8th IMC 2001, July 19 – July 25, Prague, Czech Republic, Second day

Problem 1

Let $r, s \geq 1$ be integers and $a_0, a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{s-1}$ be real non-negative numbers such that

$$(a_0 + a_1 x + \cdots + a_{r-1} x^{r-1} + x^r)(b_0 + b_1 x + \cdots + b_{s-1} x^{s-1} + x^s) = 1 + x + x^2 + \cdots + x^{r+s}.$$

Prove that each a_i and each b_j equals either 0 or 1.

Solution

Multiply the left-hand side polynomials:

$$a_0 b_0 = 1, \quad a_0 b_1 + a_1 b_0 = 1, \quad \ldots$$

From these equations, it follows that $a_0, b_0 \leq 1$. Since $a_0 b_0 = 1$, we get $a_0 = b_0 = 1$. Similarly, from $a_0 b_1 + a_1 b_0 = 1$, one of a_1, b_1 equals 0 while the other equals 1. Proceeding by induction, all a_i, b_j equal either 0 or 1.

Problem 2

Let $a_0 = \sqrt{2}$, $b_0 = 2$, and define

$$a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}, \quad b_{n+1} = \frac{2b_n}{2 + \sqrt{4 + b_n^2}}.$$

a) Prove that the sequences (a_n) and (b_n) are decreasing and converge to 0.

b) Prove that $(2^n a_n)$ is increasing, $(2^n b_n)$ is decreasing, and that these two sequences converge to the same limit.

c) Prove that there exists a constant $C > 0$ such that for all n:

$$0 < b_n - a_n < \frac{C}{8^n}.$$

Solution

Obviously $a_2 = \sqrt{2} - \sqrt{\sqrt{4 - x^2}} < \sqrt{2}$. Since the function

$$f(x) = \sqrt{2} - \sqrt{\sqrt{4 - x^2}}$$

is increasing on the interval $[0, 2]$, the inequality $a_1 > a_2$ implies that $a_2 > a_3$. Simple induction ends the proof of monotonicity of (a_n). In the same way we prove that (b_n) decreases (just notice that

$$g(x) = \frac{2x}{2 + \sqrt{4 + x^2}} = \frac{2}{2/x + \sqrt{1 + 4/x^2}}).$$

It is a matter of simple manipulation to prove that

$$2f(x) > x \quad \text{for all } x \in (0, 2),$$

this implies that the sequence $(2^n a_n)$ is strictly increasing. The inequality $2g(x) < x$ for $x \in (0, 2)$ implies that the sequence $(2^n b_n)$ strictly decreases. By an easy induction one can show that

$$a_n^2 = \frac{4b_n^2}{4 + b_n^2}$$

for positive integers n. Since the limit of the decreasing sequence $(2^n b_n)$ of positive numbers is finite we have

$$\lim 4^n a_n^2 = \lim \frac{4 \cdot 4^n b_n^2}{4 + b_n^2} = \lim 4^n b_n^2.$$

We know already that the limits $\lim 2^n a_n$ and $\lim 2^n b_n$ are equal. The first of the two is positive because the sequence $(2^n a_n)$ is strictly increasing. The existence of a number C follows easily from the equalities

$$2^n b_n - 2^n a_n = \left(4^n b_n^2 - \frac{4^{n+1} b_n^2}{4+b_n^2}\right) / \left(2^n b_n + 2^n a_n\right) = \frac{(2^n b_n)^4}{4+b_n^2} \cdot \frac{1}{4^n} \cdot \frac{1}{2^n(b_n + a_n)}$$

and from the existence of positive limits $\lim 2^n b_n$ and $\lim 2^n a_n$.

Remark. The last problem may be solved in a much simpler way by someone who is able to make use of sine and cosine. It is enough to notice that

$$a_n = 2\sin\frac{\pi}{2n+1} \quad \text{and} \quad b_n = 2\tan\frac{\pi}{2n+1}.$$

Problem 3

Find the maximum number of points on the unit sphere in $\mathbb{R}^n$ such that the distance between any two points is strictly greater than $\sqrt{2}$.

Solution

The unit sphere in $\mathbb{R}^n$ is defined by

$$S_{n-1} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_{k=1}^n x_k^2 = 1 \right\}.$$

The distance between the points $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ is:

$$d^2(X,Y) = \sum_{k=1}^n (x_k - y_k)^2.$$

We have

$$d(X,Y) > \sqrt{2} \quad \Leftrightarrow \quad d^2(X,Y) > 2$$

$$\Leftrightarrow \quad \sum_{k=1}^n x_k^2 + \sum_{k=1}^n y_k^2 - 2\sum_{k=1}^n x_k y_k > 2$$

$$\Leftrightarrow \quad 1 + 1 - 2\sum_{k=1}^n x_k y_k > 2$$

$$\Leftrightarrow \quad \sum_{k=1}^n x_k y_k < 0.$$

Taking account of the symmetry of the sphere, we can suppose that

$$A_1 = (-1, 0, \ldots, 0).$$

For $X = A_1$, the inequality $\sum_{k=1}^n x_k y_k < 0$ implies $y_1 > 0$, $\forall Y \in M_n$.

Let $X = (x_1, X')$, $Y = (y_1, Y') \in M_n \setminus \{A_1\}$, where $X', Y' \in \mathbb{R}^{n-1}$.

We have

$$\sum_{k=1}^n x_k y_k < 0 \quad \Rightarrow \quad x_1 y_1 + \sum_{k=1}^{n-1} \overline{x_k y_k} < 0 \quad \Leftrightarrow \quad \sum_{k=1}^{n-1} x'_k y'_k < 0,$$

where

$$x'_k = \frac{\overline{x_k}}{\sqrt{\sum \overline{x_k}^2}}, \quad y'_k = \frac{\overline{y_k}}{\sqrt{\sum \overline{y_k}^2}}.$$

Therefore

$$(x'_1, \ldots, x'_{n-1}), (y'_1, \ldots, y'_{n-1}) \in S_{n-2}$$

and verifies $\sum_{k=1}^n x_k y_k < 0$.

If a_n is the search number of points in $\mathbb{R}^n$, we obtain

$$a_n \leq 1 + a_{n-1}$$

and $a_1 = 2$ implies that $a_n \leq n+1$.

We show that $a_n = n+1$, giving an example of a set M_n with $(n+1)$ elements satisfying the conditions of the problem.

$$\begin{aligned}
A_1 &= (-1, 0, 0, 0, \ldots, 0, 0) \\
A_2 &= \left(\frac{1}{n}, -c_1, 0, 0, \ldots, 0, 0\right) \\
A_3 &= \left(\frac{1}{n}, \frac{1}{n-1}c_1, -c_2, 0, \ldots, 0, 0\right) \\
A_4 &= \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-1}c_2, -c_3, \ldots, 0, 0\right) \\
A_{n-1} &= \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \frac{1}{n-3}c_3, \ldots, -c_{n-2}, 0\right) \\
A_n &= \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \frac{1}{n-3}c_3, \ldots, \frac{1}{2}c_{n-2}, -c_{n-1}\right) \\
A_{n+1} &= \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \frac{1}{n-3}c_3, \ldots, \frac{1}{2}c_{n-2}, c_{n-1}\right)
\end{aligned}$$

where

$$c_k = \sqrt{\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n-k+1}\right)}, \quad k = 1, \ldots, n-1.$$

We have

$$\sum_{k=1}^{n} x_k y_k = -\frac{1}{n} < 0 \quad \text{and} \quad \sum_{k=1}^{n} x_k^2 = 1, \quad \forall X, Y \in \{A_1, \ldots, A_{n+1}\}.$$

These points are on the unit sphere in $\mathbb{R}^n$ and the distance between any two points is equal to

$$d = \sqrt{2}\sqrt{1+\frac{1}{n}} > \sqrt{2}.$$

Remark. For $n=2$ the points form an equilateral triangle in the unit circle; for $n=3$ the four points form a regular tetrahedron and in $\mathbb{R}^n$ the points form an n-dimensional regular simplex.

Problem 4

Let $A = (a_{k,\ell})_{k,\ell=1}^n$ be an $n \times n$ complex matrix such that for each $1 \leq m \leq n$ and each $1 \leq j_1 < \cdots < j_m \leq n$, the determinant

$$\det(a_{j_k,j_\ell})_{k,\ell=1}^m = 0.$$

Prove that $A^n = 0$ and that there exists a permutation $\sigma \in S_n$ such that the permuted matrix $(a_{\sigma(k),\sigma(\ell)})$ is strictly upper-triangular.

Solution.

We will only prove (2), since it implies (1). Consider a directed graph G with n vertices $V_1, \ldots, V_n$ and a directed edge from V_k to V_ℓ whenever $a_{k,\ell} \neq 0$. We shall prove that it is acyclic.

Assume that there exists a cycle and take one of minimum length m. Let $j_1 < \cdots < j_m$ be the vertices the cycle goes through and let $\sigma_0 \in S_n$ be a permutation such that $a_{j_k,j_{\sigma_0(k)}} \neq 0$ for $k = 1, \ldots, m$. Observe that for any other $\sigma \in S_n$ we have $a_{j_k,j_{\sigma(k)}} = 0$ for some $k \in \{1, \ldots, m\}$, otherwise we would obtain a different cycle through the same set of vertices and, consequently, a shorter cycle. Finally,

$$\begin{aligned}
0 &= \det(a_{j_k,j_\ell})_{k,\ell=1,\ldots,m} \\
&= (-1)^{\text{sign}\,\sigma_0} \prod_{k=1}^{m} a_{j_k,j_{\sigma_0(k)}} + \sum_{\sigma\neq\sigma_0} (-1)^{\text{sign}\,\sigma} \prod_{k=1}^{m} a_{j_k,j_{\sigma(k)}} \neq 0,
\end{aligned}$$

which is a contradiction.

Since G is acyclic there exists a topological ordering, i.e. a permutation $\sigma \in S_n$ such that $k < \ell$ whenever there is an edge from $V_{\sigma(k)}$ to $V_{\sigma(\ell)}$. It is easy to see that this permutation solves the problem.

Problem 5

Prove that there does not exist a function $f:\mathbb{R}\to\mathbb{R}$ with $f(0)>0$ satisfying

$$f(x+y)\geq f(x)+yf(f(x))\quad \forall x,y\in\mathbb{R}.$$

Solution.

Suppose that there exists a function satisfying the inequality. If $f(f(x))\leq 0$ for all x, then f is a decreasing function in view of the inequalities

$$f(x+y)\geq f(x)+yf(f(x))\geq f(x)\quad \text{for any } y\leq 0.$$

Since $f(0)>0\geq f(f(x))$, it implies $f(x)>0$ for all x, which is a contradiction. Hence there is a z such that $f(f(z))>0$. Then the inequality $f(z+x)\geq f(z)+xf(f(z))$ shows that

$$\lim_{x\to\infty} f(x)=+\infty \quad \text{and therefore} \quad \lim_{x\to\infty} f(f(x))=+\infty.$$

In particular, there exist $x,y>0$ such that $f(x)\geq 0$, $f(f(x))>1$,

$$y\geq \frac{x+1}{f(f(x))-1}$$

and $f(f(x+y+1))\geq 0$. Then

$$f(x+y)\geq f(x)+yf(f(x))\geq x+y+1,$$

and hence

$$\begin{aligned}
f(f(x+y)) &\geq f(x+y+1)+(f(x+y)-(x+y+1))f(f(x+y+1))\\
&\geq f(x+y+1)\\
&\geq f(x+y)+f(f(x+y))\\
&\geq f(x)+yf(f(x))+f(f(x+y))\\
&> f(f(x+y)).
\end{aligned}$$

This contradiction completes the solution of the problem.

Problem 6

For each positive integer n, let

$$f_n(\theta)=\sin(\theta)\sin(2\theta)\cdots\sin(2^n\theta).$$

Prove that for all real θ and all n:

$$|f_n(\theta)|\leq \frac{2}{\sqrt{3}}\,|f_n(\pi/3)|.$$

Solution.

We prove that $g(\vartheta)=|\sin\vartheta|\,|\sin(2\vartheta)|^{1/2}$ attains its maximum value $\left(\sqrt{3}/2\right)^{3/2}$ at points $2^k\pi/3$ (where k is a positive integer). This can be seen by using derivatives or a classical bound like

$$|g(\vartheta)|=|\sin\vartheta|\,|\sin(2\vartheta)|^{1/2}=\frac{\sqrt{2}}{\sqrt[4]{3}}\left(\sqrt[4]{|\sin\vartheta|\,|\sin\vartheta|\,|\sin\vartheta|\,|\sqrt{3}\cos\vartheta|}\right)^2$$

$$\leq \frac{\sqrt{2}}{\sqrt[4]{3}}\cdot\frac{3\sin^2\vartheta+3\cos^2\vartheta}{4}=\left(\frac{\sqrt{3}}{2}\right)^{3/2}.$$

Hence

$$\left|\frac{f_n(\vartheta)}{f_n(\pi/3)}\right|=\left|\frac{g(\vartheta)\cdot g(2\vartheta)^{1/2}\cdot g(4\vartheta)^{3/4}\cdots g(2^{n-1}\vartheta)^E}{g(\pi/3)\cdot g(2\pi/3)^{1/2}\cdot g(4\pi/3)^{3/4}\cdots g(2^{n-1}\pi/3)^E}\right|\cdot\left|\frac{\sin(2^n\vartheta)}{\sin(2^n\pi/3)}\right|^{1-E/2}$$

$$\leq\left|\frac{\sin(2^n\vartheta)}{\sin(2^n\pi/3)}\right|^{1-E/2}\left(\frac{1}{\sqrt[3]{2}}\right)^{1-E/2}\leq\frac{2}{\sqrt{3}}.$$

where $E=\frac{2}{3}(1-(-1/2)^n)$. This is exactly the bound we had to prove.