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# 3 Degree of Freedom Gyroscope Dynamics and Control

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# Contents

# 1 System Dynamics

## 1.1 Frames of Reference

We consider a control moment gyroscope consisting of two gimbals *(A and B)* along with a symmetric disk *(D)* as shown in the figure 1.

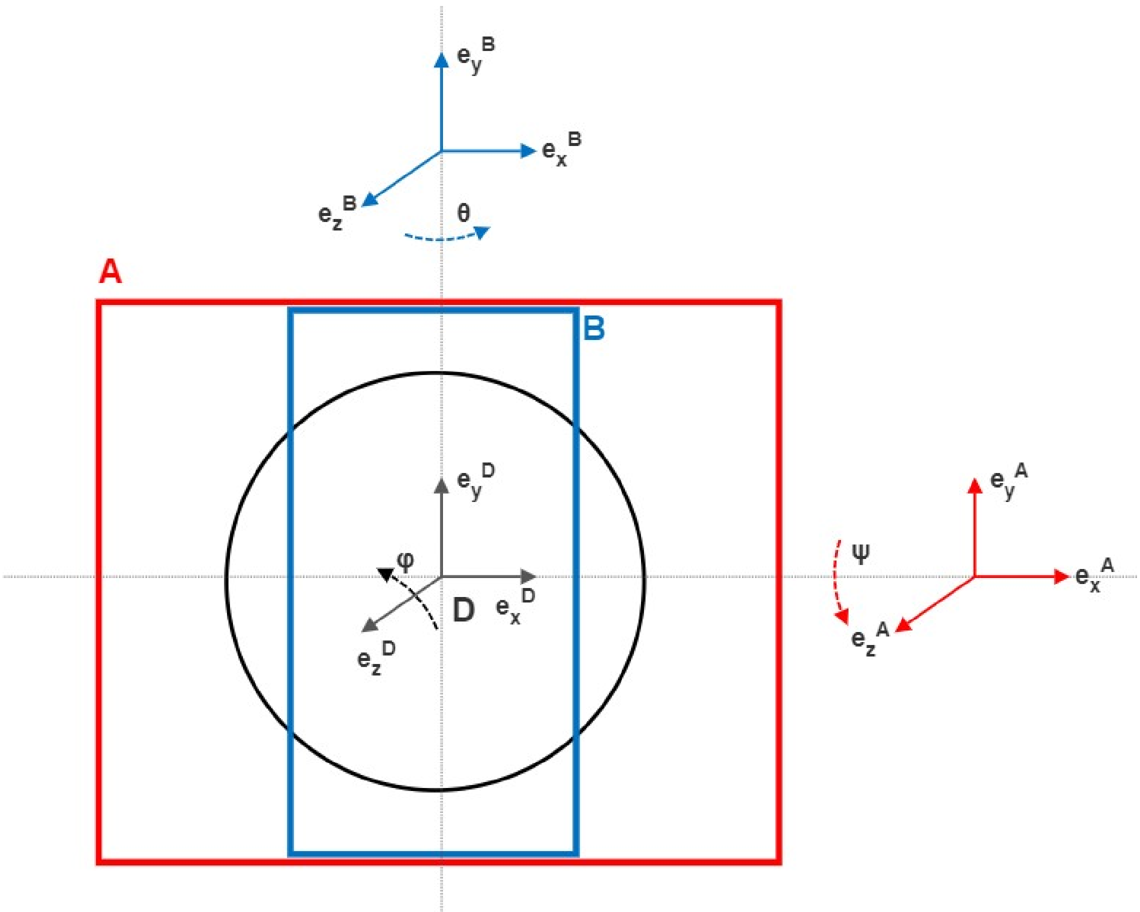


Figure 1: Gyroscope Diagram and Coordinate Axes

We take a system of right orientated orthogonal unit vectors $\mathbf{e}_\mathbf{i}^\mathbf{j}$ where:

- $i = x,\ y,\ z$
- $j = A,\ B,\ D,\ N$ where:
  - $N$ refers to the *world frame or interial frame*
  - $A,\ B,\ D$ refer to the frames of reference fixed to the bodies *(gimbals, disk)*
- The origins of the coordinate frames are located in the centre of the disk *(D)*.

We define the following angles as the positive sense of rotation for each of the defined reference frames along their respective axes of rotation:

- $\boldsymbol{\psi}$: angle of rotation around the $e_x^A$-axis w.r.t. frame $N$
- $\boldsymbol{\theta}$: angle of rotation around the $e_y^B$-axis w.r.t. frame $A$
- $\boldsymbol{\phi}$: angle of rotation around the $e_z^D$-axis w.r.t. frame $B$.

## 1.2 Rotation Matrices

We now define the following rotation matrices for an arbitrary angle $q$.

$$R_X(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(q) & -sin(q) \\ 0 & sin(q) & cos(q) \end{bmatrix}$$

$$R_Y(q) = \begin{bmatrix} cos(q) & 0 & sin(q) \\ 0 & 1 & 0 \\ -sin(q) & 0 & cos(q) \end{bmatrix}$$

$$R_Z(q) = \begin{bmatrix} cos(q) & -sin(q) & 0 \\ sin(q) & cos(q) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the rotation matrices defined above, we define the following rotation matrices between the frames where $\mathbf{R_1^2}$ denotes a rotation from 1 to 2:

- $\mathbf{R_A^N} = \mathbf{R_X}(\psi)$
- $\mathbf{R_B^A} = \mathbf{R_Y}(\theta)$
- $\mathbf{R_D^B} = \mathbf{R_Z}(\phi)$

Further we can define:

- $\mathbf{R_B^N} = \mathbf{R_A^N}\mathbf{R_B^A}$
- $\mathbf{R_D^N} = \mathbf{R_A^N}\mathbf{R_B^A}\mathbf{R_D^B}$

## 1.3 External Torque

The external torque applied to the system is the torque applied by the motors used to actuate each gimbal or to accelerate the disk. We represent the torques as follows:

1. $\boldsymbol{\tau}_\mathbf{A}$: torque applied on *gimbal A* along axis $e_x^A$ in *frame N*.
2. $\boldsymbol{\tau}_\mathbf{B}$: torque applied on *gimbal B* along axis $e_y^B$ in *frame A*.
3. $\boldsymbol{\tau}_\mathbf{D}$: torque applied on *disk D* along axis $e_z^D$ in *frame B*.

Using the above definitions and the rotation matrices derived in the previous section, we have the net external torque on the system given as:

$$\vec{M}_{net}|_N = \begin{bmatrix} \tau_A \\ 0 \\ 0 \end{bmatrix} + R_A^N \begin{bmatrix} 0 \\ \tau_B \\ 0 \end{bmatrix} + R_B^N \begin{bmatrix} 0 \\ 0 \\ \tau_D \end{bmatrix} \tag{1}$$

## 1.4 Inertia Matrix

We make the assumption that all the gimbals are massless. Therefore the moment of inertia contribution is only from the disk. Assuming axisymmetry in the disk, we can say define the inertia terms as:

$$I_{xx} = I_{yy} = I$$
$$I_{zz} = I_0$$

Therefore we can write the inertia matrix of the disk with respect to the frame $D$ as:

$$I_D^D = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_0 \end{bmatrix}$$

To obtain the net angular momentum in the $N$ frame, we have to express the angular velocities and inertia matrix in the $N$ frame.

The inertia matrix expressed in the inertial frame is given as:

$$I_D^N = R_D^N I_D^D R_D^{N^T} \tag{2}$$

## 1.5 Conservation of Angular Momentum

The net angular velocity of the system can be expressed in the inertial frame as:

$$\vec{\omega}_N = \begin{bmatrix} \dot{\psi} \\ 0 \\ 0 \end{bmatrix} + R_A^N \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + R_B^N \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \tag{3}$$

Combining equations 2 and 3, we get the net angular momentum expressed in the inertial frame as:

$$\vec{H}_{net}|_N = R_D^N I_D^D R_D^{N^T} \left( \begin{bmatrix} \dot{\psi} \\ 0 \\ 0 \end{bmatrix} + R_A^N \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + R_B^N \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \right) \tag{4}$$

By conservation of angular momentum, we have:

$$\vec{M}_{net}|_N = \frac{d}{dt} \vec{H}_{net}|_N \tag{5}$$

Substituting $\vec{M}_{net}|_N$ and $\vec{H}_{net}|_N$ from equations 1 and 5, we get the complete system dynamics.

## 2 Controller Design

### 2.1 State Space

From equations 2 and 3, we get $\vec{H}_{net}|_N$ as:

$$\vec{H}_{net}|_N = \begin{bmatrix} I + I_0 sin(\theta)^2 - I sin(\theta)^2 & 0 & I_0 sin(\theta) \\ -cos(\theta)sin(\psi)sin(\theta)(I_0 - I) & I cos(\psi) & -I_0 cos(\theta)sin(\psi) \\ cos(\psi)cos(\theta)sin(\theta)(I_0 - I) & I sin(\psi) & I_0 cos(\psi)cos(\theta) \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{6}$$

Let us define $Q$ such that:

$$\begin{gathered} \vec{H}_{net}|_N = Q \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \\ Q = \begin{bmatrix} I + I_0 sin(\theta)^2 - I sin(\theta)^2 & 0 & I_0 sin(\theta) \\ -cos(\theta)sin(\psi)sin(\theta)(I_0 - I) & I cos(\psi) & -I_0 cos(\theta)sin(\psi) \\ cos(\psi)cos(\theta)sin(\theta)(I_0 - I) & I sin(\psi) & I_0 cos(\psi)cos(\theta) \end{bmatrix} \end{gathered} \tag{7}$$

By substituting $\vec{H}_{net}|_N$ from equation 7 in equation 5 and applying the chain rule, we can write:

$$\vec{M}_{net}|_N = \dot{Q} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} + Q \begin{bmatrix} \ddot{\psi} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} \tag{8}$$

Therefore, we can write:

$$\begin{bmatrix} \ddot{\psi} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = Q^{-1} \left( \vec{M}_{net}|_N - \dot{Q} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \right) \tag{9}$$

We now define the state vector $(x)$ and control vector $(u)$ as:

$$x = \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \\ \psi \\ \theta \end{bmatrix} \qquad u = \begin{bmatrix} \tau_A \\ \tau_B \\ \tau_D \end{bmatrix}$$

With the above definitions of $x$ and $u$ and from equation 9, we can express the system dynamics in the state space form given by $\dot{x} = f(x, u)$. We get the final state space form as:

$$\begin{bmatrix} \ddot{\psi} \\ \ddot{\theta} \\ \ddot{\phi} \\ \dot{\psi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{\tau_1 \cos(\theta) - I_0 \dot{\phi} \dot{\theta} - I_0 \dot{\psi} \dot{\theta} \sin(\theta) + 2 I \dot{\psi} \dot{\theta} \sin(\theta)}{I \cos(\theta)} \\ \frac{\tau_2 + \frac{I_0 \dot{\psi}^2 \sin(2\theta)}{2} - \frac{I \dot{\psi}^2 \sin(2\theta)}{2} + I_0 \dot{\phi} \dot{\psi} \cos(\theta)}{I} \\ \frac{\frac{I \tau_1 \sin(2\theta)}{2} - \frac{I_0 \tau_1 \sin(2\theta)}{2} + {I_0}^2 \dot{\psi} \dot{\theta} + I \tau_3 \cos(\theta) - {I_0}^2 \dot{\psi} \dot{\theta} \cos(\theta)^2 + {I_0}^2 \dot{\phi} \dot{\theta} \sin(\theta) - 2 I_0 I \dot{\psi} \dot{\theta} + I_0 I \dot{\psi} \dot{\theta} \cos(\theta)^2}{I_0 I \cos(\theta)} \\ \dot{\psi} \\ \dot{\theta} \end{bmatrix}$$

## 2.2 Linearising the System

For the purpose of designing linear controllers, we need to express the state space model in the form:

$$\dot{x} = Ax + Bu$$

To obtain the $A$ and $B$ matrices, we take the Jacobian of the non-linear function $f(x, u)$ with respect to $x$ and $u$ respectively.

We linearise the system about the following equilibrium point:

$$x_e = \begin{bmatrix} \dot{\psi}_e \\ \dot{\theta}_e \\ \dot{\phi}_e \\ \psi_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega_0 \\ 0 \\ 0 \end{bmatrix} \qquad u = \begin{bmatrix} \tau_{Ae} \\ \tau_{Be} \\ \tau_{De} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Where, $\omega_0$ is the angular velocity of the disk.

On linearising we obtain:

$$A = \begin{bmatrix} 0 & -\frac{I_0\,\omega_0}{I} & 0 & 0 & 0 \\ \frac{I_0\,\omega_0}{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$B = \begin{bmatrix} \frac{1}{I} & 0 & 0 \\ 0 & \frac{1}{I} & 0 \\ 0 & 0 & \frac{1}{I_0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

## 2.3 Controllability

We check the controllability of the linearised system by constructing the controllability matrix given by:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Where $n$ is determined by $A$ given that $A$ is an $n \times n$ matrix.

For the given system, $n = 5$.

For $A$ and $B$ for the given system from equations 10 and 11, we have:

$$\mathcal{C} = \begin{bmatrix} \frac{1}{I} & 0 & 0 & 0 & -\frac{I_0\,\omega_0}{I^2} & 0 & -\frac{{I_0}^2\,{\omega_0}^2}{I^3} & 0 & 0 & 0 & \frac{{I_0}^3\,{\omega_0}^3}{I^4} & 0 & \frac{{I_0}^4\,{\omega_0}^4}{I^5} & 0 & 0 \\ 0 & \frac{1}{I} & 0 & \frac{I_0\,\omega_0}{I^2} & 0 & 0 & 0 & -\frac{{I_0}^2\,{\omega_0}^2}{I^3} & 0 & -\frac{{I_0}^3\,{\omega_0}^3}{I^4} & 0 & 0 & 0 & \frac{{I_0}^4\,{\omega_0}^4}{I^5} & 0 \\ 0 & 0 & \frac{1}{I_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{I} & 0 & 0 & 0 & -\frac{I_0\,\omega_0}{I^2} & 0 & -\frac{{I_0}^2\,{\omega_0}^2}{I^3} & 0 & 0 & 0 & \frac{{I_0}^3\,{\omega_0}^3}{I^4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{I} & 0 & \frac{I_0\,\omega_0}{I^2} & 0 & 0 & 0 & -\frac{{I_0}^2\,{\omega_0}^2}{I^3} & 0 & -\frac{{I_0}^3\,{\omega_0}^3}{I^4} & 0 & 0 \end{bmatrix}$$

For non-zero and finite values of $\omega_0$;

$$\operatorname{rank}(\mathcal{C}) = 5 = n$$

Therefore we can say that the system is **controllable**.

## 2.4 Linear Quadratic Regulator

Given a targeted state $\boldsymbol{x_d}$, the Linear Quadratic Regulator aims at finding the best control action to minimize the quadratic cost $J$ as defined in equation 12.

$$J = \int_0^\infty \boldsymbol{u} R \boldsymbol{u}^T + |\boldsymbol{x_d} - \boldsymbol{x}| Q |\boldsymbol{x_d} - \boldsymbol{x}|^T dt \tag{12}$$

Where,

- $R$ is the cost incurred for exterting the control $\boldsymbol{u}$
- $Q$ is the penalty cost for deviating from the desired state $\boldsymbol{x_d}$

The solution to the above minimization problem is the following simple feedback control:

$$\boldsymbol{u} = -K|\boldsymbol{x_d} - \boldsymbol{x}| \tag{13}$$

Where,

$$K = R^{-1} B^T P \tag{14}$$

Where, $P$ satisfies the following equation, also known as the *Riccati Equation*:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{15}$$

For the given system, we choose:

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$R = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

By solving the Riccati Equation we get:

$$K = \begin{bmatrix} 0.3188 & 0 & 0 & 0.2922 & 0.4058 \\ 0 & 0.3188 & 0 & -0.4058 & 0.2922 \\ 0 & 0 & 0.2236 & 0 & 0 \end{bmatrix} \tag{16}$$

# 3 Results

The LQR controller was simulated for various initial condition and convergence was obtained to the equilibrium point. The results for some randomised initial conditions are shown below.

## 3.1 Simulation Result 1

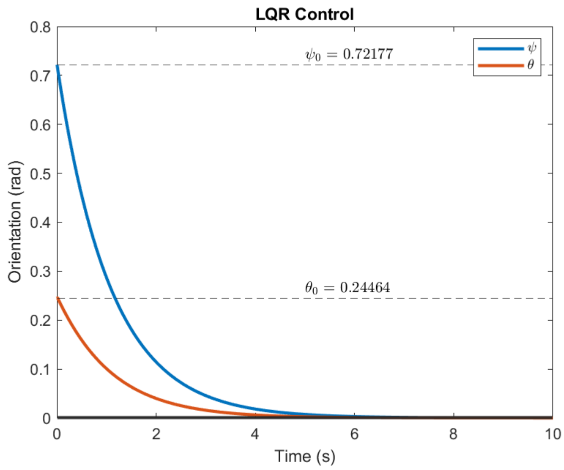


Figure 2: Variation of $\psi$ and $\theta$

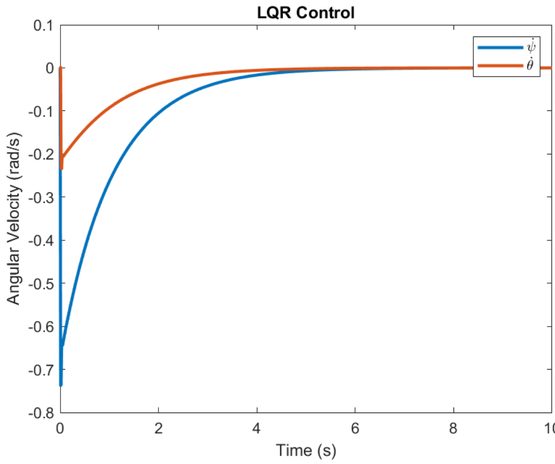


Figure 3: Variation of $\dot{\psi}$ and $\dot{\theta}$

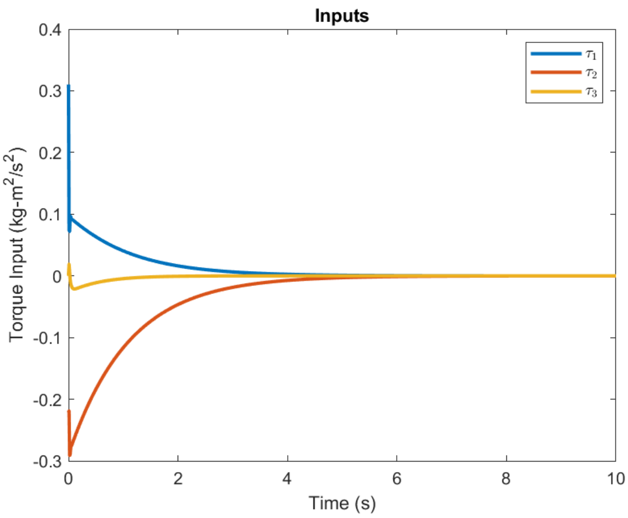


Figure 4: Variation of Input Torques

## 3.2 Simulation Result 2

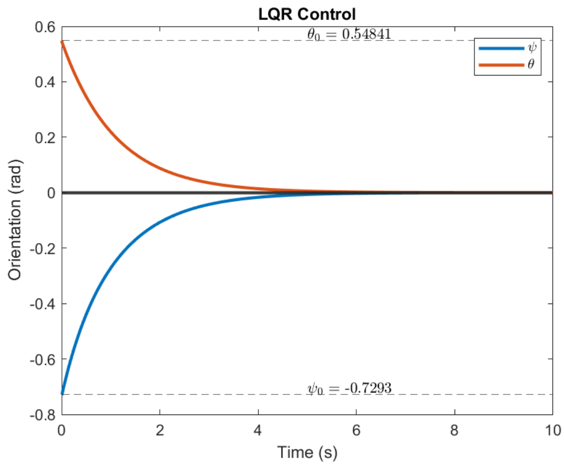


Figure 5: Variation of $\psi$ and $\theta$

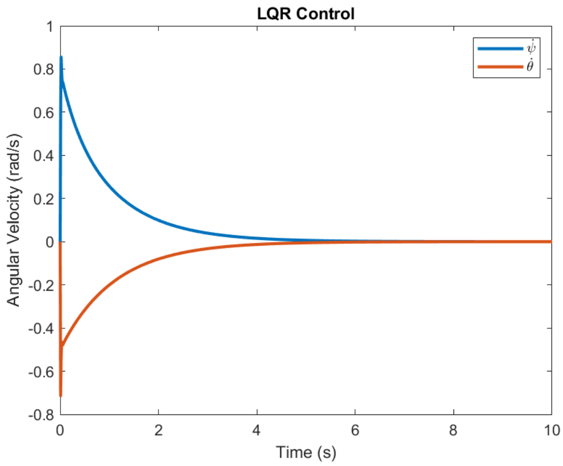


Figure 6: Variation of $\dot{\psi}$ and $\dot{\theta}$

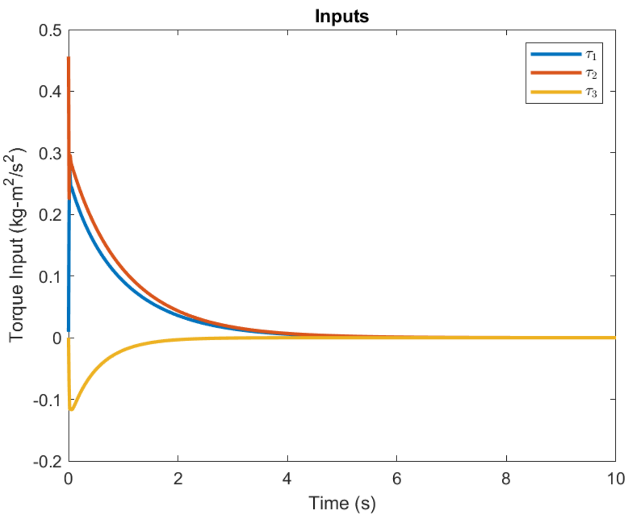


Figure 7: Variation of Input Torques

## 3.3 Simulation Result 3

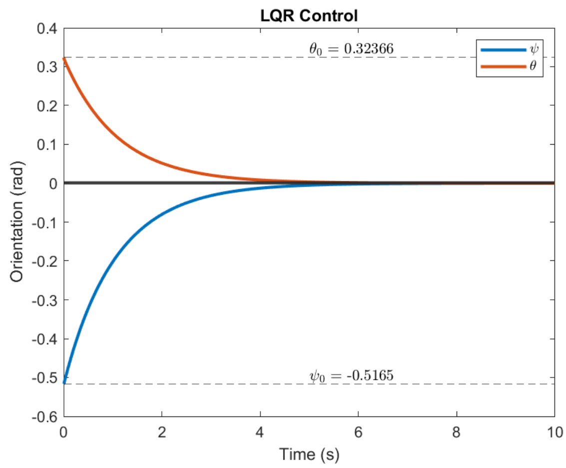


Figure 8: Variation of $\psi$ and $\theta$

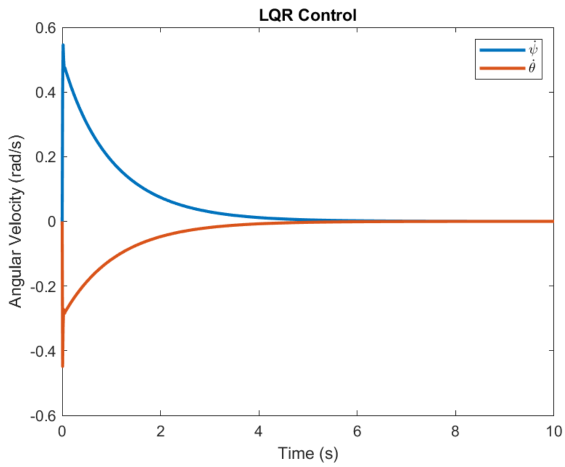


Figure 9: Variation of $\dot{\psi}$ and $\dot{\theta}$

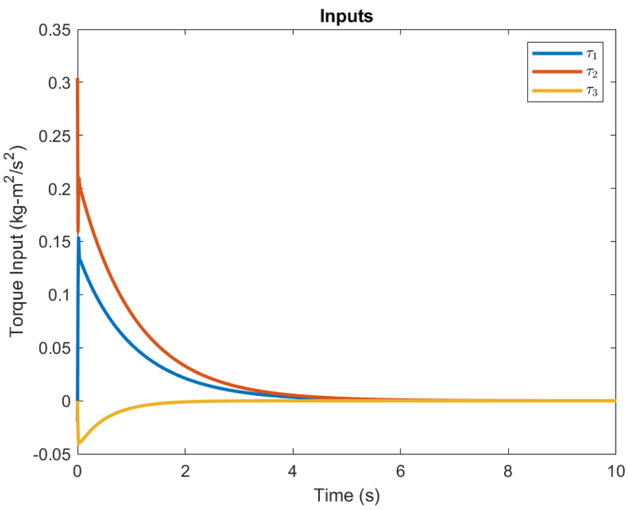


Figure 10: Variation of Input Torques