Optimal Pricing in Repeated Posted-Price Auctions with Different Patience of the Seller and the Buyer

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Setup of repeated posted-price auctions

Equal goods (e.g., ad spaces) are repeatedly offered for sale by a seller to a **single** buyer over T rounds (**the time horizon**).

› The buyer holds a private **fixed** valuation $v \in \mathbb{R}_+$ for each of those goods, and v is **unknown** to the seller.

› At each round $t = 1, \ldots, T$, a price p_t is offered by the seller, and an allocation decision $a_t \in \{0,1\}$ is made by the buyer:

$a_t = 0$, when the buyer rejects, and $a_t = 1$, when the buyer accepts.

The seller applies a **pricing algorithm** A that sets prices $\{p_t\}_{t=1}^T$ in response to buyer decisions $\mathbf{a} = \{a_t\}_{t=1}^T$ referred to as a **buyer strategy**.

The price p_t can depend only on $\{a_s\}_{s=1}^{t-1}$ and the horizon T.

Utility of the buyer

$$\mathrm{Sur}_{\gamma_B}(A, \mathbf{a}, v) := \sum_{t=1}^{T} \gamma_B^{t-1} a_t (v - p_t)$$

$\gamma_B \in (0,1]$ is the buyer's discount

Utility of the seller

$$\mathrm{Rev}_{\gamma_S}(A, \mathbf{a}) := \sum_{t=1}^{T} \gamma_S^{t-1} a_t p_t$$

$\gamma_S \in (0,1]$ is the seller's discount

Two-stage game: strategic buyer and the goal of the seller

The seller knows prior distribution of valuations D and the discount γ_B.

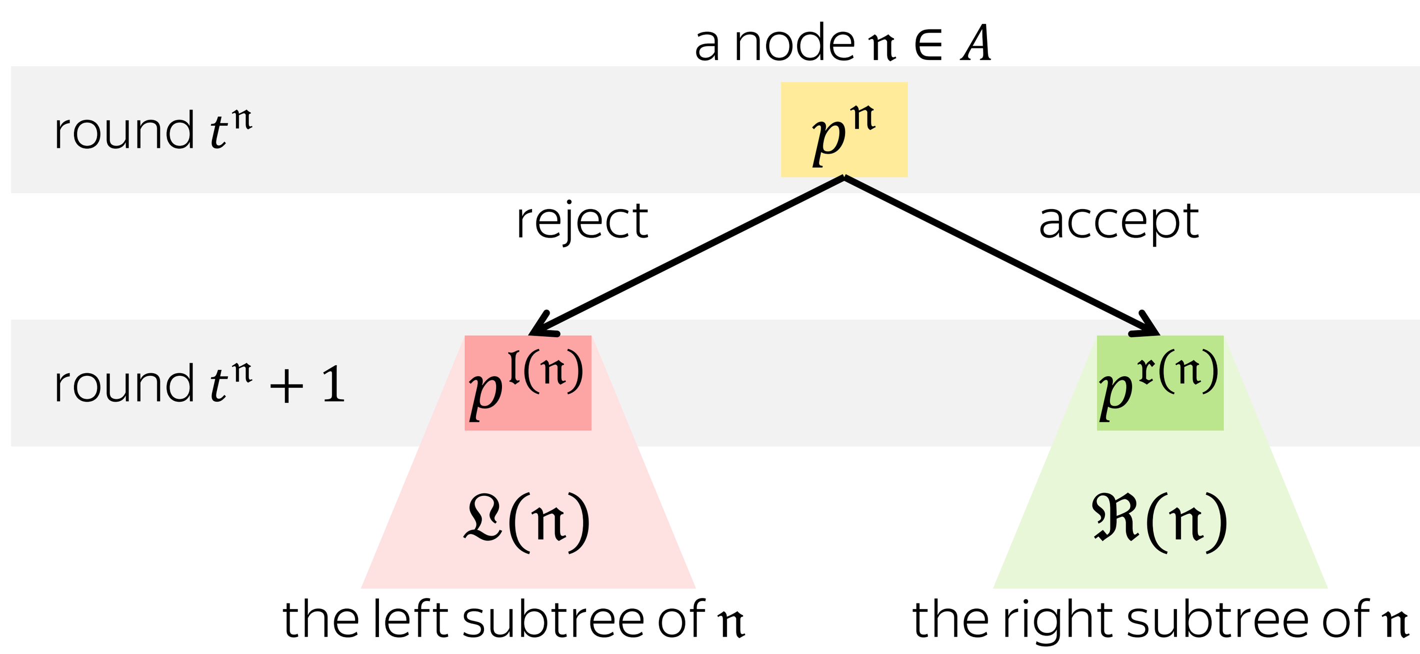
1. The seller announces a pricing algorithm A **in advance**.
2. The buyer selects an optimal strategy $\mathbf{a}^{\mathrm{Opt}}(A, v, \gamma_B) \in \mathrm{argmax}_{\mathbf{a}} \mathrm{Sur}_{\gamma_B}(A, \mathbf{a}, v)$.

The seller seeks for a pricing A^* with maximal expected strategic revenue (ESR):

$$\mathbb{E}_{v\sim D}\left[\mathrm{Rev}_{\gamma_S}\left(A, \mathbf{a}^{\mathrm{Opt}}(A, v, \gamma_B)\right)\right] \to \max_A$$

Dynamic pricing

Pricing algorithm is a labeled complete binary tree



Background: optimal static pricing

Optimal static pricing is s.t. constantly offers a price p^* that maximizes

$$H_D(p) := p\mathbb{P}_{v\sim D}[v \geq p] = p\big(1 - F_D(p)\big),$$

where $F_D(p)$ is CDF of the prior distribution of valuations D.
The price $p^*$is known as the Mayerson price.

Research questions

1. What is the optimal algorithm and its expected strategic revenue?
2. How much more is the maximal ESR than the constant Myerson's one?
3. Can the seller extract expected revenue more than in the static Myerson pricing having limits on computational resources?

Background: equal patience ($\gamma_S = \gamma_B$)

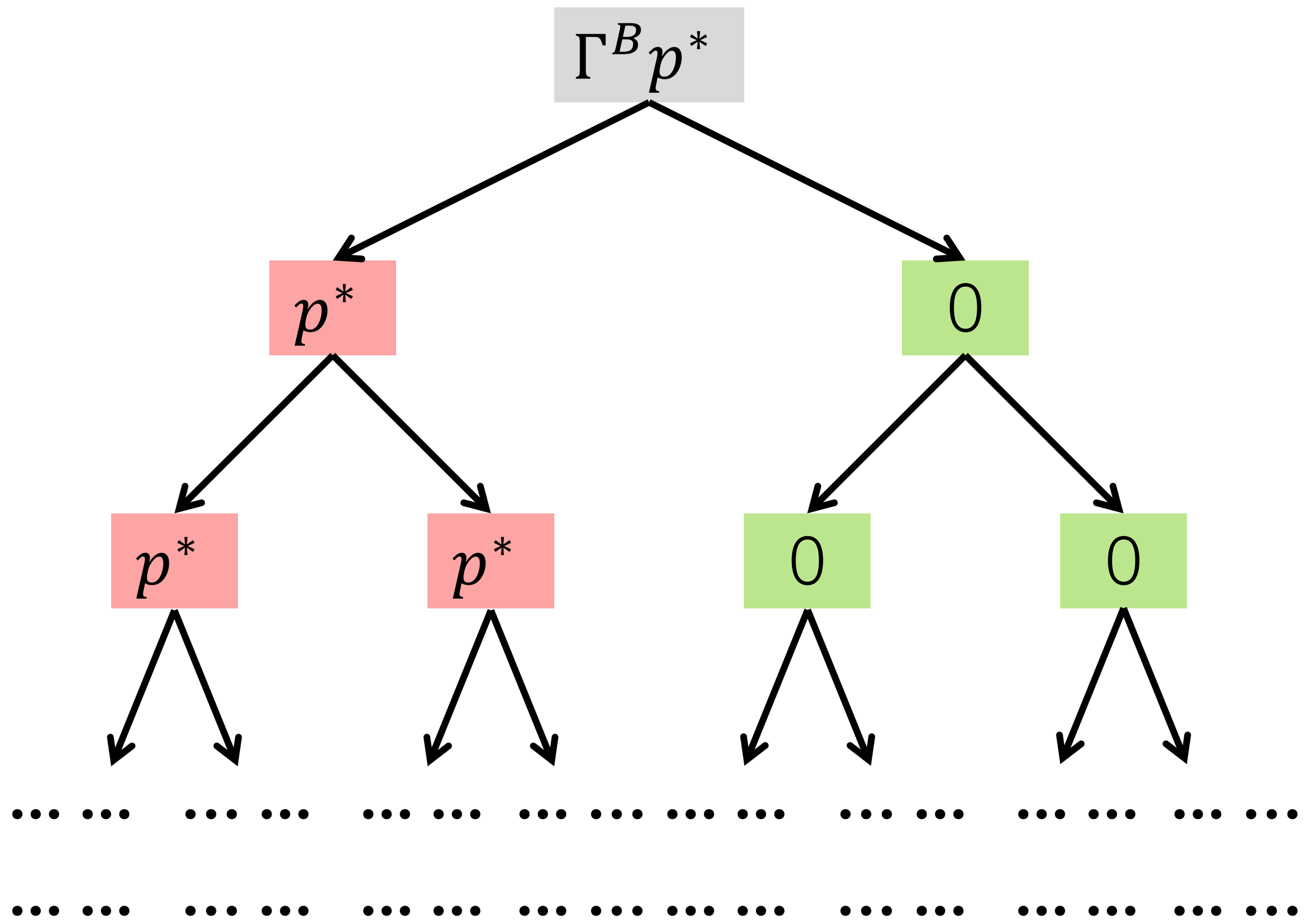
Dynamic pricing cannot help to extract more revenue than optimal static pricing.

Theorem [Devanur et al., 2015]. Let the discount rates be equal: $\gamma_S = \gamma_B = \gamma$. Then the optimal constant pricing A_1^* is optimal among all pricing algorithms and the optimal revenue is $\Gamma p^*\big(1 - F_D(p^*)\big) = \Gamma H_D(p^*)$, where $\Gamma = \sum_{t=1}^{T} \gamma^{t-1}$.

Less patient seller: the case of $\gamma_S \le \gamma_B$

"Bid-deal" algorithm

The seller "accumulates" all her revenue at the first round by proposing the buyer a "big deal" that incentivizes him to pay a large price at the first round and get all goods in the subsequent rounds for free, or, otherwise, get nothing.



Theorem. Let the discount rates be s.t. $\gamma_S \le \gamma_B$. Then the "Big-deal" algorithm A_{bd}^* is optimal among all pricing algorithms and the optimal revenue is $\Gamma^B p^*\big(1 - F_D(p^*)\big)$, where $\Gamma^B = \sum_{t=1}^{T} \gamma_B^{t-1}$.

The constant algorithm A_1^* is no longer optimal

Relative ESR of the optimal algorithm A_{bd}^* w.r.t. the optimal constant one A_1^* is

$$\frac{\Gamma^B}{\Gamma^S} = \frac{\sum_{t=1}^{T} \gamma_B^{t-1}}{\sum_{t=1}^{T} \gamma_S^{t-1}} \ge \mathbf{1}$$

All results of this section hold even for non-geometric discounts such that $\gamma_t^S \le \gamma_t^B$

Less patient buyer: the case of $\gamma_S \ge \gamma_B$

Definition. Let γ be a discount, then an algorithm A is said to be **completely active (CA) for** γ, if for any strategy $\mathbf{a}$ there exists a valuation $v \in \mathbb{R}_+$ s.t. $S_{\mathbf{a}}(v) = S(v)$, where $S_{\mathbf{a}}(u) := \mathrm{Sur}_\gamma(A, \mathbf{a}, u)$ and $S(u) := \mathrm{Sur}_\gamma\big(A, \mathbf{a}^{\mathrm{Opt}}(A, u, \gamma), u\big)$, i.e., the surplus function $S_{\mathbf{a}}$ (as a line) is tangent to the optimal surplus function S.

Proposition. Let $\gamma_S \ge \gamma_B$. Then optimal algorithm A^* can be found among completely active algorithms for γ_B.

The fundamental property of a CA algorithm:
it bijectively corresponds to the break (discontinuity) points $\{v_1, \dots, v_k\}$ of the derivative of its surplus function $S(\,)$, which is piecewise linear, $k = 2^T - 1$.

These points allow easily parametrize the expected strategic revenue:

$$\mathbb{E}_{v\sim D}\left[\mathrm{Rev}_{\gamma_S}\left(A, \mathbf{a}^{\mathrm{Opt}}(A, v, \gamma_B)\right)\right] = \sum_{i=1}^{k} \big(F_D(v_{i+1}) - F_D(v_i)\big)\mathrm{Rev}_{\gamma_S}\big(A, \mathbf{a}^i\big),$$

where $\mathrm{Rev}_{\gamma_S}(A, \mathbf{a}^i)$ can be linearly expressed in terms of the algorithm prices and, thus, in terms of the break points $\{v_1, \dots, v_k\}$.
Let

- $\boldsymbol{\Xi}_{T,\gamma_S,\gamma_B}$ be the $k\times k$ matrix that encodes these linear transformations
- $1 - \mathbf{F}_D(\mathbf{v}) = \{1 - F_D(v_i)\}_{i=1}^{k} \in \mathbb{R}^k$
- $\Delta^k = \{u_1, \dots, u_k \mid 0 \le u_1 \le \cdots \le u_k\} \subset \mathbb{R}^k$

Optimize the multidimensional bilinear-like functional

$$L_{D,\gamma_S,\gamma_B}(\mathbf{v}) := \big(1 - \mathbf{F}_D(\mathbf{v})\big)^{\mathrm{T}} \boldsymbol{\Xi}_{T,\gamma_S,\gamma_B} \mathbf{v}, \qquad \mathbf{v} \in \Delta^k$$

Properties of the functional:

- the functional is continuously differentiable as many times as the CDF F_D
- its derivatives have simple form and can be easily
- the domain Δ^k is convex and has a simple form of simplex
- the matrix $\boldsymbol{\Xi}_{T,\gamma_S,\gamma_B}$ is positive definite on the domain Δ^k

Hence, a variety of gradient methods can be used to find the solution.

When optimal break points $\{v_1, \dots, v_k\}$ are found, use them to find optimal algorithm prices (they linearly depend on the break points).

All results of this section hold even for non-geometric discounts such that $\gamma_{t+1}^S/\gamma_t^S \ge \gamma_{t+1}^B/\gamma_t^B$

Take away

If patience is not equal, dynamic pricing can help the seller boost expected revenue

- If the seller is less patient, get payments for all goods upfront in the first round
- If the seller is more patient, optimize a multidimensional functional of bilinear-like form

This functional is a multivariate analogue of the one used to determine Myerson's price and can be used

1. to find an optimal dynamic pricing, i.e., by efficient gradient-based methods;
2. to construct an optimal low-dimensional approximation to improve revenue even in the game with a large horizon.

Efficient approximations and optimal pricing algorithms with constraints (in the case of $\gamma_S \geq \gamma_B$)

Approximation by optimal τ-step pricing algorithm

The functional $L_{D,\gamma_S,\gamma_B}(\mathbf{v})$

- does not help for games with infinite horizon;
- suffers from dimensional complexity (the number of variables is $2^T - 1$).

Definition. An algorithm A is said to be a τ-step pricing algorithm, if it is constant from the τ-th round on (i.e., prices in the rounds $\tau+1,\dots,T$ are equal to the price offered in the round τ).

The optimal τ-step pricing algorithm can be found by means of optimization of the functional $L_{D,\gamma_S,\gamma_B}(\mathbf{v})$ for a reduced game with horizon τ and the discount factors $\hat{\gamma}_\tau^S = \sum_{t=\tau+1}^T \gamma_S^{t-1}$ and $\hat{\gamma}_\tau^B = \sum_{t=\tau+1}^T \gamma_B^{t-1}$ at the τ-th round.

Let $\mathfrak{A}$ denote the set of all algorithms and $\mathfrak{A}_\tau$ be the subset of τ-step algorithms.

Proposition. Let the discount rates be s.t. $\gamma_S \geq \gamma_B$ and let

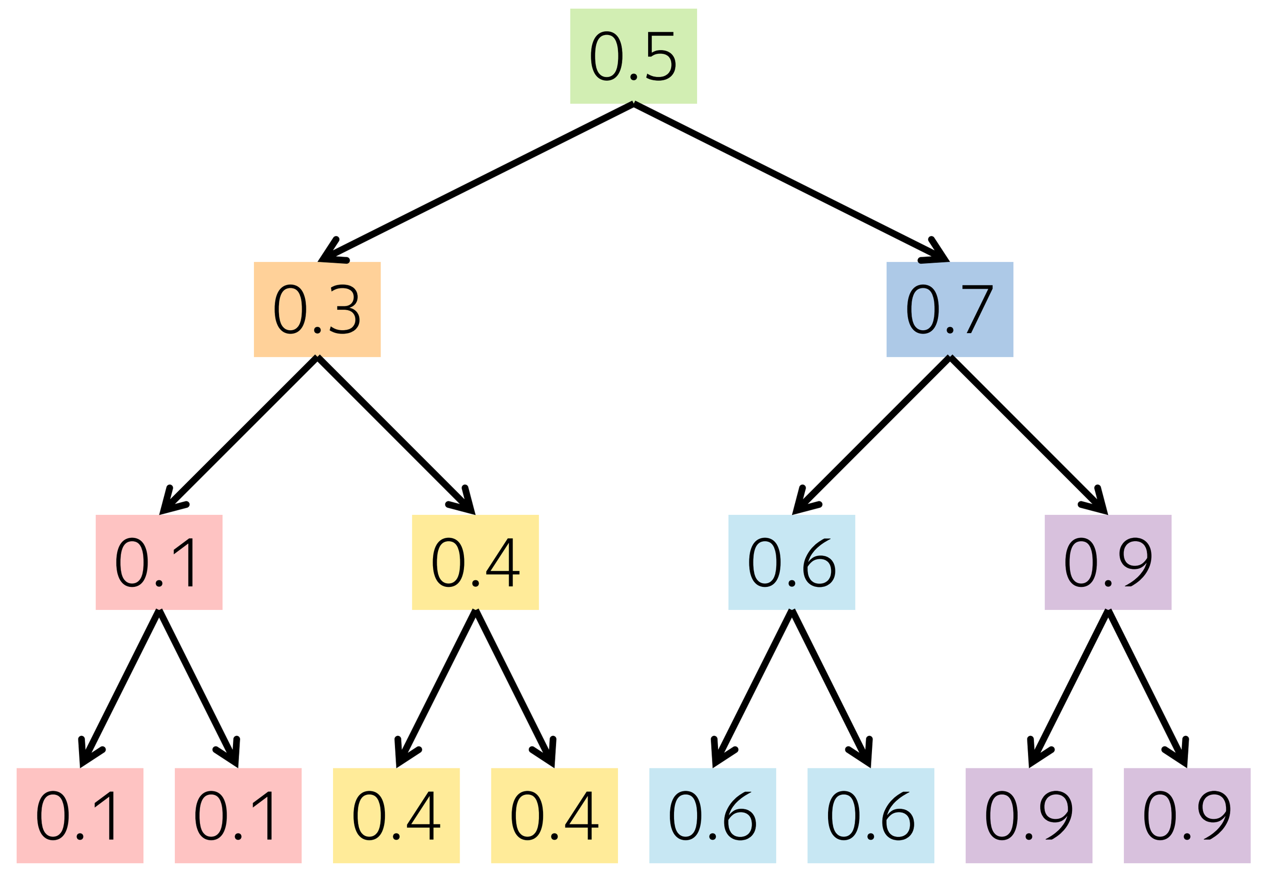
- $\mathrm{OPT} = \max_{A\in\mathfrak{A}} \mathbb{E}_{v\sim D}\left[\mathrm{Rev}_{\gamma_S}\left(A, \mathbf{a}^{\mathrm{Opt}}(A,v,\gamma_B)\right)\right]$ be the maximal ESR;
- $\mathrm{OPT}_\tau = \max_{A\in\mathfrak{A}_\tau} \mathbb{E}_{v\sim D}\left[\mathrm{Rev}_{\gamma_S}\left(A, \mathbf{a}^{\mathrm{Opt}}(A,v,\gamma_B)\right)\right]$ be the maximal ESR in the class of τ-step pricing algorithms.

Then

$$\mathrm{OPT}_\tau \leq \mathrm{OPT} \leq \mathrm{OPT}_\tau + \sum_{t=\tau+1}^{T} \gamma_S^{t-1}\, \mathbb{E}_{v\sim D}[v].$$

The parameter τ allows the seller make a trade-off between:
the achievable fraction of OPT and the computational complexity.

Example: a 3-step pricing algorithm



Optimal algorithms with constraints

One more structural insight of our functional $L_{D,\gamma_S,\gamma_B}(\mathbf{v})$:

- Optimization over the set of break points $\{v_1,\dots,v_k\}$ of the derivative of the surplus envelope $S(\,)$ allows to find optimal algorithms with constraints that can be expressed in terms of these break points.
- E.g., the seller is able to control the probability of buyer usage of each strategy $\mathbf{a}^i$ through a constraint on $F_D(v_{i+1}) - F_D(v_i)$ (e.g., setting it to 0).

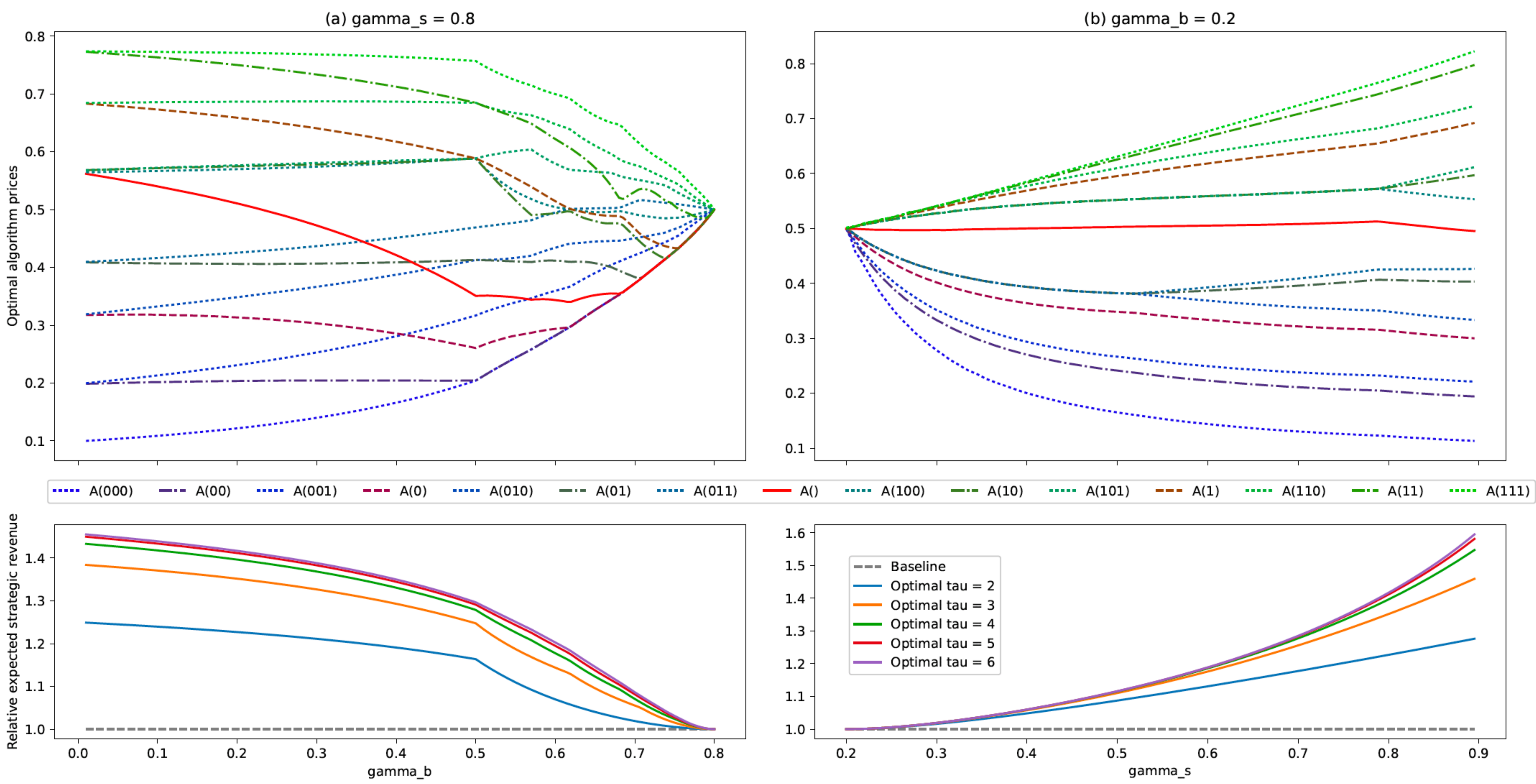
Example: the seller is looking for an algorithm s.t. strategies active with positive probability are monotone, i.e. of the form (reject n rounds, accept remaining $\mathrm{T}-n$ rounds) for some $n \leq T$. Hence, if $\mathbf{a}^i$ is not monotone, then $v_i = v_{i+1}$, i.e. the line $S_{\mathbf{a}^i}$ is tangent to the envelope S in only one point. To find an optimal algorithm among those for which $v_i = v_{i+1}$, one needs slightly update the functional L_{D,γ_S,γ_B}: replace i-th and $(i+1)$-th rows in the matrix $\Xi_{T,\gamma_S,\gamma_B}$ by their sum, do the same with i-th and $(i+1)$-th columns, and remove i-th components from the vectors $1 - \mathbf{F}_D(\mathbf{v})$ and $\mathbf{v}$. The modified optimization functional for the problem with constraints will have $\mathrm{T}+1$ variables since it is equal to the number of strategies that are active with positive probability.

Numerical experiments (in the case of $\gamma_S \geq \gamma_B$)

We find
- optimal τ-step algorithms $A^*_\tau, \tau = 2, \ldots, 6$;
- in infinite games $T = \infty$;
- with the valuation v uniformly distributed in $[0,1]$, i.e., $F_D(v) = v$ (experiment results for exponential and beta distributions are similar).

The baseline
Expected revenue of the optimal static pricing: $H_D(p^*)\Gamma^S$, where $\Gamma^S = \sum_{t=1}^{T} \gamma_S^{t-1}$.

Figure contains
- **@ the top:** prices of the optimal 4-step algorithm A^*_4 for all nodes (prices are denoted by $A(\mathbf{n})$, $\mathbf{n}$ is a node encoded by a string of 0 and 1);
- **@ the bottom:** the relative expected strategic revenue of A^*_τ (w.r.t. A^*_1);
- **@ the left:** for $\gamma_S = 0.8$ and different $0.01 \leq \gamma_B \leq 0.8$;
- **@ the right:** for $\gamma_B = 0.2$ and different $0.2 \leq \gamma_S \leq 0.995$.

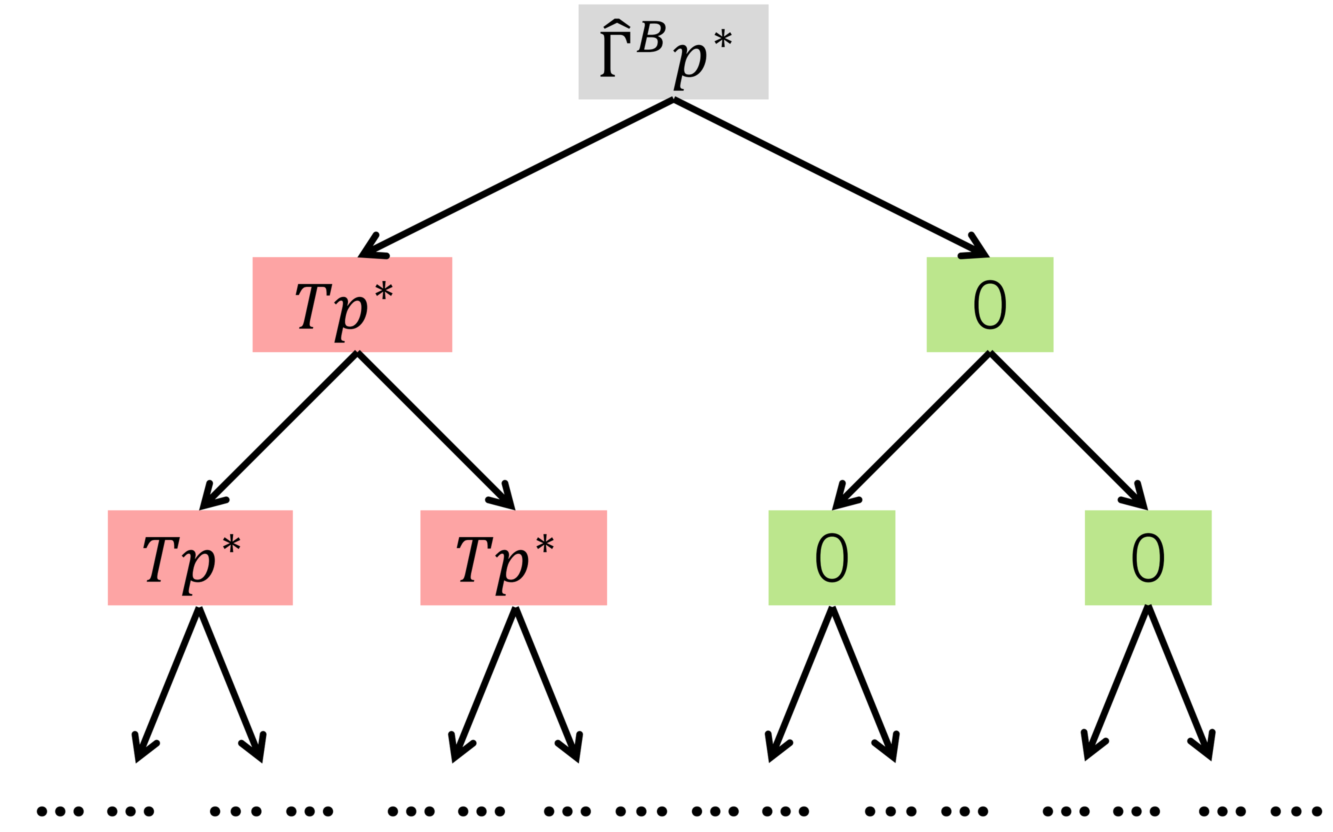


We observe that
- The constant pricing A^*_1 is not optimal: τ-step algorithms are better.
- Significant boost of revenue can be obtained even when the minimal possible step aside from the constant pricing is made: e.g., change dynamically the price only after the first round and get+20% to revenue.
- Expected revenue of A^*_τ converges quite quickly to the optimal one.
- If $|\gamma_S - \gamma_B| \to 0$, then the optimal pricing A^* converges to the optimal constant one A^*_1 (empirically showing that H_D is a special case of L_{D,γ_S,γ_B}).

Incomplete information about buyer discount

Case (1): the seller knows only a lower bound $\hat{\gamma}_B$ s.t. $\gamma_S < \hat{\gamma}_B \leq \gamma_B$

The "Big-deal" algorithm can be useful in this case as well! Just change it slightly:

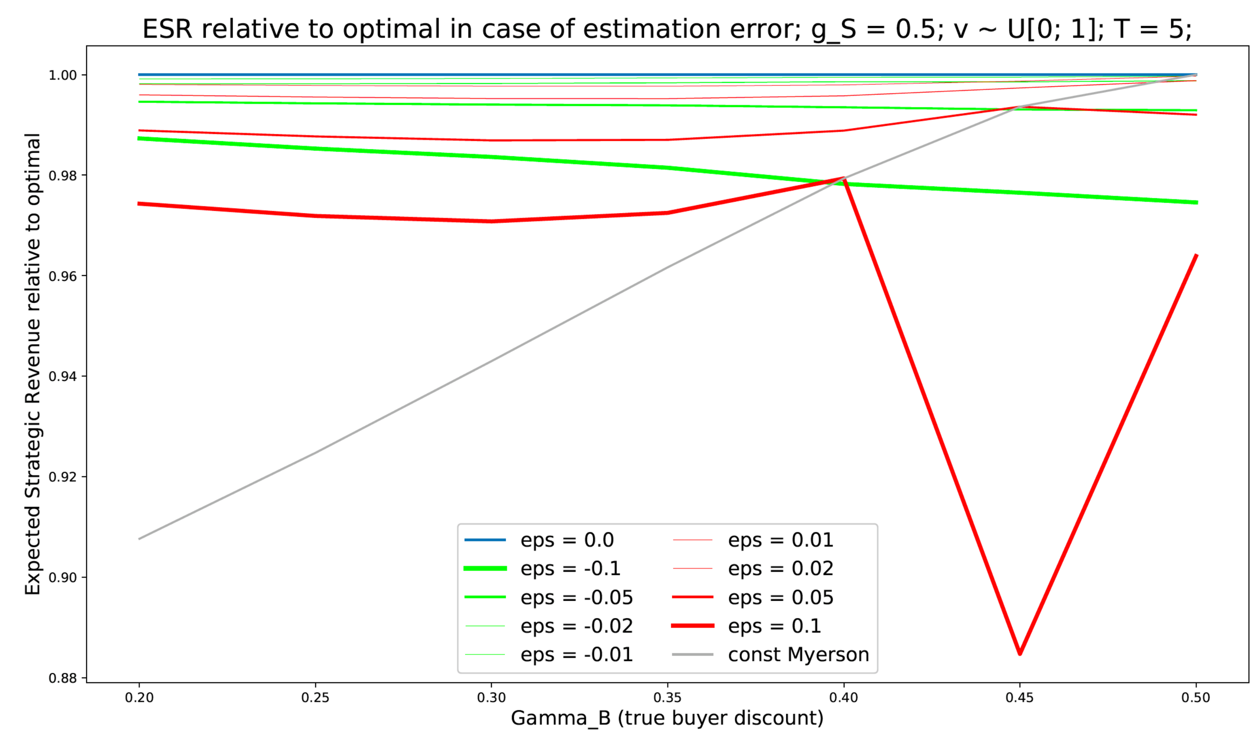


Theorem. Let the discount rates be s.t. $\gamma_S < \gamma_B$ and the seller knows only a lower bound $\hat{\gamma}_B$ s.t. $\gamma_S < \hat{\gamma}_B \leq \gamma_B$. Then the expected strategic revenue of the "Big-deal" algorithm as depicted above is at least $\hat{\Gamma}^B p^*(1 - F_D(p^*))$, where $\hat{\Gamma}^B = \sum_{t=1}^{T} \hat{\gamma}_B^{t-1}$. This revenue is better than the optimal constant one:

$$\mathrm{OPT}_1 = \Gamma^S p^*(1 - F_D(p^*)) < \hat{\Gamma}^B p^*(1 - F_D(p^*)) \leq \Gamma^B p^*(1 - F_D(p^*)) = \mathrm{OPT}$$

Case (2): the seller uses inexact $\gamma_{B,\varepsilon} = \gamma_B + \varepsilon$ and optimizes $L_{D,\gamma_S,\gamma_{B,\varepsilon}}$

The seller uses the functional $L_{D,\gamma_S,\gamma_{B,\varepsilon}}$ to find an optimal algorithm, assumes buyer's discount is $\gamma_{B,\varepsilon} = \gamma_B + \varepsilon$, but faces a buyer with true discount γ_B. We evaluate the loss in revenue by the following numerical experimentation: $T = 5$, $\mathrm{v} \sim \mathrm{D} = \mathrm{Uniform}[0,1]$, and $\gamma_S = 0.5$ (different sets of parameters give qualitatively the same results).



In figure: (a) the expected strategic revenue of this seller is divided by the ESR of a well-informed seller (i.e. s.t. $\varepsilon = 0$); (b) the ESR of Myerson's constant pricing.

We observe that
- if ε is small enough (for $\varepsilon = 0.02$, or $\varepsilon \geq 4\%$ of γ_B), then the seller is still able to extract over 99% of the optimal ESR;
- even if ε is very large (for $\varepsilon = 0.1$,or $\varepsilon \geq 20\%$ of γ_B), the seller is still able to extract over 97% of the optimal ESR for most cases ($\gamma_B \leq 0.4$);
- if the seller is able to just separate γ_B of γ_S with a decent margin, then she is able to gain extra revenue (w.r.t. the optimal constant pricing).