12th IMC 2005, July 22 – 28, Blagoevgrad

First Day

Problem 1

Let A be the $n \times n$ matrix, whose $(i,j)^{\text{th}}$ entry is $i+j$ for all $i,j = 1,2,\dots,n$. What is the rank of A?

Problem 2

For an integer $n \geq 3$ consider the sets

$$S_n = \{(x_1, x_2, \dots, x_n) : \forall i \; x_i \in \{0,1,2\}\}$$

$$A_n = \{(x_1, x_2, \dots, x_n) \in S_n : \forall i \leq n-2 \; |\{x_i, x_{i+1}, x_{i+2}\}| \neq 1\}$$

and

$$B_n = \{(x_1, x_2, \dots, x_n) \in S_n : \forall i \leq n-1 \; (x_i = x_{i+1} \Rightarrow x_i \neq 0)\}.$$

Prove that $|A_{n+1}| = 3 \cdot |B_n|$.
($|A|$ denotes the number of elements of the set A.)

Problem 3

Let $f : \mathbb{R} \to [0, \infty)$ be a continuously differentiable function. Prove that

$$\left| \int_0^1 f^3(x)\,dx - f^2(0) \int_0^1 f(x)\,dx \right| \leq \max_{0 \leq x \leq 1} |f'(x)| \left(\int_0^1 f(x)\,dx \right)^2.$$

Problem 4

Find all polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ $(a_n \neq 0)$ satisfying the following two conditions:

(i) $(a_0, a_1, \dots, a_n)$ is a permutation of the numbers $(0, 1, \dots, n)$

(ii) All roots of $P(x)$ are rational numbers.

Problem 5

Let $f : (0, \infty) \to \mathbb{R}$ be a twice continuously differentiable function such that

$$\left| f''(x) + 2x f'(x) + (x^2+1) f(x) \right| \leq 1$$

for all x. Prove that $\lim\limits_{x \to \infty} f(x) = 0$.

Problem 6

Given a group G, denote by $G(m)$ the subgroup generated by the m^{th} powers of elements of G. If $G(m)$ and $G(n)$ are commutative, prove that $G(\gcd(m,n))$ is also commutative.

($\gcd(m,n)$ denotes the greatest common divisor of m and n.)