Differential Geometry (contd..)

Metrics on Surface

- Measurements on Surface
  - Angle between two tangents
  - Length of a tangent
  - Area of a patch
- How to measure angle between two vectors in euclidean space?
- Can we generalise it to surfaces?

Surfaces (Recap)

- Continous Surface

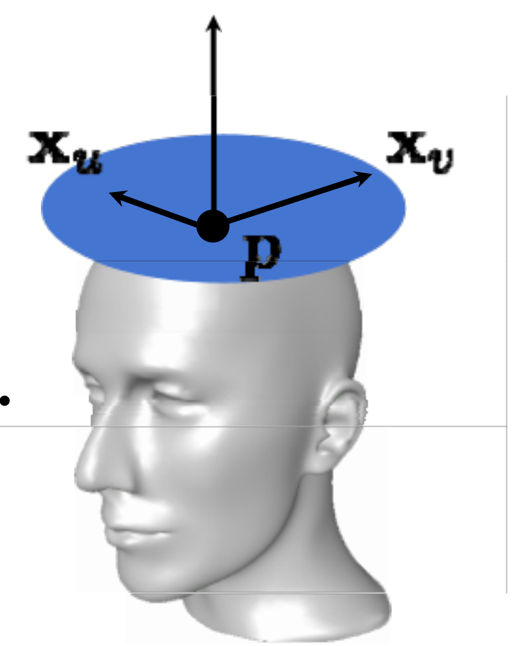
$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in \mathbb{R}^2$$

- Normal vector

$$\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v) / \|\mathbf{x}_u \times \mathbf{x}_v\|$$

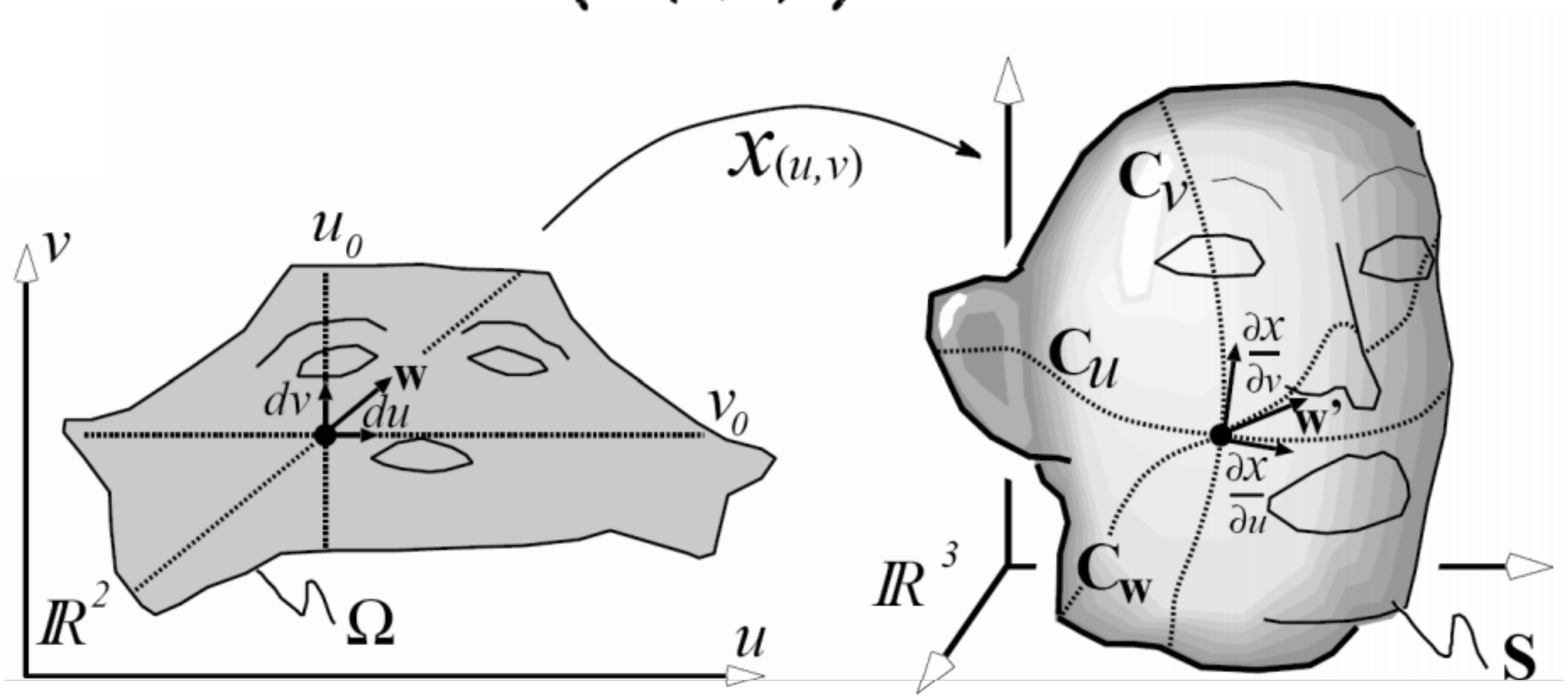
– assuming regular parameterization, i.e.

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$



First Fundamental Form

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$



First Fundamental Form

- Let us study Directional Derivatives of x
- Given a direction vector $\bar{\mathbf{w}} = (u_w, v_w)^T$ parameter domain.
- Straight line passing through (u_0 ,v_0) parameterised by *t* and oriented by $\bar{\mathbf{w}}$ is $(u, v) = (u_0, v_0) + t\bar{\mathbf{w}}$
- The same line on surface is given by $\mathbf{C}_{\mathbf{w}}(t) = \mathbf{x}(u_0 + tu_w, v_0 + tv_w)$

First Fundamental Form

- Directional derivative W of x at (u_0,v_0) relative to direction $\bar{\mathbf{w}}$ is defined to be the tangent to $\mathbf{C}_{\mathbf{w}}(t)$ at t = 0 is

$$\mathbf{w} = \partial \mathbf{C}_{\mathbf{w}}(t)/\partial t$$

- By applying chain rule, we get $\mathbf{w} = \mathbf{J}\bar{\mathbf{w}}$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \left[\mathbf{x}_u \, , \, \mathbf{x}_v\right]$$

First Fundamental Form

- The Jacobian matrix of the parameterisation function x is a linear map that transforms a vector $\bar{\mathbf{w}}$ in parameter space to the tangent vector on the surface
- Consider $\bar{\mathbf{w}}_1 \text{ and } \bar{\mathbf{w}}_2$ to be the unit direction vectors in the parameter space.
- Its scalar product: $\bar{\mathbf{w}}_1^T \bar{\mathbf{w}}_2$
- Scalar product between corresponding tangent vectors?

First Fundamental Form

$$\begin{aligned}\mathbf{w}_1^T \mathbf{w}_2 &= (\mathbf{J}\bar{\mathbf{w}}_1)^T (\mathbf{J}\bar{\mathbf{w}}_2) \\ &= \bar{\mathbf{w}}_1^T \left(\mathbf{J}^T \mathbf{J}\right) \bar{\mathbf{w}}_2\end{aligned}$$

- $J^T J$ is ***First Fundamental Form (I)***
- The first fundamental form I defines an inner product on the tangent space

$$\mathbf{I} = \mathbf{J}^T \mathbf{J} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{bmatrix}$$

First Fundamental Form

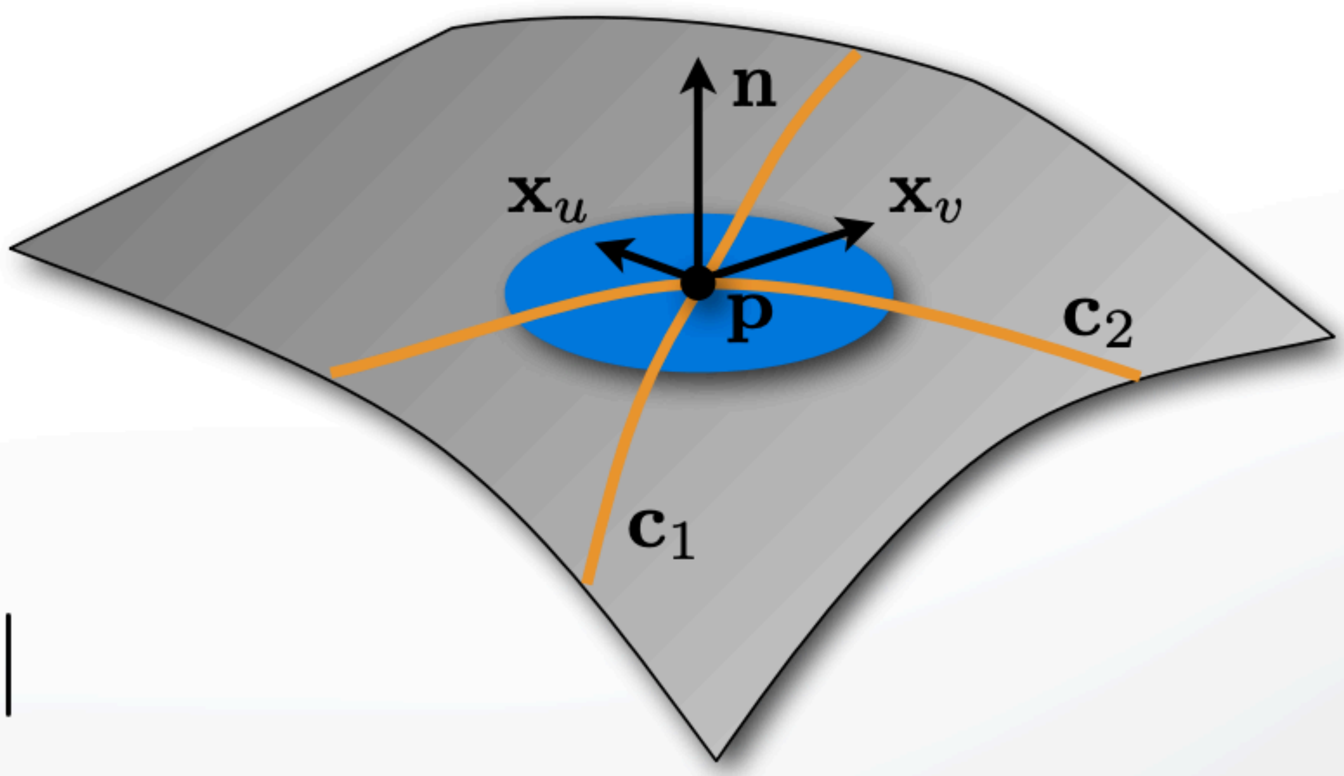
Two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ intersecting at $\mathbf{p}$

- angle of intersection?
- two tangents $\mathbf{t}_1$ and $\mathbf{t}_2$

$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

- compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos\theta \, \|\mathbf{t}_1\| \, \|\mathbf{t}_2\|$$



First Fundamental Form

Two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ intersecting at $\mathbf{p}$

$$\begin{aligned}
\mathbf{t}_1^T \mathbf{t}_2 &= (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v) \\
&= \alpha_1 \alpha_2 \mathbf{x}_u^T \mathbf{x}_u + (\alpha_1 \beta_2 + \alpha_2 \beta_1)\, \mathbf{x}_u^T \mathbf{x}_v + \beta_1 \beta_2 \mathbf{x}_v^T \mathbf{x}_v \\
&= (\alpha_1, \beta_1) \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}
\end{aligned}$$

First Fundamental Form

First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

Arc length on a surface

- Consider a curve $\mathbf{C}$ on surface X
- Its (differential) arc length at point $\mathbf{p}$ is

$$\|\dot{\mathbf{C}}\| = \|X_u \dot{u} + X_v \dot{v}\|$$

- Squaring

$$\|\dot{\mathbf{C}}\|^2 = (X_u \cdot X_u)\dot{u}^2 + 2(X_u \cdot X_v)\dot{u}\dot{v} + (X_v \cdot X_v)\dot{v}^2$$

or

$$\|\dot{\mathbf{C}}\|^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

First Fundamental Form

First fundamental form $\mathbf{I}$ allows to measure (w.r.t. surface metric)

Angles $\quad \mathbf{t}_1^\top \mathbf{t}_2 = \langle(\alpha_1, \beta_1), (\alpha_2, \beta_2)\rangle$

Length
$$\begin{aligned} \mathrm{d}s^2 &= \langle(\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v)\rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

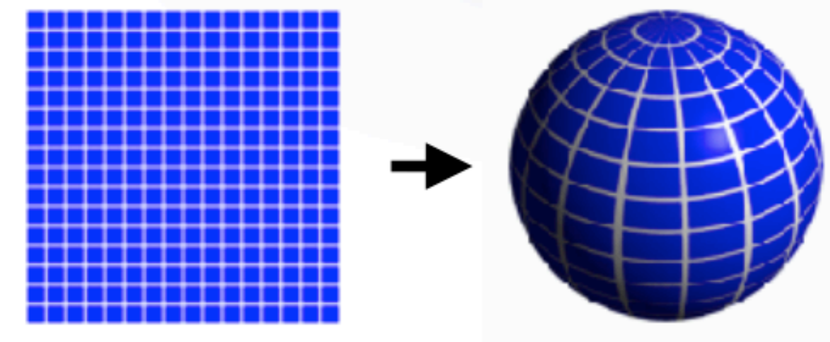
squared infinitesimal length

Area
$$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T\mathbf{x}_u \cdot \mathbf{x}_v^T\mathbf{x}_v - (\mathbf{x}_u^T\mathbf{x}_v)^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

infinitesimal Area

cross product → determinant with unit vectors → area

Sphere Example



Spherical parameterization

$$\mathbf{x}(u,v) \;=\; \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u,v) \in [0, 2\pi) \times [0, \pi)$$

Tangent vectors

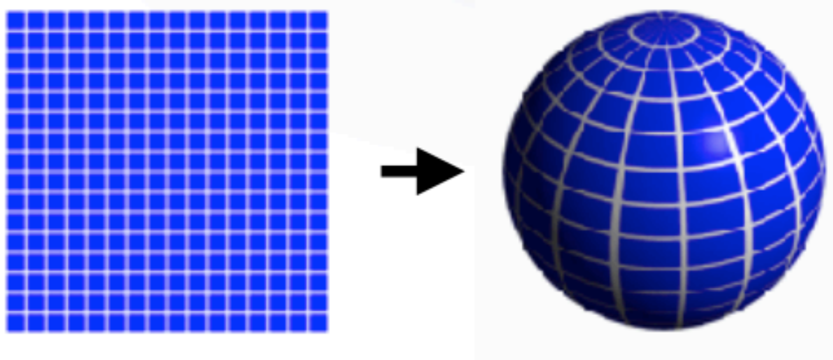
$$\mathbf{x}_u(u,v) \;=\; \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \qquad \mathbf{x}_v(u,v) \;=\; \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

First fundamental Form

$$\mathbf{I} \;=\; \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Sphere Example

Length of equator $\mathbf{x}(t, \pi/2)$



$$\int_0^{2\pi} 1 \,\mathrm{d}s \;=\; \int_0^{2\pi} \sqrt{E\left(u_t\right)^2 + 2F u_t v_t + G\left(v_t\right)^2}\,\mathrm{d}t$$

$$= \int_0^{2\pi} \sin v \,\mathrm{d}t$$

$$= \; 2\pi \sin v \;=\; 2\pi$$