9th IMC 2002, July 19 – 25, Warsaw, Poland, First day

First Day

Problem 1

A standard parabola is the graph of a quadratic polynomial $y = x^2 + ax + b$ with leading coefficient 1. Three standard parabolas with vertices V_1, V_2, V_3 intersect pairwise at points A_1, A_2, A_3. Let $A \mapsto s(A)$ be the reflection of the plane with respect to the x-axis.

Prove that standard parabolas with vertices $s(A_1), s(A_2), s(A_3)$ intersect pairwise at the points $s(V_1), s(V_2), s(V_3)$.

Solution. First we show that the standard parabola with vertex V contains point A if and only if the standard parabola with vertex $s(A)$ contains point $s(V)$.

Let $A = (a, b)$ and $V = (v, w)$. The equation of the standard parabola with vertex $V = (v, w)$ is $y = (x - v)^2 + w$, so it contains point A if and only if $b = (a - v)^2 + w$. Similarly, the equation of the parabola with vertex $s(A) = (a, -b)$ is $y = (x - a)^2 - b$; it contains point $s(V) = (v, -w)$ if and only if $-w = (v - a)^2 - b$. The two conditions are equivalent.

Now assume that the standard parabolas with vertices V_1 and V_2, V_1 and V_3, V_2 and V_3 intersect each other at points A_3, A_2, A_1, respectively. Then, by the statement above, the standard parabolas with vertices $s(A_1)$ and $s(A_2)$, $s(A_1)$ and $s(A_3)$, $s(A_2)$ and $s(A_3)$ intersect each other at points V_3, V_2, V_1, respectively, because they contain these points.

Problem 2

Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

Solution. Assume that there exists such a function. Since $f'(x) = f(f(x)) > 0$, the function is strictly monotone increasing.

By monotonicity, $f(x) > 0$ implies $f(f(x)) > f(0)$ for all x. Thus, $f(0)$ is a lower bound for $f'(x)$, and for all $x < 0$ we have

$$f(x) < f(0) + x \cdot f(0) = (1 + x)f(0).$$

Hence, if $x \leq -1$, then $f(x) \leq 0$, contradicting the property $f(x) > 0$.

So such a function does not exist.

Problem 3

Let n be a positive integer and let

$$a_k = \frac{1}{\binom{n}{k}}, \quad b_k = 2^{k-n}, \quad k = 1, 2, \ldots, n.$$

Show that

$$\sum_{k=1}^{n} \frac{a_k - b_k}{k} = 0.$$

Solution. Since

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

for all $k \geq 1$, the sum is equivalent to:

$$\frac{2^n}{n} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}} = \sum_{k=1}^{n} \frac{2^k}{k}.$$

This identity is proven by induction on n. (Detailed steps omitted for brevity.)

Problem 4

Let $f : [a, b] \to [a, b]$ be a continuous function and let $p \in [a, b]$. Define $p_0 = p$ and $p_{n+1} = f(p_n)$ for $n \geq 0$. Suppose that the set

$$T_p = \{p_n : n = 0, 1, 2, \ldots\}$$

is closed, i.e. if $x \notin T_p$ then there is a $\delta > 0$ such that for all $x' \in T_p$ we have $|x' - x| \geq \delta$. Show that T_p has finitely many elements.

Solution. If $p_m = p_n$ for some $m > n$, then T_p is finite. Otherwise, all points p_n are distinct.

There is a convergent subsequence p_{n_k} and its limit q is in T_p. Since f is continuous, $p_{n_k+1} = f(p_{n_k}) \to f(q)$, so all, except for finitely many, points p_n are accumulation points of T_p. Hence we may assume that all of them are accumulation points of T_p. Let $d = \sup\{|p_m - p_n| : m, n \geq 0\}$. Let δ_n be positive numbers such that $\sum_{n=0}^{\infty} \delta_n < \frac{d}{2}$. Let I_n be an interval of length less than δ_n centered at p_n such that there are infinitely many k's such that

$$p_k \notin \bigcup_{j=0}^{n} I_j,$$

this can be done by induction. Let $n_0 = 0$ and n_{m+1} be the smallest integer $k > n_m$ such that

$$p_k \notin \bigcup_{j=0}^{n_m} I_j.$$

Since T_p is closed, the limit of the subsequence (p_{n_m}) must be in T_p, but it is impossible because of the definition of the I_n's. Of course, if the sequence (p_{n_m}) is not convergent, we may replace it with its convergent subsequence. The proof is finished.

Remark. If $T_p = \{p_1, p_2, \ldots\}$ and each p_n is an accumulation point of T_p, then T_p is the countable union of nowhere dense sets (i.e. the single-element sets $\{p_n\}$). If T is closed, then this contradicts the Baire Category Theorem.

Problem 5

Prove or disprove the following statements:

(a) There exists a monotone function $f : [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.

(b) There exists a continuously differentiable function $f : [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x?

Solution. (a) Such a function does not exist. Each level set is either empty, a singleton, or an interval; the intervals are disjoint, hence at most countably many.

(b) Let f be such a map. Then for each value y of this map there is an x_0 such that $y = f(x)$ and $f'(x) = 0$, because an uncountable set $\{x : y = f(x)\}$ contains an accumulation point x_0 and clearly $f'(x_0) = 0$. For every $\varepsilon > 0$ and every x_0 such that $f'(x_0) = 0$ there exists an open interval I_{x_0} such that if $x \in I_{x_0}$ then $|f'(x)| < \varepsilon$. The union of all these intervals I_{x_0} may be written as a union of pairwise disjoint open intervals J_n. The image of each J_n is an interval (or a point) of length $< \varepsilon \cdot \text{length}(J_n)$ due to the Lagrange Mean Value Theorem. Thus the image of the interval $[0,1]$ may be covered with the intervals such that the sum of their lengths is $\varepsilon \cdot 1 = \varepsilon$. This is not possible for $\varepsilon < 1$.

Remarks.

1. The proof of part (b) is essentially the proof of the easy part of A. Sard's theorem about measure of the set of critical values of a smooth map.
2. If only continuity is required, there exists such a function, e.g. the first coordinate of the very well known Peano curve which is a continuous map from an interval onto a square.

Problem 6

For an $n \times n$ matrix M with real entries, let

$$\|M\| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Mx\|_2}{\|x\|_2},$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^n$.

Let A be an $n \times n$ real matrix satisfying

$$\|A^k - A^{k-1}\| \leq \frac{1}{2002k}$$

for all $k \geq 1$. Prove that $\|A^k\| \leq 2002$ for all positive integers k.

Solution. Lemma 1. Let $(a_n)_{n\geq 0}$ be a sequence of non-negative numbers such that

$$a_{2k} - a_{2k+1} \leq a_k^2, \quad a_{2k+1} - a_{2k+2} \leq a_k a_{k+1}$$

for any $k \geq 0$ and $\limsup n a_n < \frac{1}{4}$. Then

$$\limsup \sqrt[n]{a_n} < 1.$$

Proof. Let $c_l = \sup_{n\geq 2^l}(n+1)a_n$ for $l \geq 0$. We will show that $c_{l+1} \leq 4c_l^2$. Indeed, for any integer $n \geq 2^{l+1}$ there exists an integer $k \geq 2^l$ such that $n = 2k$ or $n = 2k+1$. In the first case there is:

$$a_{2k} - a_{2k+1} \leq a_k^2 \leq \frac{c_l^2}{(k+1)^2} \leq \frac{4c_l^2}{2k+1} - \frac{4c_l^2}{2k+2},$$

whereas in the second case there is:

$$a_{2k+1} - a_{2k+2} \leq a_k a_{k+1} \leq \frac{c_l^2}{(k+1)(k+2)} \leq \frac{4c_l^2}{2k+2} - \frac{4c_l^2}{2k+3}.$$

Hence a sequence $\left(a_n - \frac{4c_l^2}{n+1}\right)_{n\geq 2^{l+1}}$ is non-decreasing and its terms are non-positive since it converges to zero. Therefore:

$$a_n \leq \frac{4c_l^2}{n+1} \quad \text{for } n \geq 2^{l+1},$$

meaning that $c_{l+1}^2 \leq 4c_l^2$. This implies that a sequence $\left((4c_l)^{2^{-l}}\right)_{l\geq 0}$ is non-increasing and therefore bounded from above by some number $q \in (0,1)$ since all its terms except finitely many are less than 1. Hence $c_l \leq q^{2^l}$ for l large enough. For any n between 2^l and 2^{l+1} there is $a_n \leq \frac{c_l}{n+1} \leq q^{2^l} \leq (\sqrt{q})^n$, yielding:

$$\limsup \sqrt[n]{a_n} \leq \sqrt{q} < 1,$$

yielding $\limsup \sqrt[n]{a_n} < 1$, which ends the proof.

Lemma 2. Let T be a linear map from $\mathbb{R}^n$ into itself. Assume that

$$\limsup n\|T^{n+1} - T^n\| < \frac{1}{4}.$$

Then

$$\limsup \|T^{n+1} - T^n\|^{1/n} < 1.$$

In particular, T^n converges in the operator norm and T is power bounded.

Proof. Put $a_n = \|T^{n+1} - T^n\|$. Observe that

$$T^{k+m+1} - T^{k+m} = (T^{k+m+2} - T^{k+m+1}) - (T^{k+1} - T^k)(T^{m+1} - T^m),$$

implying that

$$a_{k+m} \leq a_{k+m+1} + a_k a_m.$$

Therefore the sequence $(a_m)_{m\geq 0}$ satisfies assumptions of Lemma 1 and the assertion of Proposition 1 follows.

Remarks.

1. The theorem proved above holds in the case of an operator T which maps a normed space X into itself; X does not have to be finite dimensional.

2. The constant $\frac{1}{4}$ in Lemma 1 cannot be replaced by any greater number since a sequence $a_n = \frac{1}{4n}$ satisfies the inequality $a_{k+m} - a_{k+m+1} \leq a_k a_m$ for any positive integers k and m whereas it does not have exponential decay.

3. The constant $\frac{1}{4}$ in Lemma 2 cannot be replaced by any number greater than $\frac{1}{e}$. Consider an operator $(Tf)(x) = xf(x)$ on $L^2([0,1])$. One can easily check that

$$\limsup \|T^{n+1} - T^n\| = \frac{1}{e},$$

whereas T^n does not converge in the operator norm. The question whether in general $\limsup n\|T^{n+1} - T^n\| < \infty$ implies that T is power bounded remains open.

Remark. The problem was incorrectly stated during the competition: instead of the inequality

$$\|A^k - A^{k-1}\| \le \frac{1}{2002^k},$$

the inequality

$$\|A^k - A^{k-1}\| \le \frac{1}{2002^n}$$

was assumed. If

$$A = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$$

then

$$A^k = \begin{pmatrix} 1 & k\varepsilon \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$A^k - A^{k-1} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix},$$

so for sufficiently small ε the condition is satisfied although the sequence $(\|A^k\|)$ is clearly unbounded.