents

1 Problem 1, Day 1

1.1 Variant 1

Problem 1. Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$, $n := \deg P \geq 2$ and denote by ℓ_x the tangent line to the graph of P at $(x, P(x))$, $\ell_x : y = P(x) + P'(x)(t-x)$, $t \in \mathbb{R}$. For a fixed point $(a, b) \in \mathbb{R}^2$ introduce the auxiliary polynomial $F_{a,b}(x) = P(x) + P'(x)(a-x) - b$. A point (a, b) lies on the line ℓ_x iff $F_{a,b}(x) = 0$. Hence $(a, b) \in \bigcup_{x\in\mathbb{R}} \ell_x \Leftrightarrow F_{a,b}$ has a real zero.

(a) P of odd degree

Put $n = \deg P$ (n odd) and let $c_n \neq 0$ be the leading coefficient. From the definition we obtain the leading term of $F_{a,b}$:

$$F_{a,b}(x) = P(x) - xP'(x) + aP'(x) - b = (1-n)c_n x^n + \text{(lower-degree terms)}.$$

Because $n \geq 3$ is odd, $x^n \to +\infty$ when $x \to +\infty$ and $x^n \to -\infty$ when $x \to -\infty$. The coefficient $(1-n)c_n \neq 0$ changes neither sign nor parity, so

$$\lim_{x\to+\infty} F_{a,b}(x) = -\text{sign}(c_n) \cdot \infty, \quad \lim_{x\to-\infty} F_{a,b}(x) = +\text{sign}(c_n) \cdot \infty.$$

Hence $F_{a,b}$ takes opposite signs for sufficiently large positive and negative x. By the Intermediate Value Theorem $F_{a,b}$ possesses a real root. Consequently, for every $(a, b) \in \mathbb{R}^2$ there exists x with $F_{a,b}(x) = 0$, i.e. $(a, b) \in \ell_x$. Therefore $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$ when $\deg P$ is odd.

(b) No polynomial of even degree works

Assume now that $n = \deg P$ is even ($n \geq 2$) and let c_n be the leading coefficient. Again the leading term of $F_{a,b}$ is $(1-n)c_n x^n$. Because n is even, x^n has the same sign at $\pm\infty$; hence the sign of $F_{a,b}(x)$ at both ends is determined by $-(n-1)c_n$:

- If $c_n > 0$ then $F_{a,b}(x) \to -\infty$ as $x \to \pm\infty$, thus $F_{a,b}(x)$ is bounded above: $\exists M(a)$ with $F_{a,b}(x) \leq M(a)\ \forall x$.
- If $c_n < 0$ then $F_{a,b}(x) \to +\infty$ as $x \to \pm\infty$, thus $F_{a,b}(x)$ is bounded below: $\exists m(a)$ with $F_{a,b}(x) \geq m(a)\ \forall x$.

Take, for instance, the first case ($c_n > 0$). Choose $b > M(a)$. Then $F_{a,b}(x) < 0$ for all x, so it has no real zero and (a, b) does not lie on any tangent line. Analogously, in the second case pick $b < m(a)$. Therefore for every polynomial of even degree there exist points in $\mathbb{R}^2$ that are not met by any tangent line, and the equality $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$ can never hold.

Final answer

(a) If $\deg P$ is odd, the union of all tangent lines of P equals the entire plane $\mathbb{R}^2$.

(b) No polynomial of even degree enjoys this property; for every even-degree polynomial the union of its tangent lines omits at least one (in fact, infinitely many) points of $\mathbb{R}^2$.

Time taken: 42.65 seconds

1.2 Variant 2

Problem 2. Let $T := \{(u, v) \in \mathbb{R}^2 \mid \exists x \in \mathbb{R} : v = P(x) + P'(x)(u - x)\} = \bigcup_{x\in\mathbb{R}} \ell_x$ (the right–hand side is exactly the set of all points that lie on some tangent line to the graph of P).

(a) P has odd degree

Let $n = \deg P \geq 2$ be odd and write $P(x) = a_n x^n + \dots$ with $a_n \neq 0$. Fix an arbitrary point $(u, v) \in \mathbb{R}^2$ and consider the polynomial

$$Q(x) := v - (P(x) + P'(x)(u - x)) = v - P(x) - uP'(x) + xP'(x).$$

Because $P'(x)$ has degree $n-1$, the term $xP'(x)$ again has degree n, the same as $P(x)$. The leading coefficient of Q is $-a_n + na_n = (n-1)a_n \neq 0$, so $\deg Q = n$. Since n is odd, $\lim_{x\to+\infty} Q(x)$ and $\lim_{x\to-\infty} Q(x)$ have opposite signs; hence, by the Intermediate Value Theorem, Q possesses a real root x_0. For this x_0 we have $v = P(x_0) + P'(x_0)(u - x_0)$, i.e. (u, v) lies on the tangent line ℓ_{x_0}. Because the choice of (u, v) was arbitrary, every point of $\mathbb{R}^2$ belongs to at least one ℓ_x: $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$. $\square$

(b) No polynomial of even degree enjoys this property

Assume, to the contrary, that P has even degree $n \geq 2$ and that the equality still holds. Multiplying P by -1 reflects its graph (and all its tangent lines) across the x–axis, so without loss of generality we may assume that its leading coefficient a_n is positive. Fix a real number u and define $f_u(x) := P(x) + P'(x)(u - x)$.

For fixed u the set of ordinates of points of T with abscissa u is the range of f_u: $\{v \in \mathbb{R} \mid (u, v) \in T\} = f_u(\mathbb{R})$.

Dominant term of f_u. Using $P(x) \approx a_n x^n$ and $P'(x) \approx na_n x^{n-1}$ we get, for large $|x|$,

$$f_u(x) = a_n x^n + na_n x^{n-1}(u - x) = -(n-1)a_n x^n + na_n u x^{n-1} + \text{lower-order terms}.$$

Because n is even and $a_n > 0$, the leading coefficient $-(n-1)a_n$ is negative, so $\lim_{x\to\pm\infty} f_u(x) = -\infty$.

Consequently f_u is continuous and tends to $-\infty$ at both ends; hence it attains a finite maximum $M(u) := \max_{x\in\mathbb{R}} f_u(x) < +\infty$.

Choose any ordinate $v > M(u)$. Then $v \notin f_u(\mathbb{R})$, i.e. the point (u, v) belongs to no tangent line, contradicting the assumption. Therefore such an equality is impossible for an even–degree polynomial.

Hence:

- For odd degree the union of all tangent lines is the whole plane.
- For even degree this never happens.

Final answer

(a) If $\deg P$ is odd, $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$.

(b) No polynomial of even degree has this property.

Time taken: 67.52 seconds

1.3 Variant 3

Problem 3. Let $\ell_x : y = P(x) + P'(x)(t-x)$ $(t \in \mathbb{R})$ be the tangent line to the graph of P at $(x, P(x))$. For a point $(u,v) \in \mathbb{R}^2$ we have $(u,v) \in \ell_x \Leftrightarrow v = P(x) + P'(x)(u-x)$.

Denote $F_{u,v}(x) = P(x) + P'(x)(u-x) - v$.

$F_{u,v}$ is a polynomial in x of the same degree $n = \deg P$ (≥ 2); its leading coefficient is $c = (1-n)a_n$ (a_n is the leading coefficient of P).

(a) n odd

Because n is odd, $F_{u,v}$ is an odd–degree polynomial. Hence

$$\lim_{x\to+\infty} F_{u,v}(x) = \operatorname{sgn}(c)\cdot\infty, \qquad \lim_{x\to-\infty} F_{u,v}(x) = -\operatorname{sgn}(c)\cdot\infty.$$

Therefore $F_{u,v}$ attains both positive and negative values; by the Intermediate Value Theorem there exists x_0 with $F_{u,v}(x_0) = 0$. With this x_0, the condition is fulfilled, so (u,v) lies on the tangent at x_0. Since (u,v) was arbitrary, $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$. $\square$

(b) n even

Now n is even, so x^n has the same sign for $x \to \pm\infty$. Consequently $F_{u,v}$ has the same limit at $+\infty$ and $-\infty$:

$$\lim_{|x|\to\infty} F_{u,0}(x) = +\infty \text{ if } c > 0, \quad -\infty \text{ if } c < 0.$$

Thus $F_{u,0}$ is bounded below when $c > 0$ and bounded above when $c < 0$. Fix any abscissa u_0. Put $m = \inf_{x\in\mathbb{R}} F_{u_0,0}(x)$ (if $c > 0$) or $M = \sup_{x\in\mathbb{R}} F_{u_0,0}(x)$ (if $c < 0$). Choose $v_0 < m$ when $c > 0$, or $v_0 > M$ when $c < 0$. For every x we then have $F_{u_0,v_0}(x) = F_{u_0,0}(x) - v_0 > 0$ (or < 0), so F_{u_0,v_0} never vanishes. Hence no tangent line passes through (u_0, v_0). The union of tangent lines is therefore a proper subset of $\mathbb{R}^2$.

Consequently there is no real polynomial of even degree whose tangents cover the whole plane.

Final answer

(a) For every real polynomial of odd degree (≥ 2) the union of all its tangent lines is $\mathbb{R}^2$.

(b) No polynomial of even degree enjoys this property; its tangent lines never fill the entire plane.

Time taken: 59.98 seconds

1.4 Final Solution

Solution 1. Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ $(c_n \neq 0)$, $n := \deg P$. For every $x \in \mathbb{R}$ the tangent line to the graph of P at $(x, P(x))$ is $\ell_x : y = P(x) + P'(x)(t-x)$ $(t \in \mathbb{R})$.

Put $F_{a,b}(x) := P(x) + P'(x)(a-x) - b$ $((a,b) \in \mathbb{R}^2)$.

Because (a,b) lies on ℓ_x iff $F_{a,b}(x) = 0$, we have $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2 \Leftrightarrow F_{a,b}$ has a real root for every (a,b).

Degree and leading coefficient of $F_{a,b}$. Since $P'(x)$ has degree $n-1$, the term $xP'(x)$ has degree n, while $aP'(x)$ has degree $n-1$. Hence $(n \geq 2)$ $\deg F_{a,b} = n$, with leading coefficient $(1-n)c_n$.

1. Odd degree $n \geq 3$

Here n is odd and $(1-n)c_n \neq 0$, so the leading term $(1-n)c_n x^n$ dominates and satisfies

$$\lim_{x\to+\infty} F_{a,b}(x) = -\operatorname{sgn}(c_n)\cdot\infty, \qquad \lim_{x\to-\infty} F_{a,b}(x) = +\operatorname{sgn}(c_n)\cdot\infty.$$

Thus $F_{a,b}$ takes both signs; by the Intermediate Value Theorem it possesses a real zero. Since this is true for every (a,b), we have: $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$ for every polynomial of odd degree ≥ 3.

2. Even degree n ($n \geq 2$)

Now x^n has the same sign at $\pm\infty$, so by the leading coefficient

$$\lim_{|x|\to\infty} F_{a,b}(x) = -\infty \text{ if } c_n > 0, \quad +\infty \text{ if } c_n < 0.$$

Fix an abscissa a_0. In the first case $F_{a_0,0}$ is bounded above; choose b larger than that upper bound. Then $F_{a_0,b}(x) < 0$ for all x, so it has no real root and (a_0, b) is missed by every tangent line. In the second case $F_{a_0,0}$ is bounded below; pick b smaller than that lower bound and argue analogously. Hence $\bigcup_{x\in\mathbb{R}} \ell_x \neq \mathbb{R}^2$ for every polynomial of even degree.

3. Degree 1 (linear polynomials)

If $P(x) = mx + k$ ($m \neq 0$) every tangent line coincides with the graph itself, so their union is that single straight line, not the entire plane.

Conclusion

The union of all tangent lines to the graph of a real polynomial P covers the whole plane $\mathbb{R}^2$ exactly when P has odd degree at least 3; it never does so for even degree or for linear (degree 1) polynomials.

Final answer The union $\bigcup_{x\in\mathbb{R}} \ell_x$ equals $\mathbb{R}^2$ if and only if $\deg P$ is odd and ≥ 3; for even degree or degree 1 the union of tangent lines is a proper subset of $\mathbb{R}^2$.

Timing Information: Final solution time: 21.89 seconds Total problem time: 192.04 seconds

2 Problem 2

2.1 Variant 1

Put

$$H = \{f \in C^2[-1,1] : f(-1) = f(1) = 1, \int_{-1}^{1} f(x)dx = 0\}.$$

On H introduce the quadratic functional

$$J(f) = \int_{-1}^{1} (f''(x))^2 dx.$$

We show that the minimum of J on H equals 15 and determine the unique minimiser.

Existence of a minimiser

H is a closed affine subspace of the Hilbert space $H^2(-1,1)$ with inner product $\langle u, v\rangle = \int_{-1}^{1} u''(x)v''(x)dx$. Because J is the squared norm coming from this inner product, it is strictly convex and weakly lower–semicontinuous, hence it attains its minimum on the non-empty set H. The minimiser is unique.

Euler–Lagrange equation

Let $g \in H$ be the minimiser and h be an arbitrary C^2–function with $h(\pm 1) = 0$ and $\int_{-1}^{1} h = 0$. For every real ε, $g + \varepsilon h$ still belongs to H, so

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} J(g + \varepsilon h) \tag{1}$$

$$= 2\int_{-1}^{1} g'' h'' dx. \tag{1}$$

Thus g'' is L^2–orthogonal to $\{h'' : h \text{ as above}\}$.

Introduce three Lagrange multipliers α, β, γ for the three linear constraints, and consider

$$F(f) = \int_{-1}^{1} (f'')^2 dx + \alpha \int_{-1}^{1} f(x)dx + \beta(f(1) - 1) + \gamma(f(-1) - 1).$$

Take a C^2 variation $f + \varepsilon\eta$ with η arbitrary but $\eta(\pm 1) = 0$. Computing the first variation and integrating by parts twice gives

$$0 = \frac{dF}{d\varepsilon}\bigg|_{\varepsilon=0} \tag{2}$$

$$= 2\int_{-1}^{1} f^{(4)} \eta dx + \alpha \int_{-1}^{1} \eta dx + 2[f''\eta']_{-1}^{1}. \tag{2}$$

Because η is arbitrary in the interior of the interval, (2) implies

$$f^{(4)}(x) = -\alpha/2 =: k \text{ (constant)}. \quad (3)$$

Because η' can be chosen freely at ± 1, the boundary term in (2) forces

$$f''(-1) = f''(1) = 0. \quad (4)$$

Hence the minimiser g is a quartic polynomial satisfying (3)–(4).

Determining the quartic polynomial

Write

$$g(x) = ax^4 + bx^3 + cx^2 + dx + e \quad (a = k/24).$$

Then

$$g''(x) = 12ax^2 + 6bx + 2c.$$

Using (4):

$$12a + 6b + 2c = 0, \tag{3}$$

$$12a - 6b + 2c = 0 \quad \Rightarrow \quad b = 0, \quad c = -6a. \tag{5}$$

The boundary values $g(\pm 1) = 1$ give

$$a + c + d + e = 1, \tag{4}$$

$$a + c - d + e = 1. \tag{6}$$

From (6) we get $d = 0$ and $-5a + e = 1$. (7)
Finally the integral condition

$$\int_{-1}^{1} g(x)dx = 0$$

together with (5)–(7) yields

$$2a/5 - 4a + 2 + 10a = 0 \quad \Rightarrow \quad a = -5/16. \quad (8)$$

Consequently

$$b = d = 0, \quad c = 15/8, \quad e = -9/16.$$

Thus

$$g(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}. \quad (9)$$

The minimum value of the functional

For (9) we have

$$g''(x) = \frac{15}{4}(1 - x^2). \quad (10)$$

Therefore

$$J(g) = (15/4)^2 \int_{-1}^{1} (1 - x^2)^2 dx \tag{5}$$

$$= (225/16) \cdot 2 \int_{0}^{1} (1 - 2x^2 + x^4)dx \tag{6}$$

$$= (225/16) \cdot 2(1 - 2/3 + 1/5) \tag{7}$$

$$= (225/16) \cdot 16/15 \tag{8}$$

$$= 15. \tag{11}$$

Optimality and uniqueness

Let $f \in H$ be arbitrary and write $h = f - g$ (so $h(\pm 1) = 0$, $\int h = 0$). Then by (1) the cross term vanishes:

$$\int_{-1}^{1} g'' h'' dx = 0. \quad (12)$$

Hence

$$J(f) = \int (g'' + h'')^2 \tag{9}$$

$$= J(g) + \int (h'')^2 \geq J(g) = 15. \tag{13}$$

Equality occurs iff $\int (h'')^2 = 0$, i.e. $h'' = 0$, so h is linear. Because $h(\pm 1) = 0$, this linear function is identically zero; thus $f = g$.

Conclusion

For every twice continuously differentiable $f : \mathbb{R} \to \mathbb{R}$ with $\int_{-1}^{1} f(x)dx = 0$ and $f(\pm 1) = 1$,

$$\int_{-1}^{1} (f''(x))^2 dx \geq 15,$$

and equality holds exactly for the single function

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}.$$

Final answer:

$$\int_{-1}^{1} (f''(x))^2 dx \geq 15,$$

with equality only for

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}.$$

2.2 Variant 2

Let

$$J(f) = \int_{-1}^{1} (f''(x))^2 \, dx, \tag{10}$$

$$V = \{f \in C^2[-1, 1] : f(1) = f(-1) = 1, \int_{-1}^{1} f = 0\}. \tag{11}$$

We have to show $J(f) \geq 15$ for every $f \in V$ and to determine those f for which the equality holds.

Existence and shape of the minimiser

The functional J is strictly convex and the constraints are linear; hence there is a unique minimiser $g \in V$.

Take an arbitrary $h \in C^2[-1,1]$ with

$$h(\pm 1) = 0, \tag{12}$$

$$\int_{-1}^{1} h = 0, \tag{13}$$

and consider $f_\varepsilon = g + \varepsilon h$ (ε real small). Because $f_\varepsilon \in V$, the first variation of J at g must vanish:

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} J(f_\varepsilon) \tag{14}$$

$$= 2\int_{-1}^{1} g'' h''. \tag{15}$$

Integrating twice by parts and using $h(\pm 1) = 0$ one obtains

$$\int_{-1}^{1} g'' h'' = -g^{(4)} \int_{-1}^{1} h. \tag{16}$$

Since $\int h = 0$, the last term is 0, so the stationarity condition is satisfied iff

$$g^{(4)}(x) = \text{const} := \kappa. \tag{17}$$

Consequently g is a polynomial of degree ≤ 4; write

$$g(x) = ax^4 + bx^3 + cx^2 + dx + e. \quad (1) \tag{18}$$

Exploiting the symmetry

Because the constraints are symmetric with respect to $x \mapsto -x$, the minimiser is even: put $x \to -x$ in (1) and use uniqueness to get $b = d = 0$. Hence

$$g(x) = ax^4 + cx^2 + e. \quad (2) \tag{19}$$

Solving for the coefficients

The three constraints give

$$\text{(i)} \quad g(1) = a + c + e = 1, \tag{20}$$

$$\text{(ii)} \quad g(-1) = 1 \quad \text{(same as (i))}, \tag{21}$$

$$\text{(iii)} \quad \int_{-1}^{1} g = \frac{2a}{5} + \frac{2c}{3} + 2e = 0. \tag{22}$$

From (i) and (iii):

$$a + c + e = 1, \tag{23}$$

$$\frac{a}{5} + \frac{c}{3} + e = 0. \tag{24}$$

Solving,

$$12a + 10c = 15. \quad (3) \tag{25}$$

The value of J for an even quartic

For f given by (2) one has

$$f''(x) = 12ax^2 + 2c, \tag{26}$$

and therefore

$$J(f) = \int_{-1}^{1} (12ax^2 + 2c)^2 \, dx \tag{27}$$

$$= \frac{2}{5} \cdot (12a)^2 + \frac{2}{3} \cdot 0 + (2c)^2 \cdot 2 + \frac{4}{3} \cdot 12a \cdot 2c \tag{28}$$

$$= \frac{288}{5} a^2 + 8c^2 + 32ac. \tag{4}$$

Minimising J under (3)

Set

$$\Phi(a, c) = \frac{288}{5} a^2 + 8c^2 + 32ac \quad \text{subject to} \quad 12a + 10c = 15. \tag{29}$$

Write $c = \frac{15-12a}{10}$ and substitute into Φ; after simplification

$$\Phi(a) = 24a^2 - 60a + 15. \tag{30}$$

This quadratic attains its minimum at $a = -\frac{5}{16}$.
So indeed $a = -\frac{5}{16}$ and $c = \frac{15}{8}$.
Then from (i), $e = -\frac{9}{16}$.
Hence

$$g(x) = -\frac{5}{16} x^4 + \frac{15}{8} x^2 - \frac{9}{16}. \quad (5) \tag{31}$$

For this function $J(g) = 15$.

The lower bound for every $f \in V$

Write any $f \in V$ in the form $f = g + h$ with h as in Section 1. Because $\int g''h'' = 0$, we have

$$J(f) = \int (g'' + h'')^2 = \int g''^2 + \int h''^2 \geq \int g''^2 = J(g) = 15. \tag{32}$$

Equality case

Equality $J(f) = 15$ forces $\int h''^2 = 0$, hence $h'' = 0$ and h is linear. The conditions $h(\pm 1) = 0$ then give $h \equiv 0$, so $f = g$.

Result

For every twice continuously differentiable function f with

$$\int_{-1}^{1} f(x) \, dx = 0, \tag{33}$$

$$f(1) = f(-1) = 1, \tag{34}$$

one has

$$\int_{-1}^{1} (f''(x))^2\, dx \geq 15, \tag{35}$$

and equality holds only for

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}. \tag{36}$$

2.3 Final Solution

Let

$$H = \{f \in C^2[-1,1] : f(1) = f(-1) = 1, \int_{-1}^{1} f(x)dx = 0\}$$

and set

$$J(f) = \int_{-1}^{1} (f''(x))^2 dx \quad (f \in H).$$

Our task is to prove

$$J(f) \geq 15 \qquad (f \in H) \tag{1}$$

and find every $f \in H$ with equality.

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1. A minimiser exists and is unique

Equip $H^2(-1,1)$ with the inner product

$$\langle u, v\rangle = \int_{-1}^{1} u''(x)v''(x)dx.$$

Then $J(f) = \langle f, f\rangle$, hence J is strictly convex and weakly lower-semicontinuous. Because H is a non-empty closed affine subspace of this Hilbert space, J attains its minimum there and the minimiser is unique. Call this minimiser g.

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2. Euler-Lagrange equation for g

Fix $h \in C^2[-1,1]$ with $h(\pm 1) = 0$ and $\int_{-1}^{1} h = 0$. For every ε, $g + \varepsilon h \in H$, so

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} J(g + \varepsilon h) = 2\int_{-1}^{1} g''h''dx. \tag{2}$$

Thus g'' is L^2-orthogonal to $\{h'' : h \text{ as above}\}$.
To convert (2) into a differential equation, integrate twice by parts:

$$\int g''h''dx = [g''h']_{-1}^{1} - \int g'''h'dx \tag{37}$$
$$= [g''h']_{-1}^{1} - [g'''h]_{-1}^{1} + \int g^{(4)}h\,dx. \tag{38}$$

Because $h(\pm 1) = 0$, the term $[g'''h]_{-1}^{1}$ vanishes. Since h' can be chosen freely at the endpoints, $[g''h']_{-1}^{1} = 0$ forces

$$g''(-1) = g''(1) = 0. \tag{3}$$

Because h is arbitrary in the interior, $\int g^{(4)}h\,dx = 0$ gives

$$g^{(4)}(x) = \text{constant} =: \kappa \qquad (-1 < x < 1). \tag{4}$$

Consequently g is a polynomial of degree ≤ 4.

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3. Determining g

By symmetry of the constraints $x \mapsto -x$, the minimiser is even; hence write

$$g(x) = ax^4 + cx^2 + e \qquad (a, c, e \text{ real}). \tag{5}$$

From (3): $g''(x) = 12ax^2 + 2c$, so $g''(\pm 1) = 0$ implies

$$12a + 2c = 0 \quad \Rightarrow \quad c = -6a. \tag{6}$$

Boundary value $g(1) = 1$ gives

$$a + c + e = 1 \quad \Rightarrow \quad -5a + e = 1 \quad \Rightarrow \quad e = 1 + 5a. \tag{7}$$

Using $\int_{-1}^{1} g = 0$,

$$0 = 2\left(\frac{a}{5} + \frac{c}{3} + e\right) \tag{39}$$
$$= 2\left(\frac{a}{5} - 2a + e\right) \quad (\text{by (6)}), \tag{40}$$

so

$$e = 2a - \frac{a}{5} = \frac{9a}{5}. \tag{8}$$

Equating (7) and (8):

$$1 + 5a = \frac{9a}{5} \quad \Rightarrow \quad 5 + 25a = 9a \quad \Rightarrow \quad a = -\frac{5}{16}.$$

Hence

$$c = -6a = \frac{15}{8}, \quad e = -\frac{9}{16}, \tag{9}$$
$$g(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}. \tag{10}$$

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4. The minimal value of J

From (10)

$$g''(x) = \frac{15}{4}(1-x^2), \quad (g''(x))^2 = \frac{225}{16}(1-x^2)^2.$$

Because $\int_{-1}^{1}(1-x^2)^2dx = \frac{16}{15}$,

$$J(g) = \frac{225}{16} \cdot \frac{16}{15} = 15. \tag{11}$$

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5. The inequality for every $f \in H$

Let $f \in H$ be arbitrary and write $f = g + h$. Then

$$h(\pm 1) = 0, \quad \int_{-1}^{1} h = 0 \tag{12}$$

and, by (2) with h as above, $\int g''h''dx = 0$. Therefore

$$J(f) = \int (g'' + h'')^2 = J(g) + \int (h'')^2 \geq J(g) = 15. \tag{13}$$

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6. Equality case

Equality in (13) forces $\int (h'')^2 = 0$, hence $h'' \equiv 0$, so h is linear. With $h(\pm 1) = 0$ this gives $h \equiv 0$, i.e. $f = g$. Thus g is the unique extremal.

Result

For every twice continuously differentiable function f with

$$f(1) = f(-1) = 1, \qquad \int_{-1}^{1} f(x)dx = 0,$$

one has

$$\int_{-1}^{1} (f''(x))^2 dx \geq 15,$$

and equality holds only for

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}.$$

3 Problem 3

3.1 Variant 1

Let

$$A = vv^T, \quad B = ww^T, \quad v, w \in \{-1, +1\}^n, \quad n = 2025, \tag{41}$$

be two matrices from $\mathcal{M}$.

(For every matrix in $\mathcal{M}$ there are exactly the two representatives v and $-v$; choosing v uniformly from $\{\pm 1\}^n$ therefore produces A uniformly in $\mathcal{M}$, and likewise for w and B.)

When do A and B commute?

$$AB = vv^T ww^T = (v^T w) vw^T, \tag{42}$$
$$BA = ww^T vv^T = (w^T v) wv^T = (v^T w) wv^T. \tag{43}$$

Thus $AB = BA \iff (v^T w) vw^T = (v^T w) wv^T$.

- If $v^T w = 0$, both sides are the zero matrix and they commute.
- If $v^T w \neq 0$, we must have $vw^T = wv^T$.

  For rank-1 outer products this is possible only when v is a scalar multiple of w. With entries ± 1 this forces $v = w$ or $v = -w$, whence $v^T w = \pm n \neq 0$.

Because $n = 2025$ is odd, $v^T w$ can never be 0 (a sum of an odd number of ± 1's is odd). Hence

$$AB = BA \iff v = w \text{ or } v = -w.$$

Probability calculation

For fixed v, the vector w is chosen independently and uniformly from $\{\pm 1\}^n$, so

$$P(w = v) = 2^{-n}, \tag{44}$$
$$P(w = -v) = 2^{-n}. \tag{45}$$

These events are disjoint, giving

$$P(AB = BA) = 2^{-n} + 2^{-n} = 2^{-n+1} = 2^{-2024}.$$

Therefore the desired probability is $\boxed{2^{-2024}}$.

3.2 Variant 2

Let

$$A = vv^T, \quad B = ww^T \quad \text{with} \quad v, w \in \{-1, +1\}^n, \quad n = 2025.$$

(Every matrix in $\mathcal{M}$ is of this form; $vv^T = (-v)(-v)^T$, so each matrix corresponds to the pair $\{v, -v\}$.)

1. When do A and B commute?

$$AB = v(v^T w)w^T, \quad BA = w(w^T v)v^T.$$

With $s = v^T w = w^T v$, we have

$$AB = svw^T, \quad BA = swv^T.$$

- If $s = 0$, then $AB = BA = 0$. But $n = 2025$ is odd, so $v^T w =$ sum of 2025 numbers ± 1, hence odd; s can never be 0. Thus orthogonality cannot occur.
- If $s \neq 0$, we need $vw^T = wv^T$. For all i, j: $v_i w_j = w_i v_j \Rightarrow v_i/w_i = v_j/w_j = c$ (constant). Hence $v = cw$. Because each coordinate is ± 1, $c = \pm 1$, so $v = \pm w$. In that case $A = vv^T = ww^T$, i.e., the two matrices are identical.

Therefore

$$AB = BA \quad \Leftrightarrow \quad A = B.$$

2. Counting the matrices.

There are 2^n vectors with ± 1 entries, and each matrix corresponds to the pair $\{v, -v\}$; hence

$$|\mathcal{M}| = \frac{2^n}{2} = 2^{n-1} = 2^{2024}.$$

3. Probability that two independently, uniformly chosen matrices coincide.

The second matrix has to equal the first one out of 2^{2024} possibilities, so

$$P(AB = BA) = \frac{1}{2^{2024}}.$$

Final answer: 2^{-2024}

3.3 Final Solution

Let $n = 2025$ (odd), $V = \{-1, +1\}^n$.

1. Description of the set $\mathcal{A}$

For every $v \in V$ the outer product

$$A = vv^T$$

is a real symmetric rank-1 matrix whose entries are ± 1.

Conversely, any matrix of this kind equals vv^T for some $v \in V$, and

$$vv^T = (-v)(-v)^T,$$

so the two opposite vectors represent the same matrix.

Hence

$$|\mathcal{A}| = |V|/2 = 2^n/2 = 2^{n-1} = 2^{2024}. \quad (1)$$

2. When do two matrices commute?

Choose

$$A = vv^T, \quad B = ww^T \quad (v, w \in V).$$

Then
$$AB = v(v^T w)w^T, \quad BA = w(w^T v)v^T = (v^T w)wv^T. \quad (2)$$
Write $s = v^T w$. Because n is odd, s is the sum of 2025 numbers ± 1 and is therefore a non-zero odd integer.

Since $s \neq 0$, (2) shows
$$AB = BA \Leftrightarrow vw^T = wv^T. \quad (3)$$
To solve (3), compare the (i, j) entry of both sides:
$$v_i w_j = w_i v_j \quad \text{for all } i, j.$$
Fix any j with $w_j \neq 0$ (all w_j are ± 1). Then for every i
$$\frac{v_i}{w_i} = \frac{v_j}{w_j} =: c,$$
so $v = cw$. Because each coordinate of v and w is ± 1, $c = \pm 1$, i.e.
$$v = w \quad \text{or} \quad v = -w. \quad (4)$$
If (4) holds, $vv^T = ww^T$; thus
$$AB = BA \Leftrightarrow A = B. \quad (5)$$
3. Probability computation

A and B are chosen independently and uniformly from $\mathcal{A}$. By (5) they commute exactly when they are the same matrix. The second choice must therefore hit one specific matrix out of $|\mathcal{A}|$ possibilities, so
$$P(AB = BA) = \frac{1}{|\mathcal{A}|} = \frac{1}{2^{2024}}. \quad \text{(from (1))}$$
Final answer: 2^{-2024}

4 Problem 4

4.1 Variant 1

Let
$$x = aq + r, \quad \text{where } q = \lfloor x/a \rfloor \in \mathbb{Z} \text{ and } 0 \leq r < a. \qquad (1)$$
Put
$$F_b(x) = (b^a + x)^{1/a}\, b^{a-1} \qquad (b = 1, 2, 3, \ldots).$$
The required identity is
$$\lfloor F_b(x) \rfloor = b^a + q \quad (q \text{ as in (1)}) \qquad (2)$$
and has to be true for every positive integer b.

A. Necessary conditions (take $b = 1$)

For $b = 1$ we have $F_1(x) = (1 + x)^{1/a}$. Equation (2) turns into

$$\lfloor (1 + x)^{1/a} \rfloor = 1 + q. \qquad (3)$$

Thus

$$1 + q \le (1 + x)^{1/a} < 2 + q \quad \Leftrightarrow \tag{46}$$
$$(1 + q)^a - 1 \le x < (2 + q)^a - 1. \qquad (4) \tag{47}$$

Intersect (4) with $aq \le x < a(q + 1)$ coming from (1).
(i) $q = -1$ Here (4) is $[-1, 0)$ and (1) gives $[-a, 0)$; their intersection is

$$-1 \le x < 0. \qquad (5)$$

(ii) $q = 0$ (4) gives $0 \le x < 2^a - 1$, while (1) gives $0 \le x < a$. Hence

$$0 \le x < a. \qquad (6)$$

(iii) $q \ge 1$
The left end of (1) is aq. Because a is even and at least 2, $(1+q)^a - 1 > aq$ except when $a = 2$ and $q = 1$, where equality holds. Hence for all even $a \ge 4$ there is no intersection when $q \ge 1$; for $a = 2$ an intersection exists only for $q = 1$, namely

$$a = 2, \quad q = 1 : \quad 3 \le x < 4. \qquad (7)$$

Combining (5), (6) and (7) we obtain the only candidates

- even $a \ge 4$: $-1 \le x < a$;
- $a = 2$: $-1 \le x < 2$ or $3 \le x < 4$. $\qquad$ (8)

B. Sufficiency
Let x satisfy (8) and write $x = aq + r$ as in (1).
B1. Case $-1 \le x < 0$ ($q = -1$). Here $-1 < x \le 0$, so $-1/b^a \le x/b^a < 0$ for every b. With $t = x/b^a$ we have

$$0 < (1 + t)^{1/a} < 1, \quad \text{so} \quad 0 \le F_b(x) < b^a.$$

Also $|x| \le 1$ implies
$$F_b(x) = b^a(1 + t)^{1/a} \ge b^a - 1.$$

Hence $b^a - 1 \le F_b(x) < b^a$, i.e. $\lfloor F_b(x) \rfloor = b^a - 1 = b^a + q$.
B2. Case $0 \le x < a$ ($q = 0$). Now $0 \le x/b^a \le a/b^a < 1$, so $1 < (1 + t)^{1/a} \le 1 + t/a$. Therefore

$$b^a < F_b(x) \le b^a + x/a < b^a + 1,$$

and $\lfloor F_b(x) \rfloor = b^a = b^a + q$.
B3. The extra interval when $a = 2$: $3 \le x < 4$ ($q = 1$). Write $x = 2 + r$ with $1 \le r < 2$. For $b = 1$, $F_1(x) = \sqrt{1 + x} \in [2, 3)$, giving floor $2 = 1 + q$. For $b \ge 2$,

$$F_b(x) = b^2\sqrt{1 + x/b^2} \tag{48}$$
$$\geq b^2(1 + x/2b^2 - x^2/8b^4) \tag{49}$$
$$\geq b^2 + 1, \qquad \text{(because } 1 \leq r < 2\text{)} \tag{50}$$

while

$$F_b(x) = b^2\sqrt{1 + x/b^2} < b^2 + x/2 \leq b^2 + 2.$$

Thus $b^2 + 1 \leq F_b(x) < b^2 + 2$ and $\lfloor F_b(x) \rfloor = b^2 + 1 = b^2 + q$.
Therefore every x listed in (8) fulfils (2).

C. Failure outside the listed sets

If $x \geq a$ (and $a \geq 4$) or $x \geq 4$ (when $a = 2$) we have $q \geq 1$ with $q \neq 1$ ($a = 2$) and the intersection in (4) is empty. Hence (3), and thus the required identity, already fails for $b = 1$. If $x < -1$ the expression $(b^a + x)^{1/a}$ is not real for $b = 1$. Thus no other x works.

Conclusion

For the given even positive integer a, the real numbers x satisfying

$$\lfloor (b^a + x)^{1/a}\, b^{a-1} \rfloor = b^a + \lfloor x/a \rfloor \quad \text{for all positive integers } b$$

are

- when $a = 2$: $-1 \leq x < 2$ or $3 \leq x < 4$;
- when $a \geq 4$ even: $-1 \leq x < a$.

4.2 Variant 2

Let

$a \in 2\mathbb{N}$, $x \in \mathbb{R}$, $b \in \mathbb{N}$.

Put

$L(b) = \lfloor b^{a-1} \sqrt[a]{b^a + x}\, \rfloor, \quad R(b) = b^a + \lfloor x/a \rfloor.$

We have to find all x for which $L(b) = R(b)$ for every positive integer b.

A necessary condition coming from $b = 1$

For $b = 1$ the equality becomes

$$\lfloor (1 + x)^{1/a} \rfloor = 1 + \lfloor x/a \rfloor. \tag{51}$$

Let $n = \lfloor x/a \rfloor$ (so $n \leq x/a < n + 1$). (1) is equivalent to

$$n + 1 \leq (1 + x)^{1/a} < n + 2 \tag{52}$$
$$\Leftrightarrow (n + 1)^a - 1 \leq x < (n + 2)^a - 1. \tag{53}$$

At the same time x belongs to the strip

$$an \leq x < a(n + 1). \tag{54}$$

Hence x must lie in the intersection of (2) and (3). Denote

$$I_n = [an, a(n+1)), \tag{55}$$
$$J_n = [(n+1)^a - 1, (n+2)^a - 1). \tag{56}$$

We need $I_n \cap J_n \neq \emptyset$.

Determination of the indices n which give a non-empty intersection

(i) $n = -1$

$I_{-1} = [-a, 0)$, $J_{-1} = [-1, 0)$. Because $a \geq 2$, I_{-1} contains J_{-1}; the intersection is the whole interval $[-1, 0)$. (The left end -1 is admissible, for it keeps $b^a + x \geq 0$ when $b = 1$.)

(ii) $n = 0$

$I_0 = [0, a)$, $J_0 = [0, 2^a - 1)$. $I_0 \subset J_0$, so we obtain the interval $[0, a)$.

(iii) $n \geq 1$ For the intersection to be non-empty we need

$$(n+1)^a - 1 < a(n+1). \tag{57}$$

Because $a \geq 2$, the function $t \mapsto t^a$ grows faster than the linear function $t \mapsto at$; consequently (4) fails once its left-hand side exceeds the right-hand side. Routine checking gives

- $a = 2$: (4) is still true for $n = 1$ (because $2 \cdot 2 - 2^2 + 1 = 1 > 0$) and false for $n \geq 2$;
- $a \geq 4$: (4) is already false for $n = 1$ (indeed $2^a > 2a + 1$).

Hence

- when $a = 2$ an extra interval occurs for $n = 1$, namely $I_1 \cap J_1 = [3, 4)$;
- when $a \geq 4$ there are no further intersections.

So (1) forces

$x \in [-1, 0) \cup [0, a)$ (all even a), and, only when $a = 2$, an additional piece $[3, 4)$.

Sufficiency: each of the obtained intervals works for every b

Write $t = x/b^a$ (so $|t| \leq 1$ because $x \geq -1$).

A convenient form of $L(b)$ is

$$L(b) = \lfloor\, b^a (1+t)^{1/a} \rfloor. \tag{58}$$

We treat separately the three possible locations of x.

$-1 \leq x < 0$ (then $\lfloor x/a \rfloor = -1$)

Here $-1 \leq t < 0$. For $0 < r < 1$ the Bernoulli inequality gives

$$(1+t)^r \geq 1 + t. \tag{59}$$

With $r = 1/a$ we deduce from (6)

$(1+t)^{1/a} \geq 1 + t \geq 1 - 1/b^a$.

Multiplying by b^a we obtain

$b^a - 1 \leq b^a(1+t)^{1/a} < b^a$.

Hence, by (5), $L(b) = b^a - 1 = R(b)$.

$0 \le x < a$ **(then** $\lfloor x/a \rfloor = 0$**)**

Now $0 \le t < a/b^a \le 1$. Concavity of $(1+t)^{1/a}$ yields

$$1 < (1+t)^{1/a} < 1 + t/a. \tag{60}$$

Multiplying by b^a and recalling $t = x/b^a$ we get
$b^a < b^a(1+t)^{1/a} < b^a + x/a < b^a + 1$.
Thus $L(b) = b^a = R(b)$.

$a = 2$**,** $3 \le x < 4$ **(here** $\lfloor x/2 \rfloor = 1$**)**

Put $y = x/b^2$ (so $0.75 \le y < 4/b^2 \le 1$). For $0 \le y \le 1$ we have the two-sided estimate

$$1 + y/2 - y^2/8 \le \sqrt{1+y} < 1 + y/2. \tag{61}$$

Multiplying (8) by b^2 gives
$b^2 + 1 < b^2\sqrt{1+y} < b^2 + 2$.
Consequently $b^2 + 1 \le L(b) \le b^2 + 1$, i.e. $L(b) = b^2 + 1 = R(b)$.

Failure outside the found intervals

If $x < -1$ the radicand $1 + x$ (when $b = 1$) becomes negative. If $x \ge a$ and $x \ne 3, 4$ when $a = 2$, then with $b = 1$ equality (1) is violated. Finally, if x is a non-zero multiple of a, then for large b the term $b^a(1+t)^{1/a}$ stays strictly below $b^a + x/a$, hence $\lfloor \cdot \rfloor$ drops by one and equality again fails.

So no other x can satisfy the required identity.

Complete description of the solution set

Let a be an even positive integer.

- If $a = 2$:
$$x \in [-1, 2) \text{ or } x \in [3, 4).$$
- If $a \ge 4$:
$$x \in [-1, a).$$

All these numbers, and only these, fulfil

$$\lfloor\, b^{a-1}\sqrt[a]{b^a + x}\,\rfloor = b^a + \lfloor x/a \rfloor$$

for every positive integer b.

4.3 Final Solution

Let a be a fixed even positive integer and let

$$F_b(x) = b^{\,a-1}\sqrt[a]{\,b^a + x} \qquad (b = 1, 2, \dots).$$

The problem asks for all real x for which

$$\lfloor F_b(x) \rfloor = b^a + \lfloor x/a \rfloor \quad \text{for every positive integer } b. \quad (\star)$$

Throughout write

$$x = aq + r \quad \text{with} \quad q := \lfloor x/a \rfloor \in \mathbb{Z}, \quad 0 \le r < a. \quad (1)$$

A necessary condition coming from $b = 1$

For $b = 1$, $(\star)$ becomes

$$\lfloor (1+x)^{1/a} \rfloor = 1 + q. \quad (2)$$

Because a is even, $(1+x)^{1/a}$ is real only if $x \ge -1$, so

$$x \ge -1. \quad (3)$$

Inequality (2) is equivalent to

$$(1+q)^a - 1 \le x < (2+q)^a - 1. \quad (4)$$

Together with (1) we need the two half-lines

$$I_q = [aq,\, a(q+1)) \quad \text{and} \quad J_q = [(1+q)^a - 1,\, (2+q)^a - 1) \quad (5)$$

to intersect.

Which q give a non-empty intersection?

$q = -1 :$ $I_{-1} = [-a, 0)$, $J_{-1} = [-1, 0) \subset I_{-1}$. Intersection: $[-1, 0)$.
$q = 0 :$ $I_0 = [0, a)$, $J_0 = [0, 2^a - 1)$. Intersection: $[0, a)$.
$q \ge 1 :$ we need $(1+q)^a - 1 < a(q+1)$. • If $a \ge 4$, this already fails for $q = 1$ because $2^a - 1 > 2a$. • If $a = 2$,

$$(1+q)^2 - 1 = q^2 + 2q \le 2q + 1 \Rightarrow q^2 \le 1 \Rightarrow q = 1.$$

For $q = 1$ the intersection is $[3, 4)$.
Hence

$$x \text{ must belong to} \quad (6)$$

$$[-1, 0) \cup [0, a) \quad (\text{all even } a)$$

$$\text{and, when } a = 2, \text{ the additional interval } [3, 4).$$

Sufficiency

Rewrite $F_b(x)$:

$$F_b(x) = b^a \left(1 + \frac{x}{b^a}\right)^{1/a}. \quad (7)$$

Put $t := x/b^a$ (so $-1 \le t < 1$ for every admissible x).

$-1 \le x < 0$ $(q = -1)$

Here $-1 \le t < 0$ and $0 < r := 1/a < 1$. Bernoulli's inequality $(1+t)^r \ge 1+t$ gives

$$1 - 1/b^a \le (1+t)^{1/a} < 1.$$

Multiplying by b^a we get $b^a - 1 \le F_b(x) < b^a$, hence $\lfloor F_b(x) \rfloor = b^a - 1 = b^a + q$.

$0 \le x < a$ $(q = 0)$

Now $0 \le t < a/b^a \le 1$. Concavity of $u \mapsto (1+u)^{1/a}$ yields

$$1 < (1+t)^{1/a} < 1 + t/a.$$

Multiplying (7) gives

$$b^a < F_b(x) < b^a + x/a < b^a + 1,$$

so $\lfloor F_b(x) \rfloor = b^a = b^a + q$.

$a = 2$, $3 \le x < 4$ $(q = 1)$

Put $y := x/b^2$ ($0 < y < 1$ for $b \ge 2$, and $y \in [3,4)$ when $b = 1$).
For $0 \le y \le 1$ the estimate $\sqrt{1+y} \ge 1 + y/2 - y^2/8$ holds, while $\sqrt{1+y} < 1 + y/2$.
(i) $b = 1$: $F_1(x) = \sqrt{1+x} \in [2,3) \Rightarrow \lfloor F_1(x) \rfloor = 2 = 1 + q$.
(ii) $b \ge 2$: Using the two-sided bound and $x < 4$,

$$b^2 + 1 \le b^2\sqrt{1+y} < b^2 + 2,$$

so $\lfloor F_b(x) \rfloor = b^2 + 1 = b^2 + q$.
Thus every x listed in (6) satisfies $(\star)$.

Impossibility of other x

If $x < -1$, the radicand $1 + x$ is negative when $b = 1$; if $x \ge a$ (and, when $a = 2$, $x \notin [3,4)$), condition (2) already fails for $b = 1$. Hence no further x work.

Solution set

For the given even positive integer a, equation $(\star)$ holds for all positive integers b iff
• $a = 2 :\ -1 \le x < 2$ or $3 \le x < 4$; • $a \ge 4$ even $:\ -1 \le x < a$.

Final answer

If $a = 2$: $x \in [-1,\ 2) \cup [3,\ 4)$. If $a \ge 4$ even: $x \in [-1,\ a)$.

5 Problem 5

5.1 Variant 1

Let n be large and $\tau \in T_n$ arbitrary. Write

$$C = \{x \in [n] : x \text{ lies on a (directed) cycle of } \tau\}, \quad m = |C|, \tag{62}$$
$$h = \max\{k \geq 0 : \tau^k(x) \notin C \text{ for some } x \in [n]\}. \tag{63}$$

Thus m is the number of cyclic points of τ and h is the largest distance (pre-period) of a point to the set C.

1. A simple counting argument

After h steps every element of $[n]$ is already on a cycle, so $\tau^{h+q} = \tau^h$ whenever q is a multiple of the least common multiple of the lengths of the cycles contained in C. Consequently

$$\mathrm{ord}(\tau) \leq h + L \quad (1) \tag{64}$$

where

$$L = \mathrm{lcm}\{\text{lengths of the cycles of } \tau\}.$$

2. Bounding L

Restrict τ to the set C ($|C| = m$). There it is a permutation, so L is the order of a permutation of m letters; hence

$$L \leq f(m) \leq f(n-h) \quad (2) \tag{65}$$

because $m \leq n - h$ and f is non-decreasing.

Combining (1) and (2),

$$\mathrm{ord}(\tau) \leq h + f(n-h) \quad (3) \tag{66}$$

It remains to show

$$h + f(n-h) < f(n) + n^{0.501}. \quad (4) \tag{67}$$

The estimate will be done separately for "small" and "large" h.

A quantitative estimate for f

Landau (1903) proved

$$\exp((1-\varepsilon)\sqrt{m\log m}) \leq f(m) \leq \exp((1+\varepsilon)\sqrt{m\log m}) \quad (5) \tag{68}$$

for every $\varepsilon > 0$ and all $m \geq m_0(\varepsilon)$. Fix $\varepsilon = 0.01$ and assume $n \geq N_0$ so that (5) is valid for every $m \leq n$.

---

3. Case $h \leq n^{0.501}$

Here $f(n-h) \leq f(n)$, so (3) gives

$$\mathrm{ord}(\tau) \leq f(n) + h \leq f(n) + n^{0.501}, \quad (6) \tag{69}$$

and (4) is proved in this range.

---

4. Case $h > n^{0.501}$
Put $\alpha = h/n$; then $\alpha \geq n^{-0.499}$. Because f is increasing, (3) and (4) are equivalent to

$$f(n) - f(n-h) \geq h - n^{0.501}. \quad (7) \tag{70}$$

We estimate the left side with (5). Using $\log(n-h) < \log n$,

$$\log f(n) - \log f(n-h) \geq 0.99\sqrt{n \log n} - 1.01\sqrt{(n-h)\log n} \tag{71}$$
$$= \sqrt{n \log n} \cdot [0.99 - 1.01\sqrt{1-\alpha}]. \tag{72}$$

For $0 < \alpha \leq 1/2$ one has $1 - \sqrt{1-\alpha} \geq \alpha/2$, whence

$$0.99 - 1.01\sqrt{1-\alpha} \geq 0.49\alpha \geq 0.49 n^{-0.499}. \quad (8) \tag{73}$$

Thus

$$\log(f(n)/f(n-h)) \geq 0.49 n^{-0.499}\sqrt{n \log n} \tag{74}$$
$$= 0.49 n^{0.001}\sqrt{\log n}. \tag{9}$$

For sufficiently large n the right side of (9) exceeds $2\log n$, hence

$$f(n) \geq n^2 f(n-h). \quad (10) \tag{75}$$

Therefore

$$f(n) - f(n-h) \geq (1 - n^{-2})f(n) \geq \frac{1}{2}f(n). \quad (11) \tag{76}$$

Because $\sqrt{n \log n} \gg \log n$, (5) implies $f(n) > n^{10}$ for large n, so

$$\frac{1}{2}f(n) > n \geq h \quad (> n^{0.501}). \quad (12) \tag{77}$$

Combining (11) and (12) we obtain

$$f(n) - f(n-h) \geq h - n^{0.501}, \quad (13) \tag{78}$$

which is exactly the desired inequality (7).

---

5. Completion of the proof
Inequality (4) has been verified in both cases. Consequently, for every $\tau \in T_n$

$$\operatorname{ord}(\tau) < f(n) + n^{0.501} \quad (14) \tag{79}$$

once n is larger than the fixed constant N_0 chosen above. Taking the maximum over τ gives

$$g(n) < f(n) + n^{0.501}$$

for all sufficiently large n, completing the proof.

Final answer.

For all sufficiently large integers n one has

$$g(n) < f(n) + n^{0.501}.$$

5.2 Variant 2

Let

- τ be an arbitrary map in T_n,
- $C(\tau)$ the set of points which lie on (directed) cycles of τ,
- $h(\tau)$ the maximal distance of a point of $[n]$ from $C(\tau)$ (“height” of the forest which is attached to the cycles),
- $\lambda(\tau)$ the least positive integer for which τ^λ acts as the identity on $C(\tau)$ (the eventual period).

$$|C(\tau)| + h(\tau) \leq n. \tag{80}$$

Indeed, the longest directed path which starts outside $C(\tau)$ and ends in $C(\tau)$ contains $h(\tau)$ different non-cyclic vertices which are not counted in $|C(\tau)|$.

The number of different powers of τ

Every power τ^k ($k \geq 0$) acts identically on all vertices which are at distance $\geq k$ from $C(\tau)$. Consequently the sequence $(\tau^k)_{k\geq 0}$ stabilises after $h(\tau)$ steps and then becomes periodic with period $\lambda(\tau)$. Thus

$$\operatorname{ord}(\tau) = h(\tau) + \lambda(\tau). \tag{81}$$

$\lambda(\tau)$ is bounded by the Landau function

Restricted to $C(\tau)$ the map τ is a permutation of $|C(\tau)|$ points whose order is $\lambda(\tau)$. Extending this permutation by fixed points on the remaining letters we obtain an element of $S_{n-h(\tau)}$ having the same order. Therefore

$$\lambda(\tau) \leq f(|C(\tau)|) \leq f(n - h(\tau)). \tag{82}$$

A convenient analytic estimate for f

Landau proved that there are positive constants $A < B$ such that for all sufficiently large m

$$A\sqrt{m \log m} \leq \log f(m) \leq B\sqrt{m \log m}. \tag{83}$$

(The lower bound can be obtained constructively from the product of the primes not exceeding $\sqrt{m \log m}$; the upper bound follows from Stirling's formula, but we shall only use the lower one.)

From (4) we shall need two easy consequences.

Lemma: Fast growth of f.

If m is large and $s \geq m^{0.501}$ then

$$f(m+s) \geq (s+1) \cdot f(m). \tag{84}$$

Proof

Put $g(x) = \sqrt{x \log x}$. For $x \geq m$ we have

$$g'(x) = \frac{\log x + 1}{2\sqrt{x \log x}} \geq \frac{1}{2} \frac{\sqrt{\log m}}{\sqrt{x}}.$$

Hence for each such x

$$g(x+1) - g(x) \geq \frac{1}{2} \frac{\sqrt{\log m}}{\sqrt{x+1}} \geq \frac{1}{2} \frac{\sqrt{\log m}}{\sqrt{2m}}.$$

Summing this inequality for $x = m, \ldots, m+s-1$ yields

$$g(m+s) - g(m) \geq s \cdot \frac{1}{2} \frac{\sqrt{\log m}}{\sqrt{2m}} \geq \frac{1}{4} s \cdot \frac{\sqrt{\log m}}{\sqrt{m}}. \tag{85}$$

If $s \geq m^{0.501}$ then the right-hand side of (6) is $\geq c \log(s+1)$ with a fixed $c > 0$, whence by (4)

$$\log f(m+s) - \log f(m) \geq \log(s+1) \quad \Rightarrow \quad f(m+s) \geq (s+1) f(m). \quad \square$$

Lemma

For all large m one has $f(m) > m^2$.

This follows at once from (4) because $\sqrt{m \log m} > 2 \log m$ when m is large.

Reduction to a one-variable maximisation

Put

$$\Phi_n(h) = h + f(n-h) \quad (0 \leq h \leq n).$$

By (2) and (3) we have $\operatorname{ord}(\tau) \leq \Phi_n(h(\tau))$; consequently

$$g(n) \leq \max_{0 \leq h \leq n} \Phi_n(h). \tag{86}$$

Evaluating the maximum

Split the range of h into two parts.

(A) $h \leq n^{0.501}$. Here $\Phi_n(h) \leq h + f(n) \leq f(n) + n^{0.501}$.

(B) $h \geq n^{0.501}$. Because of (1) we have $n - h \geq n^{0.499}$, so Lemma 2 gives $f(n-h) > n \geq h$. By Lemma 1 (with $m = n-h$ and $s = h$) we obtain

$$f(n) \geq (h+1) \cdot f(n-h) > h \cdot f(n-h). \tag{87}$$

Hence $f(n-h) \leq f(n)/h$ and therefore

$$\Phi_n(h) = h + f(n-h) \leq h + f(n)/h \leq f(n) + 1 \leq f(n) + n^{0.501}. \tag{88}$$

(The last inequality uses $h \geq n^{0.501} \geq 1$ and the fact that $f(n)$ dominates every polynomial for large n.)

Combining (8), (A) and (B) we have shown that for all sufficiently large n

$$g(n) \leq f(n) + n^{0.501}.$$

Since both $g(n)$ and $f(n)$ are integers, this inequality is in fact strict.

Final answer

For every sufficiently large positive integer n,

$$g(n) < f(n) + n^{0.501}.$$

5.3 Variant 3

Let

- τ be an arbitrary self-map of $[n]$
- $d(\tau)$ the length of the longest directed path that ends in a cycle
- $C(\tau)$ the set of the elements which lie on the cycles of τ, $|C(\tau)| = c(\tau)$.

Write $\tau^k = \tau \circ \ldots \circ \tau$ (k-times) and put $L(\tau) = \mathrm{ord}(\tau)$ (= the number of distinct maps among $\tau, \tau^2, \tau^3, \ldots$).

A general upper bound for $L(\tau)$

For every $x \in [n]$ the sequence $(\tau^k(x))_{k\geq 0}$ is eventually periodic. After $d(\tau)$ steps every element has reached a cycle, and from that moment on the map behaves like a permutation of the set $C(\tau)$. Hence

$$L(\tau) \leq d(\tau) + \mathrm{ord}(\sigma), \tag{89}$$

where σ is the permutation $\tau|_{C(\tau)}$. Call

$$f(t) = \max\{\mathrm{ord}(\pi) : \pi \in S_t\}.$$

Inequality (1) gives

$$L(\tau) \leq d(\tau) + f(c(\tau)). \tag{90}$$

Because $c(\tau) \leq n$ and f is increasing,

$$L(\tau) \leq d(\tau) + f(n). \tag{91}$$

If $d(\tau) \leq n^{0.501}$ formula (3) already yields

$$L(\tau) \leq f(n) + n^{0.501}, \tag{92}$$

so the desired theorem is proved in this case. Hence, from now on suppose

$$d := d(\tau) > n^{0.501}. \tag{93}$$

A permutation contains almost every point

Because a directed path of length d uses d vertices that are **not** on a cycle, we have

$$c := c(\tau) = n - d. \tag{94}$$

With $d > n^{0.501}$ we still have $c = n - d \geq n - n^{0.501} \to \infty$, so c is large.

Two estimates we shall use

A) (Landau, 1903) $\log f(m) = (1 + o(1))\sqrt{m \log m}$.

B) (Chebyshev/Prime Number Theorem) For every sufficiently large t there is a prime in $(\frac{t}{2}, t]$.

A prime that does not divide ord(σ)

Put $\ell := \mathrm{ord}(\sigma)$; by definition $\ell \leq f(c) \leq f(n)$.

If all primes not exceeding d divided ℓ, then the *primorial*

$$P(d) = \prod_{p \leq d} p$$

would satisfy $P(d)|\ell \leq f(n)$. But, by well-known estimates,

$$\log P(d) = (1 + o(1))d \quad \text{(Mertens)}$$

while, by (7), $\log f(n) = (1 + o(1))\sqrt{n \log n}$.

For $d > n^{0.501}$ we have $d \gg \sqrt{n \log n}$, so $P(d) > f(n)$ for large n, a contradiction. Therefore

there exists a prime $q \leq d$ that does **not** divide ℓ.

Using (8) we may (and do) choose q with

$$\frac{d}{2} < q \leq d. \tag{95}$$

Building a better map

Take q vertices among the $d = n - c$ points that do **not** lie on a cycle and connect them into one q-cycle. All remaining $(d - q)$ vertices are mapped directly to one fixed point on a cycle (so their distance to a cycle is now 1).

Call the new map $\tilde{\sigma}$. What are its properties?

- The behaviour of τ on C is untouched, so ord on C is still ℓ.
- The new q-cycle contributes the factor q to the eventual period, and because $q \nmid \ell$ we have

  ord of the permutation part $= \ell q$.
- The longest distance to a cycle in $\tilde{\sigma}$ is at most 1.

Hence

$$L(\tilde{\sigma}) \geq \ell q. \tag{96}$$

On the other hand, from (2) and (5)

$$L(\tau) \leq d + \ell. \tag{97}$$

$\tilde{\sigma}$ beats τ

Because $q \geq \frac{d}{2}$ and $q \geq 3$ (for large n) we compute

$$\begin{aligned} \ell q - (\ell + d) &= \ell(q-1) - d \qquad (98) \\ &\geq \ell\left(\frac{d}{2} - 1\right) - d \qquad (99) \\ &> 0 \qquad (100) \end{aligned}$$

(the last inequality uses $\ell \geq 2$ and $d \gg 1$).

Combining (11), (12) and (13) we get

$$L(\tilde{\sigma}) > L(\tau),$$

contradicting the choice of τ when $L(\tau) = g(n)$.

Therefore the assumption (5) is impossible; hence every map that achieves the maximal order satisfies $d \leq n^{0.501}$. Plugging this into (3) gives

$$g(n) \leq f(n) + n^{0.501}. \tag{101}$$

Conclusion

For all sufficiently large n we have proved

$$g(n) < f(n) + n^{0.501}. \quad \square$$

Final answer

For all sufficiently large n,

$$g(n) \;<\; f(n) \;+\; n^{0.501}.$$

5.4 Final Solution

Let

- $[n] = \{1, 2, \ldots, n\}$,
- S_n the set of permutations of $[n]$,
- T_n the set of all self-maps of $[n]$.

For $\tau \in T_n$ write
$$\operatorname{ord}(\tau) = |\{\tau, \tau^2, \tau^3, \ldots\}|$$
(the number of distinct iterates).

Denote
$$f(n) = \max_{\sigma \in S_n} \operatorname{ord}(\sigma), \quad g(n) = \max_{\tau \in T_n} \operatorname{ord}(\tau).$$

The goal is to prove
$$g(n) < f(n) + n^{0.501} \quad \text{for all sufficiently large } n. \quad (*)$$

The shape of an arbitrary map

For $\tau \in T_n$ let
$$C = C(\tau) = \{x : x \text{ lies on a (directed) cycle of } \tau\}, \quad m = |C|, \tag{102}$$
$$h = h(\tau) = \max\{k \geq 0 : \text{some } x \text{ satisfies } \tau^k(x) \in C \text{ but } \tau^{k-1}(x) \notin C\}. \tag{103}$$

Thus h is the largest distance of a vertex from the set of cycles (the “height” of the rooted trees that feed the cycles).

After h steps every element already sits on a cycle, so afterwards the map repeats with the period of the permutation $\tau|_C$. Consequently
$$\operatorname{ord}(\tau) \leq h + L, \quad L = \operatorname{lcm}\{\text{lengths of the cycles of } \tau|_C\}. \quad (1)$$

Restricted to C, τ is a permutation of m points, hence
$$L \leq f(m). \quad (2)$$

Because there are h non-cyclic vertices on a longest path, $m \leq n - h$; moreover f is increasing, so
$$L \leq f(n-h). \quad (3)$$

Combining (1)–(3) we obtain, for every $\tau \in T_n$,
$$\operatorname{ord}(\tau) \leq \Phi_n(h) := h + f(n-h), \quad 0 \leq h \leq n. \quad (4)$$

Landau's estimate

A classical result of Landau states that for every $\varepsilon > 0$ there is $N(\varepsilon)$ such that for all $m \geq N(\varepsilon)$
$$\exp((1-\varepsilon)\sqrt{m \log m}) \leq f(m) \leq \exp((1+\varepsilon)\sqrt{m \log m}). \quad (5)$$

Fix $\varepsilon = \frac{1}{4}$ and assume $n \geq N := N(\frac{1}{4})$.

Bounding $\Phi_n(h)$ when h is small

If $h \leq n^{0.501}$ then from (4)
$$\Phi_n(h) \leq f(n) + n^{0.501}. \quad (6)$$

Bounding $\Phi_n(h)$ when h is large

Henceforth assume $h \geq n^{0.501}$.

We first prove that

$$f(n) \geq h \cdot f(n-h). \quad (7)$$

Using (5),

$$\log f(n) - \log f(n-h) \geq \frac{3}{4}(\sqrt{n \log n} - \sqrt{(n-h)\log(n-h)}) \quad (8)$$

Put $g(t) = \sqrt{t \log t}$. For $x \geq 1$,

$$g'(x) = \frac{\log x + 1}{2\sqrt{x \log x}} \geq \frac{1}{2}\sqrt{\frac{\log x}{x}}.$$

Therefore

$$\begin{aligned}
g(n) - g(n-h) &= \int_{n-h}^{n} g'(x)dx && (104)\\
&\geq \frac{1}{2}\sqrt{\log(n-h)} \int_{n-h}^{n} \frac{dx}{\sqrt{x}} && (105)\\
&= \frac{1}{2}\sqrt{\log(n-h)} \cdot 2(\sqrt{n} - \sqrt{n-h}) && (106)\\
&\geq \frac{h\sqrt{\log(n-h)}}{2\sqrt{n}}. \quad (9) && (107)
\end{aligned}$$

Because $h \geq n^{0.501}$, we have $h/\sqrt{n} \geq n^{0.001}$. Hence the right-hand side of (9) is at least $cn^{0.001}\sqrt{\log n}$ for a fixed $c > 0$, and for large n this quantity exceeds $\log h$ (indeed, $\log h \leq \log n$). Combining (8) and (9) we get

$$\log f(n) - \log f(n-h) > \log h, \quad \text{whence } f(n) \geq hf(n-h). \quad (10)$$

Now

$$\Phi_n(h) = h + f(n-h) \leq h + \frac{f(n)}{h} \leq f(n) + h \leq f(n) + n^{0.501}. \quad (11)$$

Maximising $\Phi_n(h)$

From (6) and (11) we have $\Phi_n(h) \leq f(n) + n^{0.501}$ for every $0 \leq h \leq n$, so by (4)

$$g(n) = \max_{\tau \in T_n} \operatorname{ord}(\tau) \leq \max_{0 \leq h \leq n} \Phi_n(h) \leq f(n) + n^{0.501}. \quad (12)$$

Because both $g(n)$ and $f(n)$ are integers, inequality (12) is in fact strict once $n \geq N$.

Conclusion

There exists N_0 such that for all $n \geq N_0$

$$g(n) < f(n) + n^{0.501},$$

which is exactly inequality $(*)$.

6 Problem 6, Day 2

6.1 Variant 1

Let $F(x) = f(x) - k$, $x > 0$.

Because $f(a) = f(b) = k$ we have $F(a) = F(b) = 0$.

1. Introduce an auxiliary function

$$h : [a, b] \to \mathbb{R}, \quad h(x) = \frac{F(x)}{x} = \frac{f(x) - k}{x}$$

- $x \geq a > 0$, so the denominator never vanishes;
- f is C^1 on $(0, \infty)$, hence h is continuous on $[a, b]$ and differentiable on (a, b).

2. Evaluate h at the endpoints:

$$h(a) = \frac{0}{a} = 0, \quad h(b) = \frac{0}{b} = 0$$

Thus $h(a) = h(b)$.

3. Apply Rolle's theorem to h on $[a, b]$. There exists $\xi \in (a, b)$ such that $h'(\xi) = 0$.

4. Compute $h'(x)$:

$$h'(x) = \frac{F'(x)x - F(x)}{x^2} = \frac{xf'(x) - (f(x) - k)}{x^2} = -\frac{f(x) - xf'(x) - k}{x^2}$$

Setting $h'(\xi) = 0$ gives

$$f(\xi) - \xi f'(\xi) - k = 0 \quad \Rightarrow \quad f(\xi) - \xi f'(\xi) = k$$

Hence there exists a point ξ in (a, b) satisfying the required relation.

Final answer: There exists $\xi \in (a, b)$ such that $f(\xi) - \xi f'(\xi) = k$.

6.2 Final Solution

Let $h(x) = \frac{f(x)-k}{x}$, $x \in [a, b] \subset (0, \infty)$.

Properties of h:

- Since $a > 0$ and f is C^1 on $(0, \infty)$, h is continuous on $[a, b]$ and differentiable on (a, b).
- At the endpoints:

$$h(a) = \frac{f(a) - k}{a} = 0, \quad h(b) = \frac{f(b) - k}{b} = 0$$

Because $h(a) = h(b)$, Rolle's theorem guarantees a point $\xi \in (a, b)$ such that $h'(\xi) = 0$.

Compute h' for $x > 0$:

$$h'(x) = \frac{xf'(x) - (f(x) - k)}{x^2} = -\frac{f(x) - xf'(x) - k}{x^2}$$

Setting $x = \xi$ and using $h'(\xi) = 0$ yields:

$$f(\xi) - \xi f'(\xi) - k = 0 \quad \Rightarrow \quad f(\xi) - \xi f'(\xi) = k$$

Thus there exists a point ξ in (a, b) satisfying the required relation.

Final answer: There is $\xi \in (a, b)$ such that $f(\xi) - \xi f'(\xi) = k$.

7 Problem 7

Problem Statement

Let $M \subset \mathbb{Z}_{>0}$ be a non-empty set that satisfies

(a) $x \in M \Longrightarrow 2x \in M$;

(b) $x, y \in M,\ x + y$ even $\Longrightarrow \frac{x+y}{2} \in M$.

7.1 Variant 1

Throughout the solution call the two operations

(i) doubling: $x \mapsto 2x$;

(ii) halved sum: $x, y \mapsto \frac{x+y}{2}$ $(x + y$ even).

1. M is closed under addition If $x, y \in M$ have the same parity, $m = \frac{x+y}{2} \in M$ by (b) and $x + y = 2m \in M$ by (a). If they have opposite parity, $2x, 2y \in M$ are even, so $x + y = \frac{2x+2y}{2} \in M$. Hence

$$\text{(c) } x, y \in M \Longrightarrow x + y \in M. \tag{108}$$

Consequently every positive integral multiple of an element of M again lies in M.

2. M contains an odd integer Let $m = \min M$. If m is odd we are done. If m is even, $2m \in M$ by (a) and $x = \frac{m+2m}{2} = \frac{3m}{2}$ is an integer (because m is even). Property (b) puts x into M; x is odd, so M indeed contains at least one odd number.

3. The least odd member Denote by d the least odd element of M. We now show that every element of M is a multiple of d.

(i) Every multiple of d lies in M. First $d \in M$. Suppose $kd \in M$ $(k \geq 1)$. Then $2kd,\ 2d \in M$ by (a); these two numbers are even, hence $\frac{2kd+2d}{2} = (k + 1)d \in M$ by (b). By induction $kd \in M$ for every $k \in \mathbb{Z}_{>0}$.

(ii) No other numbers can belong to M. Assume, for a contradiction, that $x \in M$ and $d \nmid x$. Division with remainder gives $x = qd+r$ $(q \geq 1,\ 0 < r < d)$. The remainder r is necessarily odd (because d and x have opposite parities), hence $r < d$ contradicts the minimality of d once we prove $r \in M$.

To obtain r we use the already established closure properties. Because $d \in M$ and $q \geq 1$, property (c) yields $qd \in M$. Now $2x,\ qd \in M$ are both even, so

$$\frac{2x + qd}{2} = qd + r \in M.$$

Repeating the same procedure with the two even members $2x$ and $2(qd + r) = 2qd + 2r$ gives

$$\frac{2x + 2qd + 2r}{2} = qd + 2r \in M.$$

Continuing in this way we successively obtain the odd numbers $qd+r,\ qd+2r, \ldots, qd+qr = q(d+r)$ and finally r itself, a contradiction. Hence no such x exists and every element of M is a multiple of d.

Therefore $M = \{kd \mid k \in \mathbb{Z}_{>0}\}$.

4. The number d is odd By construction d is odd; thus M is the set of positive multiples of an odd integer.

5. Sufficiency Conversely, for an arbitrary odd positive integer d the set $M_d = d\mathbb{Z}_{>0}$ satisfies (a) (obvious) and (b): if $x = da,\ y = db$ and $x + y$ is even then $a + b$ is even (because d is odd), so

$$\frac{x+y}{2} = d\,\frac{a+b}{2} \in M_d.$$

Hence the required subsets are exactly the sets of all positive multiples of an odd integer.

Final answer. All non-empty subsets $M \subseteq \mathbb{Z}_{>0}$ fulfilling (a) and (b) are

$$M = \{\, kd \mid k \in \mathbb{Z}_{>0} \,\},$$

where d is any fixed odd positive integer.

7.2 Final Solution

Let

$$M \subset \mathbb{Z}_{>0}, \quad M \neq \emptyset,$$

satisfy

$$\text{(a) } x \in M \Rightarrow 2x \in M, \tag{109}$$

$$\text{(b) } x, y \in M \text{ and } x + y \text{ even} \Rightarrow (x + y)/2 \in M. \quad (\star) \tag{110}$$

0. Preliminary: M is closed under addition If x, y have the same parity, put $m := (x + y)/2 \in M$ by ($\star$); then $x + y = 2m \in M$ by (a). If x, y have opposite parity, $2x, 2y \in M$ by (a) and the previous line gives $x + y = (2x + 2y)/2 \in M$. Hence

$$\text{(c) } x, y \in M \Rightarrow x + y \in M.$$

1. The g.c.d. of M Let $g := \gcd M$. Because of (c), g is the least positive element of the additive semigroup generated by M and therefore itself belongs to M.

1.1 g cannot be even. Take g ($\in M$) and $2g$ ($\in M$ by (a)); g is even, hence $g + 2g$ is even and $(g + 2g)/2 = 3g/2 \in M$ by ($\star$). Since $3g/2$ is not divisible by g, we obtain a contradiction. Thus g is odd.

From now on g is odd.

2. If $g > 1$, then $M = g\mathbb{Z}_{>0}$

(i) Every multiple of g lies in M: starting from $g \in M$, apply (a) once to get $2g$, and combine $2g$ and g with ($\star$) to obtain $3g$; combining $2g$ and $3g$ gives $5g$, and so on. A straightforward induction yields $gk \in M$ for every $k \geq 1$.

(ii) No other numbers occur: by definition of g every element of M is a multiple of g. Consequently

$$M = \{g, 2g, 3g, \ldots\} \quad (g \text{ odd}, g > 1). \quad (1)$$

3. The case $g = 1$ Put $N := \min M$. We show that all integers $\geq N$ belong to M.

Step 1 – M already contains two consecutive integers. Because $g = 1$, M contains both an even and an odd element. Let e be even, o be odd, and assume $e < o$ (both in M). The elements $2e, 2o$ are in M by (a) and $2e + 2o$ is even, so ($\star$) gives $e + o \in M$, which lies strictly between e and o. Repeating the procedure we eventually reach $e + 1$; hence

$$N, N+1 \in M. \quad (2)$$

Step 2 – an induction. Suppose the block $\{N, N+1, \ldots, N+t\}$ ($t \geq 1$) is already in M. Two cases.

- $N + t + 1$ is even. Then $2(N + t - 1), 2(N + t + 1)$ are even and belong to M by (a); their average is $N + t$, which is in M. Averaging $N + t$ and $2(N + t) = 2N + 2t$ gives $N + t + 1 \in M$.
- $N + t + 1$ is odd. Use the same argument with $2(N + t)$ and $2(N + t + 2)$.

Thus $N + t + 1 \in M$, and induction proves

$$M = \{n \in \mathbb{Z}_{>0} : n \geq N\}. \quad (3$$

4. Sufficiency (1) is easily seen to fulfil (a) and ($\star$) because g is odd. In (3) both operations clearly keep every element $\geq N$, so (a) and ($\star$) hold as well.

Final answer. All non-empty subsets $M \subset \mathbb{Z}_{>0}$ that satisfy (a) and (b) are exactly

1. $M = \{g, 2g, 3g, \ldots\}$ for some fixed odd positive integer g;
2. $M = \{N, N+1, N+2, \ldots\}$ for some fixed positive integer N.

8 Problem 8

Let

$$J = (J_{ij}), \quad J_{ij} = \delta_{i,n+1-j} \quad (1 \leq i, j \leq n),$$

i.e. J reverses the order of the coordinates. J is real, symmetric and orthogonal: $J^T = J$, $J^2 = I$.

1. The "quarter-turn" operator

For $A \in M_n(\mathbb{R})$ define A^R by rotating the matrix through 90°:

$$(A^R)_{ij} = A_{j,n+1-i}.$$

A short computation gives

$$A^R = JA^T. \quad (1)$$

Hence the hypothesis $A = A^R$ is equivalent to

$$A = JA^T \Leftrightarrow A^T = JA. \quad (2)$$

2. Diagonalising J

Because J is real symmetric, there exists an orthogonal P such that

$$P^T JP = D = \operatorname{diag}(I_r, -I_s), \quad r + s = n. \quad (3)$$

3. The basic equation in the new basis

Put

$$B = P^T AP. \quad (4)$$

Taking transposes in (4) and using (2)–(3):

$$B = P^T AP = P^T JA^T P = DB^T. \quad (5)$$

4. Block decomposition

Write B conformally with D,

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad B_{11} \in \mathbb{R}^{r\times r}, \quad B_{22} \in \mathbb{R}^{s\times s}. \quad (6)$$

Equation (5) yields

$$B_{11} = B_{11}^T, \quad B_{12} = B_{21}^T, \quad B_{21} = -B_{12}^T, \quad B_{22} = -B_{22}^T.$$

Combining the middle two equalities gives $B_{12} = -B_{12}$; hence

$$B_{12} = B_{21} = 0. \quad (7)$$

Thus

$$B = \operatorname{diag}(S, K) \text{ with } S^T = S \text{ (real symmetric } r \times r),$$
$$K^T = -K \text{ (real skew-symmetric } s \times s). \quad (8)$$

Because P is orthogonal, A and B are similar and have the same eigenvalues.

5. Spectra of S and K

- A real symmetric matrix S has only real eigenvalues.
- A real skew-symmetric matrix K satisfies $K^T = -K$; for any eigenpair $Kx = \lambda x$ $(x \neq 0)$,

$$x^T Kx = \lambda x^T x = -x^T K^T x = -\bar{\lambda} x^T x \Rightarrow \lambda + \bar{\lambda} = 0,$$

so λ is purely imaginary (or 0).

6. Eigenvalues of A

The spectrum of A is the union of the spectra of S (real numbers) and K (purely imaginary numbers). Therefore every eigenvalue λ of A satisfies

$$\text{Re } \lambda = 0 \text{ or Im } \lambda = 0.$$

9 Problem 9

Problem Statement

Let $X_1, X_2, \dots$ be produced successively as follows: At every step write the still available positive integers in increasing order and, moving from left to right, flip a fair coin over every entry. The first number that receives a head is taken as the next value X_i. (Equivalently, the i-th smallest surviving integer is chosen with probability 2^{-i}.)

For a fixed $n \geq 1$ denote

$$Y_n = \max\{X_1, \dots, X_n\}.$$

We determine $\mathbb{E}[Y_n]$.

Solution

The distribution of the maximum

Fix $m \geq n$. Immediately after k ($0 \leq k \leq n-1$) draws have been made at most $m-1-k$ candidates not exceeding $m-1$ are still alive. The next draw stays below m iff the head appears among the first $m-k$ coins:

$$P(\text{the } (k+1)\text{-st draw} < m) = 1 - 2^{-(m-k)}.$$

Since the n draws are independent,

$$\begin{aligned} P(Y_n < m) &= \prod_{k=0}^{n-1} (1 - 2^{-(m-k)}) && (111) \\ &= \prod_{r=m-n+1}^{m} (1 - 2^{-r}), \quad (m \geq n). && (1) \end{aligned}$$

For $m \leq n$ the probability in (1) is 0. Therefore, for every integer $t \geq 1$,

$$P(Y_n \geq n + t) = 1 - \prod_{r=t}^{t+n-1} (1 - 2^{-r}). \quad (2)$$

Writing the expectation through tail probabilities

Because $\mathbb{E}[Y_n] = \sum_{k\geq 1} P(Y_n \geq k)$ and $P(Y_n \geq k) = 1$ for $k \leq n$,

$$\mathbb{E}[Y_n] = n + S_n \text{ with} \tag{112}$$

$$S_n = \sum_{t=1}^{\infty} \left[1 - \prod_{r=t}^{t+n-1} (1 - 2^{-r}) \right]. \tag{3}$$

A convenient decomposition of the summands

For fixed t define

$$q_{t,k} = 2^{-(t+k)} \prod_{r=t}^{t+k-1} (1 - 2^{-r}), \quad k = 0, 1, \dots. \quad (4)$$

$q_{t,k}$ is the probability that among the n factors $(1 - 2^{-t}), \dots, (1 - 2^{-t-n+1})$ the first one that "fails" is the $(k+1)$-st. Consequently

$$1 - \prod_{r=t}^{t+n-1} (1 - 2^{-r}) = \sum_{k=0}^{n-1} q_{t,k}. \quad (5)$$

Insert (5) into (3) and interchange the order of summation:

$$S_n = \sum_{k=0}^{n-1} S_k, \text{ where} \tag{113}$$

$$S_k = \sum_{t=1}^{\infty} q_{t,k}. \tag{6}$$

Evaluation of S_k

Put

$$S_k = \sum_{t=1}^{\infty} 2^{-(t+k)} \prod_{r=t}^{t+k-1} (1 - 2^{-r}). \quad (7)$$

To compute S_k observe that

$$\prod_{r=t}^{t+k-1} (1 - 2^{-r}) = \Pi_t / \Pi_{t+k}, \text{ with } \Pi_s = \prod_{r=s}^{\infty} (1 - 2^{-r}).$$

Since $\Pi_{s+1} = \Pi_s / (1 - 2^{-s})$, we have $\Pi_s - \Pi_{s+1} = 2^{-s} \Pi_{s+1}$. Therefore

$$2^{-(t+k)} \prod_{r=t}^{t+k-1} (1 - 2^{-r}) = (\Pi_t - \Pi_{t+1}) / \Pi_{t+k}. \quad (8)$$

Because $\Pi_t \to 0$ as $t \to \infty$, the series (7) telescopes:

$$S_k = \sum_{t=1}^{\infty} (\Pi_t - \Pi_{t+1})/\Pi_{t+k} \tag{114}$$
$$= \Pi_1/\Pi_{1+k}. \tag{9}$$

Using $\Pi_1/\Pi_{1+k} = \prod_{r=1}^{k}(1-2^{-r})$ and the elementary identity

$$\prod_{r=1}^{k}(1-2^{-r}) = \frac{1}{2^{k+1}-1}, \quad (10)$$

we obtain

$$S_k = \frac{1}{2^{k+1}-1}, \quad k = 0, 1, \ldots. \quad (11)$$

The expected maximum

Insert (11) into (6):

$$S_n = \sum_{k=0}^{n-1} \frac{1}{2^{k+1}-1} = \sum_{i=1}^{n} \frac{1}{2^i - 1}. \quad (12)$$

Finally, from (3),

$$\mathbb{E}[Y_n] = n + \sum_{i=1}^{n} \frac{1}{2^i-1} \tag{115}$$
$$= \sum_{i=1}^{n} \left(1 + \frac{1}{2^i-1}\right) \tag{116}$$
$$= \sum_{i=1}^{n} \frac{2^i}{2^i-1}. \tag{13}$$

Final Answer

$$\boxed{\mathbb{E}[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i-1}}$$

10 Problem 10

Problem Statement

Let

$$S_N = \#\{(a,b) : 1 \le a, b \le N, (a^2+a)(b^2+b) \text{ is a perfect square}\}$$

Our goal is to evaluate

$$L = \lim_{N\to\infty} S_N/N \qquad (\star)$$

10.1 Variant 1

Put

$f(n)$ = the square-free kernel of $n(n+1)$, i.e. $f(n) = \prod_{p\ (\text{prime})} p$ iff $v_p(n(n+1))$ is odd.

Because n and $n+1$ are coprime, $f(n)$ is square-free and

$(a^2+a)(b^2+b)$ is a square $\iff$ $f(a) = f(b)$. (1)

For $1 \le n \le N$ write

$m_d(N) = |\{n \le N : f(n) = d\}|$ (d square-free).

With (1)

$S_N = \sum_d m_d(N)^2 = N + \sum_d m_d(N)(m_d(N)-1)$. (2)

(the term N comes from the diagonal $a = b$).

We prove that the second summand in (2) is $o(N)$; then

$\lim_{N\to\infty} S_N/N = 1$.

---

1. Estimating $m_d(N)$

Put $x = n$, $y = \sqrt{n(n+1)/d}$. Equation $f(n) = d$ is equivalent to

$x(x+1) = dy^2$. (3)

Multiply (3) by 4 and set $X = 2x+1$, $Y = 2y$; then

$X^2 - 4dY^2 = 1, \quad X \text{ odd}, Y \text{ even}.$ (4)

(4) is a Pell equation with parameter $D = 4d$. All its positive solutions are obtained from the fundamental solution (X_1, Y_1) by

$X_k + Y_k\sqrt{D} = (X_1 + Y_1\sqrt{D})^k \quad (k = 0, 1, 2, \ldots)$.

Hence the X-components grow exponentially:

$X_k \ge (X_1 + Y_1\sqrt{D})^k/2 \ge \varepsilon_d^k, \quad \varepsilon_d := X_1 + Y_1\sqrt{D} > 1.$

Consequently the number of solutions of (4) with $X \le 2N+1$ is at most

$m_d(N) \le 1 + \log(2N+1)/\log \varepsilon_d$. (5)

Because $\varepsilon_d > \sqrt{D} = 2\sqrt{d}$ we get the useful bound

$m_d(N) \le 1 + \log(2N)/\log(2\sqrt{d}) \le C \cdot \log N/\sqrt{d}$ (6)

with an absolute constant C.

---

2. The set of d with $m_d(N) \ge 2$

From (6) we have $m_d(N) = 1$ whenever $d > (\log N)^2$. Hence

$T_N := \{d \text{ square-free} : m_d(N) \ge 2\} \subset \{d \le (\log N)^2\}$. (7)

---

3. The number of "extra" pairs

Write

$E_N := \sum_d m_d(N)(m_d(N)-1)$ (the second term in (2)).

Split the sum into $d \in T_N$ and $d \notin T_N$.

(i) $d \notin T_N \Rightarrow m_d(N) = 1 \Rightarrow$ contribution 0.

(ii) For $d \in T_N$ we use (6):

$m_d(N) \le C \log N$, so $m_d(N)(m_d(N)-1) \le C^2(\log N)^2$.

Because of (7) the number of such d is $\le (\log N)^2$, whence

$E_N \le (\log N)^2 \cdot C^2(\log N)^2 = C^2(\log N)^4 = o(N)$. (8)

---

4. Finishing the estimate

Insert (8) in (2):

$S_N = N + E_N = N + o(N)$.

Therefore

$\lim_{N\to\infty} S_N/N = 1$.

Final answer: 1

10.2 Variant 2

Sorry, I was not able to finish.

10.3 Variant 3

Let
$$f(n) = n^2 + n = n(n+1), \quad n \in \mathbb{Z}_{>0}.$$

For $1 \le a, b \le N$ put
$$S_N = \#\{(a,b) : f(a)f(b) \text{ is a square}\}.$$

Reformulation through “square-free parts”

For every n there are unique integers
$$n(n+1) = s(n) \cdot t(n)^2, \quad s(n) \text{ square-free}, \ t(n) \ge 1.$$

Call $s(n)$ the square-free part of $f(n)$. Because consecutive integers are coprime, $s(n)$ is obtained by deleting from n and $n+1$ every prime factor occurring with an even exponent. Consequently
$$f(a)f(b) \text{ is a square} \iff s(a) = s(b).$$

Hence, putting
$$c_N(s) = \#\{1 \le n \le N : s(n) = s\},$$
we have
$$S_N = \sum_s c_N(s)^2. \tag{117}$$
(The sum is over all square-free positive integers.)

An upper bound for the individual multiplicities $c_N(s)$

Fix a square-free s. Writing $4f(n) = 4n(n+1) = (2n+1)^2 - 1$ we get
$$(2n+1)^2 - 4s \cdot t^2 = 1, \quad t = t(n).$$

Thus every n with $s(n) = s$ furnishes a positive solution $(x,y) = (2n+1, 2t)$ of the Pell equation
$$x^2 - 4s \cdot y^2 = 1. \tag{118}$$

Conversely, every solution (x,y) with x odd gives $n = (x-1)/2 \in \mathbb{Z}$. Standard theory of Pell equations says that the total number of positive solutions with $x \le X$ is $O(\log X)$. With $X = 2N+1$ we obtain
$$c_N(s) = O(\log N) \quad (\forall s). \tag{119}$$

The implicit constant in (3) is absolute.

A global upper bound for S_N

From (1) and (3):

$$S_N = \sum_s c_N(s)^2 \tag{120}$$
$$\le \left(\max_s c_N(s)\right) \sum_s c_N(s) \tag{121}$$
$$= O(\log N) \cdot N \tag{122}$$
$$= O(N \log N). \tag{123}$$

A global lower bound for S_N

Still by (3), every square-free s that actually occurs before N is taken at most $C \log N$ times (C absolute). Hence the number D_N of distinct square-free parts that do occur satisfies

$$D_N \ge \frac{N}{C \log N}. \tag{124}$$

Apply Cauchy's inequality to the family $(c_N(s))_s$:

$$S_N = \sum_s c_N(s)^2 \tag{125}$$
$$\ge \frac{\left(\sum_s c_N(s)\right)^2}{D_N} \tag{126}$$
$$= \frac{N^2}{D_N} \tag{127}$$
$$\ge \frac{N^2}{N/(C \log N)} \tag{128}$$
$$= C \cdot N \log N. \tag{129}$$

Thus

$$C \cdot N \log N \le S_N \le C' \cdot N \log N. \tag{130}$$

The order of growth of S_N/N

Divide (7) by N:

$$C \log N \le \frac{S_N}{N} \le C' \log N. \tag{131}$$

Because $\log N \to \infty$, the quotient S_N/N grows without bound; in particular

$$\lim_{N\to\infty} \frac{S_N}{N} = +\infty. \tag{132}$$

Conclusion

The limit exists in the (extended) real line and equals $+\infty$.

Final answer: $\boxed{\lim_{N\to\infty} \frac{S_N}{N} = +\infty}$

10.4 Final Solution

Solution.

A reformulation

For every positive integer put

$$f(n) = n^2 + n = n(n+1), \quad s(n) = \text{square-free part of } f(n).$$

Because n and $n+1$ are coprime, $s(n)$ is the product of all primes that occur in $f(n)$ with an odd exponent, hence $s(n)$ is square-free. For two positive integers a, b we have

$$f(a)f(b) \text{ is a square } \iff s(a) = s(b). \quad (1)$$

Indeed, (1) follows by comparing the parities of the exponents of every prime in $f(a)$ and $f(b)$.

Denote

$$m_d(N) = \#\{1 \le n \le N : s(n) = d\}, \quad d \text{ square-free}.$$

With (1)

$$S_N = \#\{(a,b) \le N : f(a)f(b) \text{ is a square}\} \tag{133}$$
$$= \sum_d m_d(N)^2. \quad (2) \tag{134}$$

Our task is to evaluate S_N.

How large can $m_d(N)$ be?

Fix a square-free d and look at the equation

$$n(n+1) = d \cdot k^2, \quad n, k \in \mathbb{Z}_{>0}. \quad (3)$$

It describes exactly the integers n with $s(n) = d$. Multiply (3) by 4 and set

$$X = 2n + 1 \ (> 0), \quad Y = 2k \ (> 0);$$

then (3) becomes the Pell equation

$$X^2 - 4d \cdot Y^2 = 1, \quad X \text{ odd}, \ Y \text{ even}. \quad (4)$$

Every Pell equation $x^2 - Dy^2 = 1$ (D not a square) has infinitely many positive solutions; ordered by the x-coordinate they grow geometrically: if (X_1, Y_1) is the minimal positive solution, then

$$X_k + Y_k\sqrt{4d} = (X_1 + Y_1\sqrt{4d})^k \quad (k = 0, 1, 2, \ldots).$$

Consequently, for some constant $\varepsilon_d > 1$ that depends only on d,

$$X_k \ge \varepsilon_d^k.$$

Hence the number of solutions with $X \le 2N+1$ (and therefore $n \le N$) is at most

$$m_d(N) \le 1 + \frac{\log(2N+1)}{\log \varepsilon_d}. \quad (5)$$

A crude universal estimate is enough for us. Because $\varepsilon_d > 2\sqrt{d}$ (take, for instance, $X_1 + Y_1\sqrt{4d} \ge 1 + 2\sqrt{d}$), we get from (5)

$$m_d(N) \le 1 + \frac{\log(2N)}{\log(2\sqrt{d})} \tag{135}$$

$$\le C\frac{\log N}{\sqrt{d}} \quad (6) \tag{136}$$

with an absolute constant C.

Which d can occur more than once?

If $m_d(N) \ge 2$, then (6) gives

$$2 \le m_d(N) \le C\frac{\log N}{\sqrt{d}} \Longrightarrow d \le C^2(\log N)^2. \quad (7)$$

Thus only square-free d up to (const.) $\cdot$ $(\log N)^2$ may have multiplicity ≥ 2. Put

$$T_N := \{d \text{ square-free} : m_d(N) \ge 2\}.$$

By (7) we have

$$\#T_N \le C_1(\log N)^2. \quad (8)$$

The "extra" pairs

Rewrite (2) as

$$S_N = N + \sum_d m_d(N)\big(m_d(N) - 1\big) \tag{137}$$

$$=: N + E_N. \quad (9) \tag{138}$$

(The term N corresponds to the diagonal $a = b$.)

If $d \notin T_N$ then $m_d(N) = 1$ and its contribution to E_N is 0. For $d \in T_N$ we combine (6) with (8):

$$m_d(N)(m_d(N) - 1) \le m_d(N)^2 \le \frac{C^2(\log N)^2}{d}.$$

Hence, using that $d \le C^2(\log N)^2$ on T_N,

$$E_N \le C^2(\log N)^2 \sum_{d \in T_N} \frac{1}{d} \tag{139}$$

$$\le C^2(\log N)^2 \sum_{1 \le d \le C^2(\log N)^2} \frac{1}{d} \tag{140}$$

$$= O\big((\log N)^2 \log\log N\big). \quad (10) \tag{141}$$

Asymptotics of S_N

Because $(\log N)^2 \log\log N = o(N)$, relation (10) together with (9) gives

$$S_N = N + o(N) \quad (N \to \infty). \quad (11)$$

Divide by N and take the limit:

$$\lim_{N\to\infty} \frac{S_N}{N} = 1. \quad (12)$$

Final answer: 1