8th IMC 2001, July 19 – July 25, Prague, Czech Republic, First day

Problem 1

Let n be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, \ldots, n^2$ written in order starting top left and moving along each row in turn left–to–right. We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

Solution. The choice corresponds to a permutation $\sigma \in S_n$, with selected entries at positions $(j, \sigma(j))$. The sum is:

$$\sum_{j=1}^{n} (n(j-1) + \sigma(j)) = n \sum_{j=1}^{n} (j-1) + \sum_{j=1}^{n} \sigma(j) = n \cdot \frac{n(n-1)}{2} + \frac{n(n+1)}{2} = \frac{n(n^2+1)}{2}$$

Thus, the sum is independent of σ.

Problem 2

Let r, s, t be positive integers which are pairwise relatively prime. If a and b are elements of a commutative multiplicative group with unity element e, and $a^r = b^s = (ab)^t = e$, prove that $a = b = e$.

Does the same conclusion hold if a and b are elements of an arbitrary non-commutative group?

Solution.

1. Since $ab = ba$, and $us + vt = 1$ for some integers u, v, then:

$$ab = (ab)^{us+vt} = (ab)^{us} \cdot (ab)^{tv} = a^{us} b^{us} \cdot e = a^{us} b^{us}$$

   So $ab = a^{us}$, and:

$$b^r = (ab)^r = a^{usr} = (a^r)^{us} = e \Rightarrow b = (b^r)^x (b^s)^y = e \Rightarrow b = e, \quad \text{similarly } a = e$$

2. **Counterexample:** Let $a = (123)$, $b = (34567) \in S_7$, then $a^3 = b^5 = (ab)^7 = e$, but $a \neq e$, $b \neq e$.

Problem 3

Find

$$\lim_{t \to 1^-} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n}$$

Solution. Let $h = -\ln t$, so as $t \to 1^-$, $h \to 0^+$:

$$\sum_{n=1}^{\infty} \frac{t^n}{1+t^n} = \sum_{n=1}^{\infty} \frac{1}{1+e^{nh}} \quad \Rightarrow (1-t) \sum \to h \sum \to \int_0^{\infty} \frac{dx}{1+e^x} = \ln 2$$

Problem 4

Let $k \in \mathbb{N}$. Let $p(x)$ be a polynomial of degree n with coefficients in $\{-1, 0, 1\}$, and divisible by $(x-1)^k$. Let q be prime such that

$$\frac{q}{\ln q} < \frac{k}{\ln(n+1)}$$

Prove that all complex qth roots of unity are roots of $p(x)$.

Solution. Let $p(x) = (x-1)^k r(x)$, and let $\varepsilon_j = e^{2\pi i j/q}$. Suppose none of ε_j are roots of $r(x)$. Then:

$$\left| \prod_{j=1}^{q-1} p(\varepsilon_j) \right| \geq \left| \prod_{j=1}^{q-1} (1-\varepsilon_j)^k \right| = q^k$$

But on the other hand:

$$\left| \prod_{j=1}^{q-1} p(\varepsilon_j) \right| \leq (n+1)^{q-1} \Rightarrow (n+1)^{q-1} \geq q^k \Rightarrow \text{Contradiction}$$

Problem 5

Let A be an $n \times n$ complex matrix such that $A \neq \lambda I$ for any $\lambda \in \mathbb{C}$. Prove that A is similar to a matrix with at most one non-zero entry on the main diagonal.

Solution. *Base case:* $n = 1$ is trivial. For $n = 2$, we analyze all possible cases of matrix entries and use similarity transformations to achieve the required form. For general n, apply induction and block structure:

$$A = \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix} \Rightarrow \text{Transform } A' \text{ and apply the step recursively}$$

Even in edge cases (e.g., $n = 3$), a suitable similarity transformation reduces A to desired form by using trace and matrix similarity invariance.

Problem 6

Suppose differentiable functions $a(x), b(x), f(x), g(x) : \mathbb{R} \to \mathbb{R}$ satisfy:

- $f(x) \geq 0,\ f'(x) \geq 0,\ g(x) > 0,\ g'(x) > 0$
- $\lim_{x\to\infty} a(x) = A > 0, \lim b(x) = B > 0$
- $\lim f(x) = \lim g(x) = \infty$
- $\frac{f'}{g'} + a(x) \cdot \frac{f}{g} = b(x)$

Prove:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{B}{A+1}$$

Solution. Let $h(x) = f(x)g(x)^A$, then:

$$h'(x) = f'(x)g(x)^A + Af(x)g(x)^{A-1}g'(x) \Rightarrow \frac{h'}{(g^{A+1})'} = \frac{f'(x)}{g'(x)} + A \cdot \frac{f(x)}{g(x)}$$

Using original equation:

$$\frac{h'}{(g^{A+1})'} = b(x) - (a(x) - A)\frac{f(x)}{g(x)} \Rightarrow \lim_{x\to\infty} \frac{h'}{(g^{A+1})'} = B \Rightarrow \lim \frac{f(x)}{g(x)} = \frac{B}{A+1}$$