

# Proposed problems for the IMC 2025

**Problem 1 (Alex Avdiushenko, Neapolis University Paphos, Cyprus).** Find all strictly monotonic functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x) f^{-1}(x) = x^2.$$

**Solution.**

1. *Value at the origin.* Substituting  $x = 0$  into equation gives

$$f(0) f^{-1}(0) = 0.$$

If  $f(0) \neq 0$  then  $f^{-1}(0) = 0$ ; applying  $f$  to both sides forces  $f(0) = 0$  — a contradiction. Hence

$$f(0) = 0 \tag{1}$$

2. It is clear that either there exists an  $a$  such that  $f(a) = ka$  with  $k \neq 1$ , or else  $\forall x f(x) = x$ .

Substituting  $x = ka$  into the given equation yields  $f(ka) = k^2a$ . Proceeding by induction, one easily shows that

$$f(k^n a) = k^{n+1}a \quad (n \in \mathbb{Z}_{\geq 0}).$$

Now take a number  $x$  lying between  $a$  and  $f(a) = ka$  and set  $f(x) = lx$ . By monotonicity, the value  $f^n(x)$  lies between  $f^n(a)$  and  $f^{n+1}(a)$ . In other words, the number  $l^n x$  is always situated between  $k^n a$  and  $k^{n+1}a$ , whence it follows that  $l = k$ .

3. Because  $f$  is a strictly monotonic bijection, every real number lies between some (forward or backward) iterate of  $a$  and  $f(a) = ka$ ; thus

$$f(x) \equiv kx$$

4. *Verification.* For  $f(x) = kx$  we have  $f^{-1}(x) = x/k$ , hence

$$f(x) f^{-1}(x) = (kx) \left( \frac{x}{k} \right) = x^2.$$

If  $k > 0$  the function is strictly increasing; if  $k < 0$  it is strictly decreasing, fulfilling the monotonicity requirement.

**Problem 2 (Alex Avdiushenko, Neapolis University Paphos, Cyprus).** Find all continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that, for every  $x, y \in \mathbb{R}$ ,

$$(x - y) f(x + y) = (x + y) (f(x) - f(y)).$$

**Solution. Answer:**  $ax^2 + bx$ .

1.  $f(0) = 0$ . Put  $y = 0$  in the equation:

$$x f(x) = x(f(x) - f(0)) \implies f(0) = 0.$$

2. Consider the three special pairs  $(1, z)$ ,  $(1, z + 1)$ ,  $(2, z)$ :

Fix  $z \in \mathbb{R} \setminus \{0, 1, 2\}$ . Using original equation with the pairs listed, we successively get

$$(1, z) : f(1 + z) = \frac{1 + z}{1 - z} f(1) - \frac{1 + z}{1 - z} f(z),$$

$$(1, z + 1) : f(z + 2) = -\frac{2 + z}{z} f(1) + \frac{2 + z}{z} f(z + 1),$$

$$(2, z) : f(z + 2) = \frac{2 + z}{2 - z} f(2) - \frac{2 + z}{2 - z} f(z).$$

Substituting the first line into the second yields

$$\begin{aligned}
f(z+2) &= \frac{2+z}{z} \left( -f(1) + \frac{1+z}{1-z} f(1) - \frac{1+z}{1-z} f(z) \right) = \\
&= \frac{2+z}{z} \left( \frac{2z}{1-z} f(1) - \frac{1+z}{1-z} f(z) \right) = \\
&= \frac{2(2+z)}{z(1-z)} f(1) - \frac{(2+z)(1+z)}{z(1-z)} f(z)
\end{aligned} \tag{3}$$

3. Using the third line (for  $(2, z)$ ) and (3), we get

$$f(z+2) = \frac{2+z}{2-z} f(2) - \frac{2+z}{2-z} f(z) = \frac{2(2+z)}{z(1-z)} f(1) - \frac{(2+z)(1+z)}{z(1-z)} f(z)$$

we reduce  $(2+z)$

$$\frac{1}{2-z} f(2) - \frac{1}{2-z} f(z) = \frac{2}{z(1-z)} f(1) - \frac{1+z}{z(1-z)} f(z)$$

and group  $f(z)$

$$\left( \frac{1+z}{z(1-z)} - \frac{1}{2-z} \right) f(z) = \frac{2}{z(1-z)} f(1) - \frac{1}{2-z} f(2)$$

and multiply  $z(1-z)(2-z)$

$$((1+z)(2-z) - z(1-z)) f(z) = 2(2-z) f(1) - z(1-z) f(2)$$

$$2f(z) = 2(2-z) f(1) - z(1-z) f(2)$$

$$f(z) = 2f(1) - z \left( f(1) + \frac{f(2)}{2} \right) + \frac{z^2}{2} f(2)$$

4. That is, the function can only be quadratic, and  $f(0) = 0$ , and therefore  $f(x) = ax^2 + bx$ , which, as can and should be verified, satisfies the original equation.

**Remark.** In a similar way it can be calculated that  $f(z+2) - 2f(z+1) + f(z) = 0$ , which is discrete analogue of “second derivative = 0”.

**Problem 3 (Alex Avdiushenko, Neapolis University Paphos, Cyprus).** Let  $P, Q, R \in O(3)$  be real orthogonal  $3 \times 3$  matrices, i.e.  $P^T P = Q^T Q = R^T R = I_3$ .

Show that the matrix equation

$$P + Q = R$$

has no solutions in  $O(3)$ .

**Solution.** Assume, for contradiction, that  $P, Q, R \in O(3)$  satisfy  $P + Q = R$ .

**1. Reduce to one orthogonal matrix.** Set

$$S := P^T Q \in O(3), \quad Q = PS, \quad R = P(I_3 + S).$$

**2. Orthogonality of  $R$ .** Because  $R$  is orthogonal,

$$I_3 = R^T R = (I_3 + S)^T (I_3 + S) = I_3 + S^T + S + S^T S = I_3 + S^T + S + I_3,$$

whence

$$S + S^T = -I_3$$

**3. Minimal polynomial of  $S$ .** Multiplying the last identity on the left by  $S$  gives

$$S^2 + S + I_3 = 0.$$

Hence every eigenvalue  $\lambda$  of  $S$  satisfies  $\lambda^2 + \lambda + 1 = 0$ , i.e.

$$\lambda = \exp(\pm 2\pi i/3) \quad (\text{non-real})$$

**4. Dimension parity contradiction.** For a real matrix, non-real eigenvalues occur in complex-conjugate pairs; thus a real  $3 \times 3$  matrix with only non-real eigenvalues cannot exist (the dimension is odd). Therefore such an  $S \in O(3)$  does not exist.

**5. Conclusion.** Since  $S$  cannot exist, neither can a triple  $(P, Q, R) \in O(3)^3$  with  $P + Q = R$ . Hence the equation admits *no* solutions in  $O(3)$ .

**Remark.** The same argument generalises to any dimension  $n$ . If an orthogonal matrix  $S \in O(n)$  satisfies  $S + S^\top = -I_n$ , then it obeys the polynomial identity  $S^2 + S + I_n = 0$ ; hence its eigenvalues are the two complex cubic roots of unity

$$\lambda = \exp(\pm 2\pi i/3),$$

which are non-real. Because complex eigenvalues of a real matrix occur in conjugate pairs, such an  $S$  can exist only when the dimension  $n$  is *even*.

Consequently, the  $n = 2$  construction based on a  $120^\circ$  rotation extends to every even dimension by taking block-diagonal (direct sum) copies of the  $2 \times 2$  block, whereas for every odd  $n$  no orthogonal matrices  $P, Q, R$  satisfy  $P + Q = R$ .