SMAI-M20-L27: Illustrative Numerical Problems

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Class Review

MLP

1. Number of parameters
2. What can it solve?
3. What can it guarantee?

VC Dimension

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Recap:

- **Supervised Learning:** Formulation, Conceptual Issues, Concerns etc.
- **Classifiers:** (i) Nearest Neighbour, (ii) Notion of a Linear Classifier (iii) Perceptrons (iv) Bayesian Optimal Classifier (v) Logistic Regression (vi) Multiclass classification architectures (v) SVMs (hard margin, soft margin, kernel)(vi) MLP
- **Dimensionality Reduction and Applications:** (i) Feature Selection and Extraction (ii) PCA (iii) LDA (iv) Eigen face
- **Matrix Factorization and Applications:** (i) SVD, (ii) Eigen Decomposition (iii) Matrix Completion (iv) LSI (v) Recommendations
- **Other Topics:**
  - Linear Regression
  - Probabilistic View, Bayesian View, MLE
  - Gradient Descent: Stochastic and Batch GD
  - Loss Functions and Optimization
  - Eigen Vector based optimization
  - Neuron model, Single Layer Perceptrons
  - Kernel Functions and Kernel Matrix

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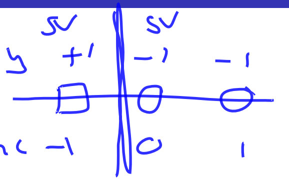
Consider a linearly separable data

x y x_2 y_2 x_3 y_3

$$(-1, +1), (0, -1), (+1, -1)$$



$wx+b=0$

1. Write the primal problem. Geometrically show the feasible region in a w, b 2D plane. Show the optimal w as -2 and b as -1. Geometrically validate the solution.
2. Write the dual objective $J(\alpha_1, \alpha_2, \alpha_3)$. And the constraint $\sum_i \alpha_i y_i = 0$. Show that the solution is $\alpha_1 = \alpha_2 = 2$ and $\alpha_3 = 0$. Which are the support vectors then?
3. Show that the dual solutiion also leads to the same w and b.

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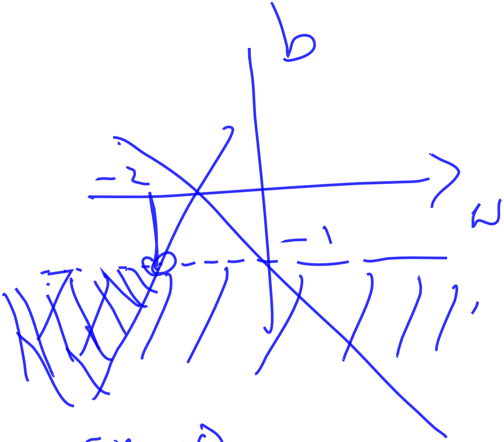
$$\min \frac{1}{2}\|w\|$$

$$y_i(w^T x_i + b) \geq 1$$

$$\min \frac{1}{2} w^2$$



$$+1(w(-1)+b) \geq 1$$

$$\boxed{-w+b \geq 1} \quad 1$$

$$(w^* \; b^*) = (-2, -1)$$

$$-2x + -1 = 0$$

$$\boxed{x = -1/2}$$

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$$\sum_{i=1}^{N} \alpha_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{d} \alpha_i \alpha_j y_i y_j x_i^T x_j$$

$$\alpha_1 + \alpha_2 + \alpha_3 - \frac{1}{2}\left(\alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3(-1)\right)$$

$$\boxed{\alpha_1 + \alpha_2 + \alpha_3 - \frac{1}{2}(\alpha_1 + \alpha_3)^2} \quad — 1$$

$$\sum_i \alpha_i y_i = 0 \implies \alpha_1 - \alpha_2 - \alpha_3 = 0$$

$$\alpha_2 = \alpha_1 - \alpha_3 \quad — 1$$

$$\boxed{2\alpha_1 - \frac{1}{2}(\alpha_1 + \alpha_3)^2} \implies \alpha_3 = 0 \quad —$$

$$\boxed{wx + b = +1}$$

3

$$2\alpha_1 - \frac{1}{2}\alpha_1^2$$

$$\boxed{\max -\frac{1}{2}(\alpha_1 - 2)^2 + 2}$$

$\alpha_1 = 2$

$$\alpha_1 = 2 \quad \alpha_2 = 2 \quad \alpha_3 = 0$$

$$\boxed{w = \sum_i \alpha_i y_i x_i}$$

$w = -2$

$b = -1$

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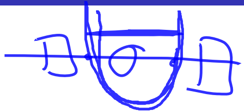
Illustrative Problem -II

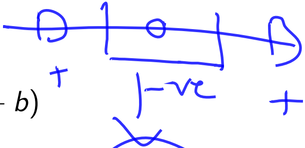
Consider a linearly in-separable 1D data



$$(-1, +1), (0, -1), (+1, +1)$$

Demonstrate how the problem can be solved with a quadratic kernel $(\mathrm{p}^T q + 1)^2$

1. Write the dual objective with the constraint $\sum_i \alpha_i y_i = 0$
2. Substitute for α_2 from the constraint in the objective. (i.e., eliminate α_2)
3. Differentiate wrt α_1 and α_3 and equate to zero. Solve to obtain $\alpha_1 = \alpha_3 = 1$ and $\alpha_2 = 2$
4. For a new sample the decision is



$$sign(\sum_i \alpha_i y_i \kappa(x_i, x) + b)$$

Simplify and get this expression as $2x^2 - 1$. (Assume $b = -1$. Indeed, you can find b yourself.)

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$$\sum_{i=1}^{N} \alpha_i - \sum_i \sum_j \alpha_i \alpha_j y_i y_j \boxed{K(x_i, x_j)}$$

$$K = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$K(p, q) = (\bar{p} q + 1)^2$$

-1, 0, $+1$

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$$\alpha_1 + \alpha_2 + \alpha_3 - \frac{1}{2}\left(4\alpha_1^2 + \alpha_2^2 + 4\alpha_3^2 - \alpha_2\alpha_1 - \alpha_2\alpha_3 \ldots\right)$$

$$\alpha_1 - \alpha_2 + \alpha_3 = 0 \qquad \sum_i \alpha_i y_i = 0$$

$$\alpha_2 = \alpha_1 + \alpha_3$$

$$2\alpha_1 + 2\alpha_3 - \left(2\alpha_1^2 + 2\alpha_3^2 - \frac{1}{2}(\alpha_1 + \alpha_3)^2\right)$$

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$\frac{\partial}{\partial \alpha_1} = 0$

$$2 - 3\alpha_1 + \alpha_3 = 0 \qquad \frac{\partial}{\partial \alpha_3} = 0$$

$$2 + \alpha_1 - 3\alpha_3 = 0$$

$(*)$

$$\boxed{\alpha_1 = 1 \qquad \alpha_3 = 1 \qquad \alpha_2 = 2}$$

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$$y_i(w^T x + b) \geq +1$$

$$\boxed{y_i(w^T + b) = +1}$$

Illustrative Problem - III

VC dimension of a function class is the largest set that can be shattered by a member of the function class.

1. Interval $[\alpha_1, \alpha_2]$ on real line. (i.e., $+$ve is inside $[\alpha_1, \alpha_2]$ else negative)
2. Axis parallel rectangles in 2D. (i.e., positive inside and negative outside)

[1]

[1] VC Dimension is d, if a set of size d can be shattered completely and no set of size $d + 1$ can be shattered.

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Illustrative Problem - IV

Consider the ExOR Problem:

x_1	x_2	y
-1	-1	-1
-1	+1	+1
+1	-1	+1
+1	+1	-1

1. Consider a kernel $(p^T q + 1)^2$ and construct the Kernel matrix
2. Write the dual objective in terms of α_1, α_2, α_3 and α_4.
3. Differentiate wrt α_1, α_2, α_3 and α_4 and equate to zero. You get four equations.
4. Solve (guess) the values for α_1, α_2, α_3 and α_4. Same?
5. We know $\phi() = [1, x_1^2, \sqrt{2}x_1x_2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2]^T$. Find w as $\sum_i \alpha_i y_i \phi(x_i)]$. Does it solve the exor? why?

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What Next:? (next three)

1. NN Architectures and NN Learning
2. Programming for Deep Learning.
3. Beyond Supervised Learning