11th IMC 2004, July 25 – 26, Skopje, North Macedonia

Second Day

Problem 1

Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Find BA.

Problem 2

Let $f, g \colon [a,b] \to [0,\infty)$ be continuous and non-decreasing functions such that for each $x \in [a,b]$ we have

$$\int_a^x \sqrt{f(t)}\,dt \le \int_a^x \sqrt{g(t)}\,dt$$

and

$$\int_a^b \sqrt{f(t)}\,dt = \int_a^b \sqrt{g(t)}\,dt.$$

Prove that

$$\int_a^b \sqrt{1+f(t)}\,dt \ge \int_a^b \sqrt{1+g(t)}\,dt.$$

Problem 3

Let D be the closed unit disk in the plane, and let $p_1, p_2, \ldots, p_n$ be fixed points in D. Show that there exists a point p in D such that the sum of the distances of p to each of $p_1, p_2, \ldots, p_n$ is greater than or equal to 1.

Problem 4

For $n \ge 1$ let M be an $n \times n$ complex matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, with multiplicities $m_1, m_2, \ldots, m_k$, respectively. Consider the linear operator L_M defined by

$$L_M(X) = MX + XM^T,$$

for any complex $n \times n$ matrix X. Find its eigenvalues and their multiplicities. (M^T denotes the transpose of M; that is, if $M = (m_{k,\ell})$, then $M^T = (m_{\ell,k})$.)

Problem 5

Prove that

$$\int_0^1 \int_0^1 \frac{dx\,dy}{x^{-1} + |\ln y| - 1} \le 1.$$

Problem 6

For $n \ge 0$ define matrices A_n and B_n as follows: $A_0 = B_0 = (1)$ and for every $n > 0$

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \quad \text{and} \quad B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & 0 \end{pmatrix}.$$

Denote the sum of all elements of a matrix M by $S(M)$. Prove that

$$S(A_n^{k-1}) = S(A_k^{n-1}) \quad \text{for every } n, k \ge 1.$$