SMAI-M20-L28: Illustrative Numerical Problems (Cont.)

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Class Review

Backpropagation

1. Convergence
2. Relationship between Loss and Acccuracy
3. Accuracy on Train Set and Test Set
4. Characterizing Optimization Problem in MLP

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Recap:

- **Supervised Learning:** Formulation, Conceptual Issues, Concerns etc.
- **Classifiers:** (i) Nearest Neighbour, (ii) Notion of a Linear Classifier (iii) Perceptrons (iv) Bayesian Optimal Classifier (v) Logistic Regression (vi) Multiclass classification architectures (v) SVMs (hard margin, soft margin, kernel)(vi) MLP (vii) BP
- **Dimensionality Reduction and Applications:** (i) Feature Selection and Extraction (ii) PCA (iii) LDA (iv) Eigen face
- **Matrix Factorization and Applications:** (i) SVD, (ii) Eigen Decomposition (iii) Matrix Completion (iv) LSI (v) Recommendations
- **Other Topics:**
  - Linear Regression
  - Probabilistic View, Bayesian View, MLE
  - Gradient Descent: Stochastic and Batch GD
  - Loss Functions and Optimization
  - Eigen Vector based optimization
  - Neuron model, Single Layer Perceptrons
  - Kernel Functions and Kernel Matrix

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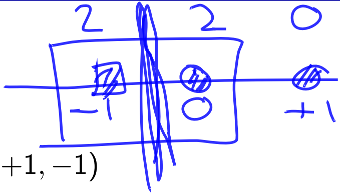
Illustrative Problem - I (Done: Q?)



Consider a linearly separable data

$$(-1,+1),(0,-1),(+1,-1)$$

$x = -1/2$

1. Write the primal problem. Geometrically show the feasible region in a w, b 2D plane. Show the optimal w as -2 and b as -1. Geometrically validate the solution.

   $wx + b = 1$

2. Write the dual objective $J(\alpha_1, \alpha_2, \alpha_3)$. And the constraint $\sum_i \alpha_i y_i = 0$. Show that the solution is $\alpha_1 = \alpha_2 = 2$ and $\alpha_3 = 0$. Which are the support vectors then?
3. Show that the dual solutiion also leads to the same w and b.

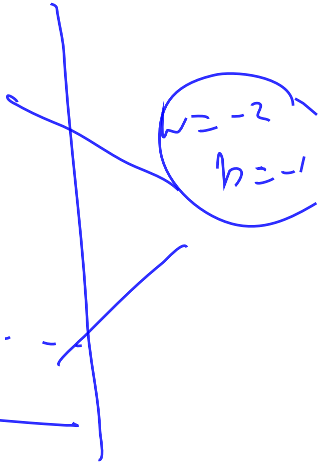
$w = \sum_{1} \alpha_1 y_1 x_1$ $\quad$ $w = -2 \quad b = -1$

- Primal

  $\min \|w\|$

- Dual

  $\max \sum_{i} \alpha_i - \sum\sum \alpha_i \cdots$



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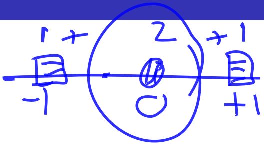
Illustrative Problem -II (Done; Q?)

Consider a linearly in-separable 1D data



$$(-1,+1),(0,-1),(+1,+1)$$

Demonstrate how the problem can be solved with a quadratic kernel $(\mathrm{p}^T q+1)^2$

if $x^2 < 0.5$ -1
else $+1$

1. Write the dual objective with the constraint $\sum_i \alpha_i y_i = 0$
2. Substitute for α_2 from the constraint in the objective. (i.e., eliminate α_2)
3. Differentiate wrt α_1 and α_3 and equate to zero. Solve to obtain $\alpha_1 = \alpha_3 = 1$ and $\alpha_2 = 2$ => All three are SV
4. For a new sample the decision is

$2x^2 - 1 = 0$
$x^2 = 1/2$

$$sign(\sum_i \alpha_i y_i \kappa(x_i, x) + b)$$

$x_i^T x$

Simplify and get this expression as $2x^2 - 1$. (Assume $b = -1$. Indeed, you can find b yourself.)

$K(x_i, x)$

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$$\text{sign}(w^T x + b) - \text{lin}$$

$$\boxed{\text{sign}\left(\sum_{i=1}^{\cancel{N}\ |sv|} \alpha_i y_i K(x_i, x) + b\right)}$$

$$w = \sum_{i=1}^{N} \alpha_i y_i x_i$$

$|sv| = 100$

100 times

slow

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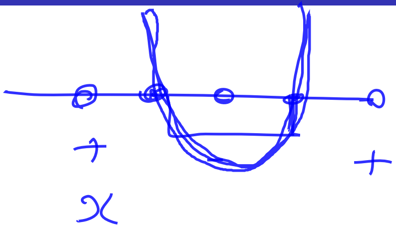
$$2x^2 - 1 = 0$$



$$x^2 = 1/2$$

$$\underline{\underline{x \longrightarrow f(x)}}$$

0	0	0
-1	0	$+1$

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Illustrative Problem - III

x_1, x_2

Consider the ExOR Problem:

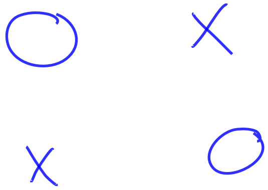
x_1	x_2	y
-1	-1	-1
-1	+1	+1
+1	-1	+1
+1	+1	-1

$+1 \quad (p^T p + 1)^2 = 9$

$-1 \quad (p^T q + 1)^2 = 1$

-1

$+1$



1. Consider a kernel $(p^T q + 1)^2$ and construct the Kernel matrix
2. Write the dual objective in terms of α_1, α_2, α_3 and α_4.
3. Differentiate wrt α_1, α_2, α_3 and α_4 and equate to zero. You get four equations.
4. Solve (guess) the values for α_1, α_2, α_3 and α_4. Same?
5. We know $\phi() = [1, x_1^2, \sqrt{2}x_1x_2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2]^T$. Find w as $\sum_i \alpha_i y_i \phi(x_i)]$. Does it solve the exor? why?

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$$\begin{bmatrix} 9 & 1 & 1 & 1 \\ 1 & 9 & 1 & 1 \\ 1 & 1 & 9 & 9 \\ 1 & 1 & 1 & 9 \end{bmatrix} 4 \times 4 \qquad \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j y_i y_j \underbrace{x_i^T x_j}_{K(x_i, x)}$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \frac{1}{2}\Big(9\alpha_1^2 - 2\alpha_1\alpha_2 - 2\alpha_1\alpha_3$$

$$+ 2\alpha_1\alpha_4 + 9\alpha_2^2 + 2\alpha_2\alpha_3 - 2\alpha_2\alpha_4$$

$$+ 9\alpha_3^2 - 2\alpha_3\alpha_4 + 9\alpha_4^2 \Big)$$

$$9\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 1$$

$$-\alpha_1 + 9\alpha_2 + \alpha_3 - \alpha_4 = 1$$

$$-\alpha_1 + \alpha_2 + 9\alpha_3 - \alpha_4 = 1$$

$$\alpha_1 - \alpha_2 - \alpha_3 + 9\alpha_4 = 1$$

$$\boxed{\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{8}}$$

$$\phi = [1,\ x_1^2,\ \sqrt{2}x_1x_2,\ x_2^2,\ \sqrt{2}x_1,\ \sqrt{2}x_2]^T$$

$$\begin{matrix} -1 & -1 \\ -1 & +1 \end{matrix}$$

$$w = \sum \alpha_i y_i \phi(x_i)$$

$$w = \frac{1}{8}\left[-1\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \\ 1 \\ -\sqrt{2} \\ -\sqrt{2} \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \\ 1 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} \cdots \right] = \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\sqrt{2}x_1x_2}$

$\underline{w^T\phi(x)}$

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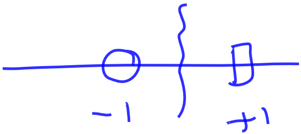
x_1	x_2	y
-1	-1	-1
-1	$+1$	$+1$
$+1$	-1	$+1$
$+1$	$+1$	-1

$\phi() \Rightarrow$

$x_3 = x_1 x_2$	y
$+1$	-1
-1	$+1$
-1	$+1$
$+1$	-1



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$$\mathrm{sign}\left(\sum_{i=1}^{|SV|} \alpha_i y_i K(x_i, x) - b\right)$$

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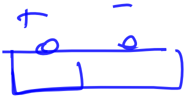
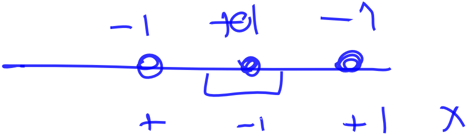
Illustrative Problem - IV

VC dimension of a function class is the largest set that can be shattered by a member of the function class.

1. Interval $[\alpha_1, \alpha_2]$ on real line. (i.e., +ve is inside $[\alpha_1, \alpha_2]$ else negative) 2
2. Axis parallel rectangles in 2D. (i.e., positive inside and negative outside) 4
3. Convex Polygons in 2D (inside positive and outside negative)
4. Union of intervals in 1D.

[1]





[1] VC Dimension is d, if a set of size d can be shattered completely and no set of size $d + 1$ can be shattered.

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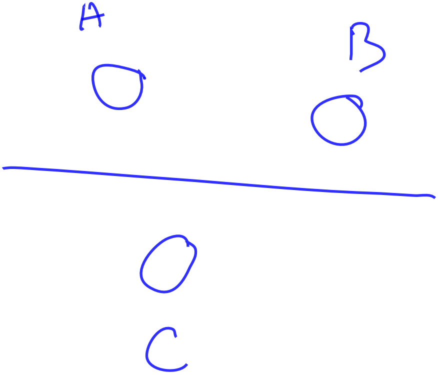

Set = 3 member

F. Class = {lin}

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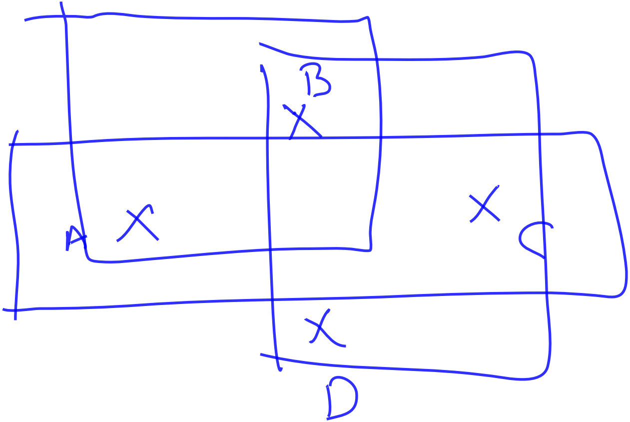
$$\boxed{\begin{cases} + & \text{if } x > \alpha \\ - & \text{else} \end{cases}} \to 1$$

+ −

o •

− +

✗ ✓



Any set of 5 can not be drawn

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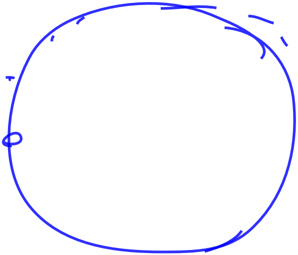
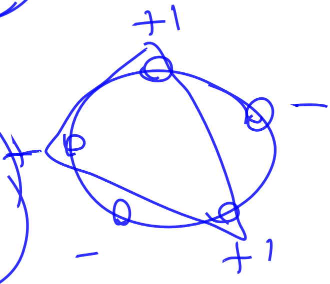


100

VC Dim d

Convex pol $= \infty$



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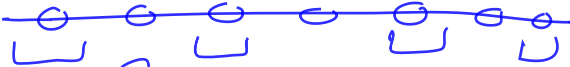
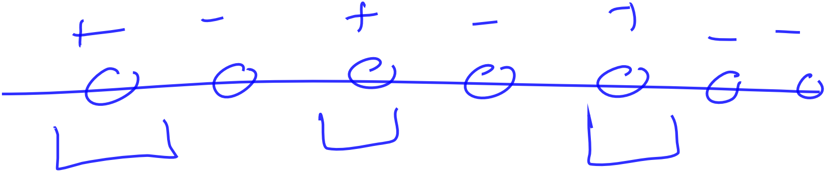


$$\bigcup_i [\alpha_i, \alpha_0]$$



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Discussion Point (Repeat, if time permits)

(Advanced; Out of Syllabus!!) We know that a new kernel can be defined in terms of existing kernels:

$$\sum_{i=1}^{K} \alpha_i \kappa(\cdot, \cdot)$$

Then why don't we formulate the overall learning problem in SVM, including that of learning these α_i

1. Discuss why it is a good idea?
2. How do we use it for "fusing" different features?
3. Why do we limit to $\sum$?

See some of the works relevant[2] and [3]. Read later.

[2] http://manikvarma.org/pubs/varma07c.pdf

[3] https://cvit.iiit.ac.in/images/ConferencePapers/2009/Rakesh09More.pdf

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Discussion Point (Repeat, if time permits)

(Advanced; Out of Syllabus!!)
We know that linear SVMs are superefficient (compared to K-SVMs).
Can we find a $\phi()$ corresponding to a Kernel and solve the problem as

$$\mathbf{w}^T \phi(x)$$

Indeed, this may become difficult for many kernels (eg. RBFs). **why?**
Can we find a finite dimensional approximation of $\phi()$? How does it help in speeding up SVM with no major reduction in accuracy?
read [4] and [5] later.

1. Discuss why it is a good idea?
2. Suggest an application where speed matters (eg. in the reference is that ofobject detection).

[4] https://cvit.iiit.ac.in/images/ConferencePapers/2010/Sreekanth10Generalized.pdf
[5] https://www.robots.ox.ac.uk/ vgg/publications/2011/Vedaldi11/vedaldi11.pdf

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What Next:?

Friday Lecture Session:

1. No formal online session
2. Submit the solution to the five problems we solved in class (11am on Friday). Write neatly with sufficient explanations and equations/pictures/sketches/details.

Next?

1. NN Architectures and NN Learning
2. Programming for Deep Learning.
3. Beyond Supervised Learning