Mobile Robot Kinematics: Part 2

Adapted from "Principles of Robot Autonomy" by D. Gammelli, J. Lorenzetti, K. Luo, G. Zardini, and M. Pavone

January 8, 2026

Source Material

Covering the following sections from "Principles of Robot Autonomy":

1.2:

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- 1.2:
  - 1.2.1 - 1.2.4 covered fully

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Both perspectives are necessary to accurately describe a robot's motion. However, we might choose to focus on only one to yield a simpler model.

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Solving dynamics equations is typically much slower than kinematic equations. Dynamics equations often require more knowledge of parameters (e.g. mass, centre-of-mass, friction coefficients, ...).

When Dynamics CANNOT be Ignored

Kinematics-only if...	Incorporate Dynamics if...
Slow/Moderate speed	High-speed maneuvers
Heavy-duty motors	Limited torque/underactuated
Wheeled/Flat ground	Legged or Aerial robots
Rigid environment	Soft or slippery terrain

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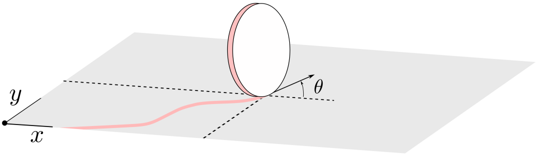
The generalized coordinates may be the same as the full system state $\mathbf{x}$ or they may represent a subset of the state (e.g. the state may include the generalized coordinates plus velocities).

Example: A Wheel



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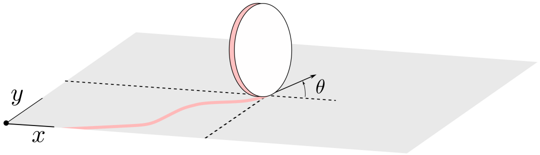

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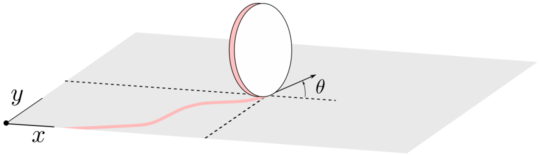

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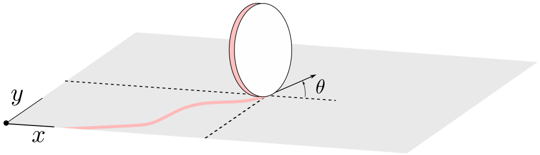

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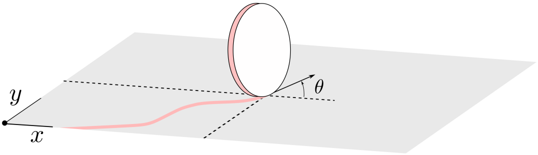

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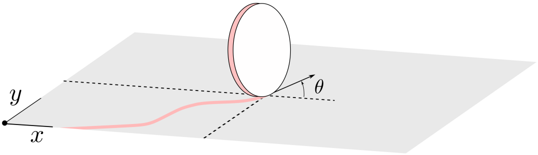

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Kinematic Constraints

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Kinematic constraints are a set of constraints imposed on the generalized coordinates, $\boldsymbol{q}$, and generalized velocities, $\dot{\boldsymbol{q}}$. We express kinematic constraints mathematically as:

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Meanwhile, this example is rather contrived. It would be better to just represent $\boldsymbol{q}$ as x.

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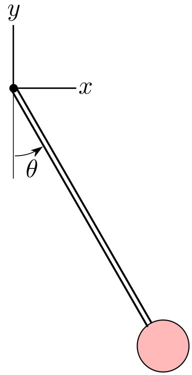

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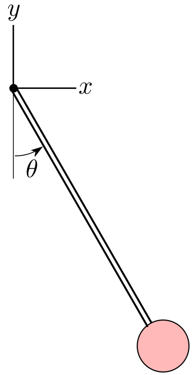

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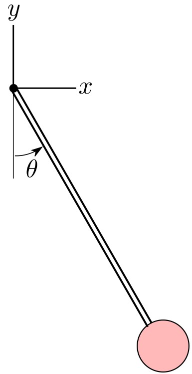

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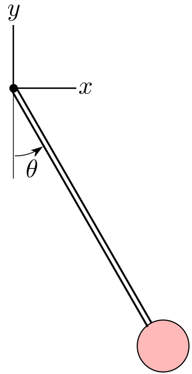

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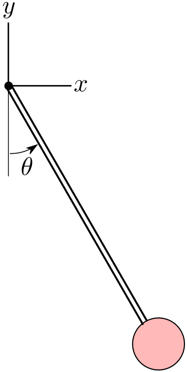

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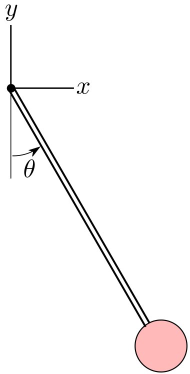

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The no-slip wheel constraint is not integrable, so it does not limit which (x, y) can be reached.

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Any system subject to at least one such constraint is nonholonomic.

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Example: For $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, any scalar multiple of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ lies in the null space because $A \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \mathbf{0}$.

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Any feasible velocity is a combination of those motions.

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Inputs $\boldsymbol{u}$ parameterize allowable motions of dimension $n - k$.

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