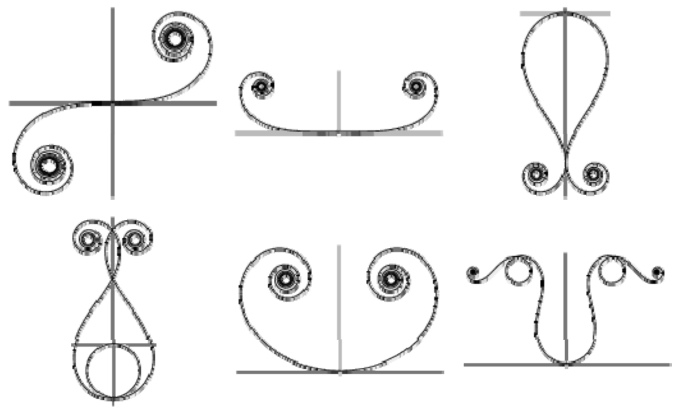
Differential Geometry

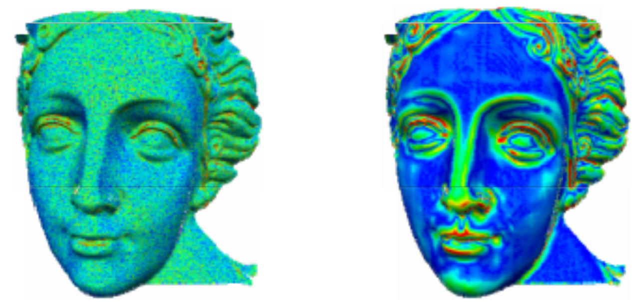


Motivation

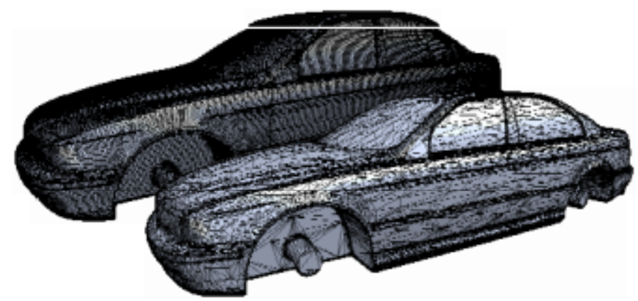
- Understand the structure of the surface
  - Properties: smoothness, curviness , important directions
- How to modify the surface to change these properties
- What properties are preserved for different modifications
- The math behind the scenes for many geometry processing applications

Motivation

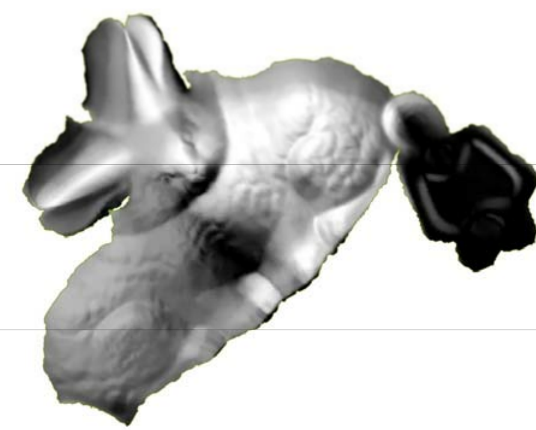
- Smoothness
  - ➡ Mesh smoothing



- Curvature
  - ➡ Adaptive simplification

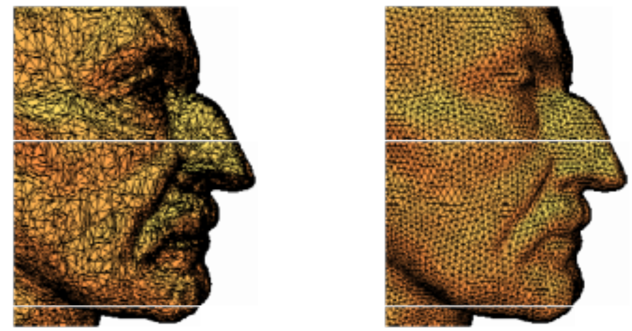


  - ➡ Parameterization

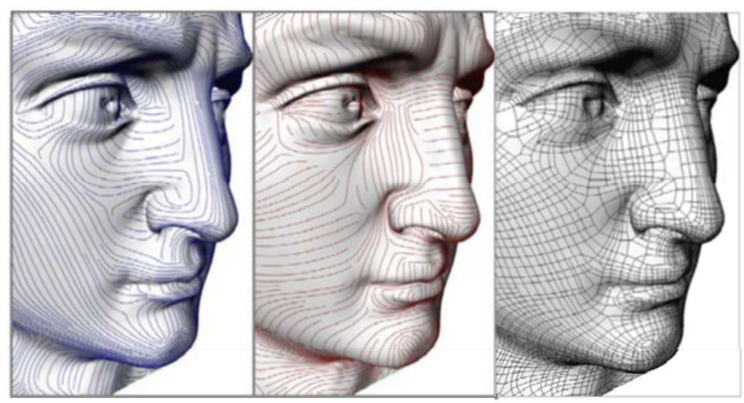


Motivation

- Triangle shape
  - ➡ Remeshing



- Principal directions
  - ➡ Quad remeshing

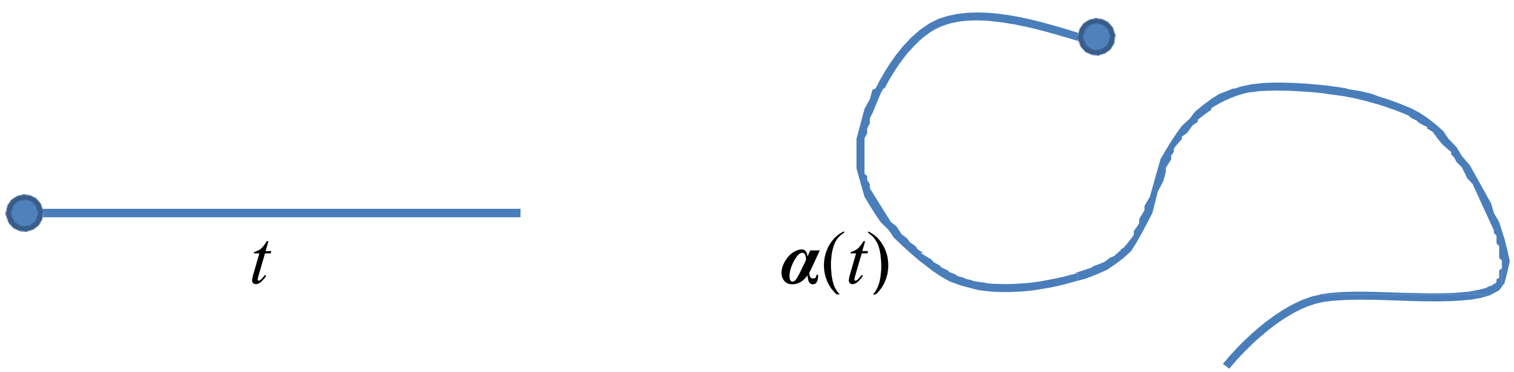


Parameterised Curves
Intution

A particle is moving in space

At time t its position is given by

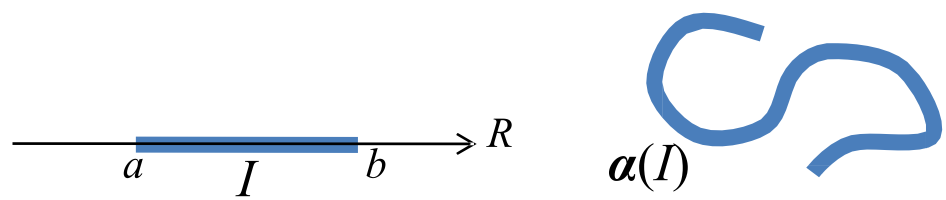
$$\boldsymbol{\alpha}(t) = (x(t), y(t), z(t))$$



Parameterised Curves

Definition

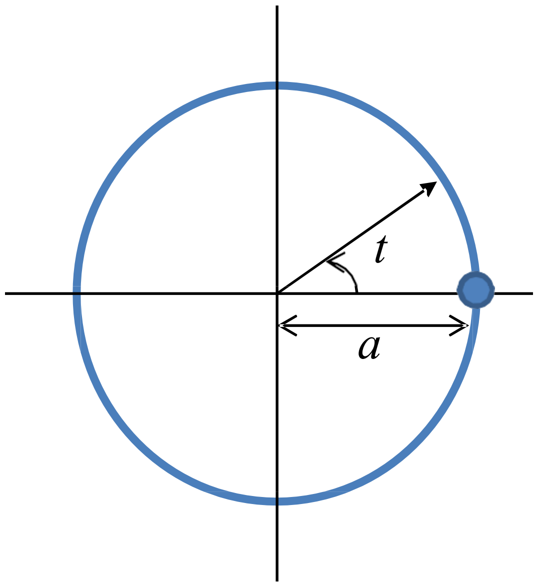
A *parameterized differentiable curve* is a differentiable map $\boldsymbol{\alpha}: I \rightarrow R^3$ of an interval $I = (\mathrm{a,b})$ of the real line R into R^3



$\boldsymbol{\alpha}$ maps $t \in I$ into a point $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)) \in R^3$ such that $x(t)$, $y(t)$, $z(t)$ are *differentiable*

A function is *differentiable* if it has, at all points, derivatives of all orders

Parameterised Curves
Example



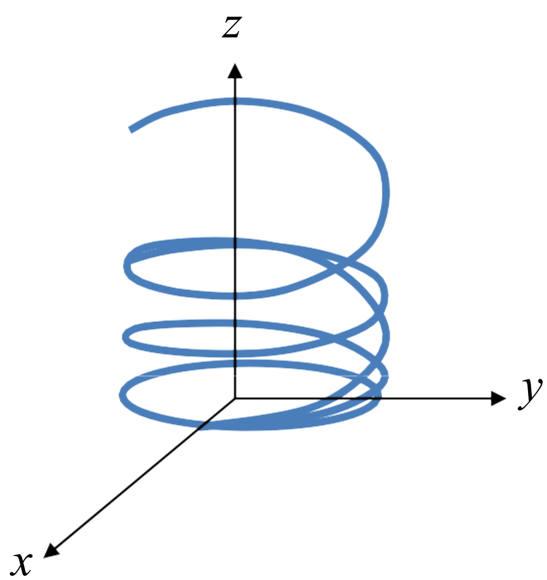
$\boldsymbol{\alpha}_1(t) = (a\cos(t),\, a\sin(t))$
$t \in [0,2\pi] = I$

$\boldsymbol{\alpha}_2(t) = (a\cos(2t),\, a\sin(2t))$

$\boldsymbol{t \in [0,\pi] = I}$

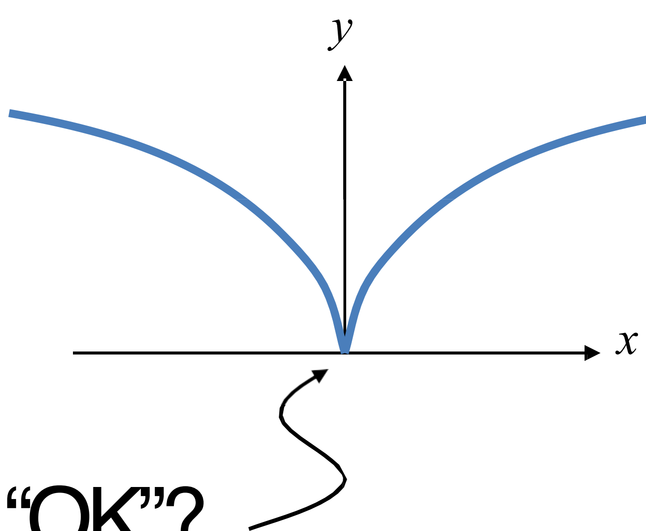
Parameterised Curves
Example

$$\boldsymbol{\alpha}(t) = (a\cos(t), a\sin(t), bt), \; t \in R$$



Parameterised Curves
Example

$$\boldsymbol{\alpha}(t) = (t^3, t^2), \;\; t \in R$$



Is this "OK"?

The Tangent Vector

Let

$$\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)) \in R^3$$

Then

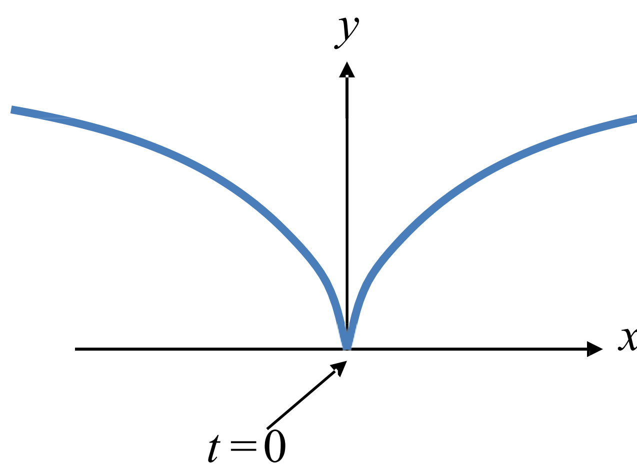
$$\boldsymbol{\alpha}'(t) = (x'(t), y'(t), z'(t)) \in R^3$$

is called the *tangent vector* (or *velocity vector*) of the curve $\boldsymbol{\alpha}$ at t

Regular Curves

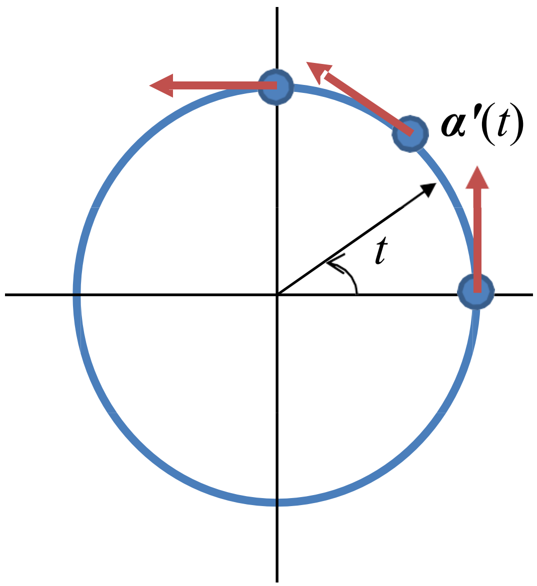
If $\boldsymbol{\alpha}'(t) = \mathbf{0}$, then t is a *singular point* of $\boldsymbol{\alpha}$.

$\boldsymbol{\alpha}(t) = (t^3, t^2), \ \ t \in R$



A parameterized differentiable curve $\boldsymbol{\alpha}: I \longrightarrow R^3$ is *regular* if $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ for all $t \in I$

Back To Circle



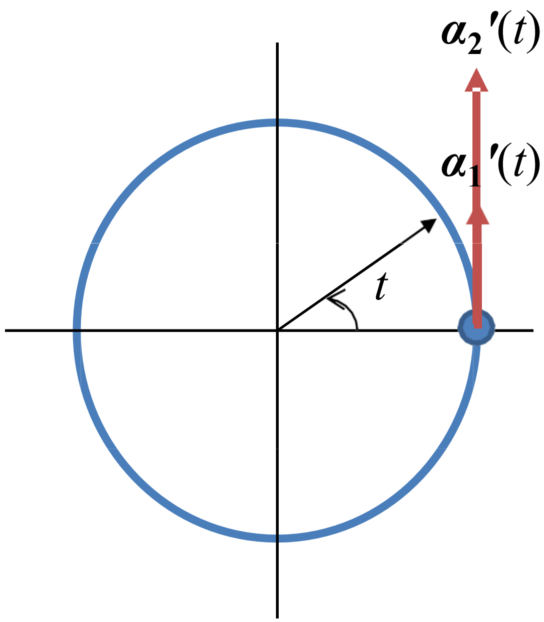
$$\boldsymbol{\alpha}(t) = (\cos(t), \sin(t))$$

$$\boldsymbol{\alpha}'(t) = (-\sin(t), \cos(t))$$

$\boldsymbol{\alpha}'(t)$ - direction of movement

$|\alpha'(t)|$ - speed of movement

Back To Circle



$$\boldsymbol{\alpha}_1(t) = (\cos(t), \sin(t))$$

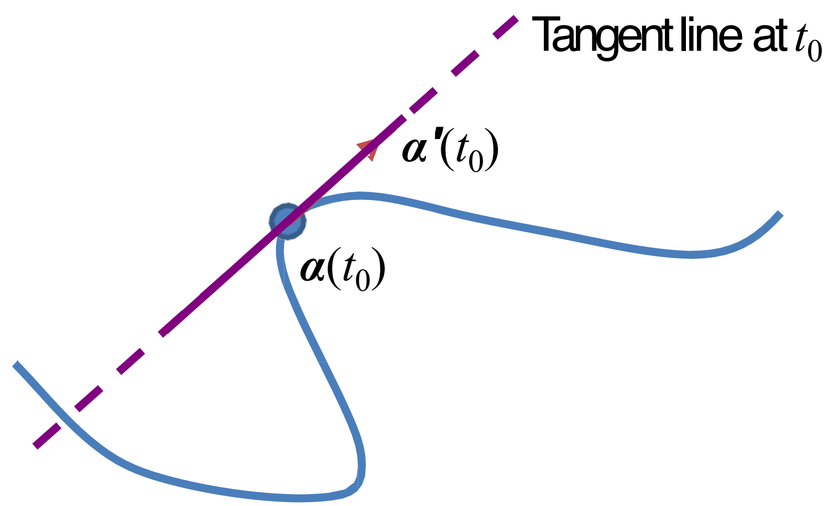
$$\boldsymbol{\alpha}_2(t) = (\cos(2t), \sin(2t))$$

Same direction, different speed

The Tangent Line

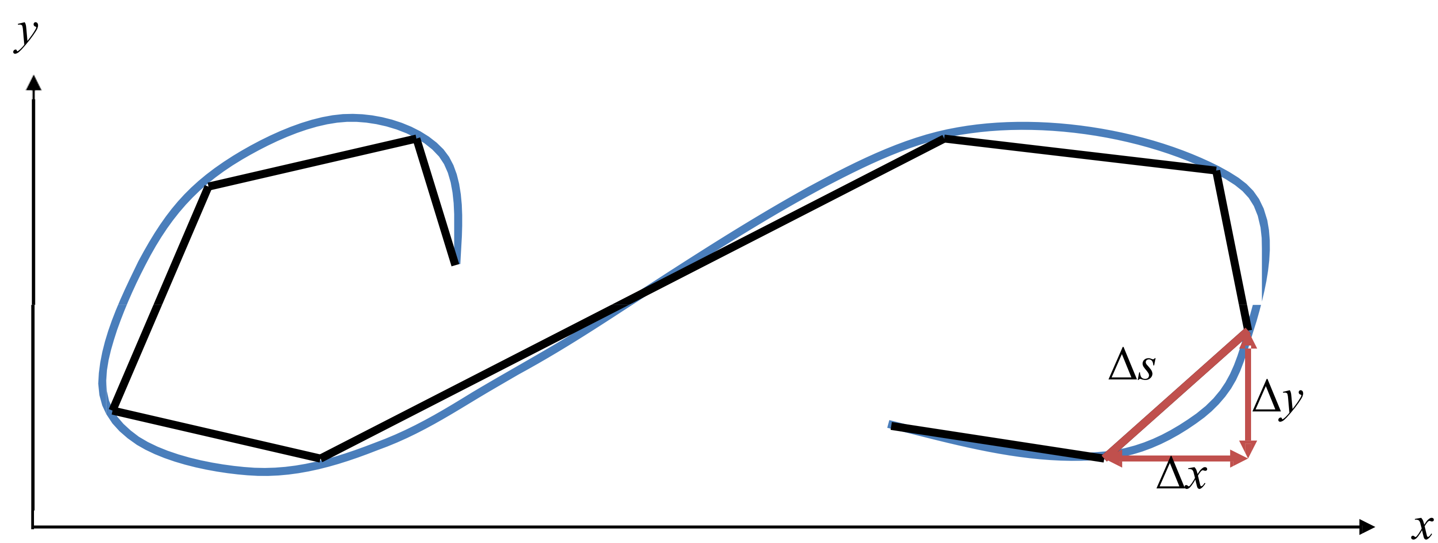
Let $\boldsymbol{\alpha}: I \rightarrow R^3$ be a parameterized differentiable curve.

For each $t \in I$ s.t. $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ the *tangent line* to $\boldsymbol{\alpha}$ at t is the line which contains the point $\boldsymbol{\alpha}(t)$ and the vector $\boldsymbol{\alpha}'(t)$



Arc Length of a curve

How long is this curve?



Approximate with straight lines

Sum lengths of lines:

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Arc Length of a curve

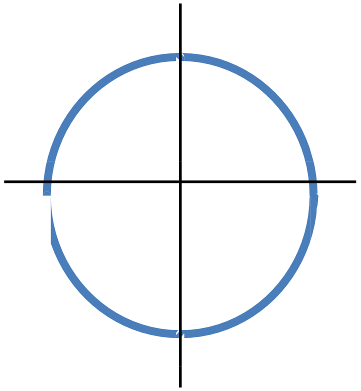
Let $\alpha: I \to R^3$ be a parameterized differentiable curve. The *arc length* of α from the point t_0 is:

$$s(t) = \int_{t_0}^{t} \left|\alpha'(t)\right| dt$$

$$= \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The arc length is an *intrinsic* property of the curve – does not depend on choice of parameterization

Examples



$\boldsymbol{\alpha}(t) = (a \cos(t), a \sin(t)), t \in [0,2\pi]$

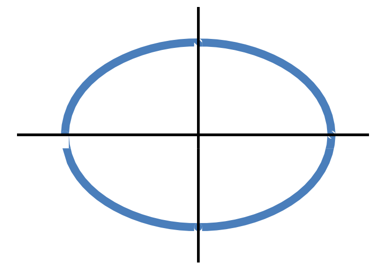
$\boldsymbol{\alpha}'(t) = (-a \sin(t), a \cos(t))$

$$L(\alpha) = \int_0^{2\pi} |\alpha'(t)| dt$$

$$= \int_0^{2\pi} \sqrt{a^2 \sin^2(t) + a^2 \cos^2(t)} dt$$

$$= a \int_0^{2\pi} dt = 2\pi a$$

Examples



$$\boldsymbol{\alpha}(t) = (a \cos(t), b \sin(t)), t \in [0,2\pi]$$

$$\boldsymbol{\alpha}'(t) = (-a \sin(t), b \cos(t))$$

$$L(\alpha) = \int_0^{2\pi} |\alpha'(t)| dt$$

$$= \int_0^{2\pi} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)} dt$$

$$= ??$$

No closed form expression for an ellipse

Arc Length Parameterization

- Re-parameterization $\mathbf{x}(u(t))$

$$\frac{d\mathbf{x}(u(t))}{dt} = \frac{d\mathbf{x}}{du}\frac{du}{dt} = \dot{\mathbf{x}}(u(t))\dot{u}(t)$$

Parameterize by arc length

A curve $\alpha: I \rightarrow R^3$ is *parameterized by arc length* if $|\alpha'(t)| = 1$, for all t

For such curves we have

$$s(t) = \int_{t_0}^{t} dt = t - t_0$$

- Re-parameterization $\mathbf{x}(u(t))$

$$\frac{d\mathbf{x}(u(t))}{dt} = \frac{d\mathbf{x}}{du}\frac{du}{dt} = \dot{\mathbf{x}}(u(t))\dot{u}(t)$$

Curves: Local Properties

Defines local properties of curves

Local = properties which depend only on behavior in neighborhood of point

We will consider only curves parameterized by arc length

Curvature

Let $\boldsymbol{\alpha}: I \longrightarrow R^3$ be a curve parameterized by arc length s.
The *curvature* of $\boldsymbol{\alpha}$ at s is defined by:

$$|\alpha''(s)| = \kappa(s)$$

$\boldsymbol{\alpha}'(s)$ – the tangent vector at s

$\boldsymbol{\alpha}''(s)$ – the *change* in the tangent vector at s

$R(s) = 1/\kappa(s)$ is called the *radius of curvature* at s.

Examples

<u>Straight line</u>

$\boldsymbol{\alpha}(s) = us + v, \quad u,v \in R^2$

$\boldsymbol{\alpha}'(s) = u$

$\boldsymbol{\alpha}''(s) = \mathbf{0} \quad \rightarrow |\alpha''(s)| = 0$

<u>Circle</u>

$\boldsymbol{\alpha}(s) = (a\cos(s/a), a\sin(s/a)), \; s \in [0, 2\pi a]$

$\boldsymbol{\alpha}'(s) = (-\sin(s/a), \cos(s/a))$

$\boldsymbol{\alpha}''(s) = (-\cos(s/a)/a, -\sin(s/a)/a) \rightarrow |\alpha''(s)| = 1/a$

Normal Vector

$|\alpha'(s)|$ is the arc length

$\alpha'(s)$ is the tangent vector

$|\alpha''(s)|$ is the curvature

$\alpha''(s)$ is ?

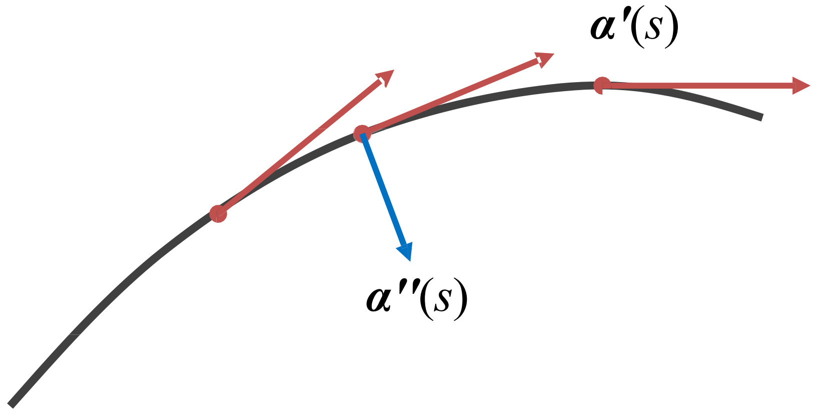
Normal Vector

$\boldsymbol{\alpha}'(s) = \boldsymbol{T}(s)$ - tangent vector

$|\boldsymbol{\alpha}'(s)|$ - arc length

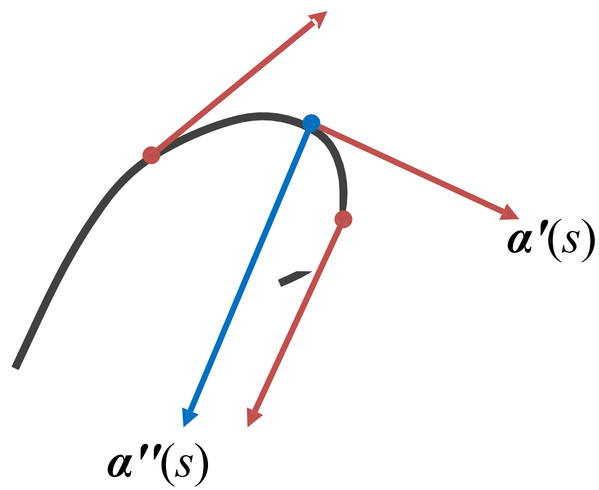
$\boldsymbol{\alpha}''(s) = \boldsymbol{T}'(s)$ - normal direction

$|\alpha''(s)|$ -- Curvature



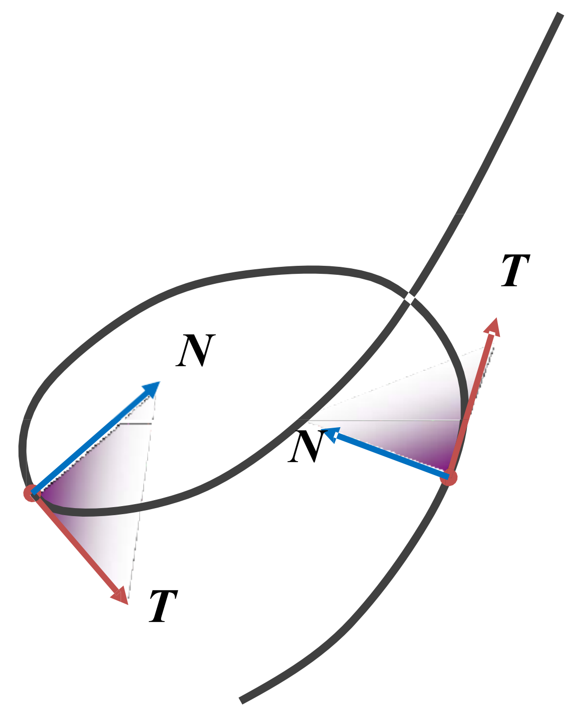
If $|\boldsymbol{\alpha}''(s)| \neq \mathbf{0}$, define $\boldsymbol{N}(s) = \boldsymbol{T}'(s)/|\boldsymbol{T}'(s)|$

Then $\boldsymbol{\alpha}''(s) = \boldsymbol{T}'(s) = \kappa(s)\boldsymbol{N}(s)$



The Osculating plane

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the *osculating plane* at s

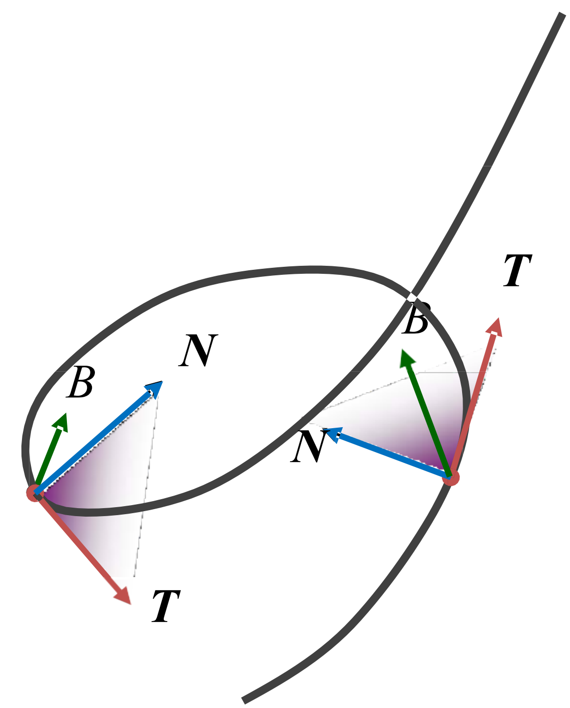


The Binormal Vector

For points s, s.t. $\kappa(s) \neq 0$, the *binormal vector* $\boldsymbol{B}(s)$ is defined as:

$$\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$$

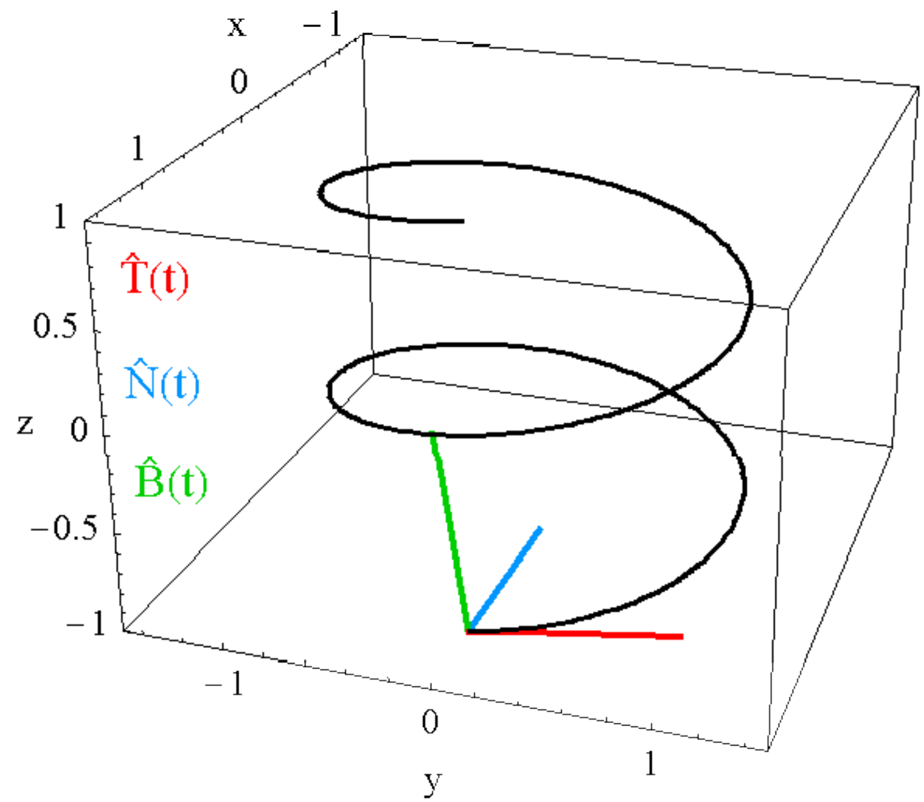
The binormal vector defines the osculating plane



The Frent Frame

$\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ form an orthonormal basis for R^3 called the *Frenet frame*

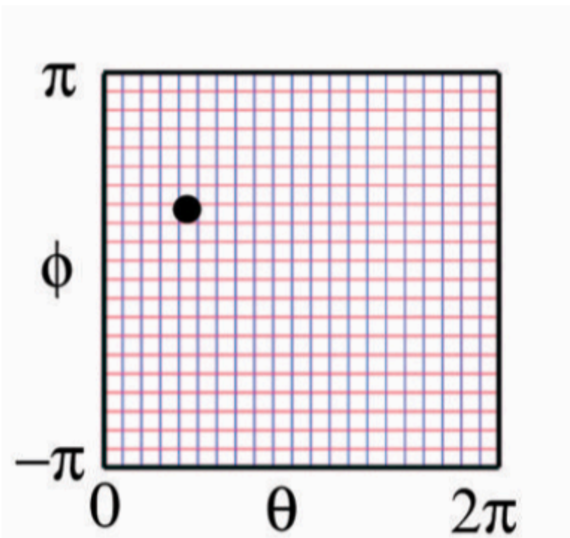
How does the frame change when the particle moves?

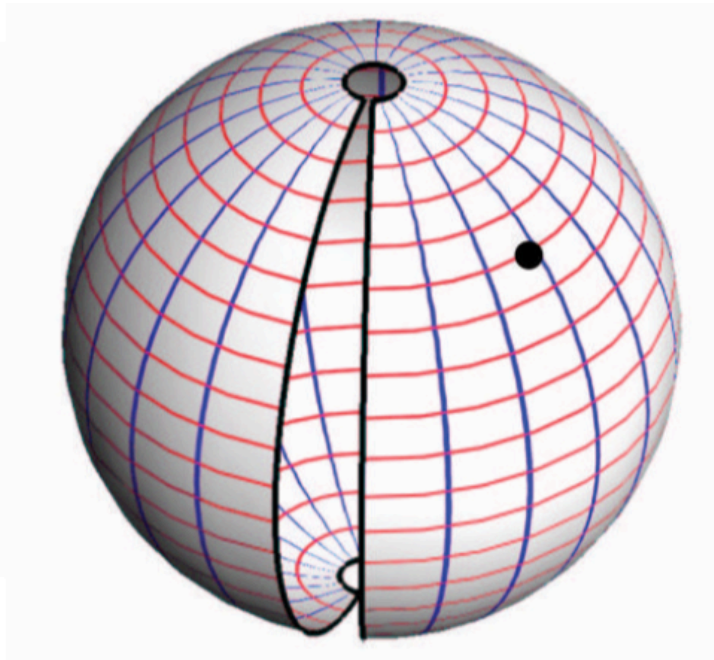


Comments

Length and curvature are invariant under rigid motion i.e. scaling, rotation, translation

Differential Geometry: Surfaces

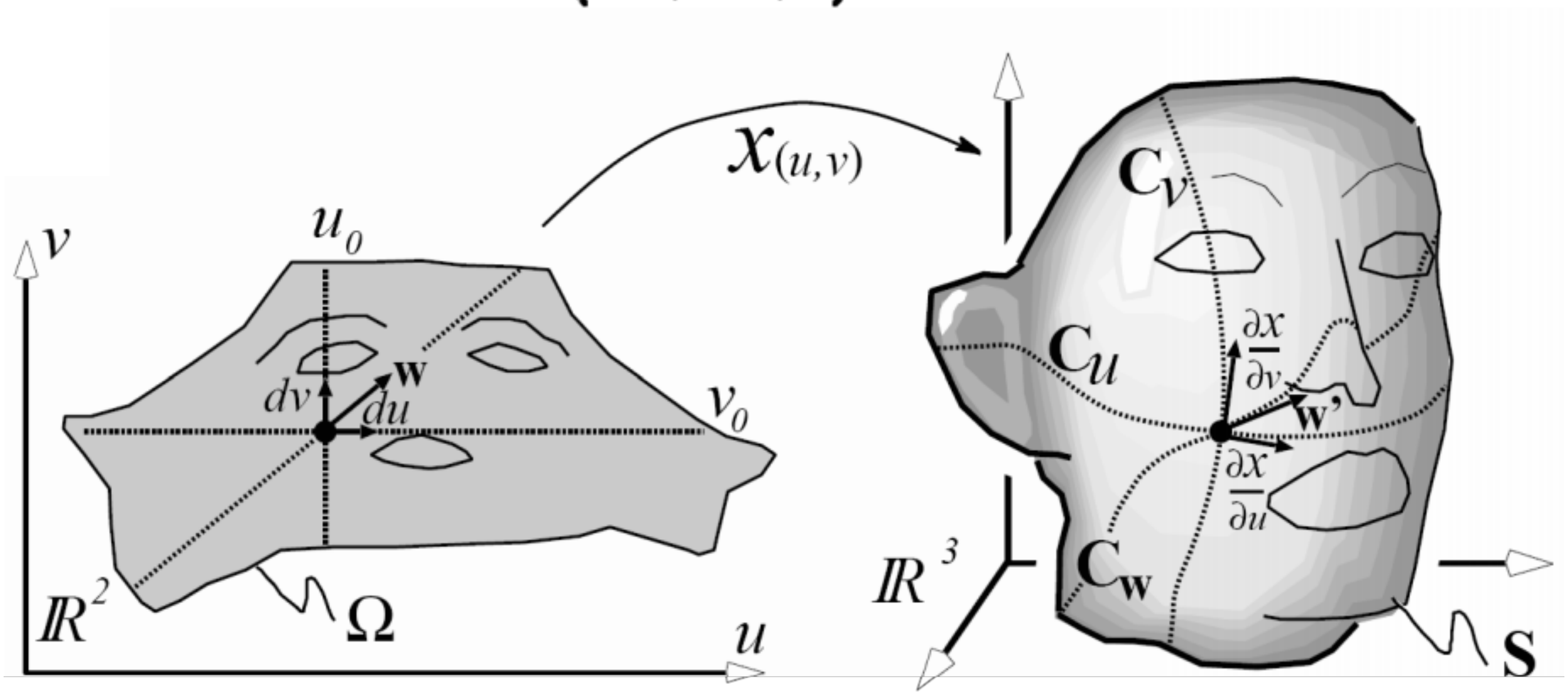




$$\mathbf{x}(\theta,\phi) = \begin{pmatrix} x(\theta,\phi) \\ y(\theta,\phi) \\ z(\theta,\phi) \end{pmatrix} = \begin{pmatrix} R\ \cos(\theta)\ \cos(\phi) \\ R\ \sin(\theta)\ \cos(\phi) \\ R\ \sin(\phi) \end{pmatrix}$$

Differential Geometry: Surfaces

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$

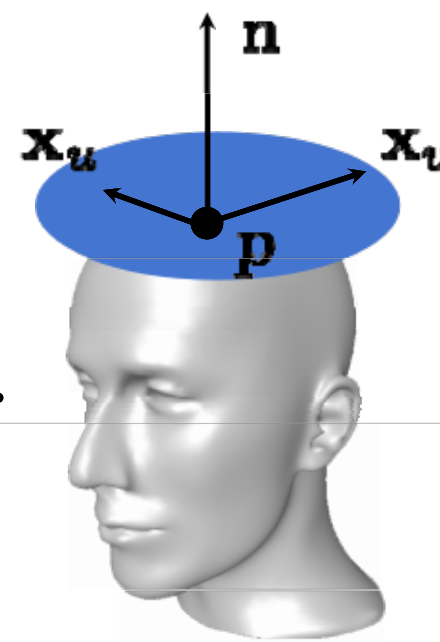


Differential Geometry: Surfaces

- Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$

- Normal vector

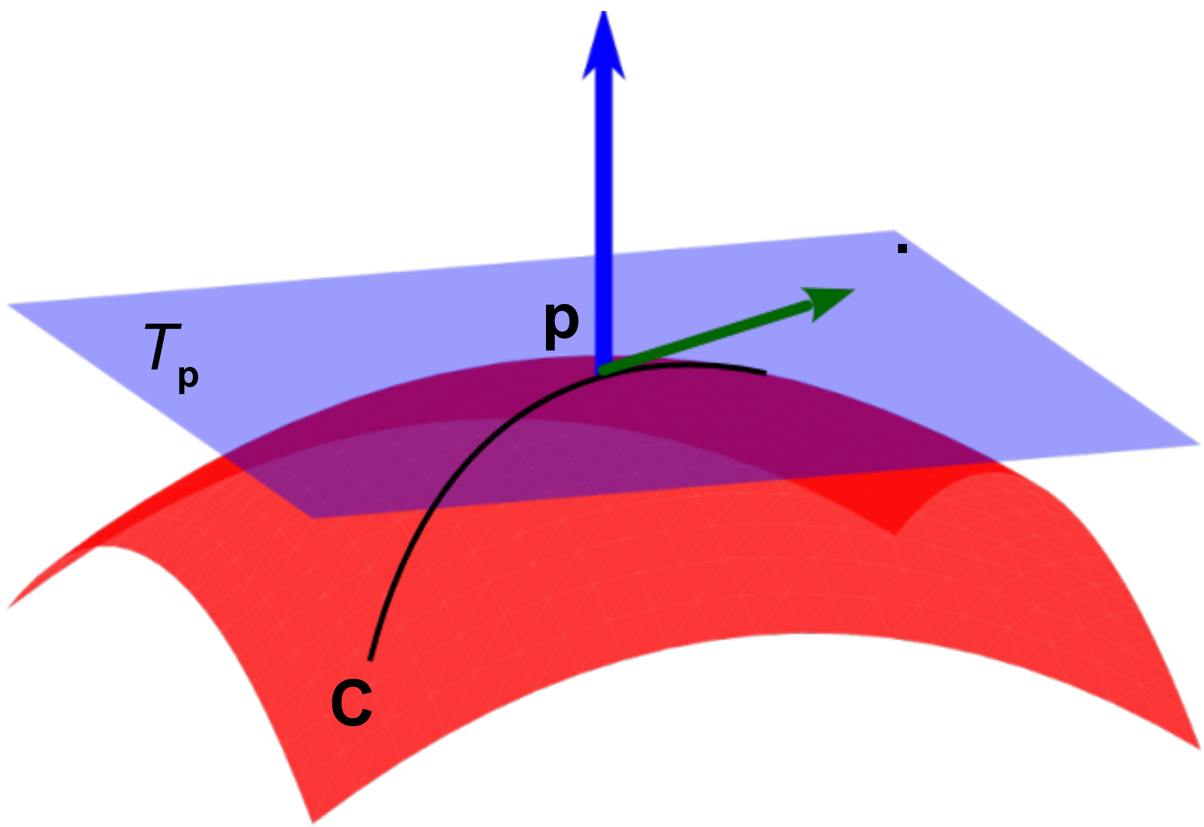


$$\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v) / \|\mathbf{x}_u \times \mathbf{x}_v\|$$

  - assuming regular parameterization, i.e.

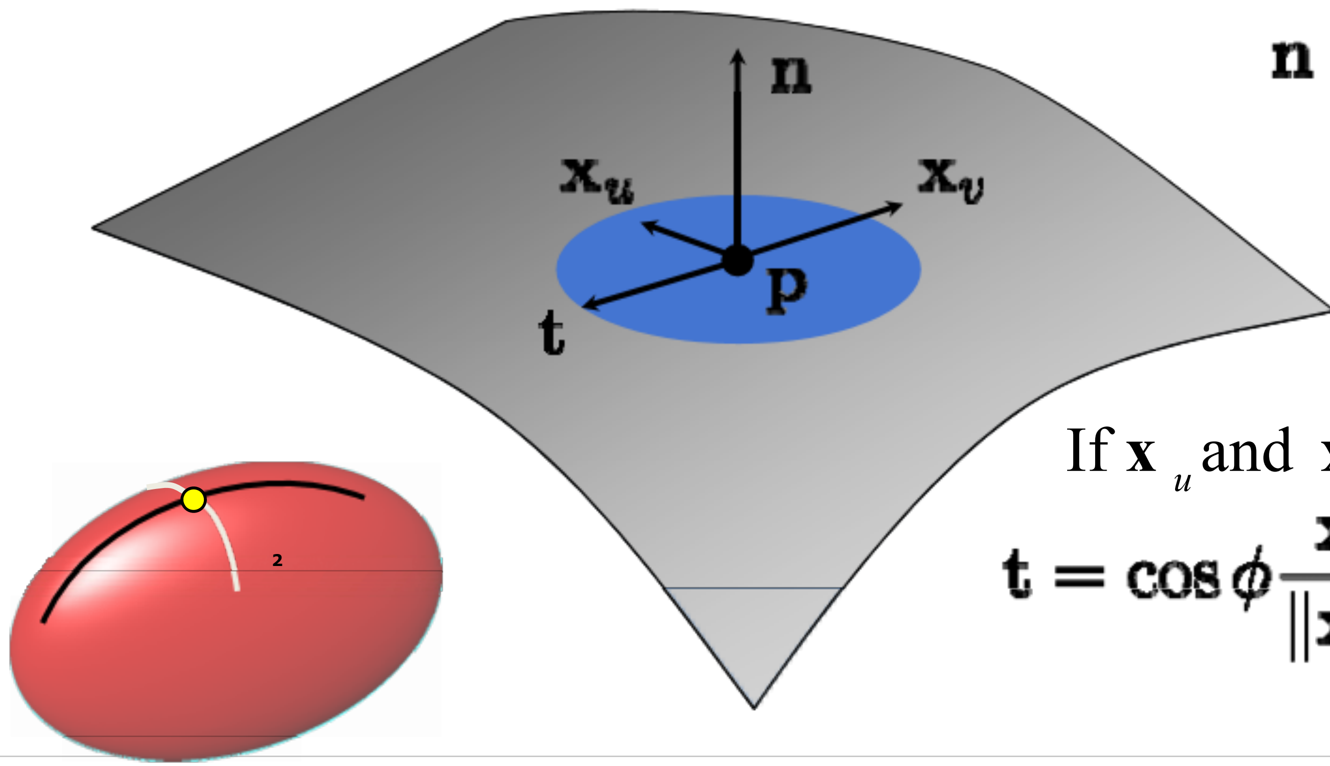
$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

Tangent space



Wikipedia

Normal Curvature



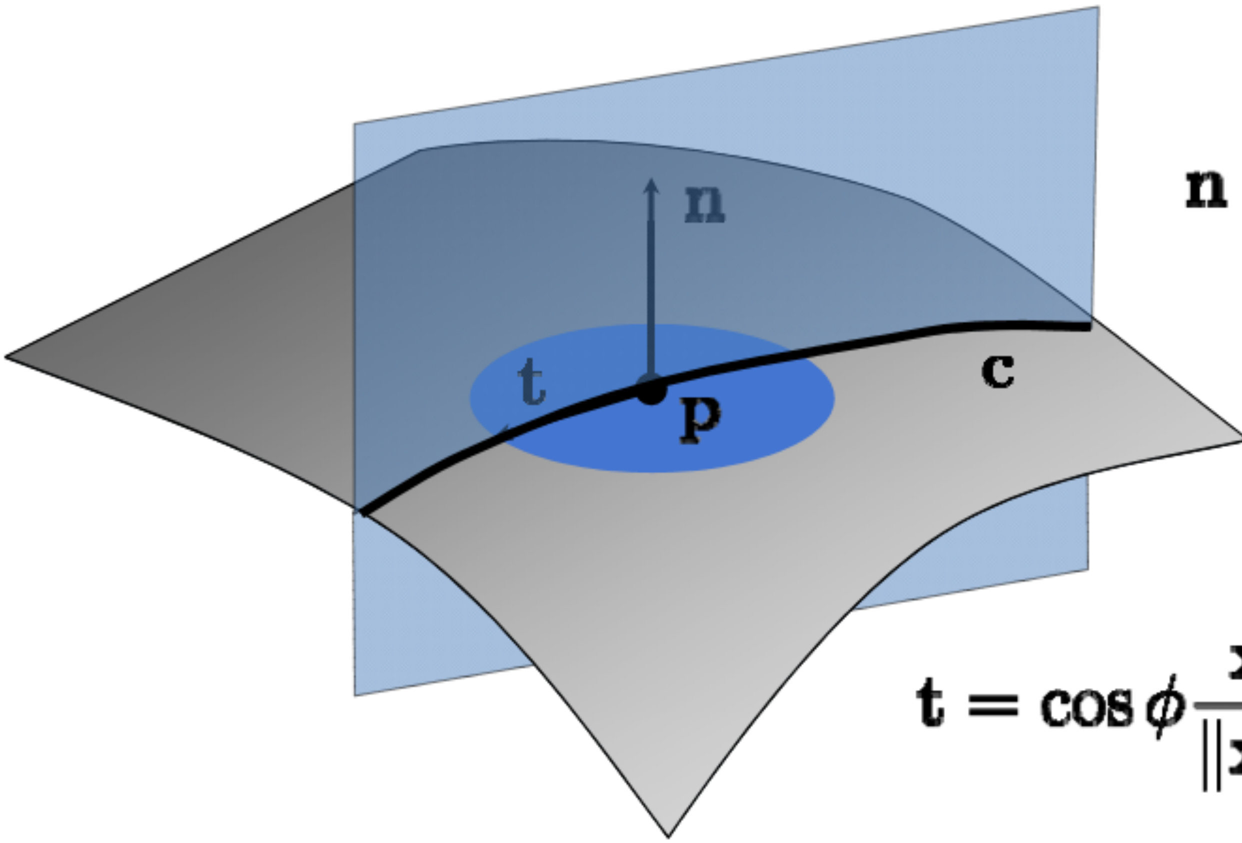
$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

If $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal:

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

elliptic

Normal Curvature

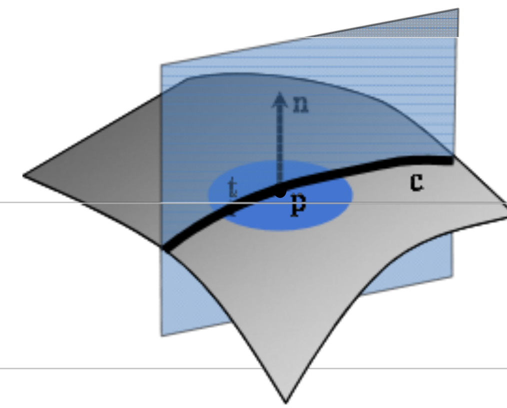


$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Surface Curvature

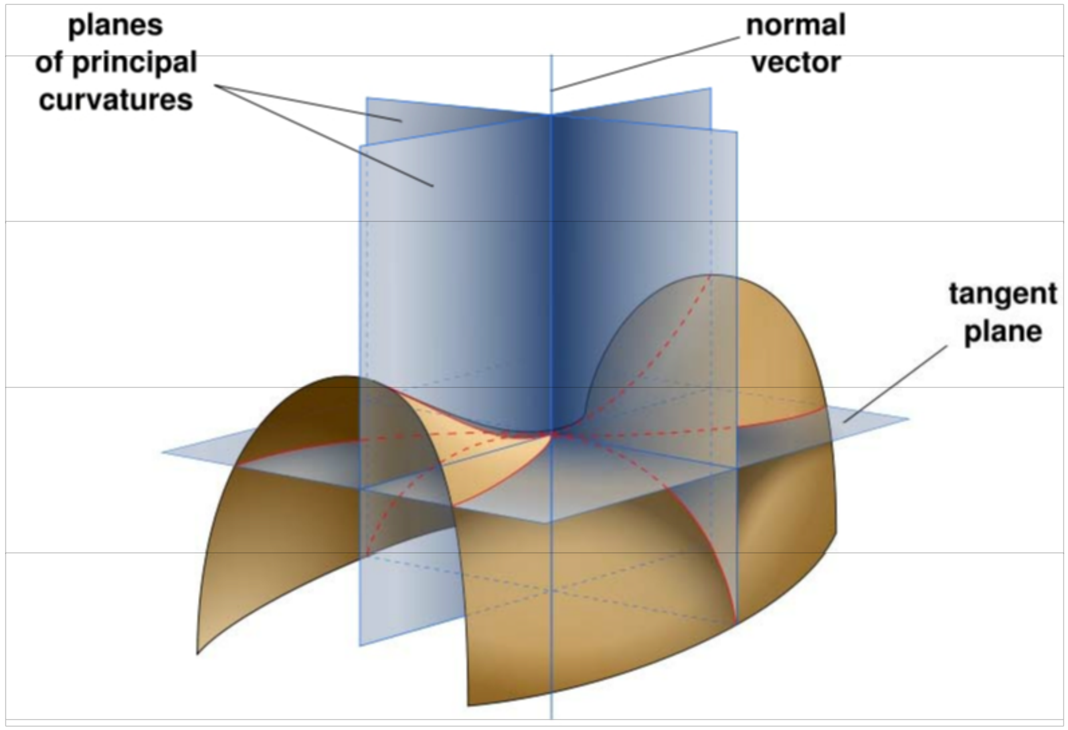
- Principal Curvatures
  - maximum curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - minimum curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$



Mean Curvature $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\phi) d\phi$

- Gaussian Curvature $K = \kappa_1 \cdot \kappa_2$

Principal Curvature

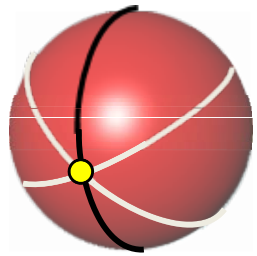


Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

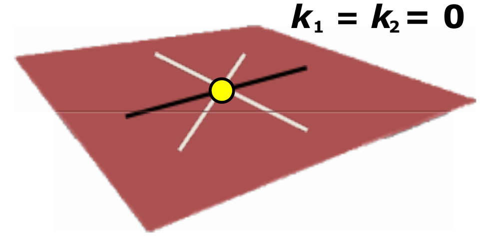
Surface Classification

Isotropic

Equal in all directions



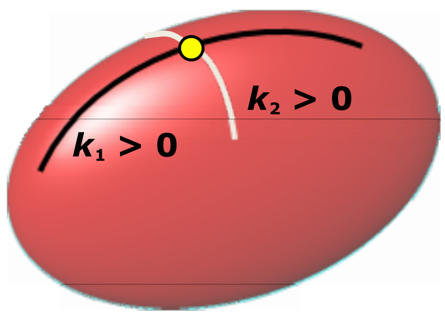
spherical



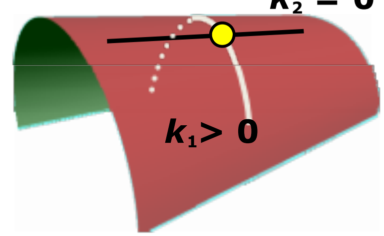
planar

Anisotropic

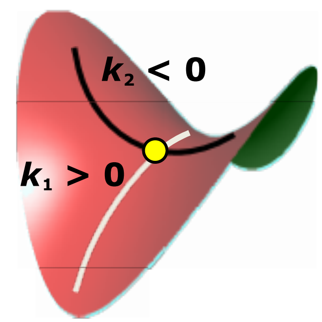
Distinct principal directions



elliptic
$K > 0$



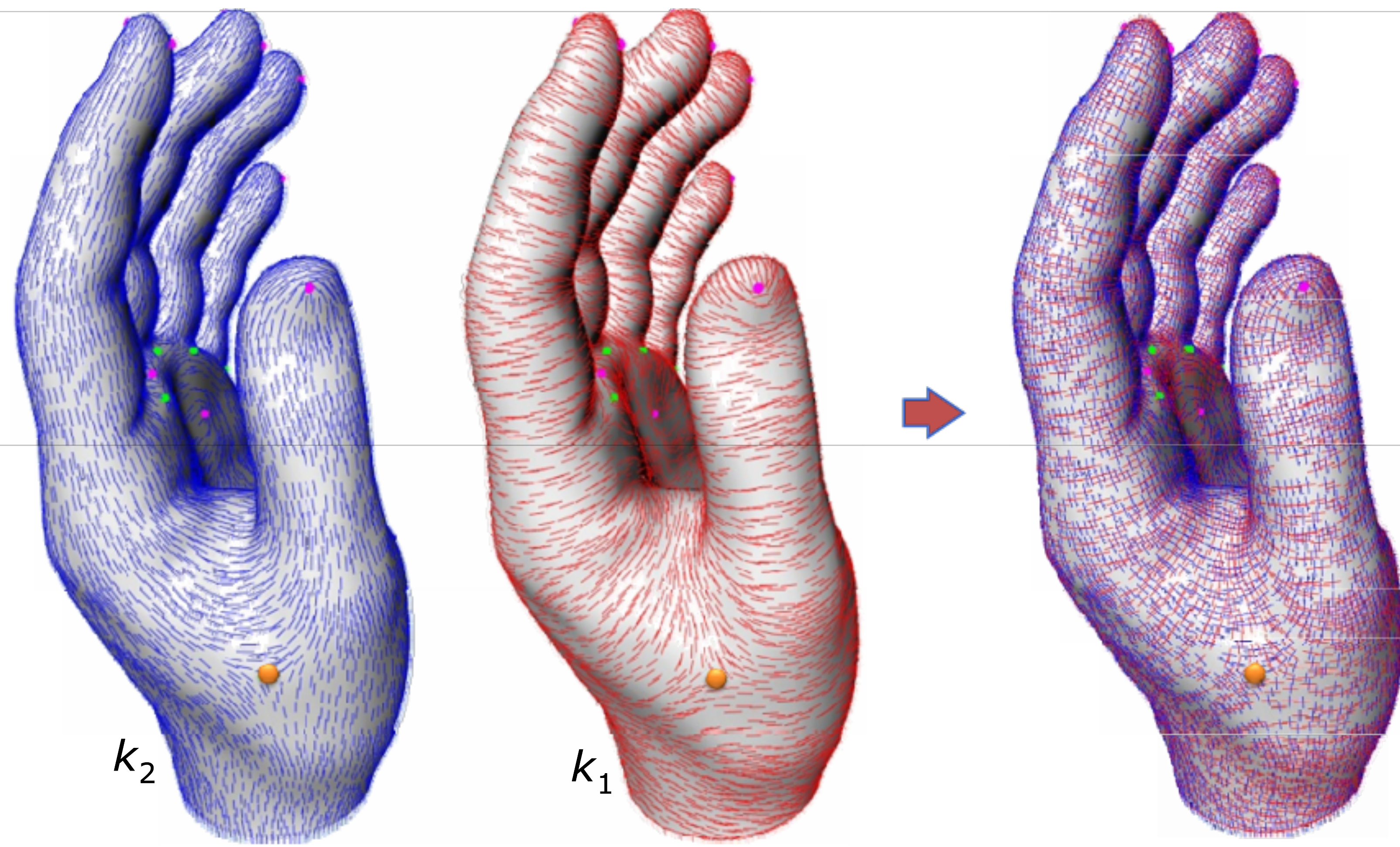
parabolic
$K = 0$



hyperbolic
$K < 0$

developable

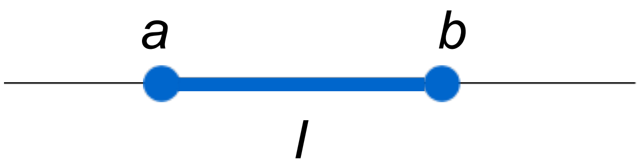
Principal Directions



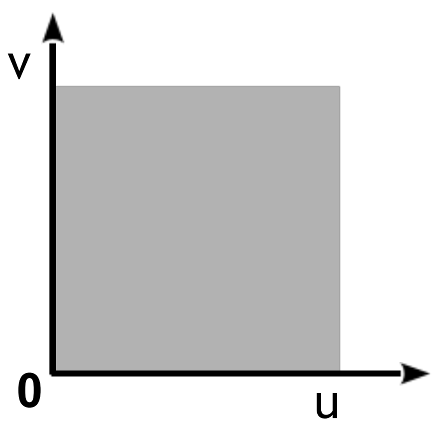
Curves and surfaces in 3D

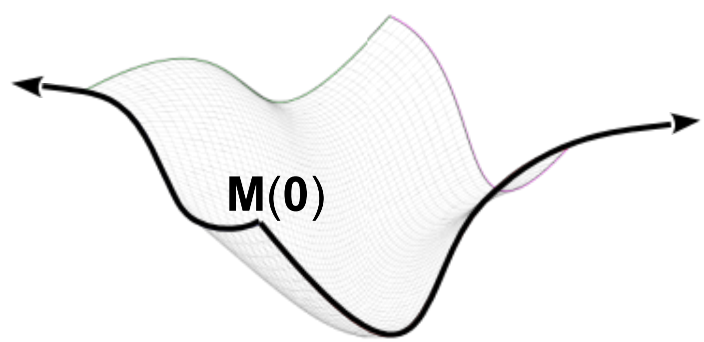
– A **curve** is a map **α** : $\mathbb{R} \rightarrow \mathbb{R}^3$ (or from some subset *I* of $\mathbb{R}$)

$\boldsymbol{\alpha}(t) = (x, y, z)$



– A **surface** is a map **M** : $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ (or from some subset Ω of $\mathbb{R}^2$)





$\mathbf{M}(u, v) = (x, y, z)$

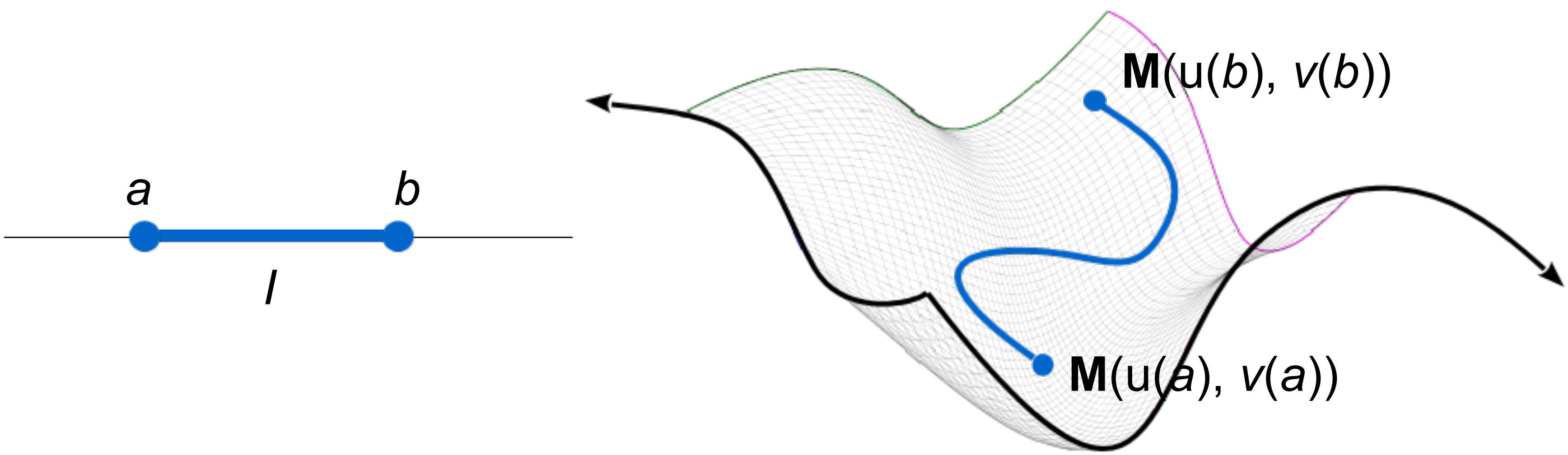
Wikipedia

Curve on a surface

- A curve **C** on surface **M** is defined as a map

$$\mathbf{C}(t) = \mathbf{M}(u(t), v(t))$$

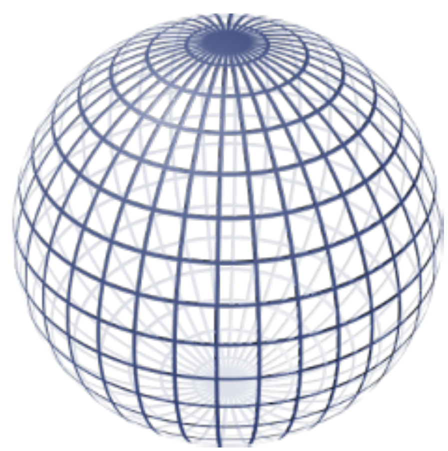
where u and v are smooth scalar functions



Wikipedia

Special cases

- The curve $\mathbf{C}(v) = (u_0, v)$ for constant u_0 is called a *u*−**curve**
- The curve $\mathbf{C}(u) = (u, v_0)$ for constant v_0 is called a *v*−**curve**
- These are collectively called **coordinate curves**
- Example: coordinate curves (θ-curves and φ-curves) on a sphere

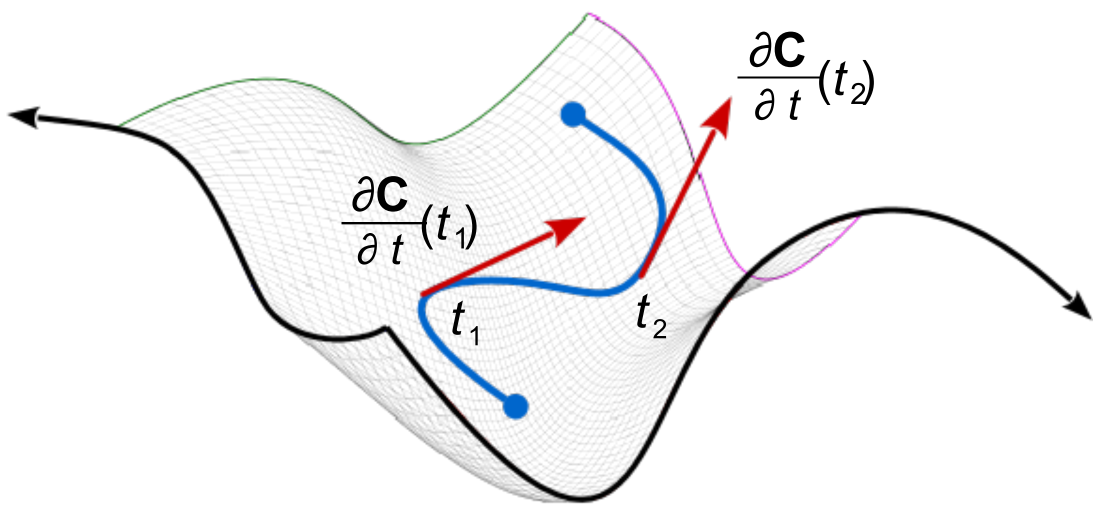


Wikipedia

Tangent vector

- The **tangent vector** to the surface curve **C** at t can be found by the chain rule

$$\frac{\partial \mathbf{C}}{\partial t} = \frac{\partial \mathbf{M}}{\partial u}\frac{d u}{d t} + \frac{\partial \mathbf{M}}{\partial v}\frac{d v}{d t}$$



Wikipedia

Thank You