Proposed problems for the IMC 2025

Problem 1 (Alex Avdiushenko, Neapolis University Paphos, Cyprus). Find all strictly monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)\, f^{-1}(x) = x^2.$$

Solution.

1. *Value at the origin.* Substituting $x = 0$ into equation gives

$$f(0)\, f^{-1}(0) = 0.$$

If $f(0) \neq 0$ then $f^{-1}(0) = 0$; applying f to both sides forces $f(0) = 0$ — a contradiction. Hence

$$f(0) = 0 \tag{1}$$

2. It is clear that either there exists an a such that $f(a) = ka$ with $k \neq 1$, or else $\forall x\, f(x) = x$.

Substituting $x = ka$ into the given equation yields $f(ka) = k^2 a$. Proceeding by induction, one easily shows that

$$f\big(k^n a\big) = k^{n+1} a \qquad (n \in \mathbb{Z}_{\geq 0}).$$

Now take a number x lying between a and $f(a) = ka$ and set $f(x) = lx$. By monotonicity, the value $f^n(x)$ lies between $f^n(a)$ and $f^{n+1}(a)$. In other words, the number $l^n x$ is always situated between $k^n a$ and $k^{n+1} a$, whence it follows that $l = k$.

3. Because f is a strictly monotonic bijection, every real number lies between some (forward or backward) iterate of a and $f(a) = ka$; thus

$$f(x) \equiv k\, x$$

4. *Verification.* For $f(x) = k\, x$ we have $f^{-1}(x) = x/k$, hence

$$f(x)\, f^{-1}(x) = \big(kx\big)\Big(\frac{x}{k}\Big) = x^2.$$

If $k > 0$ the function is strictly increasing; if $k < 0$ it is strictly decreasing, fulfilling the monotonicity requirement.

Problem 2 (Alex Avdiushenko, Neapolis University Paphos, Cyprus). Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that, for every $x, y \in \mathbb{R}$,

$$(x - y)\, f(x + y) \;=\; (x + y)\, \big(f(x) - f(y)\big).$$

Solution. Answer: $ax^2 + bx$.

1. $f(0) = 0$. Put $y = 0$ in the equation:

$$x\, f(x) = x\big(f(x) - f(0)\big) \implies f(0) = 0.$$

2. Consider the three special pairs $(1, z)$, $(1, z + 1)$, $(2, z)$:

Fix $z \in \mathbb{R} \setminus \{0, 1, 2\}$. Using original equation with the pairs listed, we successively get

$$(1, z) : f(1 + z) = \frac{1 + z}{1 - z}\, f(1) - \frac{1 + z}{1 - z}\, f(z),$$

$$(1, z + 1) : f(z + 2) = -\frac{2 + z}{z}\, f(1) + \frac{2 + z}{z}\, f(z + 1),$$

$$(2, z) : f(z + 2) = \frac{2 + z}{2 - z}\, f(2) - \frac{2 + z}{2 - z}\, f(z).$$

Substituting the first line into the second yields

$$f(z+2) = \frac{2+z}{z}\left(-f(1) + \frac{1+z}{1-z}\,f(1) - \frac{1+z}{1-z}\,f(z)\right) = $$
$$= \frac{2+z}{z}\left(\frac{2z}{1-z}\,f(1) - \frac{1+z}{1-z}\,f(z)\right) = $$
$$= \frac{2(2+z)}{z(1-z)}\,f(1) - \frac{(2+z)(1+z)}{z(1-z)}\,f(z) \tag{3}$$

3. Using the third line (for $(2, z)$) and (3), we get

$$f(z+2) = \frac{2+z}{2-z}\,f(2) - \frac{2+z}{2-z}\,f(z) = \frac{2(2+z)}{z(1-z)}\,f(1) - \frac{(2+z)(1+z)}{z(1-z)}\,f(z)$$

we reduce $(2+z)$

$$\frac{1}{2-z}\,f(2) - \frac{1}{2-z}\,f(z) = \frac{2}{z(1-z)}\,f(1) - \frac{1+z}{z(1-z)}\,f(z)$$

and group $f(z)$

$$\left(\frac{1+z}{z(1-z)} - \frac{1}{2-z}\right)\,f(z) = \frac{2}{z(1-z)}\,f(1) - \frac{1}{2-z}\,f(2)$$

and multiply $z(1-z)(2-z)$

$$((1+z)(2-z) - z(1-z))\;f(z) = 2(2-z)\,f(1) - z(1-z)\,f(2)$$
$$2f(z) = 2(2-z)\,f(1) - z(1-z)\,f(2)$$
$$f(z) = 2\,f(1) - z\left(f(1) + \frac{f(2)}{2}\right) + \frac{z^2}{2}\,f(2)$$

4. That is, the function can only be quadratic, and $f(0) = 0$, and therefore $f(x) = ax^2 + bx$, which, as can and should be verified, satisfies the original equation.

Remark. In a similar way it can be calculated that $f(z+2) - 2f(z+1) + f(z) = 0$, which is discrete analogue of "second derivative = 0".

Problem 3 (Alex Avdiushenko, Neapolis University Paphos, Cyprus). Let $P, Q, R \in O(3)$ be real orthogonal 3×3 matrices, i.e. $P^\mathsf{T}P = Q^\mathsf{T}Q = R^\mathsf{T}R = I_3$.

Show that the matrix equation

$$P + Q \;=\; R$$

has no solutions in $O(3)$.

Solution. Assume, for contradiction, that $P, Q, R \in O(3)$ satisfy $P + Q = R$.

1. Reduce to one orthogonal matrix. Set

$$S := P^\mathsf{T}Q \in O(3), \qquad Q = PS, \qquad R = P(I_3 + S).$$

2. Orthogonality of R. Because R is orthogonal,

$$I_3 = R^\mathsf{T}R = (I_3 + S)^\mathsf{T}(I_3 + S) = I_3 + S^\mathsf{T} + S + S^\mathsf{T}S = I_3 + S^\mathsf{T} + S + I_3,$$

whence

$$S + S^\mathsf{T} = -I_3$$

3. Minimal polynomial of S. Multiplying the last identity on the left by S gives

$$S^2 + S + I_3 = 0.$$

Hence every eigenvalue λ of S satisfies $\lambda^2 + \lambda + 1 = 0$, i.e.

$$\lambda = \exp\bigl(\pm 2\pi i/3\bigr) \quad \text{(non-real)}$$

4. Dimension parity contradiction. For a real matrix, non-real eigenvalues occur in complex-conjugate pairs; thus a real 3×3 matrix with only non-real eigenvalues cannot exist (the dimension is odd). Therefore such an $S \in O(3)$ does not exist.

5. Conclusion. Since S cannot exist, neither can a triple $(P, Q, R) \in O(3)^3$ with $P + Q = R$. Hence the equation admits *no* solutions in $O(3)$.

Remark. The same argument generalises to any dimension n. If an orthogonal matrix $S \in O(n)$ satisfies $S + S^{\mathsf{T}} = -I_n$, then it obeys the polynomial identity $S^2 + S + I_n = 0$; hence its eigenvalues are the two complex cubic roots of unity

$$\lambda = \exp\bigl(\pm 2\pi i/3\bigr),$$

which are non-real. Because complex eigenvalues of a real matrix occur in conjugate pairs, such an S can exist only when the dimension n is *even*.

Consequently, the $n = 2$ construction based on a $120°$ rotation extends to every even dimension by taking block-diagonal (direct sum) copies of the 2×2 block, whereas for every odd n no orthogonal matrices P, Q, R satisfy $P + Q = R$.