Instructions:

- This assignment is meant to help you grok certain concepts we will use in the course. Please don't copy solutions from any sources.
- Avoid verbosity.
- Questions marked with * are relatively difficult. Don't be discouraged if you cannot solve them right away!
- The assignment needs to be written in latex using the attached tex file. The solution for each question should be written in the solution block in space already provided in the tex file. **Handwritten assignments will not be accepted.**

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1. Suppose, a transformation matrix A, transforms the standard basis vectors of R^3 as follows :

$$\begin{bmatrix}1\\0\\0\end{bmatrix} => \begin{bmatrix}3\\8\\0\end{bmatrix};\begin{bmatrix}0\\1\\0\end{bmatrix} => \begin{bmatrix}-4\\9\\7\end{bmatrix};\begin{bmatrix}0\\0\\1\end{bmatrix} => \begin{bmatrix}-1\\2\\6\end{bmatrix}$$

   (a) If the volume of a hypothetical parallelepiped in the un-transformed space is $100units^3$ what will be volume of this parallelepiped in the transformed space?

Solution:

   (b) What will be the volume if the transformation of the basis vectors is as follows :

$$\begin{bmatrix}1\\0\\0\end{bmatrix} => \begin{bmatrix}1\\1\\3\end{bmatrix};\begin{bmatrix}0\\1\\0\end{bmatrix} => \begin{bmatrix}-1\\2\\0\end{bmatrix};\begin{bmatrix}0\\0\\1\end{bmatrix} => \begin{bmatrix}0\\2\\2\end{bmatrix}$$

Solution:

   (c) Comment on the uniqueness of the second transformation.

Solution:

2. If R^3 is represented by following basis vectors : $\begin{bmatrix}5\\2\\0\end{bmatrix}, \begin{bmatrix}8\\7\\-11\end{bmatrix}, \begin{bmatrix}-4\\-9\\3\end{bmatrix}$

(a) Find the representation of the vector $\begin{pmatrix}-3 & 1 & -2\end{pmatrix}^T$ (as represented in standard basis) in the above basis.

Solution:

(b) We know that, orthonormal basis simplifies this transformation to a great extent. What would be the representation of vector $\begin{pmatrix}-3 & 1 & -2\end{pmatrix}^T$ in the orthogonal basis represented by : $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}.$

Solution:

(c) Comment on the advantages of having orthonormal basis.

Solution:

3. A square matrix is a Markov matrix if each entry is between zero and one and the sum along each row is one. Prove that a product of Markov matrices is Markov.

Solution:

4. Give an example of a matrix A with the following three properties:
   (a) A has eigenvalues -1 and 2.
   (b) The eigenvalue -1 has eigenvector

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \tag{1}$$

   (c) The eigenvalue 2 has eigenvector

$$\begin{pmatrix}1\\1\\0\end{pmatrix} and \begin{pmatrix}0\\1\\1\end{pmatrix} \tag{2}$$

Solution:

5. Perform the Gram-Schmidt process on each of these basis for $\mathbb{R}^3$. And convert the resulting orthogonal basis into orthonormal basis.

   (a) $\langle \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \rangle$

Solution:

   (b) $\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \rangle$

Solution:

6. Suppose, every year, 4% of the birds from Canada migrate to the US, and 1% of them travel to Mexico. Similarly, every year, 6% of the birds from US migrate to Canada, and 4% to Mexico. Finally, every year 10% of the birds from Mexico migrate to the US, and 0% go to Canada.

   (a) Represent the above probabilities in a transition matrix.

Solution:

   (b) Is it possible that after some years, the number of birds in the 3 countries will become constant?

Solution:

7. (a) Show that any set of four unique vectors in $\mathbb{R}^2$ is linearly dependent.

Solution:

   (b) What is the maximum number of unique vectors that a linearly independent subset of $\mathbb{R}^2$ can have ?

Solution:

8. (a) Determine if the vectors $\{v_1, v_2, v_3\}$ are linearly independent, where
$v_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$
Justify each answer

Solution:

(b) Prove that each set {f, g} is linearly independent in the vector space of all functions from $\mathbb{R}^+$to $\mathbb{R}$.

1. f(x) = x and g(x) = $\frac{1}{x}$
2. f(x) = cos(x) and g(x) = sin(x)
3. f(x) = e^x and g(x) = ln(x)

Solution:

9. Let t_θ be

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3}$$

(a) Show that $t_{\theta_1+\theta_2} = t_{\theta_1} * t_{\theta_2}$ (* here stands for matrix multiplication).

Solution:

(b) Show that $t_\theta^{-1} = t_{-\theta}$.

Solution:

10. Given matrix has distinct eigenvalues

$$\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

(a) Diagonalize it.

Solution:

(b) Find a basis with respect to which this matrix has that diagonal representation

Solution:

(c) Find the matrices P and P^{-1} to effect the change of basis.

Solution:

11. * **Induced Matrix Norms**

In case you didn't already know, a norm $\|.\|$ is any function with the following properties:

1. $\|x\| \geq 0$ for all vectors x.
2. $\|x\| = 0 \iff x = \mathbf{0}$.
3. $\|\alpha x\| = |\alpha|\|x\|$ for all vectors x, and real numbers α.
4. $\|x + y\| \leq \|x\| + \|y\|$ for all vectors x, y.

Now, suppose we're given some vector norm $\|.\|$ (this could be L2 or L1 norm, for example). We would like to use this norm to measure the size of a matrix A. One way is to use the corresponding induced matrix norm, which is defined as $\|A\| = \sup_x\{\|Ax\| : \|x\| = 1\}$.

E.g.: $\|A\|_2 = \sup_x\{\|Ax\|_2 : \|x\|_2 = 1\}$, where $\|.\|_2$ is the standard L2 norm for vectors, defined by $\|x\|_2 = \sqrt{x^T x}$.
Note: sup stands for supremum.

Prove the following properties for an arbitrary induced matrix norm:

(a) $\|A\| \geq 0$.

Solution:

(b) $\|\alpha A\| = |\alpha|\|A\|$ for any real number α.

Solution:

(c) $\|A + B\| \leq \|A\| + \|B\|$.

Solution:

(d) $\|A\| = 0 \iff A = 0$.

Solution:

(e) $\|AB\| \leq \|A\|\|B\|$.

Solution:

(f) $\|A\|_2 = \sigma_{\max}(A)$, where $\sigma_{\max}$ is the largest singular value.

Solution:

12. Prove that the eigen vectors of a real symmetric(S_{n*n}) matrix are linearly independent and form an orthogonal basis for R^n.

Solution:

13. **RAYLEIGH QUOTIENT**
Let A be an n×n real symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ and corresponding orthonormal eigenvectors $v_1, \ldots, v_n$.

(a) Show that

$$\lambda_1 = \min_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2} \quad and \quad \lambda_n = \max_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2}. \tag{4}$$

Also, show directly that if $v \neq 0$ minimizes $\frac{\langle x, Ax \rangle}{\|x\|^2}$, then v is an eigenvector of A corresponding to the minimum eigenvalue of A.

Solution:

(b) show that

$$\lambda_2 = \min_{x \perp v_1, x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2}. \tag{5}$$

14. An m × n matrix has full row rank if its row rank is m, and it has full column rank if its column rank is n. Show that a matrix can have both full row rank and full column rank only if it is a square matrix.

Solution:

15. Let A be a $m \times n$ matrix, and suppose $\vec{v}$ and $\vec{w}$ are orthogonal eigenvectors of $A^T A$. Show that $A\vec{v}$ and $A\vec{w}$ are orthogonal.

Solution:

16. Let $u_1, u_2, ..., u_n$ be a set of n orthonormal vectors. Similarly let $v_1, v_2, ..., v_n$ be another set of n orthonormal vectors.

    (a) Show that $u_1 v_1^T$ is a rank-1 matrix.

Solution:

    (b) Show that $u_1 v_1^T + u_2 v_2^T$ is a rank-2 matrix.

Solution:

    (c) Show that $\sum_{i=1}^{n} u_i v_i^T$ is a rank-n matrix.

Solution: