Second Olympiad for NUP team selection

May 2025

Problem 1. (10 points) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any real numbers $a < b$, the image $f([a, b])$ is a closed interval of length $b - a$.

Problem 2. (10 points) Let A be an $n \times n$ real matrix such that $3A^3 = A^2 + A + I$ (I is the identity matrix). Show that the sequence A^k converges to an idempotent matrix. (A matrix B is called idempotent if $B^2 = B$.)

Problem 3. (10 points) Let $a_1, a_2, \ldots, a_{51}$ be non-zero elements of a field. We simultaneously replace each element with the sum of the 50 remaining ones. In this way we get a sequence $b_1, \ldots, b_{51}$. If this new sequence is a permutation of the original one, what can be the characteristic of the field? (The characteristic of a field is p, if p is the smallest positive integer such that

$$\underbrace{x + x + \cdots + x}_{p} = 0$$

for any element x of the field. If there exists no such p, the characteristic is 0.)

Problem 4. (10 points) Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f(b) - f(a) = (b - a)f'\left(\sqrt{ab}\right) \quad \text{for all} \quad a, b > 0.$$

Problem 5. (10 points) Let $f(x)$ be a polynomial with real coefficients of degree n. Suppose that $\frac{f(k)-f(m)}{k-m}$ is an integer for all integers $0 \leq k < m \leq n$. Prove that $a - b$ divides $f(a) - f(b)$ for all pairs of distinct integers a and b.