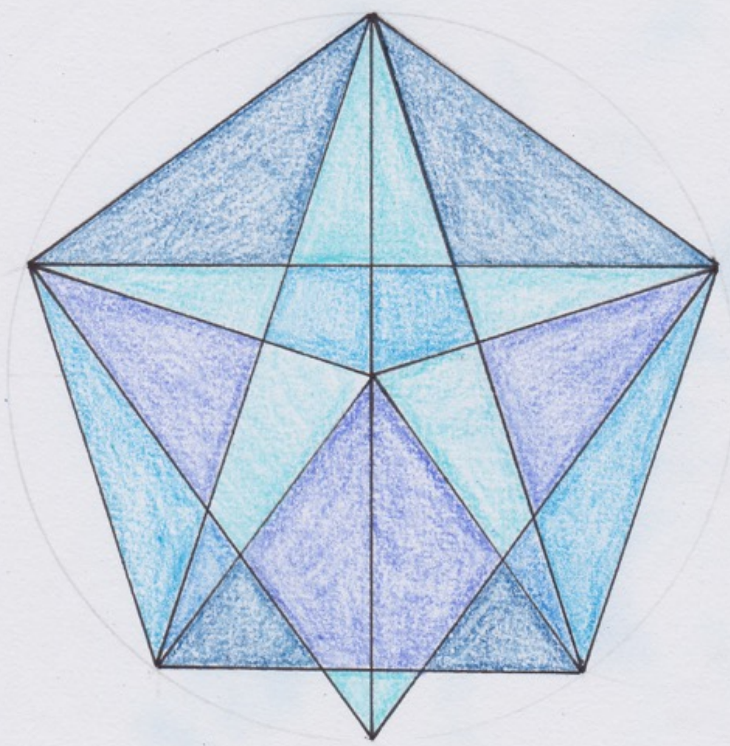
# Trigonometry

with Mr. Marsch



10th Grade

October 2014

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# Trigonometric Ratios for Right Triangles

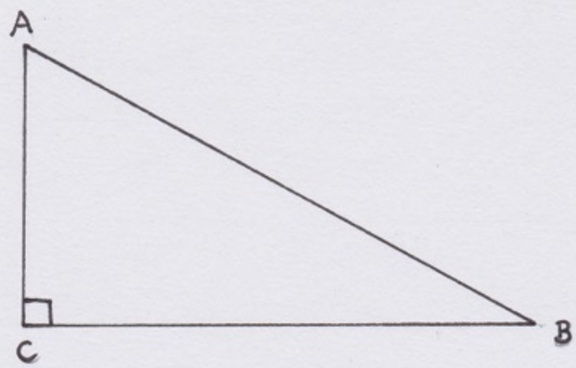
In any right triangle, the sine of an angle is the ratio of the length of the opposite side to the length of the hypotenuse. The cosine of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse. The tangent of an angle is the ratio of the length of the opposite side to the length of the adjacent side.

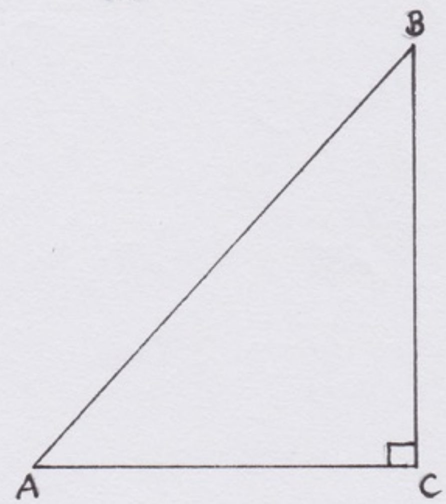
In $\triangle ABC$:

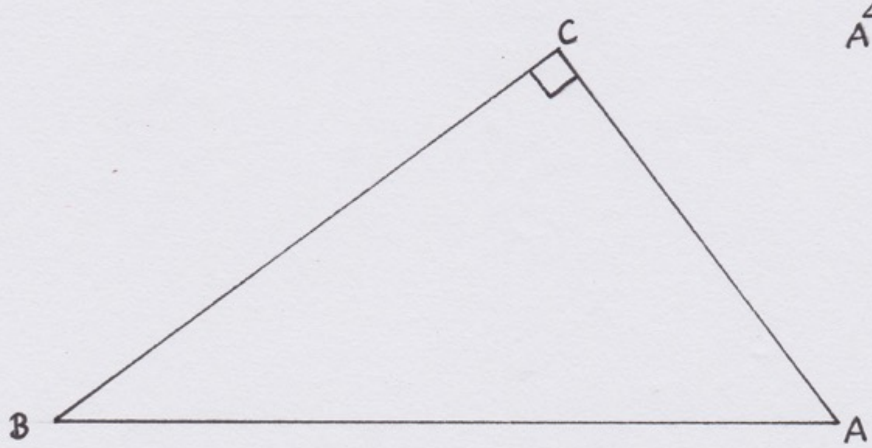
$$\sin A = \frac{BC}{AB} \qquad \sin B = \frac{AC}{AB}$$

$$\cos A = \frac{AC}{AB} \qquad \cos B = \frac{BC}{AB}$$

$$\tan A = \frac{BC}{AC} \qquad \tan B = \frac{AC}{BC}$$

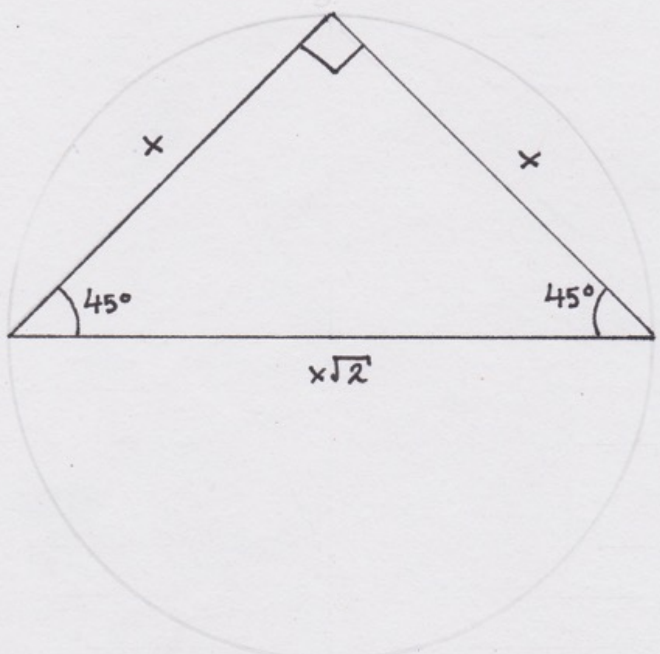


## Trigonometric Ratios for 45° and 60°

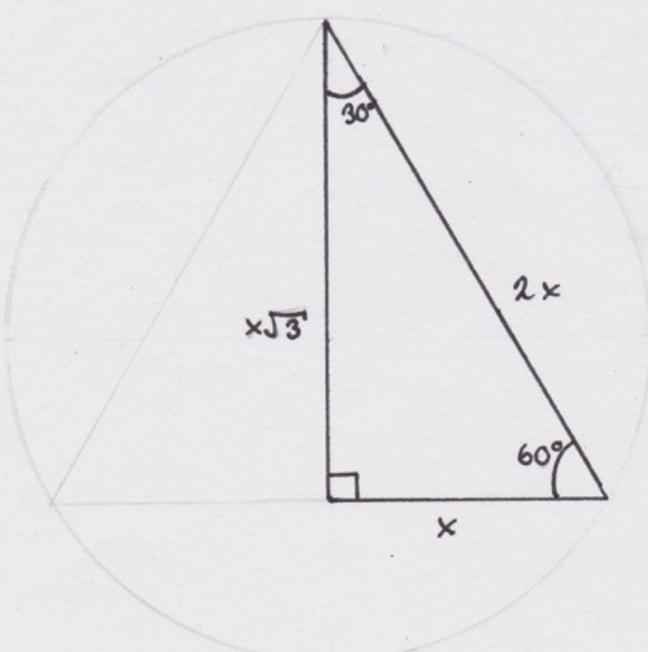
The right isosceles triangle and the half equilateral triangle provide a basis for the most commonly used and easily derived sine, cosine, and tangent values.




$$\sin 45° = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{x}{x} = 1$$




$$\sin 30° = \frac{x}{2x} = \frac{1}{2}$$

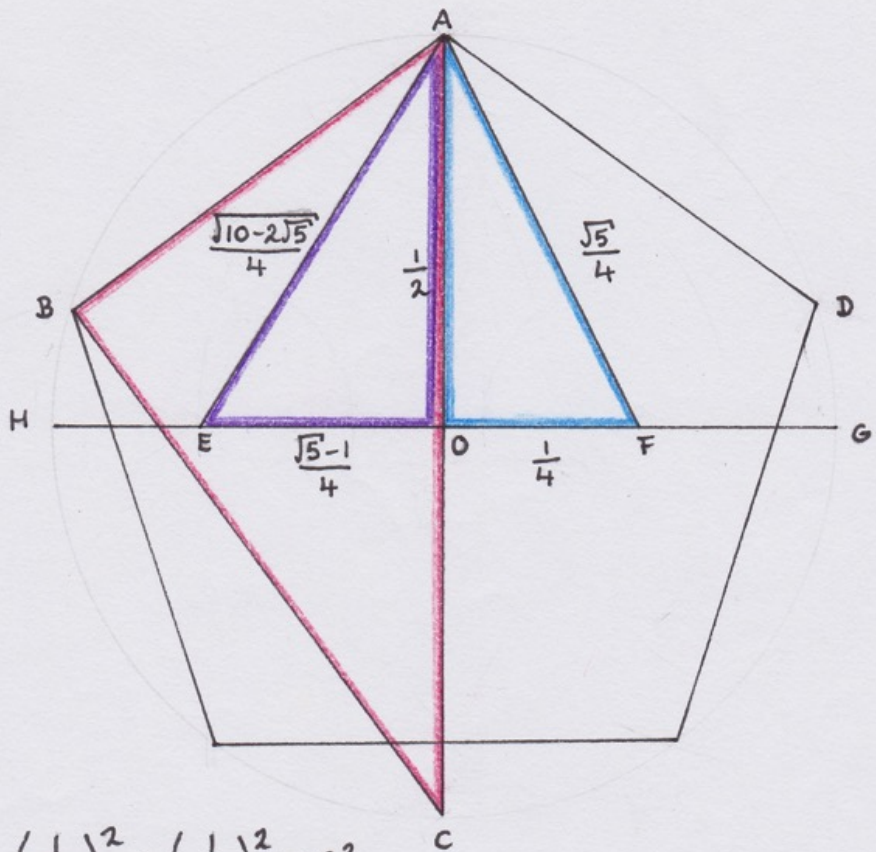
$$\cos 30° = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$$

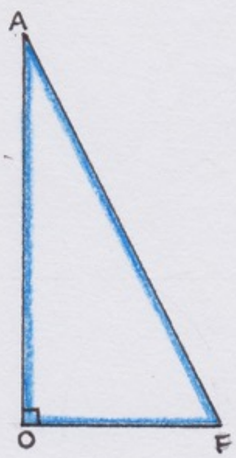
$$\tan 30° = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sin 60° = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{x}{2x} = \frac{1}{2}$$

$$\tan 60° = \frac{x\sqrt{3}}{x} = \sqrt{3}$$




$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 = c^2$$

$$\frac{1}{4} + \frac{1}{16} = c^2$$

$$\frac{5}{16} = c^2$$

$$\frac{\sqrt{5}}{4} = c$$

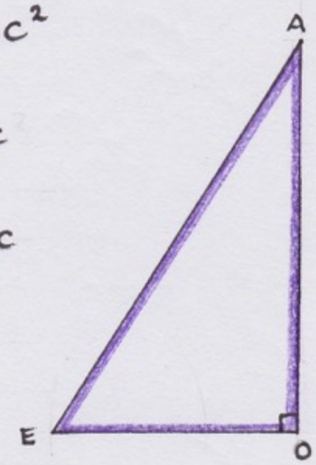
$$\left(\frac{\sqrt{5}-1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 = c^2$$

$$\sqrt{\frac{6-2\sqrt{5}}{16} + \frac{1}{4}} = c$$

$$\sqrt{\frac{6-2\sqrt{5}}{16} + \frac{4}{16}} = c$$

$$\sqrt{\frac{10-2\sqrt{5}}{16}} = c$$

$$\frac{\sqrt{10-2\sqrt{5}}}{4} = c$$





$$\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2 + b^2 = 1^2$$

$$\frac{10-2\sqrt{5}}{16} + b^2 = 1$$

$$b = \sqrt{\frac{16}{16} - \frac{10-2\sqrt{5}}{16}}$$

$$b = \frac{\sqrt{6+2\sqrt{5}}}{4}$$

# Trigonometric Ratios for 36° and 54°

We can discover sine, cosine, and tangent values for 36° and 54° by considering ratios of $\Delta ABC$ on the previous page in the regular pentagon figure. Beginning with a circle of arbitrary diameter, which we can set as 1 unit to simplify calculation, we derive trigonometric ratios for the 36-54-90 triangle after three applications of the Pythagorean Theorem.

By construction, $OG = \frac{1}{2}$ and $OF = \frac{1}{4}$. Therefore $OG = OH$. Since, according to the pentagon construction, $\Delta AEF$ is isosceles, $AF = EF$ as well. Then, by the second use of the Pythagorean Theorem $AE = AB$, which is also the measure of $AD$ and any side length of the pentagon. Since the pentagon is oriented with one vertex angle bisected by $AC$, $m\angle BAC = 54°$; since $\angle ABC$ is inscribed in the circle and its rays intercept $AC$ it is a right angle. Therefore $m\angle ACB = 36°$. It follows immediately that $\sin 36° = \cos 54° = \frac{\sqrt{10-2\sqrt{5}}}{4}$ because $HG = 1$. From one more application of the Pythagorean Theorem we can find that $BC = \frac{\sqrt{6+2\sqrt{5}}}{4}$ and this gives us $\sin 54° = \cos 36° = \frac{\sqrt{6+2\sqrt{5}}}{4}$. The tangents follow as well: $\tan 36° = \frac{\sqrt{10-2\sqrt{5}}}{\sqrt{6+2\sqrt{5}}}$ and $\tan 54° = \frac{\sqrt{6+2\sqrt{5}}}{\sqrt{10-2\sqrt{5}}}$.

$$\sin 36° = \cos 54° = \frac{\sqrt{10-2\sqrt{5}}}{4} \approx .588$$

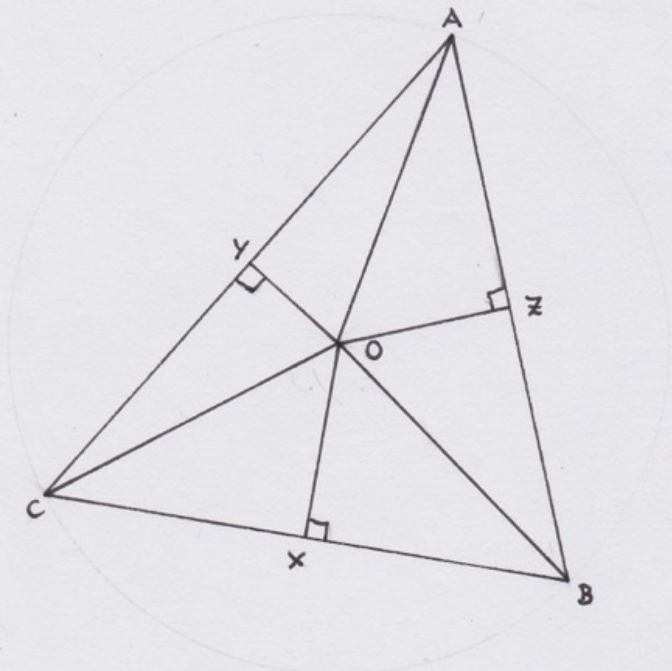
$$\sin 54° = \cos 36° = \frac{\sqrt{6+2\sqrt{5}}}{4} \approx .809$$

$$\tan 36° = \frac{\sqrt{10-2\sqrt{5}}}{\sqrt{6+2\sqrt{5}}} \approx .727$$

$$\tan 54° = \frac{\sqrt{6+2\sqrt{5}}}{\sqrt{10-2\sqrt{5}}} \approx 1.376$$

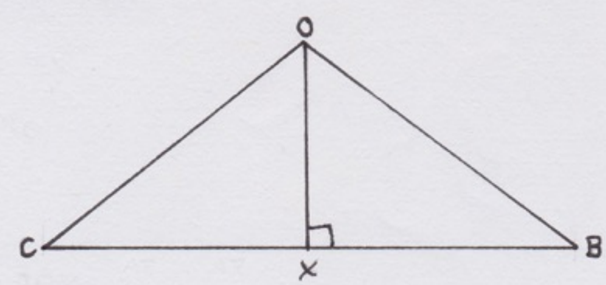
# The Law of Sines

The first step in developing trigonometry beyond the right triangle is to discover the Law of Sines, which states that in any triangle, the ratio of any side length to the sine value of the angle opposite is equal to either of the other two side-to-opposite-sine ratios in the triangle. Below we verify this relationship, beginning from a triangle inscribed in a circle. In the diagram OA, OB, and OC




are radii, and OX, OY, and OZ are constructed altitudes. Since $\triangle AOB$, $\triangle BOC$, and $\triangle AOC$ are isosceles, these three allitudes are also perpendicular bisectors and angle bisectors. Therefore $AY = \frac{1}{2}AC$, $m\angle AOY = \frac{1}{2}m\angle AOC$. We also note that $m\angle BOX = m\angle A$, $m\angle COY = m\angle B$, and $m\angle AOZ = m\angle C$ since in each case the matching pairs of angles show the 2:1 ratio between inscribed and central angles.

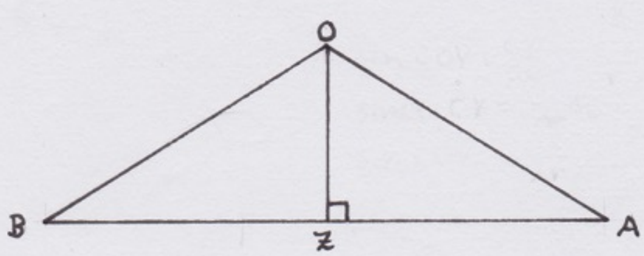
For right triangles AOZ, COY, and BOX we consider the sines of the angles with vertexes at the center O.




$$\sin BOX = \frac{BX}{OB} = \frac{BX}{r}$$

$$\text{since } BX = \frac{1}{2}BC$$

$$\sin BOX = \frac{BC}{2r}$$




$$\sin AOZ = \frac{AZ}{OA} = \frac{AZ}{r}$$

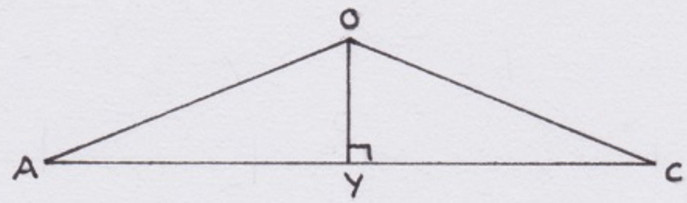
$$\text{since } AZ = \frac{1}{2}AB$$

$$\sin AOZ = \frac{AB}{2r}$$

$$\sin COY = \frac{CY}{OC} = \frac{CY}{r}$$

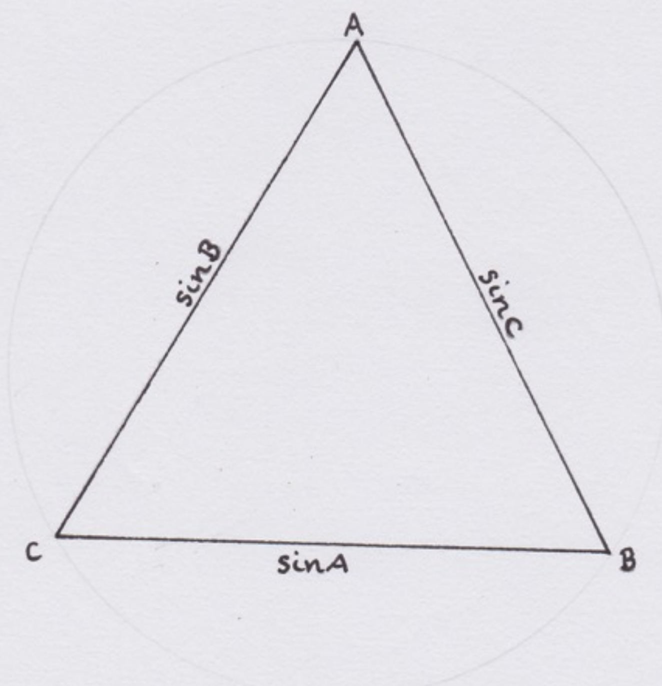
$$\text{since } CY = \frac{1}{2}AC$$

$$\sin COY = \frac{AC}{2r}$$




Since OX, OY, and OZ are angle bisectors, $BOX = m\angle A$, $COY = m\angle B$, and $AOZ = m\angle C$. Thus we have $\sin A = \frac{BC}{2r}$, $\sin B = \frac{AC}{2r}$, and $\sin C = \frac{AB}{2r}$, and so $2r = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, which is the Law of Sines.

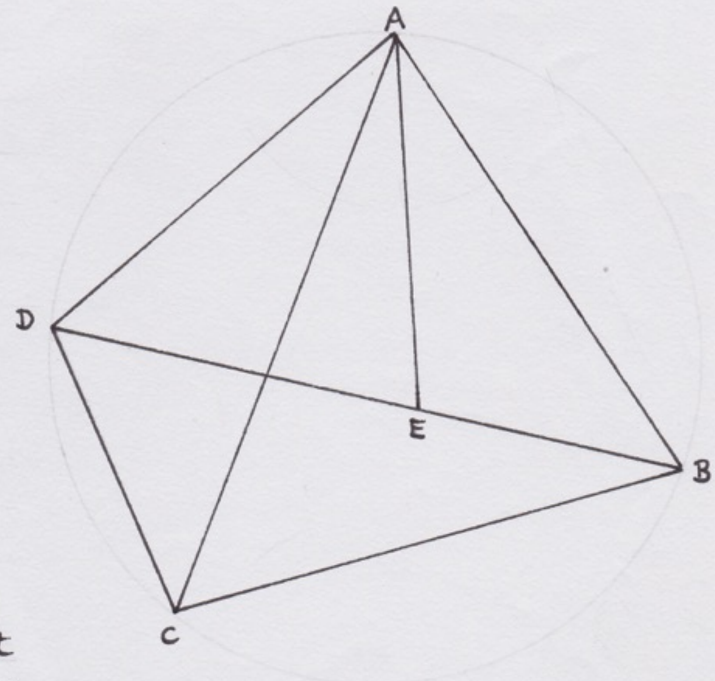
A corollary is that if the circle has a unit diameter, then $2r = 1$, and $\sin A = a$, $\sin B = b$, and $\sin C = c$.



# Ptolemy's Theorem

We verify this theorem by analizing quadrilateral ABCD.

Ptolemy's Theorem allows us to discover a relationship between sides and diagnals of quadrilaterals that provide a basis for new trigonometric formulas that don't depend on triangles at all. Stated completely: Given any quadrilateral inscribed in a circle, the product of the diagnals are equal to the sum of the products of the opposite sides.




In the figure, we constructed $\angle A$ so that $m\angle CAD = m\angle BAE$. Since $\angle ACD$ and $\angle ABD$ intercept the same arc, they measure the same; thus $\Delta ADC \sim \Delta AEB$ since they share two sets of corresponding angles that are congruent. Also, $\Delta AED \sim \Delta ABC$ for the same reason since $m\angle ADB = m\angle ACB$ because they intercept arc AB and since $m\angle DAE = m\angle BAC$ because each adds $m\angle CAE$ to the angles that measured the same by construction

We form two equations from the corresponding sides of the pairs of similar triangles and combine them into a system.

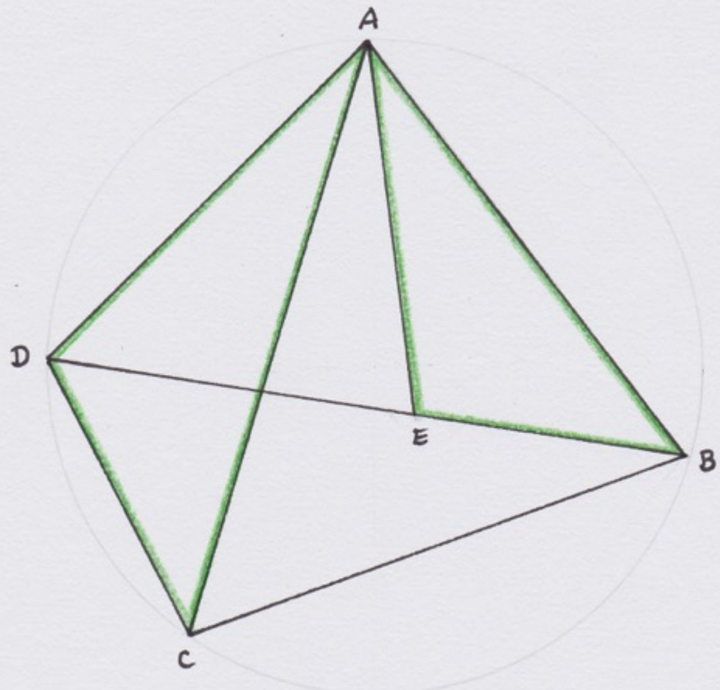
$$\frac{DC}{AC} = \frac{BE}{AB} \qquad \frac{AC}{BC} = \frac{AD}{DE}$$

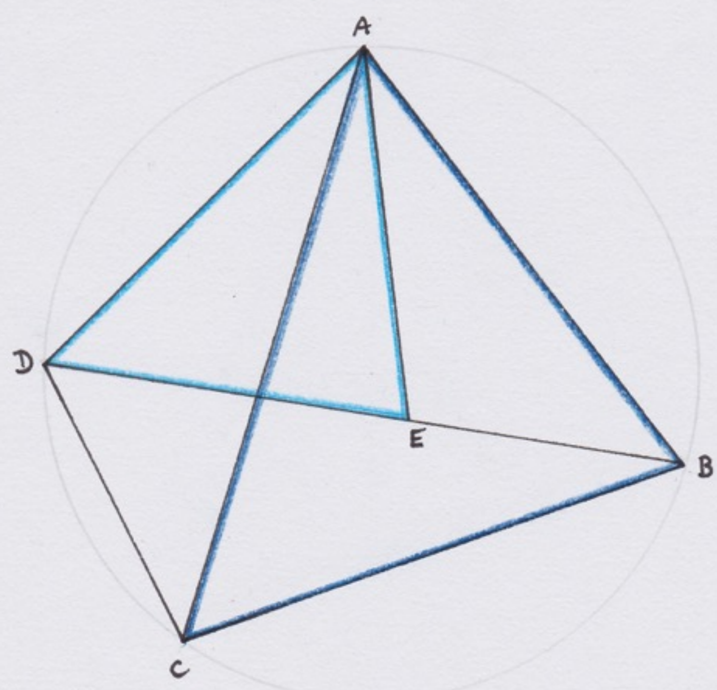
$$AB \cdot DC = AC \cdot BE \qquad AC \cdot DE = AD \cdot BC$$

$$AB \cdot DC + AD \cdot BC = AC \cdot DE + AC \cdot BE$$

$$AB \cdot DC + AD \cdot BC = AC\,(DE + BE)$$

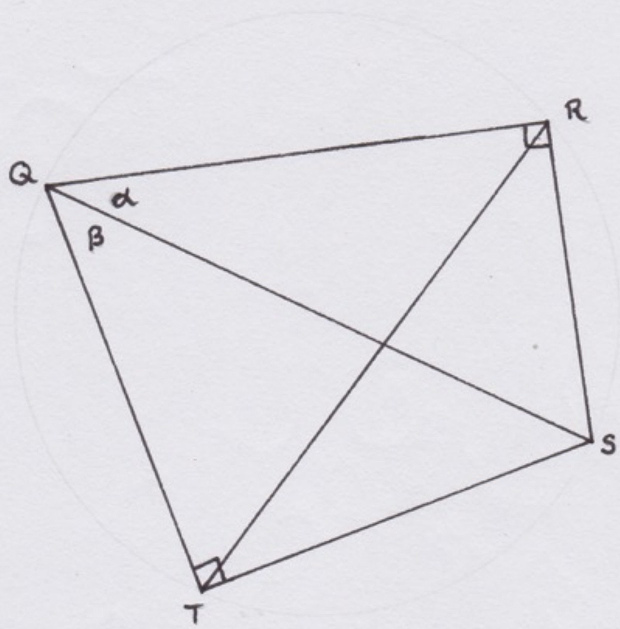
$$AB \cdot DC + AD \cdot BC = AC \cdot DB$$


A
D
E
B
C


A
D
E
B
C

# Angle Addition Formulas

From Ptolemy's Theorem and Law of Sines applied to a unit diameter circle, we can derive formulas to calculate sines, cosines, and tangents of angles $\alpha+\beta$ or $\alpha-\beta$, provided that we know the trigonometric function values for $\alpha+\beta$. In the diagram let QS be a diameter with length 1 unit. Then QRS and QTS are right triangles.

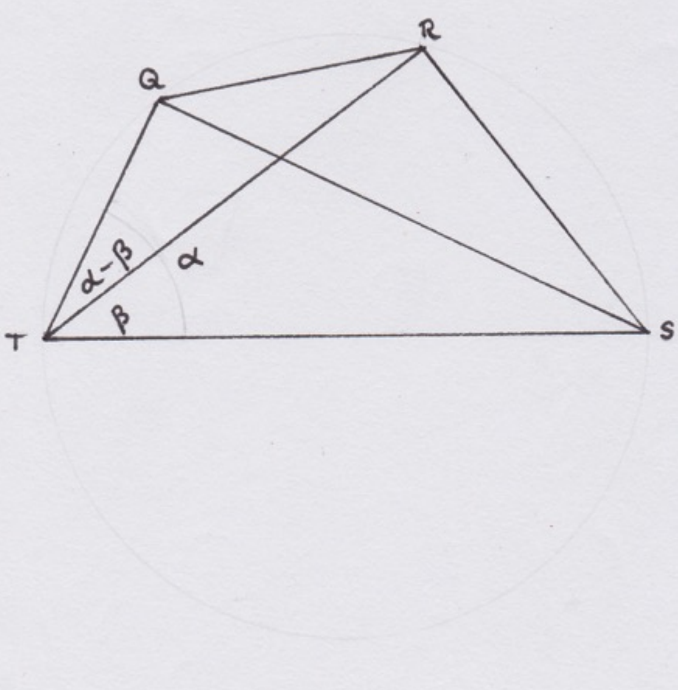



Since $QS=1$, $\sin\alpha=\frac{RS}{QS}=RS$, $\cos\alpha=\frac{QR}{QS}=QR$, $\sin\beta=\frac{ST}{QS}=ST$, and $\cos\beta=\frac{QT}{QS}=QT$. According to the corollary to the Law of Sines on page 5, $RT=\sin(\alpha+\beta)$. Because QRST is a quadrilateral inscribed in a circle Ptolemy's Theorem allows us to form the equation:

$$QS\cdot RT = RS\cdot QT + QR\cdot ST$$

$$\sin(\alpha+\beta)\cdot 1 = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$$

In the second diagram let $ST=1$, and then TQS and TRS are right triangles. Therefore




$\sin\alpha=\frac{QS}{ST}=QS$, $\cos\alpha=\frac{QT}{ST}=QT$, $\sin\beta=\frac{RS}{ST}=RS$, and $\cos\beta=\frac{RT}{ST}=RT$. By Ptolemy's Theorem, we can form another equation.

$$QS\cdot RT = QR\cdot ST + RS\cdot QT$$

$$\sin\alpha\cdot\cos\beta = \sin(\alpha-\beta)\cdot 1 + \sin\beta\cdot\cos\alpha$$

$$\sin(\alpha-\beta) = \sin\alpha\cdot\cos\beta - \sin\beta\cdot\cos\alpha$$

These two formulas together with conclusions from page 2 allow us to calculate sines for 15° and 75°.

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cdot \cos 30^\circ - \cos 45^\circ \cdot \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cdot \cos 30^\circ + \cos 45^\circ \cdot \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

# Sine, Cosine, and Tangent Values

Using the angle addition formulas and the identities $\cos\theta = \sin(90-\theta)$ and $\tan \frac{\sin\theta}{\cos\theta}$, we can generate new trigonometric values systematically from ones we already know. Eventually, we can calculate these for every $3°$ between $0°$ and $90°$. We show the first few derivations below and then collect the cumulative result into a table on the next page.

$$\sin 24° = \sin(54°-30°) = \sin 54° \cdot \cos 30° - \cos 54° \cdot \sin 30°$$

$$= \frac{\sqrt{6+2\sqrt{5}}}{4} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{10-2\sqrt{5}}}{4} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3(6+2\sqrt{5})}}{8} - \frac{\sqrt{10-2\sqrt{5}}}{8}$$

$$= \frac{\sqrt{18+6\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{8}$$

$$\sin 66° = \sin(36°+30°) = \sin 36° \cdot \cos 30° + \cos 36° \cdot \sin 30°$$

$$= \frac{\sqrt{10-2\sqrt{5}}}{4} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{6+2\sqrt{5}}}{4} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3(10-2\sqrt{5})}}{8} + \frac{\sqrt{6+2\sqrt{5}}}{8}$$

$$= \frac{\sqrt{30-6\sqrt{5}} + \sqrt{6+2\sqrt{5}}}{8}$$

| | sine | cosine | tangent |
|---|---|---|---|
| 0 | $0$ | $1$ | $0$ |
| 6 | $\frac{\sqrt{30-6\sqrt5}-\sqrt{6+2\sqrt5}}{8} \approx .105$ | $\frac{\sqrt{18+6\sqrt5}+\sqrt{10-2\sqrt5}}{8} \approx .995$ | $\frac{\sqrt{30-6\sqrt5}-\sqrt{6+2\sqrt5}}{\sqrt{18+6\sqrt5}+\sqrt{10-2\sqrt5}} \approx .105$ |
| 9 | $\frac{\sqrt{3+\sqrt5}-\sqrt{5-\sqrt5}}{4} \approx .156$ | $\frac{\sqrt{3+\sqrt5}+\sqrt{5-\sqrt5}}{4} \approx .988$ | $\frac{\sqrt{3+\sqrt5}-\sqrt{5-\sqrt5}}{\sqrt{3+\sqrt5}+\sqrt{5-\sqrt5}} \approx .158$ |
| 15 | $\frac{\sqrt6-\sqrt2}{4} \approx .259$ | $\frac{\sqrt6+\sqrt2}{4} \approx .966$ | $\frac{\sqrt6+\sqrt2}{\sqrt6-\sqrt2} \approx .268$ |
| 24 | $\frac{\sqrt{18+6\sqrt5}-\sqrt{10-2\sqrt5}}{8} \approx .407$ | $\frac{\sqrt{30-6\sqrt5}+\sqrt{6+2\sqrt5}}{8} \approx .914$ | $\frac{\sqrt{18+6\sqrt5}-\sqrt{10-2\sqrt5}}{\sqrt{30-6\sqrt5}+\sqrt{6+2\sqrt5}} \approx .445$ |
| 30 | $\frac{1}{2} = .5$ | $\frac{\sqrt3}{2} \approx .866$ | $\frac{\sqrt3}{3} \approx .577$ |
| 36 | $\frac{\sqrt{10-2\sqrt5}}{4} \approx .588$ | $\frac{\sqrt{6+2\sqrt5}}{4} \approx .809$ | $\frac{\sqrt{10-2\sqrt5}}{\sqrt{6+2\sqrt5}} \approx .727$ |
| 45 | $\frac{\sqrt2}{2} \approx .707$ | $\frac{\sqrt2}{2} \approx .707$ | $1$ |
| 54 | $\frac{\sqrt{6+2\sqrt5}}{4} \approx .809$ | $\frac{\sqrt{10-2\sqrt5}}{4} \approx .588$ | $\frac{\sqrt{6+2\sqrt5}}{\sqrt{10-2\sqrt5}} \approx 1.376$ |
| 60 | $\frac{\sqrt3}{2} \approx .866$ | $\frac{1}{2} = .5$ | $\sqrt3 \approx 1.732$ |
| 66 | $\frac{\sqrt{30-6\sqrt5}+\sqrt{6+2\sqrt5}}{8} \approx .914$ | $\frac{\sqrt{18+6\sqrt5}-\sqrt{10-2\sqrt5}}{8} \approx .407$ | $\frac{\sqrt{30-6\sqrt5}+\sqrt{6+2\sqrt5}}{\sqrt{18-6\sqrt5}-\sqrt{10-2\sqrt5}} \approx 2.246$ |
| 75 | $\frac{\sqrt6+\sqrt2}{4} \approx .966$ | $\frac{\sqrt6-\sqrt2}{4} \approx .259$ | $\frac{\sqrt6+\sqrt2}{\sqrt6-\sqrt2} \approx 3.732$ |
| 81 | $\frac{\sqrt{3+\sqrt5}+\sqrt{5-\sqrt5}}{4} \approx .988$ | $\frac{\sqrt{3+\sqrt5}-\sqrt{5-\sqrt5}}{4} \approx .156$ | $\frac{\sqrt{3+\sqrt5}+\sqrt{5-\sqrt5}}{\sqrt{3+\sqrt5}-\sqrt{5-\sqrt5}} \approx 6.314$ |
| 84 | $\frac{\sqrt{18+6\sqrt5}+\sqrt{10-2\sqrt5}}{8} \approx .995$ | $\frac{\sqrt{30-6\sqrt5}-\sqrt{6+2\sqrt5}}{8} \approx .105$ | $\frac{\sqrt{18+6\sqrt5}+\sqrt{10-2\sqrt5}}{\sqrt{30-6\sqrt5}-\sqrt{6+2\sqrt5}} \approx 9.514$ |
| 90 | $1$ | $0$ | $\infty$ |

# Some Trigonometric Identities

Below we collect and derive some of the analytical truths that follow from core definitions and previous conclusions.

On page we verified the angle addition formulas for the sine. The companion formulas for cosine are:

$$\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$

From these four patterns we can derive "double-angle" formulas:

$$\begin{aligned} \sin 2\theta &= \sin(\theta+\theta) \\ &= \sin\theta\cdot\cos\theta + \sin\theta\cdot\cos\theta \\ &= 2\sin\theta\cdot\cos\theta \\ \cos 2\theta &= \cos(\theta+\theta) \\ &= \cos\theta\cdot\cos\theta - \sin\theta\cdot\sin\theta \\ &= (\cos\theta)^2 - (\sin\theta)^2 \end{aligned}$$

There are two other versions of $\cos 2\theta$, which we can discover via the Pythagorean identity $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

We can prove this latter from the initial right triangle definitions: in any right triangle $ABC$, $a^2+b^2=c^2$ and $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$, which gives $(\sin A)^2 + (\cos A)^2 = 1$.

$$\begin{aligned} \cos 2\theta &= \left(1-(\sin\theta)^2\right) - (\sin\theta)^2 \\ &= 1 - 2(\sin\theta)^2 \end{aligned} \qquad \text{OR} \qquad \begin{aligned} \cos 2\theta &= (\cos\theta)^2 - \left(1-(\cos\theta)^2\right) \\ &= 2(\cos\theta)^2 - 1 \end{aligned}$$

These latter two identities for $\cos 2\theta$ allow us to generate "half-angle" formulas for sine and cosine.

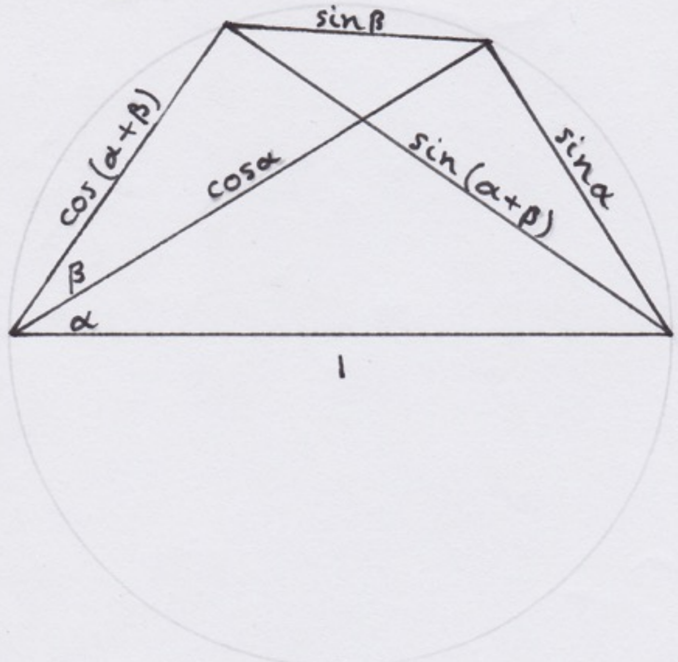
We let $\theta = \frac{\varphi}{2}$

$$\begin{aligned} \cos 2\theta &= 2(\cos\theta)^2 - 1 \\ \varphi &= 2(\cos\theta)^2 - 1 \\ \frac{\cos\varphi+1}{2} &= \left(\cos\frac{\varphi}{2}\right)^2 \\ \pm\sqrt{\frac{\cos\varphi+1}{2}} &= \cos\left(\frac{\varphi}{2}\right) \end{aligned} \qquad \begin{aligned} \cos 2\theta &= 1 - 2(\sin\theta)^2 \\ \varphi &= 1 - 2(\sin\theta)^2 \\ \frac{\cos\varphi-1}{2} &= \left(\sin\frac{\varphi}{2}\right)^2 \\ \pm\sqrt{\frac{\cos\varphi-1}{2}} &= \sin\left(\frac{\varphi}{2}\right) \end{aligned}$$

We mention a few other essential identities without proof.

$$\sin(90-\theta) = \cos\theta$$
$$\cos(90-\theta) = \sin\theta$$
$$\frac{\sin\theta}{\cos\theta} = \text{tangent}\,\theta$$

# Angle Addition Formulas for Cosines




$$\cos(\alpha+\beta)\cdot\sin\alpha + \sin\beta\cdot 1 = \sin(\alpha+\beta)\cdot\cos\alpha$$

$$\cos(\alpha+\beta)\cdot\sin\alpha = (\sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta)\cdot\cos\alpha - \sin\beta$$

$$\cos(\alpha+\beta)\cdot\sin\alpha = \sin\alpha\cdot\cos\beta\cdot\cos\alpha + (\cos\alpha)^2\cdot\sin\beta - \sin\beta$$

$$\cos(\alpha+\beta) = \frac{\sin\alpha\cdot\cos\beta\cdot\cos\alpha + (\cos\alpha)^2\cdot\sin\beta - \sin\beta}{\sin\alpha}$$

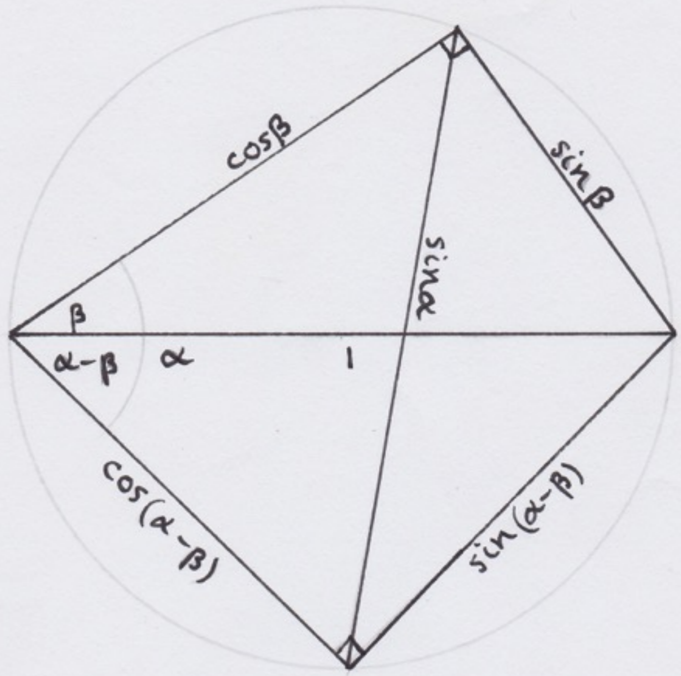
$$\cos(\alpha+\beta) = \cos\beta\cdot\cos\alpha + \frac{(\cos\alpha)^2\cdot\sin\beta}{\sin\alpha} - \frac{\sin\beta}{\sin\alpha}$$

$$\cos(\alpha+\beta) = \cos\beta\cdot\cos\alpha + \frac{(\cos\alpha)^2\cdot\sin\beta - \sin\beta}{\sin\alpha}$$

$$\cos(\alpha+\beta) = \cos\beta\cdot\cos\alpha + \frac{((\cos\alpha)^2 - 1)\sin\beta}{\sin\alpha}$$

$$\cos(\alpha+\beta) = \cos\beta\cdot\cos\alpha + \frac{(1-(\sin\alpha)^2 - 1)\sin\beta}{\sin\alpha}$$

$$\cos(\alpha+\beta) = \cos\beta\cdot\cos\alpha - \sin\alpha\cdot\sin\beta$$




$$\cos(\alpha-\beta)\cdot\sin\beta+\sin(\alpha-\beta)=\sin\alpha\cdot 1$$

$$\cos(\alpha-\beta)\cdot\sin\beta=\sin\alpha-\cos\beta(\sin\alpha\cdot\cos\beta-\sin\beta\cdot\cos\alpha)$$

$$\cos(\alpha-\beta)\cdot\sin\beta=\sin\alpha-\sin\alpha(\cos\beta)^2+\sin\beta\cdot\cos\beta\cdot\cos\alpha$$

$$\cos(\alpha-\beta)=\frac{\sin\alpha}{\sin\beta}-\frac{\sin\alpha(\cos\beta)^2}{\sin\beta}+\cos\beta\cdot\cos\alpha$$

$$\cos(\alpha-\beta)=\frac{\sin\alpha(1-(\cos\beta)^2)}{\sin\beta}+\cos\beta\cdot\cos\alpha$$

$$\cos(\alpha-\beta)=\frac{\sin\alpha(1-(\cos\beta)^2\cdot\sin\beta}{(\sin\beta)^2}+\cos\beta\cdot\cos\alpha$$

$$\cos(\alpha-\beta)=\frac{\sin\alpha(\sin\beta)^2\cdot\sin\beta}{(\sin\beta)^2}+\cos\beta\cdot\cos\alpha$$

$$\cos(\alpha-\beta)=\sin\alpha\sin\beta+\cos\beta\cdot\cos\alpha$$

# The Law of Tangents

In addition to the Law of Sines, there is a Law of Cosines and there is a Law of Tangents. Each of these has immediate uses in solving triangles (i.e. finding missing sides and angles, given any three components), and they have varied uses across different areas of math and science, typically in substitutions of one variable or another. We verify the Law of Tangents below, proving two lemmas first.

We state the law symbolically. Given $\Delta ABC$,

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$

Lemma 1: $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{a+b}{a-b}$

We prove this from left to right using the Law of Sines.

$$\text{since } \frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\text{then } \frac{\sin A}{\sin B} = \frac{a}{b}$$

Dividing the left side of the lemma by $\frac{\sin B}{\sin B}$ we have

$$\frac{\frac{\sin A + \sin B}{\sin B}}{\frac{\sin A - \sin B}{\sin B}}$$

$$= \frac{\frac{\sin A}{\sin B} + 1}{\frac{\sin A}{\sin B} - 1}$$

$$= \frac{\frac{a}{b} + 1}{\frac{a}{b} - 1}$$

$$= \frac{\frac{a+b}{b}}{\frac{a-b}{b}}$$

$$= \frac{a+b}{a-b}$$

Lemma 2: $\sin A \pm \sin B = 2\left(\sin\frac{A\pm B}{2}\right)\cos\left(\frac{A\pm B}{2}\right)$

We prove the "top" version of this on the next page.

From the angle addition and subtraction formulas:

$$2\left(\sin \frac{A}{2}+\frac{B}{2}\right)\cdot \cos\left(\frac{A}{2}-\frac{B}{2}\right)$$

$$= 2\left(\left(\sin \frac{A}{2}\cdot \cos\frac{B}{2} + \cos\frac{A}{2}\cdot \sin\frac{B}{2}\right)\left(\cos\frac{A}{2}\cdot\cos\frac{B}{2} + \sin\frac{A}{2}\cdot\sin\frac{B}{2}\right)\right)$$

$$= 2\left(\left(\cos\frac{B}{2}\right)^2 \sin\frac{A}{2}\cdot\cos\frac{A}{2} + \left(\sin\frac{A}{2}\right)^2\cos\frac{B}{2}\cdot\sin\frac{B}{2} + \left(\cos\frac{A}{2}\right)^2 \sin\frac{B}{2}\cdot\cos\frac{B}{2} + \left(\sin\frac{B}{2}\right)^2\cos\frac{A}{2}\cdot\sin\frac{A}{2}\right)$$

$$= 2\left(\left(\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(\left(\cos\frac{B}{2}\right)^2 + \left(\sin\frac{B}{2}\right)^2\right) + \cos\frac{B}{2}\sin\frac{B}{2}\left(\left(\sin\frac{A}{2}\right)^2 + \left(\cos\frac{A}{2}\right)^2\right)\right)$$

$$= 2\left(\sin\frac{A}{2}\cos\frac{A}{2} + \cos\frac{B}{2}\sin\frac{B}{2}\right)$$

$$= 2\sin\frac{A}{2}\cos\frac{A}{2} + 2\cos\frac{B}{2}\sin\frac{B}{2}$$

$$= \sin 2\left(\frac{A}{2}\right) + \cos 2\left(\frac{B}{2}\right)$$

$$= \sin A + \sin B$$

At this point we have:

$\sin A+\sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ and $\sin A-\sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

Dividing "equals by equals",

$$\frac{\sin A+\sin B}{\sin A-\sin B} = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}$$

$$\frac{\sin A+\sin B}{\sin A-\sin B} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$

From Lemma 1,

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$