IMC-2000 Problems

Day 1

Problem 1. Is it true that if $f : [0,1] \to [0,1]$ is

a. monotone-increasing

b. monotone-decreasing

then there exists an $x \in [0,1]$ for which $f(x) = x$?

Problem 2. Let $p(x) = x^5 + x$ and $q(x) = x^5 + x^2$. Find all pairs (w, z) of complex numbers with $w \neq z$ for which $p(w) = p(z)$ and $q(w) = q(z)$.

Problem 3. A and B are square complex matrices of the same size and

$$\mathrm{rank}(AB - BA) = 1.$$

Show that $(AB - BA)^2 = 0$.

Problem 4. a) Show that if (x_i) is a decreasing sequence of positive numbers then

$$\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}.$$

b) Show that there is a constant C so that if (x_i) is a decreasing sequence of positive numbers then

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left(\sum_{i=m}^{\infty} x_i^2\right)^{1/2} \leq C \sum_{i=1}^{\infty} x_i.$$

Problem 5. Let R be a ring of characteristic zero (not necessarily commutative). Let e, f and g be idempotent elements of R satisfying $e + f + g = 0$. Show that $e = f = g = 0$.

(R is of characteristic zero means that, if $a \in R$ and n is a positive integer, then $na \neq 0$ unless $a = 0$. An idempotent x is an element satisfying $x = x^2$.)

Problem 6. Let $f : \mathbb{R} \to (0, \infty)$ be an increasing differentiable function for which

$$\lim_{x \to \infty} f(x) = \infty$$

and f' is bounded.

Let $F(x) = \int_0^x f$. Define the sequence (a_n) inductively by

$$a_0 = 1, \quad a_{n+1} = a_n + \frac{1}{f(a_n)},$$

and the sequence (b_n) simply by $b_n = F^{-1}(n)$. Prove that

$$\lim_{n \to \infty} (a_n - b_n) = 0.$$

Day 2

Problem 1.
a) Show that the unit square can be partitioned into n smaller squares for any n if n is large enough.
b) Let $d \geq 2$. Show that there is a constant $N(d)$ such that, whenever $n \geq N(d)$, a d-dimensional unit cube can be partitioned into n smaller cubes.

Problem 2. Let f be continuous and nowhere monotone on $[0,1]$. Show that the set of points on which f attains local minima is dense in $[0,1]$.

(A function is nowhere monotone if there exists no interval where the function is monotone. A set is dense if each non-empty open interval contains at least one element of the set.)

Problem 3. Let $p(z)$ be a polynomial of degree $n \geq 1$ with complex coefficients. Prove that there exist at least $n+1$ complex numbers z for which $p(z)$ is 0 or 1.

Problem 4. Suppose the graph of a polynomial of degree 6 is tangent to a straight line at 3 points A_1, A_2, A_3, where A_2 lies between A_1 and A_3.
a) Prove that if the lengths of the segments A_1A_2 and A_2A_3 are equal, then the areas of the figures bounded by these segments and the graph of the polynomial are equal as well.
b) Let $k = \dfrac{A_2A_3}{A_1A_2}$, and let K be the ratio of the areas of the appropriate figures. Prove that

$$\frac{2}{7}k^5 < K < \frac{7}{2}k^5.$$

Problem 5. Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$,

$$f(x)f(yf(x)) = f(x+y).$$

Problem 6. For an $m \times m$ real matrix A, e^A is defined as

$$\sum_{n=0}^{\infty} \frac{1}{n!}A^n.$$

(The sum is convergent for all matrices.)

Prove or disprove that for all real polynomials p and $m \times m$ real matrices A and B, $p(e^{AB})$ is nilpotent if and only if $p(e^{BA})$ is nilpotent. (A matrix A is nilpotent if $A^k = 0$ for some positive integer k.)