Instructions:

- This assignment is meant to help you grok certain concepts we will use in the course. Please don't copy solutions from any sources.
- Avoid verbosity.
- Questions marked with * are relatively difficult. Don't be discouraged if you cannot solve them right away!
- The assignment needs to be written in latex using the attached tex file. The solution for each question should be written in the solution block in space already provided in the tex file. **Handwritten assignments will not be accepted.**

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1. Suppose, a transformation matrix A, transforms the standard basis vectors of R^3 as follows :

$$\begin{bmatrix}1\\0\\0\end{bmatrix} => \begin{bmatrix}3\\8\\0\end{bmatrix};\begin{bmatrix}0\\1\\0\end{bmatrix} => \begin{bmatrix}-4\\9\\7\end{bmatrix};\begin{bmatrix}0\\0\\1\end{bmatrix} => \begin{bmatrix}-1\\2\\6\end{bmatrix}$$

(a) If the volume of a hypothetical parallelepiped in the un-transformed space is $100 units^3$ what will be volume of this parallelepiped in the transformed space?

Solution: Let, a,b,c are transformed basis vectors.
Volume of the parallelepiped spanned by transformed basis vectors is:

$$\begin{aligned}
&= \left|(a \times b) \cdot c\right| * 100\, units^3 \\
&= \left| det \left(\begin{bmatrix} 3 & -4 & -1 \\ 8 & 9 & 2 \\ 0 & 7 & 6 \end{bmatrix} \right) \right| * 100\, units^3 \\
&= \left|(3(9*6-7*2) - 8(-4*6+7))\right| * 100\, units^3 \\
&= 25600\, units^3
\end{aligned}$$

(b) What will be the volume if the transformation of the basis vectors is as follows :

$$\begin{bmatrix}1\\0\\0\end{bmatrix} => \begin{bmatrix}1\\1\\3\end{bmatrix};\begin{bmatrix}0\\1\\0\end{bmatrix} => \begin{bmatrix}-1\\2\\0\end{bmatrix};\begin{bmatrix}0\\0\\1\end{bmatrix} => \begin{bmatrix}0\\2\\2\end{bmatrix}$$

Solution: Let, a,b,c are transformed basis vectors.
Volume spanned by transformed basis vectors is:

$$\begin{aligned}
&= \left|(a \times b) \cdot c\right| * 100\, units^3 \\
&= \left| det \left(\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 2 \\ 3 & 0 & 2 \end{bmatrix} \right) \right| * 100\, units^3 \\
&= \left|1(2*2) - (-1)(2 - 3*2)\right| * 100\, units^3 \\
&= 0 * 100\, units^3 \\
&= 0\, units^3
\end{aligned}$$

(c) Comment on the uniqueness of the second transformation.

Solution: In the second transformation, given three basis vectors v_1, v_2, v_3 are not linearly independent.

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$$

$$\therefore \lambda_1 = \frac{-2}{3}, \lambda_2 = \frac{-2}{3}, \lambda_3 = 1$$

is non-zero solution of the given equation.

There are two independent vectors. And vectors span parallelogram in two-dimensional plane. So, the volume of the parallelepiped is zero.

2. If R^3 is represented by following basis vectors : $\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 7 \\ -11 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 3 \end{bmatrix}$

(a) Find the representation of the vector $\begin{pmatrix} -3 & 1 & -2 \end{pmatrix}^T$ (as represented in standard basis) in the above basis.

Solution: The vector $e = \begin{pmatrix} -3 & 1 & -2 \end{pmatrix}^T$ in the form of given basis vectors v_1,v_2 and v_3 can be found as :

$$\begin{aligned}
b_1 v_1 + b_2 v_2 + b_3 v_3 &= e \\
Ab &= e
\end{aligned}$$

$$\therefore b = A^{-1}e$$

where, A is matrix whose columns are basis vector v_1,v_2 and v_3. b is vector represented in given basis and e is vector represented in standard basis

Matrix A is $\begin{bmatrix} 5 & 8 & -4 \\ 2 & 7 & -9 \\ 0 & -11 & 3 \end{bmatrix}$.

$$\begin{aligned} \therefore b &= \begin{bmatrix} 5 & 8 & -4 \\ 2 & 7 & -9 \\ 0 & -11 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \\ &= \frac{-1}{350} \begin{bmatrix} -78 & 20 & -44 \\ -6 & 15 & 37 \\ -22 & 55 & 19 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{171}{175} \\ \frac{41}{350} \\ -\frac{83}{350} \end{bmatrix} \end{aligned}$$

(b) We know that, orthonormal basis simplifies this transformation to a great extent. What would be the representation of vector $\begin{pmatrix} -3 & 1 & -2 \end{pmatrix}^T$ in the orthogonal basis represented by : $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Solution: The vector $e = \begin{pmatrix} -3 & 1 & -2 \end{pmatrix}^T$ in the form of given orthogonal basis vectors v_1,v_2 and v_3 can be found by :

$$b_1v_1 + b_2v_2 + b_3v_3 = e \tag{1}$$

vector $b = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$ is vector in the form of given orthogonal basis.
Since, vectors v_1,v_2 and v_3 are orthogonal,

$$\langle v_i, v_j \rangle = 0, \, if \, i \neq j$$

Do dot product with v_1 both sides of the equation 1,

$$b_1\langle v_1, v_1\rangle + b_2\langle v_1, v_2\rangle + b_3\langle v_1, v_3\rangle = \langle e, v_1\rangle$$

$$\begin{aligned} b_1 &= \frac{\langle e, v_1\rangle}{\langle v_1, v_1\rangle} \\ &= \frac{\begin{bmatrix}-3 & 1 & -2\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}} \\ &= \frac{-3-1}{1+1} = -2 \end{aligned}$$

Do dot product with v_2 both sides of the equation 1, we will get

$$\begin{aligned} b_2 &= \frac{\langle e, v_2\rangle}{\langle v_2, v_2\rangle} \\ &= \frac{\begin{bmatrix}-3 & 1 & -2\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}} \\ &= \frac{-3+1}{1+1} = -1 \end{aligned}$$

Do dot product with v_2 both sides of the equation 1, we will get

$$\begin{aligned} b_2 &= \frac{\langle e, v_2\rangle}{\langle v_2, v_2\rangle} \\ &= \frac{\begin{bmatrix}-3 & 1 & -2\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}}{\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}} \\ &= \frac{-2}{1} = -2 \end{aligned}$$

vector $b = \begin{pmatrix} -2 & -1 & -2 \end{pmatrix}^T$ is vector in the form of given orthogonal basis.

(c) Comment on the advantages of having orthonormal basis.

Solution: If orthonormal basis $B = \{v_1, v_2, \cdots, v_n\}$ is given then the unique coordinate representation of a vector w with respect to B is vector $b = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$ given by:

$$\langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2 + \cdots + \langle w, v_n \rangle v_n = w$$

(we can generalize this equation from question solution of part(b))
Where, $b_1 = \langle w, v_1 \rangle$, $b_2 = \langle w, v_2 \rangle$ and so on.

If basis are not orthonormal then to derive the coordinate representation of a vector, we have to compute the inverse or solve the system of linear equation which has time complexity $O(n^3)$.

But if basis are orthonormal then we can derive the coordinate representation of a vector by simple dot product of vectors or by transpose of matrix since for orthonormal matrix $A^{-1} = A^T$ which has time complexity $O(n^2)$.

Hence orthonormal basis makes computation easy.

3. A square matrix is a Markov matrix if each entry is between zero and one and the sum along each row is one. Prove that a product of Markov matrices is Markov.

Solution: A square markov matrix has each entry between zero and one and the sum along each row is one.
Let, $A = [a_{ij}]$ be an $n \times n$ markov matrix. vector $e = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T$ of size n whose all elements are equal to 1.
we know that,

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad for\ i = 1 \cdots n$$

$$\therefore Ae = e$$

$B = [b_{ij}]$ be an $n \times n$ markov matrix.

$$\therefore Be = e$$

Now, Product of two markov matrix is markov matrix if sum along each row of result matrix is one. i.e. $(AB)e = e$.

$$ABe = A(Be) = Ae = e$$

Hence we can say that product of two Markov matrix is Markov.

4. Give an example of a matrix A with the following three properties:
   (a) A has eigenvalues -1 and 2.
   (b) The eigenvalue -1 has eigenvector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{1}$$

   (c) The eigenvalue 2 has eigenvector

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} and \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{2}$$

Solution: An $n \times n$ matrix with n independent eigenvectors can be expressed as $A = PDP^{-1}$, where D is the diagonal matrix $diag(\lambda_1 \lambda_2 \ldots \lambda_n)$ and P is the matrix $(v_1|v_2|\ldots|v_n)$ where v_i is the eigenvector corresponding to eigenvalue λ_i.

$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

So,

$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1 & 2 & 0 \\ -2 & 2 & 2 \\ -3 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{-3}{2} & \frac{3}{2} & \frac{-1}{2} \end{bmatrix} \\ \therefore A &= \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & \frac{-3}{2} \\ -3 & 5 & -3 \\ \frac{-9}{2} & \frac{9}{2} & \frac{-5}{2} \end{bmatrix} \end{aligned}$$

matrix A satisfies all three properties.

5. Perform the Gram-Schmidt process on each of these basis for $\mathbb{R}^3$. And convert the resulting orthogonal basis into orthonormal basis.

(a) $\langle \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \rangle$

Solution: Let, u_1, u_2 and u_3 are given basis for $\mathbb{R}^3$. Resulting orthogonal basis v_1, v_2 and v_3 can be found as follows:

Let, $u_1 = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}^T$

$$\therefore v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{12}} \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}^T$$

Let, $u_2 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$

$$\begin{aligned}
\therefore w_2 &= u_2 - \langle u_2, v_1 \rangle v_1 \\
&= \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T - 0 * v_1 \\
&= \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T \\
\therefore v_2 &= \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T
\end{aligned}$$

Let, $u_3 = \begin{pmatrix} 0 & 3 & 1 \end{pmatrix}^T$

$$\begin{aligned}
\therefore w_3 &= u_3 - \langle u_3, v_1 \rangle v_1 - \langle u_3, v_2 \rangle v_2 \\
&= u_3 - \frac{4}{\sqrt{3}} * v_1 + \frac{1}{\sqrt{2}} * v_2 \\
&= \begin{pmatrix} 0 & 3 & 1 \end{pmatrix}^T - \frac{2}{3} * \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}^T + \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T \\
&= \begin{pmatrix} -\frac{5}{6} & \frac{5}{3} & -\frac{5}{6} \end{pmatrix}^T \\
\therefore v_3 &= \frac{w_3}{\|w_3\|} = \frac{5}{\sqrt{6}} \begin{pmatrix} -\frac{5}{6} & \frac{5}{3} & -\frac{5}{6} \end{pmatrix}^T
\end{aligned}$$

Vector v_1 , v_2 and v_3 are orthonormal basis for $\mathbb{R}^3$.

(b) $\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \rangle$

Solution: Let, u_1, u_2 and u_3 are given basis for $\mathbb{R}^3$. Resulting orthogonal basis v_1, v_2 and v_3 can be found as follows:

Let, $u_1 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T$

$$\therefore v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T$$

Let, $u_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$

$$\begin{aligned} \therefore w_2 &= u_2 - \langle u_2, v_1 \rangle v_1 \\ &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T + \frac{1}{2} * \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^T \\ \therefore v_2 &= \frac{w_2}{\|w_2\|} = \sqrt{2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^T \end{aligned}$$

Let, $u_3 = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}^T$

$$\begin{aligned} \therefore w_3 &= u_3 - \langle u_3, v_1 \rangle v_1 - \langle u_3, v_2 \rangle v_2 \\ &= u_3 + \frac{1}{\sqrt{2}} * v_1 - \frac{5}{2}\sqrt{2} * v_2 \\ &= \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}^T + \frac{1}{2} * \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T - 5 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^T \\ &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \\ \therefore v_3 &= \frac{w_3}{\|w_3\|} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \end{aligned}$$

Vector v_1 , v_2 and v_3 are orthonormal basis for $\mathbb{R}^3$.

6. Suppose, every year, 4% of the birds from Canada migrate to the US, and 1% of them travel to Mexico. Similarly, every year, 6% of the birds from US migrate to Canada, and 4% to Mexico. Finally, every year 10% of the birds from Mexico migrate to the US, and 0% go to Canada.

(a) Represent the above probabilities in a transition matrix.

Solution: we have three countries: Canada, US and Mexico are the transition states 1,2 and 3 respectively.
Let Transition matrix is $P = [p_{ij}]$ where p_{ij} represent probability of bird migrate from j^{th} state to i^{th} state.
So Transition matrix will be:

$$P = \begin{bmatrix} 0.95 & 0.06 & 0 \\ 0.04 & 0.90 & 0.10 \\ 0.01 & 0.04 & 0.90 \end{bmatrix}$$

(b) Is it possible that after some years, the number of birds in the 3 countries will become constant?

Solution: The number of birds in the 3 countries can be represented by vector $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$.
We can say that the number of birds in the 3 countries will become constant if $Px = x$ where P is transition matrix represents probability of bird migrate from j^{th} state to i^{th} state.

$$Px = x$$
$$\therefore Px - x = 0$$
$$\therefore (P - I)x = 0$$
$$\begin{bmatrix} 0.95 - 1 & 0.06 & 0 \\ 0.04 & 0.90 - 1 & 0.10 \\ 0.01 & 0.04 & 0.90 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -0.05 & 0.06 & 0 \\ 0.04 & -0.10 & 0.10 \\ 0.01 & 0.04 & -0.10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By solving this system of equations using Gaussian elimination method we get,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{30}{13}x_3 \\ \frac{25}{13}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{30}{13} \\ \frac{25}{13} \\ 1 \end{bmatrix} x_3$$

Since we can get the vector x such that $Px = x$, It is possible that after years the number of birds in the 3 countries will become constant.

7. (a) Show that any set of four unique vectors in $\mathbb{R}^2$ is linearly dependent.

Solution: Let $V = \{v_1, v_2, v_3, v_4\}$ set of four unique vectors in $\mathbb{R}^2$.
case 1 : vector v_1 and v_2 are linearly dependent.
Then there are λ_1, λ_2 not both zero and $\lambda_1 v_1 + \lambda_2 v_2 = 0$.

case 2 : vector v_1 and v_2 are linearly independent.
so every vector $v_i = (u_1, u_2)$ is in the span of v_1, v_2. i.e. $v_i = \lambda_1 v_1 + \lambda_2 v_2$.

Set is linearly dependent if one of λ have non zero solution for

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = 0$$

we can write v_3 and v_4 in terms of v_1, v_2

$$\begin{aligned} \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3(a_1 v_1 + a_2 v_2) + \lambda_4(a_3 v_1 + a_4 v_2) &= 0 \\ (\lambda_1 + a_1\lambda_3 + a_3\lambda_4)v_1 + (\lambda_2 + a_2\lambda_3 + a_4\lambda_4)v_2 &= 0 \end{aligned}$$

Since v_1 and v_2 are linearly independent,

$$\begin{aligned} \lambda_1 + a_1\lambda_3 + a_3\lambda_4 &= 0 \\ \lambda_2 + a_2\lambda_3 + a_4\lambda_4 &= 0 \end{aligned}$$

Here, we have four variable and two equation (more variable and than equation) hence this homogeneous system has infinitely many solutions, in particular, it has a nonzero solution $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Hence any set of four unique vectors in $\mathbb{R}^2$ is linearly dependent.

(b) What is the maximum number of unique vectors that a linearly independent subset of $\mathbb{R}^2$ can have ?

Solution: Maximum number of unique vectors that a linearly independent subset of $\mathbb{R}^2$ can have is 2.

Since $v_1 = (a, b)$ and $v_2 = (c, d)$ vectors are linearly independent, every vector $v_3 = (u_1, u_2)$ is in the span of v_1, v_2.

$$\lambda_1 v_1 + \lambda_2 v_2 = v_3$$

Then vectors v_1, v_2 and v_3 are

$$\lambda_1 v_1 + \lambda_2 v_2 + (-1)v_3 = 0$$

linearly dependent set.

8. (a) Determine if the vectors $\{v_1, v_2, v_3\}$ are linearly independent, where
$v_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$
Justify each answer

Solution: Vectors v_1, v_2, v_3 are called linearly independent if they satisfy a relation $r_1v_1 + r_2v_2 + r_3v_3 = 0$, where the coefficients r_1, r_2, r_3 are all equal to zero.
Let, matrix V whose columns are given vectors v_1, v_2, v_3 and vector r of the coefficients r_1, r_2, r_3.

$$\therefore Vr = 0$$
$$\begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gaussian elimination method,

$$\begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\therefore r_3 = 0$$
$$2r_2 + 4r_3 = 0$$
$$\therefore r_2 = 0$$
$$5r_1 + 7r_2 + 9r_3 = 0$$
$$\therefore r_1 = 0$$

The coefficients r_1, r_2, r_3 are all equal to zero, thus vector v_1, v_2, v_3 are linearly independent.

(b) Prove that each set {f, g} is linearly independent in the vector space of all functions from $\mathbb{R}^+$to $\mathbb{R}$.

1. f(x) = x and g(x) = $\frac{1}{x}$
2. f(x) = cos(x) and g(x) = sin(x)
3. f(x) = e^x and g(x) = ln(x)

Solution: Let, f and g be differentiable on $[a,b]$. If Wronskian $W(f,g)(t_0)$ is nonzero for some t_0 in $[a,b]$ then f and g are linearly independent on $[a,b]$. If f and g are linearly dependent then the Wronskian is zero for all t in $[a,b]$.

The Wronskian of two differentiable functions f and g is $W(f,g) = fg' - gf'$.

1. f(x) = x and g(x) = $\frac{1}{x}$

$$\therefore f'(x) = 1 \qquad\qquad g'(x) = \frac{-1}{x^2}$$

   The Wronskian is

$$\begin{aligned} W(f,g)(x) &= f(x)g'(x) - g(x)f'(x) \\ &= x * \frac{-1}{x^2} - \frac{1}{x} * 1 \\ &= -\frac{1}{x} - \frac{1}{x} \\ &= -\frac{2}{x} \end{aligned}$$

   Now, put $x = 1$ to get

$$W(f,g)(1) = -2$$

   which is nonzero. We can conclude that f(x) and g(x) are linearly independent.

2. f(x) = cos(x) and g(x) = sin(x)

$$\therefore f'(x) = -sin(x) \qquad\qquad g'(x) = cos(x)$$

   The Wronskian is

$$\begin{aligned} W(f,g)(x) &= f(x)g'(x) - g(x)f'(x) \\ &= cos(x) * cos(x) - sin(x) * -sin(x) \\ &= cos^2(x) + sin^2(x) \\ &= 1 \end{aligned}$$

   Now, for any value of x, $W(f,g)(x) = 1$ which is nonzero. We can conclude that f(x) and g(x) are linearly independent.

3. f(x) = e^x and g(x) = ln(x)

$$\therefore f'(x) = e^x \qquad\qquad g'(x) = \frac{1}{x}$$

The Wronskian is

$$\begin{aligned} W(f,g)(x) &= f(x)g'(x) - g(x)f'(x) \\ &= e^x * \frac{1}{x} - ln(x) * e^x \\ &= e^x(\frac{1}{x} - ln(x)) \end{aligned}$$

Now, put $x = 1$ to get

$$W(f,g)(1) = e(1-0) = e$$

which is nonzero. We can conclude that f(x) and g(x) are linearly independent.

9. Let t_θ be

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3}$$

(a) Show that $t_{\theta_1+\theta_2} = t_{\theta_1} * t_{\theta_2}$ (* here stands for matrix multiplication).

Solution: Given,

$$t_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Now,

$$\begin{aligned} t_{\theta_1+\theta_2} &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{bmatrix} \end{aligned} \tag{1}$$

And,

$$t_{\theta_1} * t_{\theta_2} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{bmatrix} \tag{2}$$

from result (1) and (2), we can say that $t_{\theta_1+\theta_2} = t_{\theta_1} * t_{\theta_2}$.

(b) Show that $t_\theta^{-1} = t_{-\theta}$.

Solution: Given,

$$t_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Now,

$$t_\theta^{-1} = \frac{adj(t_\theta)}{det(t_\theta)}$$

$$= \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

And,

$$t_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

Since, cos is even function. $\therefore \cos(-\theta) = \cos(\theta)$. and sin is odd function. $\therefore \sin(-\theta) = -\sin(\theta)$.

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

from result (1) and (2), we can say that $t_\theta^{-1} = t_{-\theta}$.

10. Given matrix has distinct eigenvalues

$$\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

(a) Diagonalize it.

Solution: The characteristic polynomial $p(\lambda)$ of a given matrix A is:

$$\begin{aligned} p(\lambda) &= det(A - \lambda I) \\ &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)((-1-\lambda)^2 - 0) - 2(6(-1-\lambda) - 0) + 1(-12 - (-1)(-1-\lambda)) \\ &= -\lambda^3 - \lambda^2 + 12\lambda \\ &= -\lambda(\lambda^2 + \lambda - 12) \\ p(\lambda) &= -\lambda(\lambda+4)(\lambda-3) \end{aligned}$$

From the characteristic equation, the eigenvalues are:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -4 \\ \lambda_3 &= 3 \end{aligned}$$

Now, eigenvectors x_1, x_2 and x_3 corresponding to eigenvalues λ_1, λ_2 and λ_3 can be calculate as follows:

$$Ax_1 = \lambda_1 x_1$$

$$\therefore (A - \lambda_1 I)x_1 = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = 0$$

By solving this,

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} \frac{-1}{13} \\ \frac{-6}{13} \\ 1 \end{bmatrix}$$

$$Ax_2 = \lambda_2 x_2$$

$$\therefore (A - \lambda_2 I)x_2 = 0$$

$$\begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = 0$$

By solving this,

$$x_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$Ax_3 = \lambda_3 x_3$$

$$\therefore (A - \lambda_3 I)x_3 = 0$$

$$\begin{bmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = 0$$

By solving this,

$$x_3 = \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{-3}{2} \\ 1 \end{bmatrix}$$

The Diagonal matrix is:

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The matrix with the eigenvectors (x_1, x_2, x_3) as its columns:

$$P = \begin{bmatrix} \frac{-1}{13} & -1 & -1 \\ \frac{-6}{13} & 2 & \frac{-3}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore P^{-1} = \begin{bmatrix} \frac{13}{12} & 0 & \frac{13}{12} \\ \frac{-9}{28} & \frac{2}{7} & \frac{3}{28} \\ \frac{-16}{21} & \frac{-2}{7} & \frac{-4}{21} \end{bmatrix}$$

So, Diagonalize matrix D will be $P^{-1}AP$.

$$D = \begin{bmatrix} \frac{13}{12} & 0 & \frac{13}{12} \\ \frac{-9}{28} & \frac{2}{7} & \frac{3}{28} \\ \frac{-16}{21} & \frac{-2}{7} & \frac{-4}{21} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} \frac{-1}{13} & -1 & -1 \\ \frac{-6}{13} & 2 & \frac{-3}{2} \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) Find a basis with respect to which this matrix has that diagonal representation

Solution: The eigenvectors of matrix is the basis for which the matrix A can be represented by a diagonal matrix. We have found eigenvectors in previous question. So,

$\left\langle \begin{pmatrix} \frac{-1}{13} \\ \frac{-6}{13} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ \frac{-3}{2} \\ 1 \end{pmatrix} \right\rangle$ is the basis.

(c) Find the matrices P and P^{-1} to effect the change of basis.

Solution: matrix P and p^{-1} to effect the change of basis can be given as:

$$P = \begin{bmatrix} \frac{-1}{13} & -1 & -1 \\ \frac{-6}{13} & 2 & \frac{-3}{2} \\ 1 & 1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{13}{12} & 0 & \frac{13}{12} \\ \frac{-9}{28} & \frac{2}{7} & \frac{3}{28} \\ \frac{-16}{21} & \frac{-2}{7} & \frac{-4}{21} \end{bmatrix}$$

11. * **Induced Matrix Norms**

In case you didn't already know, a norm $\|.\|$ is any function with the following properties:

1. $\|x\| \geq 0$ for all vectors x.
2. $\|x\| = 0 \iff x = \mathbf{0}$.
3. $\|\alpha x\| = |\alpha|\|x\|$ for all vectors x, and real numbers α.
4. $\|x + y\| \leq \|x\| + \|y\|$ for all vectors x, y.

Now, suppose we're given some vector norm $\|.\|$ (this could be L2 or L1 norm, for example). We would like to use this norm to measure the size of a matrix A. One way is to use the corresponding induced matrix norm, which is defined as $\|A\| = \sup_x\{\|Ax\| : \|x\| = 1\}$.

E.g.: $\|A\|_2 = \sup_x\{\|Ax\|_2 : \|x\|_2 = 1\}$, where $\|.\|_2$ is the standard L2 norm for vectors, defined by $\|x\|_2 = \sqrt{x^T x}$.
Note: sup stands for supremum.

Prove the following properties for an arbitrary induced matrix norm:

(a) $\|A\| \geq 0$.

Solution: We know that,

$$\|A\| = \sup_{\|x\|=1} \{\|Ax\|\} \tag{1}$$

Since $\|Ax\| \geq 0$ (from vector norm property (1) because Ax is vector),

$$\therefore \|A\| \geq 0$$

(b) $\|\alpha A\| = |\alpha|\|A\|$ for any real number α.

Solution:

$$\|\alpha A\| = \sup_{\|x\|=1} \{\|\alpha Ax\|\}$$

here, Ax is vector. so from vector norm property (3) we can write $\|\alpha Ax\| = |\alpha|\|Ax\|$,

$$\begin{aligned}\|\alpha A\| &= \sup_{\|x\|=1} \{|\alpha|\|Ax\|\} \\ &= |\alpha| \sup_{\|x\|=1} \{\|Ax\|\}\end{aligned}$$

from equation (1) we can write as,

$$\therefore \|\alpha A\| = |\alpha|\|A\|$$

(c) $\|A + B\| \leq \|A\| + \|B\|$.

Solution:

$$\begin{aligned}\|A + B\| &= \sup_{\|x\|=1} \{\|(A + B)x\|\} \\ &= \sup_{\|x\|=1} \{\|Ax + Bx\|\}\end{aligned}$$

here, Ax and Bx are vectors. so from vector norm property (4) we can write

$\|Ax + Bx\| \;\le \|Ax\| + \|Bx\|,$

$$\begin{aligned}\|A+B\| &\le \sup_{\|x\|=1}\{\|Ax\| + \|Bx\|\}\\ &\le \sup_{\|x\|=1}\{\|Ax\|\} + \sup_{\|x\|=1}\{\|Bx\|\}\end{aligned}$$

from equation (1) we can write as,

$$\therefore \|A+B\| \le \|A\| + \|B\|$$

(d) $\|A\| = 0 \iff A = 0.$

Solution: We have to prove two statements.

1. $\|A\| = 0 \implies A = 0.$

$$\|A\| = \sup_{\|x\|=1}\{\|Ax\|\} = 0$$

here, Ax is vector. from vector norm property (2) we can say that,

$$\begin{aligned}\|Ax\| = 0 &\implies Ax = \mathbf{0}\\ &\implies A = \mathbf{0} \qquad\qquad (\because x \neq \mathbf{0})\end{aligned}$$

Hence $\|A\| = 0 \implies A = \mathbf{0}$ is proved.

2. $A = 0 \implies \|A\| = 0.$
$A = 0$ then $Ax = 0$.
here, Ax is vector. from vector norm property (2) we can say that,

$$\begin{aligned}Ax = \mathbf{0} &\implies \|Ax\| = 0\\ &\implies \|A\| = 0 \qquad\qquad \left(\because \|A\| = \sup_{\|x\|=1}\{\|Ax\|\}\right)\end{aligned}$$

Hence $A = \mathbf{0} \implies \|A\| = 0$ is proved.

From both statements, $\|A\| = 0 \iff A = 0.$

(e) $\|AB\| \le \|A\|\|B\|.$

Solution: Matrix norm defined as $\|A\| = \sup_{\|x\|\neq 0}\{\frac{\|Ax\|}{\|x\|}\}$.

$$\|A\| = \sup_{\|x\|\neq 0}\{\frac{\|Ax\|}{\|x\|}\} \geq \frac{\|Ay\|}{\|y\|} \qquad (for\, an\, arbitrary\, y)$$
$$\therefore \|Ay\| \leq \|A\|\|y\| \qquad (property\, I)$$

Now,

$$\|AB\| = \sup_{\|x\|=1}\{\|ABx\|\}$$

here, A is matrix and Bx is vector. so from property I we can write $\|ABx\| \leq \|A\|\|Bx\|$,

$$\begin{aligned}\|AB\| &\leq \sup_{\|x\|=1}\{\|A\|\|Bx\|\} \\ &\leq \|A\| \sup_{\|x\|=1}\{\|Bx\|\}\end{aligned}$$

from equation (1) we can write as,

$$\therefore \|AB\| \leq \|A\|\|B\|$$

(f) $\|A\|_2 = \sigma_{\max}(A)$, where $\sigma_{\max}$ is the largest singular value.

Solution: The singular value decomposition of Matrix A of the form,

$$A = U\Sigma V^* \qquad (1)$$

where, U and V are an unitary matrix, Σ is a diagonal matrix with non-negative real numbers on the diagonal, and V^* is the conjugate transpose of V.
Now,

$$\begin{aligned}\|A\| = \sup_{\|x\|=1}\{\|Ax\|\} &= \sup_{\|x\|=1}\{\|U\Sigma V^*x\|\} \qquad (from\,(1)) \\ &= \sup_{\|x\|=1}\{\|\Sigma V^*x\|\}\end{aligned}$$

since U is unitary, that is, $\|Ux\|^2 = x^TU^TUx = x^Tx = \|x\|^2$, for some vector x.

Let $y = V^*x$. By the same argument above, $\|y\|^2 = \|V^*x\|^2 = \|x\|^2 = 1$ since V is unitary.

$$\sup_{\|x\|=1}\{\|\Sigma V^*x\|\} = \sup_{\|y\|=1}\{\|\Sigma y\|\}$$

$$
\begin{aligned}
&= \sup_{\|y\|=1} \{(\sum_i \sigma_i^2 y_i^2)^{\frac{1}{2}}\} \\
&= \sigma_i
\end{aligned}
$$

Where, σ_i is maximum singular value.
The maximum for the above is attained when $y = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$ where σ_1 is maximum value.
so, $\|A\| = \sigma_{\max}(A)$ is proved.

12. Prove that the eigen vectors of a real symmetric(S_{n*n}) matrix are linearly independent and form an orthogonal basis for R^n.

Solution: Assume that S_{n*n} is real symmetric matrix, and x and y are eigenvectors of S corresponding to distinct eigenvalues λ and μ. Then,

$$
\begin{aligned}
Sx &= \lambda x \\
\therefore (Sx)^T &= x^t S^t = x^T S = \lambda x^T && (1) \\
Sy &= \mu y && (2) \\
\lambda \langle x, y \rangle &= \lambda x^T y \\
&= x^T S y && (from\,(1)) \\
&= x^T (\mu y) && (from\,(2)) \\
&= \mu x^T y \\
&= \mu \langle x, y \rangle \\
\therefore (\lambda - \mu) \langle x, y \rangle &= 0
\end{aligned}
$$

Since $(\lambda - \mu) \neq 0$, then $\langle x, y \rangle = 0$, i.e., $x \perp y$.
Since vectors are orthogonal they are also linearly independent.

Hence the eigen vectors of a real symmetric matrix are linearly independent and form an orthogonal basis for R^n.

13. **RAYLEIGH QUOTIENT**
Let A be an n×n real symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ and corresponding orthonormal eigenvectors $v_1, \ldots, v_n$.
   (a) Show that

$$
\lambda_1 = \min_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2} \quad and \quad \lambda_n = \max_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2}. \tag{4}
$$

Also, show directly that if $v \neq 0$ minimizes $\frac{\langle x, Ax\rangle}{\|x\|^2}$, then v is an eigenvector of A corresponding to the minimum eigenvalue of A.

Solution: Let A be an n×n real symmetric matrix with orthonormal eigenvectors $v_1, v_2, \ldots, v_n$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$.

matrix A can be represent as,

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^T \tag{1}$$

for any vector $x \in \mathbb{R}^n$, we can express it as, $x = \sum_{i=1}^{n} k_i v_i$

$$\begin{aligned}
\langle x, Ax\rangle &= x^T A x \\
&= (\sum_{i=1}^{n} k_i v_i)^T (\sum_{i=1}^{n} k_i A v_i) \\
&= (\sum_{i=1}^{n} k_i v_i)^T (\sum_{i=1}^{n} k_i \lambda_i v_i) \\
\therefore \langle x, Ax\rangle &= \sum_{i=1}^{n} {k_i}^2 \lambda_i \qquad (2) \\
\|x\|^2 &= x^T x \\
&= (\sum_{i=1}^{n} k_i v_i)^T (\sum_{i=1}^{n} k_i v_i) \\
&= \sum_{i,j} k_i k_j v_i^T v_j \\
\therefore \|x\|^2 &= \sum_{i=1}^{n} {k_i}^2 \qquad (3) \\
R_A(x) &= \frac{x^T A x}{\|x\|^2} \\
&= \frac{\sum_{i=1}^{n} {k_i}^2 \lambda_i}{\sum_{i=1}^{n} {k_i}^2} \qquad (4) \\
& \qquad (from\,(2)\,and\,(3))
\end{aligned}$$

Since $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$

$$R_A(x) \geq \frac{\sum_{i=1}^{n} {k_i}^2 \lambda_1}{\sum_{i=1}^{n} {k_i}^2}$$

$$
\begin{aligned}
&\geq \lambda_1 \frac{\sum_{i=1}^{n} {k_i}^2}{\sum_{i=1}^{n} {k_i}^2} \\
&\geq \lambda_1 \\
\therefore \min_{x \neq 0} R_A(x) &= \min_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2} = \lambda_1 \qquad (5)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
R_A(x) &\leq \frac{\sum_{i=1}^{n} {k_i}^2 \lambda_n}{\sum_{i=1}^{n} {k_i}^2} \\
&\leq \lambda_n \frac{\sum_{i=1}^{n} {k_i}^2}{\sum_{i=1}^{n} {k_i}^2} \\
&\leq \lambda_n \\
\therefore \max_{x \neq 0} R_A(x) &= \max_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2} = \lambda_n \qquad (6)
\end{aligned}
$$

If vector v minimizes $R_A(x)$ then,

$$
\begin{aligned}
\min_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2} &= \frac{\langle v, Av \rangle}{\|v\|^2} \\
&= \frac{v^T A v}{v^T v} \\
&= \frac{v^T \lambda v}{v^T v} \\
&= \lambda \frac{v^T v}{v^T v} \\
&= \lambda
\end{aligned}
$$

From equation (5) we know that $\min_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2} = \lambda_1$

$$
\therefore \lambda = \lambda_1
$$

Hence v is an eigenvector of A corresponding to the minimum eigenvalue of A.

(b) show that

$$
\lambda_2 = \min_{x \perp v_1, x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2}. \qquad (5)
$$

Solution: This will be same as previous proof except the fact that all the vectors x are orthogonal to the eigenvector v_1.

For any vector $x \in \mathbb{R}^n$, we can express it as linear combination of eigenvectors $v_2, v_3, \cdots, v_n$ and the component of v_1 will be 0, $x = \sum_{i=2}^{n} k_i v_i$

$$\begin{aligned}
\langle x, Ax \rangle &= x^T A x \\
&= (\sum_{i=2}^{n} k_i v_i)^T (\sum_{i=2}^{n} k_i A v_i) \\
&= (\sum_{i=2}^{n} k_i v_i)^T (\sum_{i=2}^{n} k_i \lambda_i v_i) \\
\therefore \langle x, Ax \rangle &= \sum_{i=2}^{n} {k_i}^2 \lambda_i && (1) \\
\|x\|^2 &= x^T x \\
&= (\sum_{i=2}^{n} k_i v_i)^T (\sum_{i=2}^{n} k_i v_i) \\
&= \sum_{i,j} k_i k_j v_i^T v_j \\
\therefore \|x\|^2 &= \sum_{i=2}^{n} {k_i}^2 && (2) \\
R_A(x) &= \frac{x^T A x}{\|x\|^2} \\
&= \frac{\sum_{i=2}^{n} {k_i}^2 \lambda_i}{\sum_{i=2}^{n} {k_i}^2} && (3) \\
& && (from\,(1)\,and\,(2))
\end{aligned}$$

Since $\lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$

$$\begin{aligned}
R_A(x) &\geq \frac{\sum_{i=1}^{n} {k_i}^2 \lambda_2}{\sum_{i=1}^{n} {k_i}^2} \\
&\geq \lambda_2 \frac{\sum_{i=1}^{n} {k_i}^2}{\sum_{i=1}^{n} {k_i}^2} \\
&\geq \lambda_2 \\
\therefore \min_{x \perp v_1, x \neq 0} R_A(x) &= \min_{x \perp v_1, x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2} = \lambda_2 && (4)
\end{aligned}$$

14. An m × n matrix has full row rank if its row rank is m, and it has full column rank if its column rank is n. Show that a matrix can have both full row rank and full column rank only if it is a square matrix.

Solution: A matrix is full row rank when each of the rows of the matrix are linearly independent and full column rank when each of the columns of the matrix are linearly independent.

For a non-square matrix with m rows and n columns, there will always be the case that either the rows or columns (whichever is larger in number) are linearly dependent because number of row/column vectors will be greater than dimension. so matrix can not have both both full row rank and full column rank.

A square matrix can have all rows and columns linearly independent.

Hence a matrix can have both full row rank and full column rank only if it is a square matrix.

15. Let A be a $m \times n$ matrix, and suppose $\vec{v}$ and $\vec{w}$ are orthogonal eigenvectors of $A^T A$. Show that $A\vec{v}$ and $A\vec{w}$ are orthogonal.

Solution: Let A be a $m \times n$ matrix.
$\vec{v}$ and $\vec{w}$ are orthogonal eigenvectors of $A^T A$.
Suppose λ_1 and λ_2 are corresponding eigenvalues.

$$\therefore A^T A v = \lambda_1 v \qquad (1)$$
$$\therefore A^T A w = \lambda_2 w \qquad (2)$$

Now, to show that Av and Aw are orthogonal we have to prove that $(Av)^T(Aw) = 0$.

$$(Av)^T(Aw) = v^T A^T A w$$

From equation (2),

$$(Av)^T(Aw) = v^T \lambda_2 w$$

Since, λ_2 is scalar we can write it as:

$$(Av)^T(Aw) = \lambda_2 v^T w$$

We know that, vector v and w are orthogonal eigenvectors so $v^T w = 0$.

$$\therefore (Av)^T(Aw) = 0$$

Hence $A\vec{v}$ and $A\vec{w}$ are orthogonal.

16. Let $u_1, u_2, ..., u_n$ be a set of n orthonormal vectors. Similarly let $v_1, v_2, ..., v_n$ be another set of n orthonormal vectors.

(a) Show that $u_1 v_1^T$ is a rank-1 matrix.

Solution: Let $A = u_1 v_1^T$
If $x \in \mathbb{R}^m$ then matrix A can be used to define a linear transformation $L_A \colon \mathbb{R}^m \to \mathbb{R}^n$ given by $L_A(x) = Ax$,

$$Ax = u_1 v_1^T x = \langle x, v_1 \rangle u_1$$

($\because v_1^T x$ is dot product of x and v_1 and it is scalar.)

Thus, matrix A maps every vector in R^m to a scalar multiple of vector u_1. Hence,

$$\begin{aligned} rank(u_1 v_1^T) = rank(A) &= rank(L_A) \\ &= dim(im(A)) \\ &= rank(u_1) \\ &= 1 \end{aligned}$$

(b) Show that $u_1 v_1^T + u_2 v_2^T$ is a rank-2 matrix.

Solution: Let, $A = u_1 v_1^T + u_2 v_2^T$
We can find the SVD of matrix A, which has the form:

$$A = U \Sigma V^T$$

Where U and V are orthogonal matrix.Now,

$$\begin{aligned} u_1 v_1^T + u_2 v_2^T &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \\ &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \\ &= U I_2 V \end{aligned}$$

Which is singular value decomposition of matrix A.

The rank of a matrix is equal to the number of non-zero singular values which is 2.
Hence rank of $u_1 v_1^T + u_2 v_2^T$ is 2.

(c) Show that $\sum_{i=1}^{n} u_i v_i^T$ is a rank-n matrix.

Solution: Let, $A = \sum_{i=1}^{n} u_i v_i^T$
We can find the SVD of matrix A, which has the form:

$$A = U\Sigma V^T$$

Where U and V are orthogonal matrix.Now,

$$\begin{aligned}\sum_{i=1}^{n} u_i v_i^T &= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \\ &= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \\ &= U I_n V\end{aligned}$$

Which is singular value decomposition of matrix A.

The rank of a matrix is equal to the number of non-zero singular values in I_n which is n.
Hence rank of $\sum_{i=1}^{n} u_i v_i^T$ is n.