Discrete Math Olympiad Problems

March 2025

Problem 1. Suppose that in a not necessarily commutative ring R the square of any element is 0. Prove that $abc + abc = 0$ for any three elements a, b, c.

Problem 2. We throw a dice (which selects one of the numbers $1, 2, \dots, 6$ with equal probability) n times. What is the probability that the sum of the values is divisible by 5?

Problem 3. Assume that $x_1, \dots, x_n \geq -1$ and $\sum_{i=1}^n x_i^3 = 0$. Prove that $\sum_{i=1}^n x_i \leq \frac{n}{3}$.

Problem 4. Let α be a real number, $1 < \alpha < 2$.
a) Show that α has a unique representation as an infinite product

$$\alpha = \left(1 + \frac{1}{n_1}\right)\left(1 + \frac{1}{n_2}\right)\dots$$

where each n_i is a positive integer satisfying

$$n_i^2 \leq n_{i+1}.$$

b) Show that α is rational if and only if its infinite product has the following property:
For some m and all $k \geq m$,

$$n_{k+1} = n_k^2.$$

Problem 5. Suppose that F is a family of finite subsets of $\mathbb{N}$ and for any two sets $A, B \in F$ we have $A \cap B \neq \emptyset$.
a) Is it true that there is a finite subset Y of $\mathbb{N}$ such that for any $A, B \in F$ we have $A \cap B \cap Y \neq \emptyset$?
b) Is the statement a) true if we suppose in addition that all of the members of F have the same size?
Justify your answers.

Problem 6. Let X be an arbitrary set, let f be a one-to-one function mapping X onto itself. Prove that there exist mappings $g_1, g_2 : X \to X$ such that $f = g_1 \circ g_2$ and $g_1 \circ g_1 = id = g_2 \circ g_2$, where id denotes the identity mapping on X.

Problem 7. Prove that the following proposition holds for $n = 3$ and $n = 5$, and does not hold for $n = 4$.
"For any permutation π_1 of $\{1, 2, \dots, n\}$ different from the identity, there is a permutation π_2 such that any permutation π can be obtained from π_1 and π_2 using only compositions (for example, $\pi = \pi_1 \circ \pi_1 \circ \pi_2 \circ \pi_1$)."

Problem 8. Let P be an algebraic polynomial of degree n having only real zeros and real coefficients.
a) Prove that for every real x the following inequality holds:

$$(n-1)(P'(x))^2 \geq nP(x)P''(x)$$

b) Examine the cases of equality.

Problem 9. Let S be the set of all words consisting of the letters x, y, z, and consider an equivalence relation $\sim$ on S satisfying the following conditions: for arbitrary words $u, v, w \in S$

(i) $uu \sim u$;

(ii) if $v \sim w$, then $uv \sim uw$ and $vu \sim wu$.

Show that every word in S is equivalent to a word of length at most 8.

Problem 10. *a)* Show that if (x_i) is a decreasing sequence of positive numbers then

$$\left(\sum_{i=1}^n x_i^2\right)^{1/2} \leq \sum_{i=1}^n \frac{x_i}{\sqrt{i}}.$$

b) Show that there is a constant C so that if (x_i) is a decreasing sequence of positive numbers then

$$\sum_{m=1}^\infty \frac{1}{\sqrt{m}} \left(\sum_{i=m}^\infty x_i^2\right)^{1/2} \leq C \sum_{i=1}^\infty x_i.$$