Mobile Robot Kinematics: Part 1

Adapted from "Principles of Robot Autonomy" by D. Gammelli, J. Lorenzetti, K. Luo, G. Zardini, and M. Pavone

January 8, 2026

Source Material

Covering the following sections from “Principles of Robot Autonomy”:

1.1:

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- 1.1:
  - 1.1.1 covered lightly (ignoring linearity and time-invariance)

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  - 1.1.2 covering only Example 1.1.3

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Many robots are subject to *kinematic constraints* which are crucial to understand.

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A state space model is a framework to describe a system (e.g. a moving robot) evolving over time.

State

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Note: $\mathbf{x}$ is a vector, whereas x is a scalar.

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Mobile robot

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

...with linear (v) and angular (ω) speeds

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \theta \\ v \\ \omega \end{bmatrix}$$

...with landmarks

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \theta \\ l_{1,x} \\ l_{1,y} \\ l_{2,x} \\ l_{2,y} \\ \vdots \end{bmatrix}$$

Arm with 6 joint angles

$$\mathbf{x} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_6 \\ \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_6 \end{bmatrix}$$

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The process of inferring the state from the system's output is known as *state estimation*.

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Observations equations have this form:

$$\mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t))$$

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$D(t)$ is the direct matrix ($q \times m$).

If $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are actually time-invariant, we can express the model more compactly,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t).\end{aligned}$$

We will often leave off the "(t)" even when it applies (e.g. $\mathbf{x}$ is always going to be a function of time, otherwise we don't need the model).

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u}\\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}.\end{aligned}$$

Example: Double-Integrator

Consider a mass in space (i.e. no friction, no gravity). We can push the mass in a fixed direction, through an input force u.

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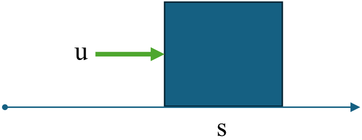

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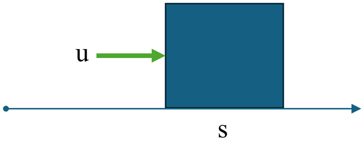

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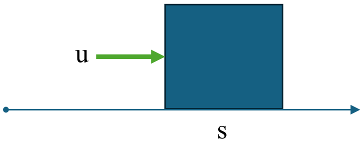

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To recover s we would need to integrate $\ddot{s}$ twice, hence this is referred to as a double-integrator system.

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We can now write this in matrix form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

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