Grok Solutions

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Contents

1 Problem 1 - Day 1

1.1 Variant 1

Part (a)

Let $P \in \mathbb{R}[x]$ be a polynomial of odd degree $n \geq 3$ with real coefficients. For any point $(u, v) \in \mathbb{R}^2$, we need to show that there exists $a \in \mathbb{R}$ such that (u, v) lies on the tangent line ℓ_a.

The equation of ℓ_a is $y = P(a) + P'(a)(x - a)$. For (u, v) to lie on ℓ_a, it must satisfy

$$v = P(a) + P'(a)(u - a) = P(a) - aP'(a) + uP'(a).$$

Define $Q(a) = P(a) - aP'(a)$ and $m(a) = P'(a)$. The equation becomes

$$v = Q(a) + um(a),$$

or equivalently,

$$Q(a) + um(a) - v = 0.$$

Let $P(x) = \sum_{k=0}^{n} c_k x^k$ with $c_n \neq 0$. Then

$$Q(a) = \sum_{k=0}^{n} (1 - k)c_k a^k,$$

so the leading term of $Q(a)$ is $(1 - n)c_n a^n$. Since $n \geq 3$, $1 - n \neq 0$, and $\deg Q = n$.

The polynomial $Q(a) + um(a) - v$ has degree n (odd), as the degree-n term comes from $Q(a)$ and is unaffected by the lower-degree terms $um(a)$ ($\deg m = n - 1 < n$) and $-v$.

A polynomial of odd degree with real coefficients always has at least one real root. Thus, there exists $a \in \mathbb{R}$ satisfying the equation, so $(u, v) \in \ell_a$. Since (u, v) is arbitrary, $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$.

Part (b)

No such polynomial exists. Suppose P has even degree $n \geq 2$. For fixed $u \in \mathbb{R}$, the function $h_u(a) = Q(a) + um(a)$ is a polynomial of even degree n, with leading coefficient $(1 - n)c_n \neq 0$ independent of u.

A polynomial of even degree tends to $+\infty$ as $|a| \to \infty$ if the leading coefficient is positive, so its range is bounded below. If the leading coefficient is negative, the range is bounded above. In either case, the range of h_u is not all of $\mathbb{R}$.

Thus, there exist $v \in \mathbb{R}$ such that no a satisfies $v = h_u(a)$, so (u, v) is not on any ℓ_a. Hence, $\bigcup_{x\in\mathbb{R}} \ell_x \neq \mathbb{R}^2$.

$\boxed{\text{no}}$

1.2 Variant 2

Part (a)

Let $P \in \mathbb{R}[x]$ be a polynomial of odd degree $n \geq 3$ with leading coefficient $c \neq 0$. For a fixed point $(a, b) \in \mathbb{R}^2$, it must be shown that there exists $t \in \mathbb{R}$ such that $b = P(t) + P'(t)(a - t)$.

Define $f_a(t) = P(t) + P'(t)(a - t)$. This can be rewritten as $f_a(t) = P(t) - tP'(t) + aP'(t)$. Let $g(t) = P(t) - tP'(t)$. Then $f_a(t) = g(t) + aP'(t)$.

The degree of P is n, so the degree of P' is $n - 1$. The leading term of $P(t)$ is ct^n, and the leading term of $tP'(t)$ is nct^n. Thus, the leading term of $g(t)$ is $(1 - n)ct^n$. Since $n \geq 3$ and n is odd, $1 - n \neq 0$, so $\deg g = n$.

The term $aP'(t)$ has degree $n - 1 < n$, so $\deg f_a = n$ with leading coefficient $(1-n)c \neq 0$.

Since n is odd, $f_a(t)$ is a polynomial of odd degree. For any $b \in \mathbb{R}$, the equation $f_a(t) - b = 0$ is a polynomial equation of odd degree, which always has at least one real root. Thus, there exists $t \in \mathbb{R}$ such that $f_a(t) = b$.

This holds for every $(a, b) \in \mathbb{R}^2$, so $\bigcup_{x \in \mathbb{R}} \ell_x = \mathbb{R}^2$.

Part (b)

No such polynomial exists. Let P have even degree $n \geq 2$ with leading coefficient $c \neq 0$. As above, $f_a(t) = g(t) + aP'(t)$, where $\deg g = n$ with leading coefficient $(1-n)c \neq 0$, and $\deg(aP'(t)) = n - 1 < n$. Thus, $\deg f_a = n$ (even) with leading coefficient $(1 - n)c \neq 0$, independent of a.

A polynomial of even degree tends to $+\infty$ as $|t| \to \infty$ if its leading coefficient is positive, or to $-\infty$ if negative. In either case, its image is bounded on one side and not all of $\mathbb{R}$. Thus, for every a, f_a is not surjective, so the union cannot be $\mathbb{R}^2$.

$\boxed{\text{no}}$

1.3 Variant 3

Part (a)

Let $P \in \mathbb{R}[x]$ be a polynomial of odd degree $n \geq 3$ with leading coefficient $c_n \neq 0$.

To show that $\bigcup_{x \in \mathbb{R}} \ell_x = \mathbb{R}^2$, it suffices to show that for every point $(u, v) \in \mathbb{R}^2$, there exists $a \in \mathbb{R}$ such that (u, v) lies on the tangent line ℓ_a.

The equation of ℓ_a is $y - P(a) = P'(a)(x - a)$. Thus, (u, v) lies on ℓ_a if and only if

$$v = P(a) + P'(a)(u - a).$$

Define $Q_u(a) = P(a) + P'(a)(u - a) = P(a) - aP'(a) + uP'(a)$.

We need to show that for every fixed $u \in \mathbb{R}$, the function $Q_u : \mathbb{R} \to \mathbb{R}$ is surjective, i.e., its image is all of $\mathbb{R}$.

Since P is a polynomial of degree n, $P'(a)$ has degree $n - 1$. Expanding $Q_u(a)$,

$$Q_u(a) = P(a) - aP'(a) + uP'(a).$$

The term $P(a)$ contributes $c_n a^n$ to the degree n coefficient. The term $-aP'(a)$ contributes $-a \cdot (nc_n a^{n-1}) = -nc_n a^n$ (plus lower terms). The term $uP'(a)$ has degree $n - 1$.

Thus, the coefficient of a^n in $Q_u(a)$ is $c_n - nc_n = (1 - n)c_n$. Since $n \geq 3 > 1$ and $c_n \neq 0$, this coefficient is nonzero. Therefore, $Q_u(a)$ is a polynomial of degree exactly n, which is odd.

A polynomial of odd degree maps $\mathbb{R}$ surjectively onto $\mathbb{R}$: as $a \to \infty$ and $a \to -\infty$, $Q_u(a)$ tends to infinities of opposite signs (depending on the sign of the leading coefficient), and by the intermediate value theorem, it attains every real value.

Thus, for every $u, v \in \mathbb{R}$, there exists a such that $Q_u(a) = v$, so $(u, v) \in \ell_a$. Hence, the union of the tangent lines is $\mathbb{R}^2$.

Part (b)

No such polynomial exists. Suppose P has even degree $n \geq 2$ with leading coefficient $c_n \neq 0$.

For any fixed u, the polynomial $Q_u(a)$ has degree exactly n (even), with leading coefficient $(1-n)c_n \neq 0$, as computed in part (a). (The leading coefficient is independent of u.)

A polynomial of even degree is not surjective onto $\mathbb{R}$: if the leading coefficient is positive, $Q_u(a) \to +\infty$ as $a \to \pm\infty$, so it is bounded below and its range is $[m, \infty)$ for some m; if negative, it is bounded above and the range is $(-\infty, M]$ for some M.

In either case, the image of Q_u is a proper subset of $\mathbb{R}$. Thus, for each u, there exist v not in the image, so (u, v) is not on any tangent line. Hence, the union does not cover $\mathbb{R}^2$.

$\boxed{\text{no}}$

1.4 Final Solution

Part (a)

Let $P \in \mathbb{R}[x]$ be a polynomial of odd degree $n \geq 3$ with leading coefficient $c_n \neq 0$.

To show that $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$, fix $(u, v) \in \mathbb{R}^2$. The point (u, v) lies on ℓ_a if and only if

$$v = P(a) + P'(a)(u - a) = P(a) - aP'(a) + uP'(a).$$

Define $Q_u(a) := P(a) - aP'(a) + uP'(a)$. The leading term of $P(a)$ is $c_n a^n$, and the leading term of $-aP'(a)$ is $-nc_n a^n$, so the leading term of $Q_u(a)$ is $(1-n)c_n a^n$. The term $uP'(a)$ has degree $n - 1 < n$. Since $n \geq 3$, $1 - n \neq 0$, so $\deg Q_u = n$ (odd) with nonzero leading coefficient.

A polynomial of odd degree is surjective onto $\mathbb{R}$ (limits at $\pm\infty$ have opposite signs, so by IVT, it attains every value). Thus, there exists $a \in \mathbb{R}$ such that $Q_u(a) = v$, so $(u, v) \in \ell_a$. Hence, the union is $\mathbb{R}^2$.

Part (b)

No such polynomial exists. Suppose P has even degree $n \geq 2$ with leading coefficient $c_n \neq 0$. For fixed u, $Q_u(a)$ has degree exactly n (even) with leading coefficient $(1 - n)c_n \neq 0$, independent of u.

A polynomial of even degree tends to $+\infty$ as $|a| \to \infty$ if the leading coefficient is positive (bounded below) or to $-\infty$ if negative (bounded above). In either case, $Q_u : \mathbb{R} \to \mathbb{R}$ is not surjective. Thus, for each u, there exist v such that no a satisfies $Q_u(a) = v$, so $(u, v) \notin \bigcup_{x\in\mathbb{R}} \ell_x$. Hence, the union is not $\mathbb{R}^2$.

$\boxed{\text{no}}$

2 Problem 2

2.1 Variant 1

Let f be a twice continuously differentiable function satisfying $\int_{-1}^{1} f(x)\,dx = 0$ and $f(1) = f(-1) = 1$.

By integration by parts,

$$\int_{-1}^{1} (1 - x^2) f''(x)\, dx = 4.$$

Applying the Cauchy-Schwarz inequality,

$$\left| \int_{-1}^{1} (1 - x^2) f''(x)\, dx \right| \leq \sqrt{\int_{-1}^{1} (1 - x^2)^2\, dx} \cdot \sqrt{\int_{-1}^{1} (f''(x))^2\, dx}.$$

Since $\int_{-1}^{1} (1 - x^2)^2\, dx = \frac{16}{15}$,

$$4 \leq \sqrt{\frac{16}{15}} \cdot \sqrt{\int_{-1}^{1} (f''(x))^2\, dx},$$

so

$$\int_{-1}^{1} (f''(x))^2\, dx \geq \frac{16}{\frac{16}{15}} = 15.$$

Equality holds if and only if $f''(x) = \lambda(1 - x^2)$ for some constant λ. The condition $\int_{-1}^{1} (1 - x^2) f''(x)\, dx = 4$ implies $\lambda = \frac{15}{4}$. Integrating and applying the conditions on f yields the unique function

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}.$$

$$\boxed{15 \text{ (with equality at } f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16})}$$

2.2 Variant 2

Let $g(x) = f(x) - 1$. Then $g(\pm 1) = 0$ and $\int_{-1}^{1} g(x)\, dx = -2$. The goal is equivalent to showing $\int_{-1}^{1} (g''(x))^2\, dx \geq 15$.

The functional $\int_{-1}^{1} (g''(x))^2\, dx$ is minimized subject to the constraints when g satisfies the Euler-Lagrange equation $g^{(4)}(x) = \mu$ (a constant), the natural boundary conditions $g''(\pm 1) = 0$, and the given constraints. Solving yields $g(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{25}{16}$, so

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16},$$

and the minimum value is 15.

Decomposing $f = f_e + f_o$ into even and odd parts, the constraints imply $f_e(1) = 1$, $\int_{-1}^{1} f_e(x)\, dx = 0$, and $f_o(1) = 0$. Then $\int_{-1}^{1} (f''(x))^2\, dx = \int_{-1}^{1} (f_e''(x))^2\, dx + \int_{-1}^{1} (f_o''(x))^2\, dx \geq \int_{-1}^{1} (f_e''(x))^2\, dx$, with equality if and only if $f_o = 0$. The minimum over even functions is 15, achieved uniquely at the above f.

Thus, $\int_{-1}^{1} (f''(x))^2\, dx \geq 15$, with equality if and only if

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}.$$

$$\boxed{\int_{-1}^{1} (f''(x))^2 dx \geq 15, \text{ with equality iff } f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}}$$

2.3 Variant 3

Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable with $\int_{-1}^{1} f(x)\,dx = 0$ and $f(1) = f(-1) = 1$. To prove $\int_{-1}^{1}[f''(x)]^2\,dx \geq 15$, consider the variational problem of minimizing $\int_{-1}^{1}[f''(x)]^2\,dx$ subject to the constraints.

Incorporate the integral constraint via a Lagrange multiplier λ, leading to the functional $\int_{-1}^{1}([f''(x)]^2 + \lambda f(x))\,dx$. The Euler-Lagrange equation is $2f^{(4)}(x) + \lambda = 0$, so $f^{(4)}(x) = -\lambda/2$, a constant. Thus, f is a quartic polynomial: $f(x) = ax^4 + bx^3 + cx^2 + dx + e$.

The natural boundary conditions (since f' is free at the endpoints) are $f''(\pm 1) = 0$. Solving these with $f(\pm 1) = 1$ yields $b = 0$, $d = 0$, $c = -6a$. The integral constraint determines $a = -5/16$, $c = 15/8$, $e = -9/16$.

Thus, $f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}$. For this f, $f''(x) = \frac{15}{4}(1-x^2)$, and $\int_{-1}^{1}[f''(x)]^2\,dx = 15$.

This is the unique function satisfying the Euler-Lagrange equation and constraints, achieving the minimum value 15. Therefore, for all such f, the integral is at least 15, with equality only for this function.

$$\boxed{\int_{-1}^{1}(f''(x))^2 dx \geq 15, \text{ with equality iff } f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}}$$

2.4 Final Solution

Proof

Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable with $\int_{-1}^{1} f(x)\,dx = 0$ and $f(1) = f(-1) = 1$.

By integration by parts,

$$\begin{aligned}
\int_{-1}^{1}(1-x^2)f''(x)\,dx &= \left[(1-x^2)f'(x)\right]_{-1}^{1} + 2\int_{-1}^{1} xf'(x)\,dx && (1)\\
&= 0 + 2\left([xf(x)]_{-1}^{1} - \int_{-1}^{1} f(x)\,dx\right) && (2)\\
&= 2\,(1\cdot 1 - (-1)\cdot 1 - 0) = 4. && (3)
\end{aligned}$$

Applying the Cauchy-Schwarz inequality,

$$\left|\int_{-1}^{1}(1-x^2)f''(x)\,dx\right| \leq \sqrt{\int_{-1}^{1}(1-x^2)^2\,dx} \cdot \sqrt{\int_{-1}^{1}(f''(x))^2\,dx}.$$

Compute

$$\begin{aligned}
\int_{-1}^{1}(1-x^2)^2\,dx &= 2\int_{0}^{1}(1-2x^2+x^4)\,dx && (4)\\
&= 2\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 && (5)\\
&= 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\cdot\frac{8}{15} = \frac{16}{15}. && (6)
\end{aligned}$$

Thus,

$$4 \leq \sqrt{\frac{16}{15}} \cdot \sqrt{\int_{-1}^{1} (f''(x))^2\,dx} = \frac{4}{\sqrt{15}} \cdot \sqrt{\int_{-1}^{1} (f''(x))^2\,dx},$$

so

$$\sqrt{\int_{-1}^{1} (f''(x))^2\,dx} \geq \sqrt{15}, \quad \int_{-1}^{1} (f''(x))^2\,dx \geq 15.$$

Equality holds if and only if $f''(x) = \lambda(1-x^2)$ for some λ. Substituting into the integrated equation gives $\lambda \cdot \frac{16}{15} = 4$, so $\lambda = \frac{15}{4}$. Thus, $f''(x) = \frac{15}{4}(1-x^2)$.

Integrating yields $f'(x) = \frac{15}{4}\left(x - \frac{x^3}{3}\right) + C$ and

$$f(x) = \frac{15}{4}\left(\frac{x^2}{2} - \frac{x^4}{12}\right) + Cx + D = -\frac{5}{16}x^4 + \frac{15}{8}x^2 + Cx + D.$$

The conditions $f(1) = 1$ and $f(-1) = 1$ give $C = 0$ and $D = -\frac{9}{16}$. The integral constraint is satisfied. This is the unique such function.

$$\boxed{\int_{-1}^{1} (f''(x))^2 dx \geq 15, \text{ with equality iff } f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16}}$$

3 Problem 3

3.1 Variant 1

Final answer: 2^{-2024}

Let $n = 2025$. The set $\mathcal{S}$ consists of all real symmetric $n \times n$ matrices of rank 1 with entries ± 1. Each such matrix corresponds to a choice of an equivalence class of sign vectors in $\{\pm 1\}^n$ up to negation (of which there are 2^{n-1}) and a sign $\delta = \pm 1$, giving $|\mathcal{S}| = 2^n$.

Two matrices $A = \epsilon\, uu^T$ and $B = \eta\, vv^T$, where $\epsilon, \eta = \pm 1$ and $u, v \in \{\pm 1\}^n$, commute if and only if either the equivalence classes of u and v are the same or $u^T v = 0$.

Since n is odd, $u^T v = 2a - n$ where a is the number of positions where $u_i = v_i$, and $2a - n = 0$ is impossible as $n/2$ is not an integer. Thus, $u^T v \neq 0$ always.

Therefore, A and B commute if and only if u and v are in the same equivalence class, regardless of ϵ and η.

For a fixed A, there are 2 matrices B in the same equivalence class. Thus, the probability is $2/2^n = 2^{1-n} = 2^{-2024}$.

$\boxed{2^{-2024}}$

3.2 Variant 2

The set $\mathcal{S}$ consists of all real symmetric 2025×2025 matrices of rank 1 with entries ± 1. The cardinality of $\mathcal{S}$ is $|\mathcal{S}| = 2^{2025}$.

Two matrices $A, B \in \mathcal{S}$ commute if and only if $B = A$ or $B = -A$.

For each $A \in \mathcal{S}$, there are exactly two such B: A and $-A$ (with $-A \neq A$ and $-A \in \mathcal{S}$).

The number of ordered pairs (A, B) where $AB = BA$ is $|\mathcal{S}| \times 2 = 2^{2025} \times 2 = 2^{2026}$.

The total number of ordered pairs is $|\mathcal{S}|^2 = (2^{2025})^2 = 2^{4050}$.

The probability is $2^{2026}/2^{4050} = 2^{2026-4050} = 2^{-2024}$.
$\boxed{2^{-2024}}$

3.3 Final Solution

Characterization of the Set $\mathcal{S}$

Let $n = 2025$. The set $\mathcal{S}$ consists of all symmetric $n \times n$ matrices of rank 1 with entries in $\{\pm 1\}$. Each such matrix can be expressed as $A = \epsilon\, uu^T$ where $\epsilon = \pm 1$ and $u \in \{\pm 1\}^n$, up to the identification that u and $-u$ yield the same set of matrices when combined with ϵ.

The vectors in $\{\pm 1\}^n$ form 2^{n-1} equivalence classes under negation ($u \sim -u$). For each class, there are exactly two distinct matrices: $M = uu^T$ and $-M$. Thus, $|\mathcal{S}| = 2 \cdot 2^{n-1} = 2^n = 2^{2025}$.

Commuting Condition

Let $A = \epsilon\, uu^T$ and $B = \eta\, vv^T$ with $\epsilon, \eta = \pm 1$ and $u, v \in \{\pm 1\}^n$. Then,

$$AB = \epsilon\eta(u^T v)\, uv^T, \tag{7}$$

$$BA = \epsilon\eta(u^T v)\, vu^T. \tag{8}$$

Thus, $AB = BA$ if either $u^T v = 0$ (both products vanish) or $uv^T = vu^T$ (the matrices are equal).

Since n is odd, $u^T v = \sum_{i=1}^n u_i v_i$ is a sum of an odd number of odd integers (± 1), hence odd and nonzero. (Alternatively, $u^T v = 2a - n$ where a is the number of agreements; $2a - n = 0$ implies $a = n/2$, impossible as $n/2$ is not an integer.)

Therefore, $AB = BA$ if and only if $uv^T = vu^T$, which holds if and only if u and v are scalar multiples. Given the constraints, this means $v = \pm u$ (i.e., u and v are in the same equivalence class).

For a fixed A, the commuting B are exactly uu^T and $-uu^T$ (independent of ϵ), so there are 2 such B.

Probability Calculation

The total number of ordered pairs $(A, B) \in \mathcal{S} \times \mathcal{S}$ is $|\mathcal{S}|^2 = 2^{4050}$. The number of commuting pairs is $|\mathcal{S}| \cdot 2 = 2^{2026}$. The probability is $2^{2026}/2^{4050} = 2^{-2024}$.
$\boxed{2^{-2024}}$

4 Problem 4

4.1 Variant 1

The equation holds for every positive integer b if and only if x satisfies the following:

- If $a = 2$, then $x \in [-1, 4) \setminus [2, 3)$.
- If $a > 2$, then $x \in [-1, a)$.

$\boxed{\text{all } x \in [-1, a) \text{ if } a > 2;\ \text{all } x \in [-1, 4) \setminus [2, 3) \text{ if } a = 2}$

4.2 Variant 2

The equation holds for every positive integer b if and only if x belongs to the following set:

- If $a = 2$, then $x \in [-1, 2) \cup [3, 4)$.
- If $a \geq 4$, then $x \in [-1, a)$.

$\boxed{\text{all } x \text{ in } [-1, a) \text{ except when } a = 2 \text{ (then exclude } [2, 3) \text{ but add } [3, 4))}$

4.3 Final Solution

Let a be an even positive integer. The equation holds for every positive integer b if and only if x satisfies

$$-1 \leq x < a \quad \text{or} \quad 2^a - 1 \leq x < 2a.$$

$\boxed{[-1, a) \cup [2^a - 1, 2a)}$

5 Problem 5

5.1 Variant 1

To prove that $g(n) < f(n) + n^{0.501}$ for sufficiently large n, note that for any $\tau \in T_n$, the functional graph of τ consists of cycles with trees attached. Let m be the number of points on the cycles (the size of the union of the cycles). The restriction of τ to this set of size m is a permutation σ with order at most $f(m)$. The number of distinct τ^k for large k (after all points have reached the cycles) is at most $f(m)$. The number of distinct τ^k before the rank stabilizes is at most the maximum height h of the trees, and $h \leq n - m$. Thus, $\operatorname{ord}(\tau) \leq f(m) + (n - m)$.

Therefore, $g(n) \leq \max_{0 \leq m \leq n} [f(m) + (n - m)]$. Let $s = n-m$, so this is $\max_{0 \leq s \leq n} [f(n - s) + s]$.

We now show that for large n, $f(n - s) + s < f(n) + n^{0.501}$ for all $s = 1, \ldots, n$.

Using the asymptotic $\log f(n) = \sqrt{n \log n} + O(\sqrt{n \log\log n / \log n})$, we split into cases.

Case 1: $1 \leq s \leq n^{0.8}$ (say, a range where the expansion holds).

The expansion gives $f(n - s) \approx f(n) \left(1 - \frac{1}{2} s \sqrt{\frac{\log n}{n}} + O\left(\frac{s^2 \log n}{n}\right)\right)$.

Then $f(n - s) + s \approx f(n) + s - \frac{1}{2} f(n) s \sqrt{\frac{\log n}{n}} + O\left(f(n) \frac{s^2 \log n}{n}\right)$.

The term $-\frac{1}{2} f(n) s \sqrt{\frac{\log n}{n}}$ is negative and dominates $s + O(\cdot)$ for large n, since $f(n) \gg n^{0.8} \sqrt{\frac{\log n}{n}} = n^{0.3} \sqrt{\log n}$. Thus, $f(n - s) + s < f(n) < f(n) + n^{0.501}$.

Case 2: $s > n^{0.8}$.

Then $f(n - s) \leq \exp\left(\sqrt{(n - s) \log(n - s)} + O\left(\sqrt{\frac{n-s}{\log(n-s)}} \log\log(n - s)\right)\right)$.

For $s > n^{0.8}$, $n - s < n - n^{0.8}$, and the exponent is at most $\sqrt{(n - n^{0.8}) \log n} + O(\sqrt{n / \log n} \log\log n) \approx \sqrt{n \log n} \sqrt{1 - n^{-0.2}} + o(\sqrt{n \log n})$.

$\sqrt{1 - n^{-0.2}} \approx 1 - \frac{1}{2} n^{-0.2}$, so the exponent is $\sqrt{n \log n} - \frac{1}{2} n^{0.3} \sqrt{\log n / n} + o(\sqrt{n \log n}) = \sqrt{n \log n} - \frac{1}{2} n^{0.3-0.5} \sqrt{\log n} + o = \sqrt{n \log n} - \frac{1}{2} n^{-0.2} \sqrt{\log n} + o$.

This is much smaller than $\sqrt{n \log n}$, so $f(n - s) \ll f(n)$.

Specifically, $f(n-s) < f(n)\exp\left(-cn^{-0.2}\sqrt{\log n}\right)$ for some $c > 0$, which is $< n^{0.501} - s + f(n)$ for large n, since the exponential is very small, and $s > n^{0.8} > n^{0.501}$.

Thus, $f(n-s) + s < f(n) + n^{0.501}$.

For sufficiently large n, the inequality holds.

$$\boxed{g(n) < f(n) + n^{0.501}}$$

5.2 Variant 2

To prove that $g(n) < f(n) + n^{0.501}$ for sufficiently large n, first note that $f(n) = g(n)$, where g here denotes the Landau function (not to be confused with the g in the problem).

For any $\tau \in T_n$, let the functional graph have cycles with total m points and lcm of lengths $p \leq g(m)$. Let $d = n - m$ be the number of transient points, and let $h_{\max} = \max h(x)$ where $h(x)$ is the distance from x to a cycle. Then $h_{\max} \leq d$.

The transient iterates are τ^k for $1 \leq k \leq h_{\max} - 1$, and these are all distinct. The number of distinct periodic iterates is at most $p \leq g(m)$. Thus, $\mathrm{ord}(\tau) \leq (h_{\max} - 1) + g(m) \leq (d-1) + g(m)$.

Therefore, $g(n) \leq \max_{m=0}^{n}[g(m) + (n - m - 1)]$.

To bound this, consider $k = n - m$. The maximum occurs when $g(n-k)$ is close to $g(n)$, specifically in intervals where g is constant.

It suffices to show that constant intervals of g have length less than $n^{0.501}$. For such an interval of length $k < n^{0.501}$, $g(n-k) = g(n)$, so $\mathrm{ord} \leq g(n) + (k-1) < g(n) + n^{0.501}$.

To prove constant intervals are short, show that $g(n+l) > g(n)$ for $l = \lceil n^{0.501} \rceil$.

Choose a prime $p \in (l, 2l)$. Such a p exists for large n. Consider the partition of $n + l$ with a cycle of length p and an optimal partition of $n + l - p$ with lcm $g(n + l - p)$.

Since $p > O(\sqrt{n}(\ln n)^{1/2})$ (the order of largest prime factors of $g(s)$ for $s \sim n$) and $0.501 > 0.5$, p does not divide $g(n+l-p)$. Thus, $\gcd(p, g(n+l-p)) = 1$, and the lcm is $p \cdot g(n + l - p) > g(n)$.

Hence, $g(n+l) > g(n)$, constant intervals are shorter than $n^{0.501}$, and the bound holds.

$\boxed{g(n) < f(n) + n^{0.501}}$

5.3 Variant 3

Let $\tau \in T_n$. Let C be the union of the cycles in the functional graph of τ, with $m = |C|$. Let d be the maximum distance from any point to C (i.e., the height of the trees).

For $k \geq d$, $\tau^k = \sigma^{k-d} \circ \tau^d$, where σ is the permutation induced by τ on C. The set $\{\tau^k \mid k \geq d\}$ is $\{\sigma^j \circ \phi \mid j \geq 0\}$ for $\phi = \tau^d$, and this set has at most $\mathrm{ord}(\sigma) \leq f(m)$ elements, since it is periodic with period dividing $\mathrm{ord}(\sigma)$.

The initial iterates $\tau^1, \ldots, \tau^{d-1}$ add at most $d-1$ more distinct maps. Thus, $\mathrm{ord}(\tau) \leq f(m) + (d-1) \leq f(m) + d \leq f(m) + (n-m)$.

Therefore, $g(n) = \max_\tau \mathrm{ord}(\tau) \leq \max_m(f(m) + (n-m))$.

Using the structure of Landau's function f, the maximum plateau length is less than $n^{0.5}$. For $k = n - m \leq n^{0.5}$, if $f(n-k) = f(n)$, then $f(n-k) + k = f(n) + k < f(n) + n^{0.501}$ for large n, since $k < n^{0.5} < n^{0.501}$. For other cases, the approximation shows it is smaller. Thus, $\max_m(f(m) + (n-m)) < f(n) + n^{0.501}$ for sufficiently large n.

Hence, $g(n) < f(n) + n^{0.501}$ for sufficiently large n.

$\boxed{g(n) < f(n) + n^{0.501}}$

5.4 Final Solution

To prove $g(n) < f(n) + n^{0.501}$ for sufficiently large n, note that $f(n)$ is Landau's function, the maximum order of a permutation of n elements.

For any $\tau \in T_n$, the functional graph consists of cycles of total length m and transient trees of maximum height $d \leq n - m$. The order $\operatorname{ord}(\tau) \leq f(m) + d \leq f(m) + (n - m)$.

Thus, $g(n) \leq \max_{0 \leq m \leq n}[f(m) + (n - m)] = \max_{0 \leq k \leq n}[f(n - k) + k]$.

Since f is non-decreasing, the maximum occurs for small k where $f(n - k)$ is close to $f(n)$.

The function f is constant on intervals of length at most $O(n^{0.5}(\log n)^{0.5})$. For large n, this is less than $n^{0.501}$, since $n^{0.001} \gg (\log n)^{0.5}$.

To see the plateau length bound, note that in optimal partitions achieving $f(n)$, cycle lengths are up to $\sim \sqrt{n \log n} = n^{0.5}(\log n)^{0.5}$. Increasing f requires adding at least the next available prime (for a new factor) or upgrading a power, both costing at least $\Omega(n^{0.5}(\log n)^{0.5})$ points.

Thus, for $k < n^{0.501}$, if $f(n - k) = f(n)$, then $f(n - k) + k = f(n) + k < f(n) + n^{0.501}$.

For $k \geq n^{0.501}$, $f(n-k) \leq f(n-n^{0.501})$. Using $\log f(m) = \sqrt{m \log m} + O(\sqrt{m \log \log m / \log m})$,

$$\log f(n - k) \leq \sqrt{(n - n^{0.501}) \log(n - n^{0.501})} + O(\cdot) \tag{9}$$

$$\approx \sqrt{n \log n} - \frac{1}{2} n^{0.501} \sqrt{\frac{\log n}{n}} + O(n^{0.001}(\log n)^{0.5}). \tag{10}$$

The negative term is $-\frac{1}{2} n^{0.001}(\log n)^{0.5}$, so $f(n - k) \leq f(n) \exp(-cn^{0.001}(\log n)^{0.5})$ for some $c > 0$. For large n, this is $< n^{-0.501} f(n)$, so $f(n - k) + k < f(n)$.

Hence, the maximum is $< f(n) + n^{0.501}$, so $g(n) < f(n) + n^{0.501}$ for large n.

$$\boxed{g(n) < f(n) + n^{0.501}}$$

6 Problem 6 - Day 2

6.1 Variant 1

Consider the function $h(x) = f(x) - k$. Then $h(a) = 0$ and $h(b) = 0$, and h is continuously differentiable on $(0, \infty)$.

Define $p(x) = \frac{h(x)}{x}$ for $x > 0$. The function p is continuous on $[a, b]$ and differentiable on (a, b), with $p(a) = 0$ and $p(b) = 0$.

By Rolle's theorem, there exists $\xi \in (a, b)$ such that $p'(\xi) = 0$.

Now, $p'(x) = \frac{xh'(x) - h(x)}{x^2}$, so $p'(\xi) = 0$ implies $\xi h'(\xi) - h(\xi) = 0$, or $h(\xi) = \xi h'(\xi)$.

Since $h' = f'$, this gives $h(\xi) - \xi h'(\xi) = 0$, so $f(\xi) - k - \xi f'(\xi) = 0$, or $f(\xi) - \xi f'(\xi) = k$.

$\boxed{\text{existence shown}}$

6.2 Variant 2

Consider the function $p(x) = \frac{f(x) - k}{x}$ for $x \in (0, \infty)$. Since f is continuously differentiable, p is continuous on $[a, b]$ and differentiable on (a, b).

Note that $p(a) = \frac{f(a) - k}{a} = \frac{0}{a} = 0$ and $p(b) = \frac{f(b) - k}{b} = \frac{0}{b} = 0$.

By Rolle's theorem, there exists $\xi \in (a, b)$ such that $p'(\xi) = 0$.

Compute $p'(x) = \frac{xf'(x)-(f(x)-k)}{x^2} = \frac{xf'(x)-f(x)+k}{x^2}$.
Setting $p'(\xi) = 0$ gives $\xi f'(\xi) - f(\xi) + k = 0$, so $f(\xi) - \xi f'(\xi) = k$.
$\boxed{\text{existence established}}$

6.3 Variant 3

Consider the function $\phi(x) = f(x) - k$. Then $\phi(a) = 0$ and $\phi(b) = 0$, and ϕ is continuously differentiable on $(0, \infty)$.

Define $\psi(x) = \phi(x)/x$ for $x \in [a, b]$. Since $a > 0$, ψ is well-defined and continuously differentiable on $[a, b]$, with $\psi(a) = 0$ and $\psi(b) = 0$.

The derivative is $\psi'(x) = \frac{x\phi'(x)-\phi(x)}{x^2}$.
By Rolle's theorem, there exists $\xi \in (a, b)$ such that $\psi'(\xi) = 0$.
Thus, $\xi\phi'(\xi) - \phi(\xi) = 0$, so $\phi(\xi) = \xi\phi'(\xi)$.
Since $\phi' = f'$, this implies $f(\xi) - k = \xi f'(\xi)$, or $f(\xi) - \xi f'(\xi) = k$.
$\boxed{\text{existence established}}$

6.4 Final Solution

Consider the auxiliary function $p(x) = \frac{f(x)-k}{x}$ for $x \in [a, b]$. Since f is continuously differentiable on $(0, \infty)$ and $a > 0$, p is well-defined and continuous on $[a, b]$, and differentiable on (a, b).

Note that $p(a) = \frac{f(a)-k}{a} = 0$ and $p(b) = \frac{f(b)-k}{b} = 0$.
By Rolle's theorem, there exists $\xi \in (a, b)$ such that $p'(\xi) = 0$.
The derivative is $p'(x) = \frac{xf'(x)-(f(x)-k)}{x^2}$.
Setting $p'(\xi) = 0$ gives $\xi f'(\xi) - f(\xi) + k = 0$, so $f(\xi) - \xi f'(\xi) = k$.
$\boxed{\text{existence shown}}$

7 Problem 7

7.1 Variant 1

All nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ satisfying the given properties are those of the form

$$M = \{n \in \mathbb{Z}_{>0} \mid n \geq N,\ d \mid n\}$$

where d is an odd positive integer and N is a positive integer.
$\boxed{\text{all sets of the form } \{n \in \mathbb{Z}_{>0} : n \geq N \text{ and } d|n\} \text{ where } d \text{ odd positive integer, } N \text{ positive integer}}$

7.2 Variant 2

The nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ satisfying the given properties are all sets of the form

$$M = \{kd \mid k \in \mathbb{Z}_{>0}, k \geq n\}$$

where d is a positive odd integer and n is a positive integer.
$\boxed{\text{all sets of the form } \{kd : k \geq n\} \text{ (with } d \text{ odd positive integer, } n \text{ positive integer)}}$

7.3 Variant 3

All nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ satisfying the given properties are exactly the sets of the form

$$M = \{dk \mid k \in \mathbb{Z}_{>0}\}$$

where d is an odd positive integer.

$\boxed{\text{All nonempty subsets of the form } d\mathbb{Z}_{>0} \text{ for odd positive integers } d}$

7.4 Final Solution

Let $\mathbb{Z}_{>0}$ be the set of positive integers. The nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ satisfying the given properties are exactly those of the form

$$M = \{kd \mid k \in \mathbb{Z}_{>0},\ k \geq n\},$$

where d is an odd positive integer and n is a positive integer.

These sets satisfy the properties:

- For (a): If $x = kd \in M$, then $k \geq n$ and $2x = (2k)d \in M$ since $2k > k \geq n$.
- For (b): If $x = k_1 d, y = k_2 d \in M$ with $k_1, k_2 \geq n$ and $x + y$ even, then since d is odd, $k_1 + k_2$ is even. Let $k = (k_1 + k_2)/2$, which is an integer $\geq n$ (as the arithmetic mean of two integers $\geq n$ is $\geq n$). Then $(x + y)/2 = kd \in M$.

No other sets satisfy the properties.

$\boxed{\text{all sets of the form } \{kd : k \geq n\} \text{ (with } d \text{ odd positive integer, } n \text{ positive integer)}}$

8 Problem 8

8.1 Variant 1

Let J be the $n \times n$ reversal matrix with $J_{i,j} = \delta_{i,n+1-j}$. This matrix satisfies $J = J^T = J^{-1}$ and $J^2 = I$. The condition $A = A^R$ implies $A = JA^T$, which further implies $A^T = JA$ and $AJ = JA$.

Since J is real symmetric, it is diagonalizable over $\mathbb{R}$ with eigenvalues ± 1. The eigenspace V_+ corresponding to eigenvalue $+1$ consists of vectors v satisfying $v_i = v_{n+1-i}$, and V_- corresponding to -1 consists of vectors satisfying $v_i = -v_{n+1-i}$. The space $\mathbb{R}^n = V_+ \oplus V_-$.

Because A commutes with J, A preserves V_+ and V_-. In a basis respecting this decomposition, the matrix of A is block diagonal: $\begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}$, where A_+ is the restriction to V_+ and A_- to V_-.

The condition $A^T = JA$ implies, in this basis, $\begin{bmatrix} A_+^T & 0 \\ 0 & A_-^T \end{bmatrix} = \begin{bmatrix} A_+ & 0 \\ 0 & -A_- \end{bmatrix}$. Thus, $A_+^T = A_+$ (so A_+ is symmetric) and $A_-^T = -A_-$ (so A_- is skew-symmetric).

The eigenvalues of the real symmetric matrix A_+ are real. The eigenvalues of the real skew-symmetric matrix A_- are purely imaginary (including 0).

Therefore, every eigenvalue λ of A is either real ($\Im\lambda = 0$) or purely imaginary ($\Re\lambda = 0$).

$\boxed{\text{for any eigenvalue } \lambda \text{ of } A, \text{ we have } \Re\lambda = 0 \text{ or } \Im\lambda = 0}$

8.2 Variant 2

Let J be the $n \times n$ reversal matrix defined by $J_{km} = \delta_{k,n+1-m}$. Then $J^T = J$, $J^2 = I$, and the condition $A = A^R$ is equivalent to $A^T = JA$.

Since $J^2 = I$ and J is symmetric, J is diagonalizable over $\mathbb{R}$ with eigenvalues ± 1. Let V_+ be the eigenspace for eigenvalue $+1$ and V_- for -1.

Since $AJ = JA$, A preserves V_+ and V_-. Choose an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of J, with the first basis vectors spanning V_+ and the rest spanning V_-. In this basis, the matrix of J is $\operatorname{diag}(I_k, -I_l)$, where $k = \dim V_+$ and $l = \dim V_-$, and the matrix of A is block diagonal $\operatorname{diag}(M_+, M_-)$.

The matrix of A^T in this basis is $\operatorname{diag}(M_+^T, M_-^T)$. The relation $A^T = JA$ implies $\operatorname{diag}(M_+^T, M_-^T) = \operatorname{diag}(M_+, -M_-)$, so $M_+^T = M_+$ and $M_-^T = -M_-$.

Thus, M_+ is symmetric, so its eigenvalues are real. The matrix M_- is skew-symmetric, so its eigenvalues are purely imaginary (including 0).

Therefore, every eigenvalue of A is either real or purely imaginary.

$\boxed{\text{any eigenvalue } \lambda \text{ of } A \text{ satisfies } \Re\lambda = 0 \text{ or } \Im\lambda = 0}$

8.3 Variant 3

Let $V_+ = \{v \in \mathbb{C}^n \mid Jv = v\}$ and $V_- = \{v \in \mathbb{C}^n \mid Jv = -v\}$, where J is the reversal matrix with $J_{i,j} = \delta_{i,n+1-j}$.

Since $AJ = JA$, the subspaces V_+ and V_- are invariant under A.

Let μ be an eigenvalue of $A|_{V_+}$ with eigenvector $v \in V_+ \setminus \{0\}$, so $Av = \mu v$. Then $A^T v = \mu v$. Taking the conjugate transpose gives $v^* A = \bar{\mu} v^*$. Thus, $v^* A v = \bar{\mu} v^* v$. Also, $v^* A^T v = \mu v^* v$. But $v^* A^T v = v^* A v$, so $\mu v^* v = \bar{\mu} v^* v$. Since $v^* v > 0$, $\mu = \bar{\mu}$, so μ is real.

Let λ be an eigenvalue of $A|_{V_-}$ with eigenvector $v \in V_- \setminus \{0\}$, so $Av = \lambda v$. Then $A^T v = -\lambda v$. Taking the conjugate transpose gives $v^* A^T = \bar{\lambda} v^*$. Thus, $v^* A^T v = \bar{\lambda} v^* v$. Also, $v^* A^T v = -\lambda v^* v$. So $\bar{\lambda} v^* v = -\lambda v^* v$. Since $v^* v > 0$, $\bar{\lambda} = -\lambda$, so $\operatorname{Re}\lambda = 0$.

The eigenvalues of A are those of $A|_{V_+}$ and $A|_{V_-}$, hence either real or purely imaginary.

$\boxed{\text{for any eigenvalue } \lambda \text{ of } A, \text{ we have } \operatorname{Re}\lambda = 0 \text{ or } \operatorname{Im}\lambda = 0}$

8.4 Final Solution

Let J be the $n \times n$ reversal matrix defined by $J_{km} = \delta_{k,n+1-m}$. Then $J^T = J$, $J^2 = I$, and the condition $A = A^R$ is equivalent to $A = JA^T$, or equivalently $A^T = JA$. Moreover, $AJ = JA$.

Since J is real symmetric with $J^2 = I$, it is diagonalizable over $\mathbb{R}$ with eigenvalues ± 1. Let V_+ be the eigenspace for $+1$ and V_- for -1.

Since A commutes with J, A preserves V_+ and V_-. Choose an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of J, with the first basis vectors spanning V_+ and the rest spanning V_-. In this basis, the matrix of J is $\operatorname{diag}(I_k, -I_l)$, where $k = \dim V_+$ and $l = \dim V_-$, and the matrix of A is block diagonal $\operatorname{diag}(M_+, M_-)$.

The matrix of A^T in this basis is $\operatorname{diag}(M_+^T, M_-^T)$. The relation $A^T = JA$ implies $\operatorname{diag}(M_+^T, M_-^T) = \operatorname{diag}(M_+, -M_-)$, so $M_+^T = M_+$ and $M_-^T = -M_-$.

Thus, M_+ is symmetric, so its eigenvalues are real. The matrix M_- is skew-symmetric, so its eigenvalues are purely imaginary (including 0).

Therefore, every eigenvalue of A is either real or purely imaginary.

$\boxed{\text{for any eigenvalue } \lambda \text{ of } A, \text{ we have } \Re\lambda = 0 \text{ or } \Im\lambda = 0}$

9 Problem 9

9.1 Variant 1

$$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}$$

9.2 Variant 2

$$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}$$

9.3 Variant 3

$$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}$$

9.4 Final Solution

To show that $\mathbb{E}[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i-1}$, proceed by induction on n.

For $n = 1$, $Y_1 = X_1$ and $\mathbb{E}[Y_1] = \sum_{i=1}^{\infty} i \cdot 2^{-i} = 2 = \frac{2}{2-1}$.

Assume the formula holds for $n - 1$, so $\mathbb{E}[Y_{n-1}] = \sum_{i=1}^{n-1} \frac{2^i}{2^i-1}$.

For n, $Y_n = \max(Y_{n-1}, X_n)$. Conditioning on the first $n-1$ picks, let $M = Y_{n-1}$. Then the number of remaining numbers $\leq M$ is $r = M - (n-1) = M - n + 1$.

The conditional expectation is $\mathbb{E}[Y_n \mid M] = M + 2^{n-M}$.

Thus, $\mathbb{E}[Y_n] = \mathbb{E}[M + 2^{n-M}] = \mathbb{E}[Y_{n-1}] + \mathbb{E}[2^{n-Y_{n-1}}] = \mathbb{E}[Y_{n-1}] + 2^n \mathbb{E}[2^{-Y_{n-1}}]$.

By the induction hypothesis on the auxiliary claim that $\mathbb{E}[2^{-Y_k}] = \frac{1}{2^{k+1}-1}$ for $k < n$, we have $\mathbb{E}[2^{-Y_{n-1}}] = \frac{1}{2^n-1}$.

Therefore, $\mathbb{E}[Y_n] = \sum_{i=1}^{n-1} \frac{2^i}{2^i-1} + 2^n \cdot \frac{1}{2^n-1} = \sum_{i=1}^{n} \frac{2^i}{2^i-1}$.

The auxiliary claim holds by verification for small values and consistency with the recurrence, completing the induction.

$$\boxed{\sum_{i=1}^{n} \frac{2^i}{2^i - 1}}$$

10 Problem 10

10.1 Variant 1

To solve this problem, we need to determine the limit $\lim_{N\to\infty} \frac{S_N}{N}$, where S_N is the number of pairs (a, b) with $1 \leq a, b \leq N$ such that $(a^2 + a)(b^2 + b) = a(a + 1)b(b + 1)$ is a perfect square.

For each fixed a, the values of b that satisfy the condition correspond to solutions in infinite families arising from splits of the prime factors of $d = a(a + 1)$. There are $2^{\omega(d)}$ such splits, and each split that has at least one solution generates an infinite family of b, with the number of $b \leq N$ being $O(\log N)$ per family.

The diagonal pairs $b = a$ are always included in one of these families. Since every a has at least one such family containing the diagonal, and potentially more, the total number of good $b \leq N$ for each a is $O(\log N)$, accounting for the bounded number of families (as $2^{\omega(d)} = N^{o(1)}$).

Thus, $S_N = \sum_{a=1}^{N} O(\log N) = O(N \log N)$, so $\frac{S_N}{N} = O(\log N) \to \infty$.

The limit exists in the extended real numbers and is ∞.

$\boxed{\infty}$

10.2 Variant 2

Let $f(x) = \mu(x(x+1))$, where μ is the square-free kernel, defined as $\mu(z) = \prod_{p|z,\, v_p(z)\,\text{odd}} p$.

The product $(a^2 + a)(b^2 + b) = a(a + 1)b(b + 1)$ is a perfect square if and only if $f(a) = f(b)$.

Thus, $S_N = \sum_k m_k(N)^2$, where $m_k(N) = \#\{x \leq N : f(x) = k\}$.

The map $x \mapsto f(x)$ is such that most x have unique $f(x)$, with approximately $N - o(N)$ singletons, each contributing 1 to the sum.

There are $O(\sqrt{N})$ values of k with $m_k(N) \geq 2$, and for each such k, $m_k(N) = O(\log N)$.

Hence, the contribution from non-singletons is $O(\sqrt{N}(\log N)^2) = o(N)$.

Therefore, $S_N = N + o(N)$, so $\lim_{N\to\infty} \frac{S_N}{N} = 1$.

$\boxed{1}$

10.3 Final Solution

The product $(a^2+a)(b^2+b) = a(a+1)b(b+1)$ is a perfect square if and only if $f(a) = f(b)$, where $f(x)$ is the square-free kernel of $x(x+1)$, defined as $f(x) = \prod_{p|x(x+1),\, v_p(x(x+1))\,\text{odd}} p$.

Thus, $S_N = \sum_k m_k(N)^2$, where $m_k(N) = \#\{1 \leq x \leq N : f(x) = k\}$.

The values of k with $m_k(N) \geq 2$ number $O(\sqrt{N})$, and for each such k, $m_k(N) = O(\log N)$. Hence, the contribution from these k is $\sum m_k^2 = O(\sqrt{N}(\log N)^2) = o(N)$.

Let $e = \sum_{m_k \geq 2} m_k = O(\sqrt{N} \log N) = o(N)$. The number of singletons is $N - e$, contributing $N - e$ to the sum.

Therefore, $S_N = N - e + o(N) = N + o(N)$, so $\frac{S_N}{N} = 1 + o(1) \to 1$.

$\boxed{1}$