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# Some Properties of Well-Based Sequences

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**Abstract**—S. V. Kitaev posed the problem of finding the number of well-based sequences and the problem of existence of a bijection between these objects and the sets associated with the sequence (A103580). The well-based sequences define the class of the graphs whose independent sets are enlisted by S. V. Kitaev (2006). In this paper, the desirable bijection is obtained, and it is proved that the number of well-based sequences increases as $\Theta(2^{n/2})$.

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## INTRODUCTION

Consider a set of positive integers $\{a_1, a_2, \dots, a_k\}$ ordered by increasing and having cardinality $k \geq 2$. Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be the set of words of the kind $A_i = 1\underbrace{0\dots0}_{a_i-1}1$. Assume that each word $A_i'$ obtained from the word $A_i \in \mathcal{A}$ by substituting some subset of 0s in $A_i$ for 1s includes $A_j \in \mathcal{A}$ as a subword, where $j < i$. Call $\mathcal{A}$ a *well-based set*; and call the corresponding sequence $a_i$, a *well-based sequence*. Note that each well-based set should include the word 11, and each well-based sequence includes 1. Indeed, after substitution of 1s for all 0s, the word $A_2$ includes only 11 as a subword of the kind $1\underbrace{0\dots0}_{l\geq0}1$, which means that 11 belongs to $\mathcal{A}$ and each well-based sequence begins with 1. Thus, the unique well-based set of cardinality 1 is $\mathcal{A} = \{11\}$.

It is easy that, in the definition of well-based set, we can replace the substitution of an arbitrary set of 0s with the substitution of some single 0. This definition is equivalent to the property that $a_1 = 1$ and, for every $a_i$, $i \geq 2$, and every partition $a_i = t + s$, where $t \in \mathbb{N}$ and $s \in \mathbb{N}$, we have $t = a_j$ or $s = a_j$, where $j < i$. In the sequel, we use this definition of well-based sequence.

Let $W(n)$ be the set of well-based sequences whose elements are at most $n$, and let $w(n)$ denote the number of these sequences.

Some examples of well-based sequences are as follows: $\langle 1, 2, 3\rangle$, $\langle 1, 2, 5\rangle$, $\langle 1, 3, 5, 7, \dots\rangle$.

Consider some sequence that includes $\{1, 2, \dots, k\}$ and an arbitrary subset of $\{k + 2, \dots, 2k + 1\}$. This sequence will be well-based since, for every number from $\{1, 2, \dots, 2k + 1\}$, each partition into two positive integer summands has a summand belonging to $\{1, 2, \dots, k\}$. There are exactly $2^k$ such sequences. It is also obvious that for different $k$ the sequences are distinguished, since the first positive integers not included into these sequences are different. Hence, we have some lower bound for the number of well-based sequences:

$$w(2n + 1) \geq \sum_{m=0}^{n} 2^m + 1 = 2^{n+1}, \qquad w(2n) \geq \sum_{m=0}^{n-1} 2^m + 1 = 2^n$$

(addition of 1 corresponds to the well-based sequences $\langle 1, 2, \dots, 2n + 1\rangle$ and $\langle 1, 2, \dots, 2n\rangle$ respectfully); i.e., $w(n) \geq 2^{[n+1/2]}$. Both inequalities are strong for $n \geq 5$, since, in both cases, the well-based sequence $\langle 1, 3, 5\rangle$ was not counted.

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## 1. WELL-BASED SEQUENCES AND BASES

Let us call a nonempty set $B \subseteq \{1, 2, \dots, n\} = [n]$ a *base* if none of the elements $g$ of $B$ equals some nonnegative linear combination of the elements of $B \backslash g$. Denote the set of the bases considered in [5, A103580] by $B(n)$.

**Theorem 1.** *There is a bijection between $W(n)$ and $B(n)$.*

*Proof.* Start with introducing the definition of sum of subsets of $[n]$. Let $C \subseteq [n]$ and $D \subseteq [n]$. Then

$$C + D = \{f \in [n] \mid f = c + d, \ \ c \in C \cup \{0\}, \ \ d \in D \cup \{0\}\}.$$

Call a nonempty set $Z \subseteq [n]$ *closed under summation* or simply *closed* if $Z + Z = Z$.

Let us show that if $W \in W(n)$ and $W \neq [n]$ then $Z = [n] \backslash W$ is closed. By assumption, $Z$ is nonempty. Hence, if it is unclosed then there are $x \in Z$ and $y \in Z$ such that $u = x + y \in W$. On the other hand, neither $x$ nor $y$ belongs to the well-based sequence $W$, which contradicts the definition of well-based sequence. Thus, the addition $W \neq [n]$ to each well-based sequence $[n]$ is a closed set.

Consider a closed set $Z \neq [n]$ and show now that $W = [n] \backslash Z$ is well-based. If not, then for some $u \in W$ we have $u = x + y$, where neither $x$ nor $y$ belongs to $W$; i.e., $x \in Z$ and $y \in Z$, but $x + y \in W$; a contradiction to the completeness of $Z$.

Thus, it is proved that, for every set $V \subseteq [n]$, $V \neq \varnothing$ and $V \neq [n]$, if $V$ is a well-based sequence then its addition is closed; and if $V$ is closed then the addition is well-based. Now, for each well-based sequence, define some closed set associated with this sequence: If $W = [n]$ then we take the closed set $Z = [n]$, otherwise, the closed set $[n] \backslash W$. Thus, it remains to construct a bijection between closed sets and bases.

Let $Z(n)$ denote the set of all closed sets. It is easy to show that the completeness of $P$ is equivalent to the claim that $P$ includes all nonnegative linear combinations of its elements in $[n]$. Indeed, if $P$ is closed then $P + P = P$; and, by induction, it is easy to prove that

$$\underbrace{P + \dots + P}_{k} = P$$

for every integer $k \geq 2$. Therefore, $P$ includes all finite sums of its elements, and, as a consequence, all nonnegative linear combinations of the elements. Suppose to the contrary that $P$ includes all nonnegative linear combinations of its elements in $[n]$. Then $P$ includes the sums of every two of its elements; and so $P + P = P$; i.e., $P$ is closed.

Given an arbitrary $Y \subseteq [n]$, define the *linear span* of $Y$ as

$$L(Y) = \left\{ y \in [n] \ \middle| \ y = \sum_{k=1}^{m} \alpha_k y_k, \ \ \alpha_k \in \mathbb{N} \cup \{0\}, \ \ y_k \in Y \right\}.$$

Let a mapping $\varphi : B(n) \to Z(n)$ satisfy $\varphi(B) = L(B)$, where $B \in B(n)$. We will prove the following

**Lemma.** *The mapping $\varphi$ is an injection.*

*Proof.* Assume that $L(B_1) = L(B_2)$, where $B_1$ and $B_2$ are some distinguished bases. It is simple that the first elements of these bases are identical since (otherwise) the smallest of them would not belong to the linear span of the other base. Consider now the maximal quantity of the matching initial elements (this set is nonempty); i.e.,

$$p_1 = q_1, \qquad \dots, \qquad p_m = q_m, \qquad p_{m+1} < q_{m+1}.$$

Then, since $p_{m+1} \in L(B_2)$ and $p_{m+1} < q_{m+1}$, we have $p_{m+1} \in L(\{q_1, \dots, q_m\})$, and consequently $p_{m+1} \in L(\{p_1, \dots, p_m\})$. This contradicts the inclusion $\{p_1, \dots, p_{m+1}\} \subseteq B_1$. The case $p_{m+1} > q_{m+1}$ can be considered by analogy. Thus, all bases correspond to different linear spans. The proof of Lemma 1 is complete. □

Consider an arbitrary set $R$ that includes all linear nonnegative combinations of its elements. Construct a base in $R$ as follows: Let $r_1$ be the smallest element of $R$. Consider $L(r_1)$. If $L(r_1) = R$ then $r_1$ is the desired base; otherwise, consider $r_2$ which is the smallest element of $R \backslash L(r_1)$. If $L(r_1, r_2) = R$ then $r_1, r_2$ is the desired base; otherwise, consider $r_3$ that is the smallest element of the set $R \backslash L(r_1, r_2)$, and so on. In result, by finiteness of $R$, we find a base of $R$. Hence, the mapping $\varphi$ is surjective. By Lemma 1, $\varphi$ is bijection. The proof of Theorem 1 is complete. □

**Example.** For $n = 3$, we have the following bijection:
$\varphi(\{1\}) = \{1, 2, 3, 4\}$,
$\varphi(\{2\}) = \{2, 4\}$, $\{1, 2, 3, 4\} \backslash \{2, 4\} = \{1, 3\}$,
$\varphi(\{3\}) = \{3\}$, $\{1, 2, 3, 4\} \backslash \{3\} = \{1, 2, 4\}$,
$\varphi(\{4\}) = \{4\}$, $\{1, 2, 3, 4\} \backslash \{4\} = \{1, 2, 3\}$,
$\varphi(\{2, 3\}) = \{2, 3, 4\}$, $\{1, 2, 3, 4\} \backslash \{2, 3, 4\} = \{1\}$,
$\varphi(\{3, 4\}) = \{3, 4\}$, $\{1, 2, 3, 4\} \backslash \{3, 4\} = \{1, 2\}$.

## 2. ASYMPTOTIC ENUMERATION OF WELL-BASED SEQUENCES

A subset $H$ of the integers is called a *sum-free set* if, for all $a, b \in H$, the number $a + b$ does not belong to $H$. Let $S(n)$ denote the family of all sum-free subsets $H \subseteq [n]$. Put $s(n) = |S(n)|$. In 1988, P. Cameron and P. Erdos assumed [2] that $s(n) = O(2^{n/2})$. The Cameron–Erdos Conjecture on the number of sum-free sets on the interval $[1, n]$ was independently proved in 2003 by B. J. Green [3] and A. A. Sapozhenko [1].

Note that each base is a sum-free set. The opposite is false since, for example, every subset of the odd numbers is sum-free. Nevertheless, if its cardinality is at least 2 and it does include 1 then it is not a base.

Thus, $B(n) \subset S(n)$ and $w(n) \leq s(n) - 2^{[(n-1)/2]}$ hold. Consequently, $w(n) = O(2^{n/2})$ and, by the above-obtained lower bound, we have

**Theorem 2.** $w(n) = \Theta(2^{n/2})$.

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